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Hung-Ju Chen^a & Ming-Chia Li^b

^a Department of Economics, National Taiwan University, Taipei, Taiwan

^b Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan

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Human capital externality and chaotic equilibrium dynamics

Hung-Ju Chen^{a*} and Ming-Chia Li^b

^aDepartment of Economics, National Taiwan University, Taipei, Taiwan; ^bDepartment of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan

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This study develops a two-period overlapping generations model in which adults undertake educational investment decisions on behalf of young agents. In addition to educational investment, we argue that the accumulation of human capital is also dependent upon the externality from average human capital within the economy. In a departure from the previous literature in this area, we assume that there is a reduction in the overall productivity of human capital accumulation brought about by human capital externality, and show that complicated dynamics will emerge under this circumstance. In addition to displaying the chaotic dynamics in the sense of Li and Yorke, we also verify the existence of Devaney's chaos and Smale's chaos.

Keywords: chaotic dynamics; externality; human capital accumulation

1. Introduction

There has been increasing interest over recent years surrounding the study of chaotic behaviour during the overall process of economic development. Utilizing a standard neoclassical model with capital accumulation in order to investigate the possibility of complicated dynamics, Day [1,2] showed that chaotic trajectories would emerge under certain conditions on savings and productivity, and indeed, chaotic motion is an important element in the study of economic development because it suggests that future economic performance cannot be predicted from a prior developmental pattern. As such, the tiniest of differences between two initial conditions will result in very different trajectories.

Following Day's consideration of a model with a negative capital externality in [1,2],¹ Boldrin *et al.* [3] subsequently went on to develop a two-sector endogenous growth model with positive capital externality which demonstrated that chaotic equilibrium will exist within such an environment. Adopting the model in [3] as an example, Mitra [4] provided a sufficient condition for topological chaos which is applicable to endogenous models when the Li–Yorke criterion in [5] is not satisfied.

Although the growth literature relating to chaotic dynamics has tended to focus on models with capital accumulation, Lucas [6] and Becker *et al.* [7] argued that human capital also plays an important role in economic growth. Our aim in this article is therefore to develop a model with human capital accumulation within which complex behaviour will be an inherent factor. We develop a two-period overlapping generations (OLG) model in

*Corresponding author. Email: hjc@ntu.edu.tw

which adults make educational investment decisions on behalf of young agents, and argue that in addition to educational investment, the accumulation of human capital is also dependent upon the externality from average human capital within the economy. The average human capital here refers to the common knowledge and information that exists within the economy. Notice that we do not distinguish between terminologies relating to knowledge, information and human capital in this article; however, Dasgupta and David [8] noted that knowledge is the product of research, and that information is the codification of such knowledge.

Galor and Tsiddon [9] set up an accumulation function of human capital with (local) home environment and (global) technological externalities in order to study the linkage between economic growth and income distribution. Their work captured local externalities in the form of educational investment and parental human capital, whereas the global externality was represented by the average human capital within the economy. Positive global externality was assumed by introducing a non-decreasing function of average human capital. Such a positive externality, from average human capital to human capital accumulation, was also used by De la Croix and Doepke [10,11] in order to study the role of differential fertility during the overall process of economic growth.²

Although a high level of average human capital can contribute to the overall accumulation of human capital, we argue that it can also cause a reduction in the level of productivity of human capital accumulation. Stephan [12, Subsection 9A] argued that in discovery, excessive knowledge is a bad thing because it ‘encumbers’ researchers. Moreover, when there is an increase in the average human capital, people need to spend more time on examining and digesting the knowledge or information they receive. Since the invention of the internet, people nowadays can get information and knowledge easily and even instantly. However, too much information and knowledge may cause the problem of ‘information pollution’.³ Also, the wrong information and knowledge may be widespread through the internet and will lower the individual’s human capital if one accepts the wrong information or knowledge.⁴ Hence, in contrast to the previous literature on endogenous growth theory, we assume that with a rise in average human capital, there will be a corresponding increase in the overall degradation of human capital accumulation productivity.

Numerous works have concentrated on the study of the complicated dynamics that are present in OLG models. Based upon the assumption that children inherit their consumption tastes from their parents, De la Croix [16] showed that this would generate endogenous oscillations. Nishimura and Shimomura [17] extended the model in [16] to a trade model and went on to argue that such child–parent externalities, along with international trade, will together generate chaos. Chen and Li [18] and Chen *et al.* [19] showed that for OLG models, expectation formation is an important determinant to the occurrence of chaos. Medio and Negroni [20] examined the complicated dynamics that occurred in a two-dimensional OLG model with production function, whereas Yokoo [21] subsequently proposed a two-dimensional OLG model with government debt.

When analysing the one-dimensional dynamical system, the Li–Yorke criterion has been the general focus of the literature studying chaotic motion in economic models. Day [1] and Boldrin *et al.* [3] examined Li–Yorke chaos in growth models. Complex dynamics can also easily arise in a monetary economy. Auray *et al.* [22] also used the Li–Yorke criterion to examine the presence of chaotic motion in a ‘cash-in-advance’ model with habit persistence based upon *catching up with the Joneses* literature. A further contribution of this article is that, in addition to demonstrating chaotic dynamics in the sense of Li and Yorke, we also use the first-order nonlinear difference equation in human

capital, generalized by our model, to verify the presence of Devaney's chaos and Smale's chaos.

The remainder of this article is organized as follows. The next section presents a simple model with human capital accumulation. We then derive the law of motion of human capital in an economy occupied by homogeneous agents. In Section 3, we show that with human capital externality, the chaotic equilibrium dynamics in the sense of Li and Yorke will present. Besides the Li–Yorke chaos, we also illustrate the presence of Devaney's and Smale's chaos in Section 4. A numerical example is also given in this section. The final section provides the conclusions drawn from this study.

2. The model

We adopt an infinite-horizon, discrete time OLG model within which agents live for two periods, corresponding to childhood (young agents) and adulthood (old agents). Each adult gives birth to a single child, there is no population growth, and we normalize the population size to one. Earnings for an adult are equal to his/her *human capital*, h_t .

2.1. Schools

We assume that parents make educational investment decisions q_t for their children,⁵ that is they decide how much tuition they want to pay. Moreover, we make the following 'assumptions of schools' (AS):

- (AS1) For any level of educational expenditure chosen by a parent, there always exists a school charging the tuition at the same amount to accept his/her child.
- (AS2) A school will charge the same tuition fees for all types of students.
- (AS3) Schools earn zero profit.
- (AS4) School quality is measured by its expenditure per student.

(AS1) implies that every young agent has a school to attend. (AS2) indicates that there is no price discrimination amongst students within a school; therefore, private schools can be perfectly segregated by their tuition fees.⁶ (AS3) along with (AS4) illustrate that school quality can be represented by its tuition.

2.2. Human capital accumulation function

What makes this article different from the previous literature of human capital is the accumulation function of the human capital. The key to the model is that there is a negative externality of the average human capital when forming human capital accumulation. The justification given for this is the remark of Stephan [12, p. 1220] in discovery:

Fourth, there is anecdotal evidence that 'too' much knowledge can be a bad thing in discovery in the sense that it 'encumbers' the researcher. There is the suggestion, for example, that exceptional research may at times be done by the young because the young 'know' less than their elders and hence are less encumbered in their choice of problems and in the way they approach a questions.

Besides, a negative externality of the average human capital may arise when some of knowledge/information is wrong. One good example is the problem of the information

quality on the internet. The invention of the internet provides another way for agents to exchange information and to accumulate human capital. However, the discovery of the internet makes users overwhelmed with information. When there is too much information, agents need to spend lots of time on digesting it, not to mention that some of the information might be wrong and the wrong information would lower the human capital accumulation. During his interview by the BBC [14], Jakob Nielsen mentioned that:

the entire ideology of information technology for the last 50 years has been that more information is better, that mass producing information is better. But the net is now so much a machine with all the answers instantly, it has mutated into 'procrastination apparatus' which spews information without much prioritisation.

He defined 'information pollution as information overload taken to the extreme' and likened it to pollution in the physical environment. He also argued that 'information pollution can become an impediment to your ability to get your work done'. This indicates that information pollution may reduce working productivity.

We then incorporate the ideas of Stephan [12] and Nielsen [14] when forming the human capital accumulation function. One can think that there is a saturation level of the average human capital representing the situation in which there is so much knowledge/information within the economy that people cannot distinguish between right and wrong information/knowledge and hence an increase in the average human capital contributes nothing to (or even reduces) the accumulation of human capital.

Following the literature, we assume that human capital is accumulated according to a Cobb–Douglas learning technology:⁷

$$h_{t+1} = Aq_t^\eta H_t^\delta (\mu - H_t)^\beta, \quad (2.1)$$

where $A > 0$ is the productivity of human capital accumulation, q_t is the educational investment, H_t is the average human capital for the society, $\mu > 0$ is the saturation level of H_t . The parameters $\eta, \delta, \beta \in [0, 1]$ are the corresponding elasticity of q_t, H_t and $(\mu - H_t)$ to future human capital.

Equation (2.1) is a variation of the traditional human capital accumulation function by including the externality term. As we have explained in the introduction, there is a saturation level of the average human capital H_t in economy, which is represented by μ . When H_t is much less than μ , we have that higher average human capital is beneficial for the accumulation of human capital by the factor H_t^δ . By contrast, when H_t is close to μ , the benefit no longer exists. The factor $(\mu - H_t)^\beta$ represents the impact of reduction in the productivity of human capital accumulation caused by excessive information. As H_t approaches μ , this factor approaches zero. Thus, as H_t becomes larger the human capital h_{t+1} accumulates at a lower rate. If H_t is large, the human capital declines. We restrict all factors devoted to the accumulation of human capital to exhibit diminishing or constant returns to scale.

2.3. The maximization problem for households

We assume parents care about their consumption c_t and their children's human capital h_{t+1} . All agents have the same utility function over their life cycle, which is:

$$\ln c_t + \omega \ln h_{t+1}, \quad (2.2)$$

where $\omega > 0$ represents preference towards human capital.

Parents need to decide how to allocate their income between consumption and educational investment for their children. Hence, the budget constraint for adults is

$$c_t + q_t = h_t. \tag{2.3}$$

2.4. Equilibrium

Given h_1 , an equilibrium comprises sequences of individual human capital stock $\{h_t\}_{t \geq 1}$, the average human capital stock $\{H_t\}_{t \geq 1}$, and individual decisions $\{c_t, q_t\}_{t \geq 1}$ such that:

- (1) the household maximization problem is solved by $\{c_t, q_t\}_{t \geq 1}$, maximizing the utility function subject to Equations (2.1) and (2.3); and
- (2) young agents will accumulate human capital following Equation (2.1).

2.5. The law of motion of human capital

It is easy to see that for the maximization problem, the optimal choice of educational investment that parents choose for their children is

$$q_t = \frac{\eta\omega}{1 + \eta\omega} h_t. \tag{2.4}$$

Equation (2.4) shows that investment in education increases along with any increase in parental human capital and is a linear function of h_t . By substituting Equation (2.4) within the human capital accumulation function of Equation (2.1), we have the law of motion of human capital:

$$h_{t+1} = A \left(\frac{\eta\omega}{1 + \eta\omega} \right)^\eta h_t^\eta H_t^\delta (\mu - H_t)^\beta. \tag{2.5}$$

Equation (2.5) indicates that the human capital accumulation function is an increasing function of parental human capital, all other things being held constant. Under an economy occupied with homogeneous agents, the law of motion of human capital becomes

$$H_{t+1} = A \left(\frac{\eta\omega}{1 + \eta\omega} \right)^\eta H_t^{\eta+\delta} (\mu - H_t)^\beta. \tag{2.6}$$

Equation (2.6) shows that the economy can be represented by a one-dimensional dynamical system in human capital.

3. Chaotic dynamics: Li–Yorke chaos

In this section, we study the dynamical behaviour of the average human capital based on the law of motion of human capital (2.6). Without loss of generality, we may assume that

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$\mu = 1$. Let $\lambda = A\left(\frac{\eta\omega}{1+\eta\omega}\right)^\eta$, $\alpha = \eta + \delta$, and $x = H_t$, then the model (2.6) turns into the family of functions $f_{\lambda,\alpha,\beta} : [0,1] \rightarrow \mathbb{R}$ defined by

$$f_{\lambda,\alpha,\beta}(x) = \lambda x^\alpha(1-x)^\beta, \tag{3.1}$$

where $\lambda > 0$, $0 < \alpha \leq 2$ and $0 < \beta \leq 1$ are real parameters. For simplicity, we write $f = f_{\lambda,\alpha,\beta}$, denote the identity function by f^0 , and inductively define $f^n = f \circ f^{n-1}$ for positive integer n .

Figure 1 indicates that the dynamic behaviour of Equation (3.1) with $\alpha = 2$ and $\beta = 1$ varies from simple dynamics to chaotic dynamics as λ increases.

Before proving the existence of chaotic dynamics, we give elementary analysis on the model (3.1). By calculating the derivative f' , we find that the maximum of f occurs at the critical point $\frac{\alpha}{\alpha+\beta}$. Moreover, f is strictly increasing on $[0, \frac{\alpha}{\alpha+\beta}]$ and strictly decreasing on $[\frac{\alpha}{\alpha+\beta}, 1]$. Hence, $f(\frac{\alpha}{\alpha+\beta}) \leq 1$ if and only if $f([0,1]) \subset [0,1]$. By computing the second derivative f'' , one has that f' is strictly increasing on $[0,a]$ for some $0 < a < 1$ and is strictly decreasing on $[a,1]$ if $\alpha + \beta - 1 > 0$, and f' is strictly decreasing on $[0,1]$ otherwise. By the intermediate value theorem, $f(\frac{\alpha}{\alpha+\beta}) > \frac{\alpha}{\alpha+\beta}$ implies that there exist $0 < p_- < \frac{\alpha}{\alpha+\beta} < p < 1$ such that

$$f(p_-) = f(p) = p. \tag{3.2}$$

3.1. Li-Yorke chaos

Following the article of Li and Yorke [5], we define the Li-Yorke chaos.

Definition 3.1: Let $h : I \rightarrow I$ be a map, where I is an interval. We say that h exhibits Li-Yorke chaos on I if:

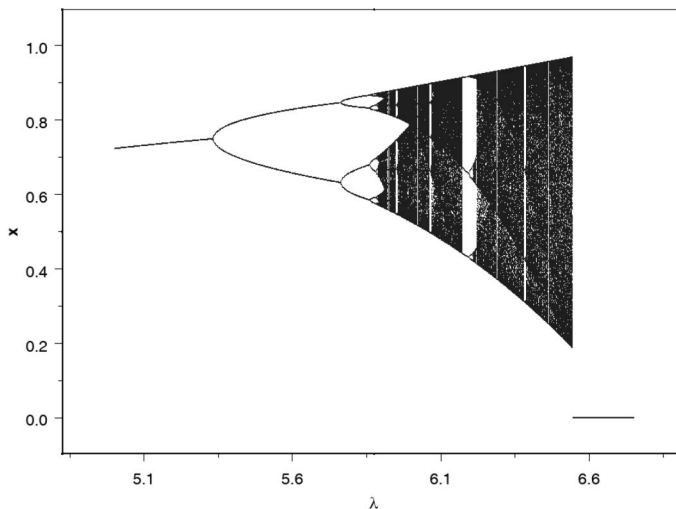


Figure 1. The bifurcation diagram of $f_{\lambda,2,1}(x)$ in λ .

- (1) h has periodic points of all periods; here by a *periodic point p of period n* , we mean that $f^n(p) = p$ and $f^i(p) \neq p$ for $0 < i < n$;
- (2) there exists an uncountable set $S \subset I$ such that

(i) if $x, x, y \in S$ with $x \neq y$ then

$$\limsup_{n \rightarrow \infty} |h^n(x) - h^n(y)| > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} |h^n(x) - h^n(y)| = 0,$$

(ii) if $x \in S$ and $y \in I$ is periodic then

$$\limsup_{n \rightarrow \infty} |h^n(x) - h^n(y)| > 0.$$

We recall some related theorems. The *Li-Yorke Theorem* in [5, Theorem 1] says that any continuous map on an interval with a periodic point of period three exhibits Li-Yorke chaos. A periodic point of period one is also called a *fixed point*. The existence of a fixed point is guaranteed by the well-known *fixed point theorem*: if I is a closed interval, $h : I \rightarrow \mathbb{R}$ is a continuous function, and $h(I) \supset I$, then f has a fixed point in I .

By using the theorems mentioned above, we establish the existence of Li-Yorke chaos for our model for the case when the maximum of f is equal to one.

Theorem 3.2: *Let $f = f_{\lambda, \alpha, \beta}$ be given by Equation (3.1). If $f(\frac{\alpha}{\alpha+\beta}) = 1$, then f has periodic orbits of all periods and exhibits Li-Yorke chaos on $[0,1]$.*

Proof: Let $I_1 = [0, \frac{\alpha}{\alpha+\beta}]$ and $I_2 = [\frac{\alpha}{\alpha+\beta}, 1]$. Because $f(0) = 0$, $f(1) = 0$, and $f(\frac{\alpha}{\alpha+\beta}) = 1$, $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$. Because $f(I_1) \supset I_1$, there is a closed subinterval A_1 of I_1 such that $f(A_1) = I_1$. Because $f(I_2) \supset I_1 \supset A_1$, there is a closed subinterval A_2 of I_2 such that $f(A_2) = A_1$. Again, because $f(I_1) \supset I_2 \supset A_2$, there is a closed subinterval A_3 of I_1 such that $f(A_3) = A_2$. Hence $f^3(A_3) = f^2(A_2) = f(A_1) = I_1 \supset A_3$. Because f^3 is continuous, the fixed point theorem implies that f^3 has a fixed point, namely z , in A^3 . Then $f(z) \in A_2$ and $f^3(z) = z$. Because the common point of I_1 and I_2 is $\frac{\alpha}{\alpha+\beta}$ and $f^3(\frac{\alpha}{\alpha+\beta}) = 0$, $z \neq \frac{\alpha}{\alpha+\beta}$. Therefore, z is a periodic point of period three for f . By the Li-Yorke Theorem, f exhibits Li-Yorke chaos. □

Before the maximum $f(\frac{\alpha}{\alpha+\beta})$ attains the number one, we can have that f^2 exhibits Li-Yorke chaos. Although similar results can be found in [4, Proposition 2.3], our methodology is very different from his. We use the method of interval covering to prove the existence of Li-Yorke chaos for f^2 .

Theorem 3.3: *Let $f = f_{\lambda, \alpha, \beta}$ be given by Equation (3.1), where λ, α, β satisfy $f(\frac{\alpha}{\alpha+\beta}) > \frac{\alpha}{\alpha+\beta}$, and let p_- be given by Equation (3.2). If $f^2(\frac{\alpha}{\alpha+\beta}) \leq p_-$, then f has periodic orbits of all even periods on $[0,1]$ and f^2 exhibits Li-Yorke chaos on $[0,1]$.*

Proof: Let $I_1 = [p_-, \frac{\alpha}{\alpha+\beta}]$ and $I_2 = [\frac{\alpha}{\alpha+\beta}, p_-]$. Because $f^2(p_-) = f^2(p) = p$ and $f^2(\frac{\alpha}{\alpha+\beta}) \leq p_-$, the continuity of f^2 implies that $f^2(I_1) \supset I_1 \cup I_2$ and $f^2(I_2) \supset I_1 \cup I_2$. By the same argument as in the proof of Theorem 3.2, f^6 has a fixed point, namely z , in I_1 , and $f^2(z) \in I_2$. Because $f^2(\frac{\alpha}{\alpha+\beta}) \leq p_-$, $z \neq \frac{\alpha}{\alpha+\beta}$. Therefore, the point z is a periodic

point of period three for f^2 and of period six for f . By the Li–Yorke Theorem, f^2 exhibits Li–Yorke chaos. \square

One can see from Figure 2 that the model $f_{\lambda,\alpha,\beta}$ in Equation (3.1), with $\lambda = 6.5$, $\alpha = 2$ and $\beta = 1$, satisfies the conditions of Theorem 3.3.

If the dynamics of Equation (2.6) exhibits Li–Yorke chaos, then irregular cycles will emerge with the development of economics. Hence, unlike [1] and [3], which concentrated on the possibility of endogenous fluctuations in an economy with capital accumulation, our result enriches this line of studying by showing that it is also likely to obtain Li–Yorke chaos in an economy with human capital accumulation.

4. Other types of chaos

Besides verifying the possibility of the presence of chaos in the sense of Li and Yorke when there is negative externality of human capital, in this section we show that this nonlinear first-order difference equation in human capital can also exhibit Devaney's and Smale's chaos under certain conditions. Although most theoretical studies of complex dynamics in economic models tend to focus on the examination of Li–Yorke chaos due to the mathematical convenience, we provide the other two alternative considerations of chaos. A numerical example is given at the end of the section.

4.1. Devaney's chaos

In his popular textbook, Devaney [29] gives the following definition for chaos.

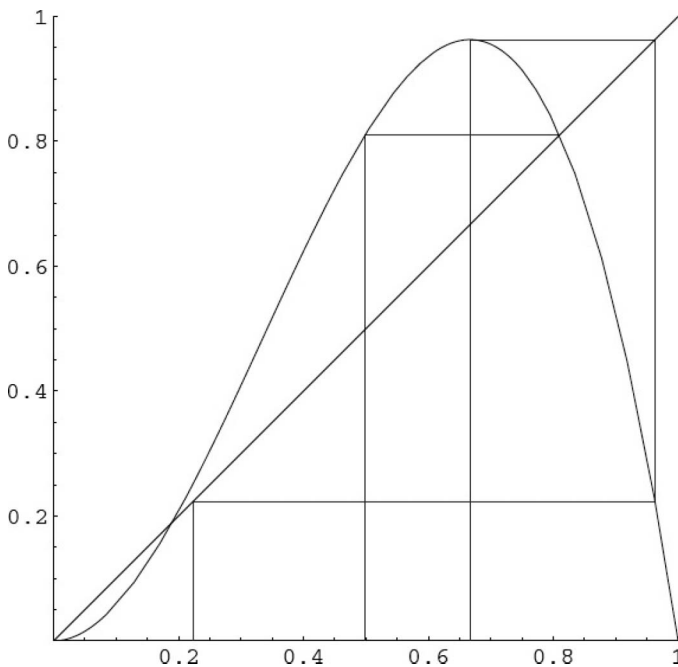


Figure 2. The graph of $f_{6.5,2,1}(x)$.

Definition 4.1: Let $h : I \rightarrow I$ be a map, where I is a closed interval. We say that h exhibits Devaney’s chaos on I if the following conditions are satisfied:

- (1) the set of periodic points is dense in I ;
- (2) the map h is *topologically transitive*, i.e. for any given pair of nonempty open sets U and V in I , there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$; and
- (3) the map h has *sensitive dependence on initial conditions*, i.e. there exists $\alpha > 0$ such that for any $x \in I$ and any $\epsilon > 0$, there are $y \in I$ and $n \in \mathbb{N}$ such that $|x - y| < \epsilon$ and $|h^n(x) - h^n(y)| > \alpha$.

We also need the following definition. For a C^3 map $h : I \rightarrow I$, where I is an interval, the *Schwarzian derivative* of h is defined by

$$S_h(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2} \left(\frac{h''(x)}{h'(x)} \right)^2$$

for $x \in I$ with $h'(x) \neq 0$. By using the chain rule, one has that $S_h < 0$ implies $S_{h^2} < 0$. Thus, we have the following property that

$$\text{if } S_h < 0, \text{ then } S_{h^n} < 0 \text{ for all } n \geq 1. \tag{4.1}$$

Moreover, $S_h < 0$ implies that h' cannot have a positive local minimum or a negative local maximum. Indeed, if c is a critical point of h' , then $h'''(c)/h'(c) = S_h(c) < 0$ and hence $h'''(c)$ and $h'(c)$ have opposite signs. Therefore, by continuity of h' , we have that if $h' \neq 0$ and $S_h < 0$ on $[a, b]$ then for any $x \in (a, b)$,

$$\text{either } h'(x) > \min \{h'(a), h'(b)\} > 0 \text{ or } h'(x) < \max \{h'(a), h'(b)\} < 0. \tag{4.2}$$

Return to our study on the model $f = f_{\lambda, \alpha, \beta}$ in Equation (3.1). Assume $\alpha \leq 1$ and $f(\frac{\alpha}{\alpha+\beta}) = 1$. Then there are $p_- < p$ as defined in Equation (3.2). Because f' is strictly decreasing on $[0, 1]$, the mean value theorem implies that 0 and p are the only fixed points of f . Because $f^2(p_-) = f^2(p) = p$ and $f^2(\frac{\alpha}{\alpha+\beta}) = 0 < p_-$, the intermediate value theorem implies that there exist $p_- < \ell_1 < r_1 < p$ such that

$$f^2(\ell_1) = f^2(r_1) = p_-. \tag{4.3}$$

In fact, such a pair ℓ_1 and r_1 satisfying Equation (4.3) is unique due to the monotonicity of f on $[p_-, \frac{\alpha}{\alpha+\beta}]$ and $[\frac{\alpha}{\alpha+\beta}, 1]$.

For the case when the maximum attains one, we have shown that the model (3.1) exhibits the Li–Yorke chaos in Theorem 3.2. Furthermore, Devaney’s chaos may exist.

Theorem 4.2: Let $f = f_{\lambda, \alpha, \beta}$ be given by Equation (3.1) with $\alpha \leq 1$ and $f(\frac{\alpha}{\alpha+\beta}) = 1$ and let $p_- < \ell_1 < r_1 < p$ be as in Equations (3.2) and (4.3). If $\min\{|f'(p_-)|, |f'(p)|\} > 1$, $\max\{r_1 - \frac{\alpha}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} - \ell_1\} < p_-$ and the Schwarzian derivative $S_f(x) < 0$ for $x \in [0, 1]$, then f exhibits Devaney’s chaos on $[0, 1]$.

Proof: Let $J = [p_-, p] \setminus \left\{ \frac{\alpha}{\alpha+\beta} \right\}$. For $x \in J$, define $\tau(x) = \min\{n \in \mathbb{N} : f^n(x) \in [p_-, p]\}$. Then $\tau(x)$ is well defined. Indeed, let $x \in J$ then $f(x) \in [p, 1)$ and so $f^2(x) \in (0, p]$. Because $f(y) > y$ for all $y \in (0, p_-)$, there exists a positive integer n such that $f^n(x) \in [p_-, p]$.

First, we claim

$$|(f^{\tau(x)})'(x)| > 1 \text{ for all } x \in J. \quad (4.4)$$

For $n \geq 1$, let $I_n = \left\{ x \in \left(\frac{\alpha}{\alpha+\beta}, p \right] : \tau(x) = n \right\}$ and $\hat{I}_n = \left\{ x \in \left[p_-, \frac{\alpha}{\alpha+\beta} \right) : \tau(x) = n \right\}$. Then $J = \bigcup_{n=1}^{\infty} (I_n \cup \hat{I}_n)$, $I_1 = \{p\}$, $\hat{I}_1 = \{p_-\}$, and $|f'(x)| > 1$ for $x \in I_1 \cup \hat{I}_1$. Consider $n \geq 2$. The continuity of f implies that $I_n = [r_{n-2}, r_{n-1})$ and $\hat{I}_n = (\ell_{n-2}, \ell_{n-1}]$ for some $p_- = \ell_0 \leq \ell_{n-2} < \ell_{n-1} < r_{n-2} < r_{n-1} \leq r_0 = p$. It is easy to check that $f^n(r_{n-2}) = p_-$, $f^n(r_{n-1}) = p$, and f^n maps $\left[\frac{\alpha}{\alpha+\beta}, r_{n-2} \right]$ and $[r_{n-2}, r_{n-1}]$ homeomorphically onto $[0, p_-]$ and $[p_-, p]$, respectively. By the mean value theorem applied to f^n on $\left[\frac{\alpha}{\alpha+\beta}, r_{n-2} \right]$ and $[r_{n-2}, r_{n-1}]$, respectively, one gets that there exist $y_n \in \left(\frac{\alpha}{\alpha+\beta}, r_{n-2} \right)$ and $z_n \in (r_{n-2}, r_{n-1})$ such that $(f^n)'(y_n) = \frac{p_- - \frac{\alpha}{\alpha+\beta}}{r_{n-2} - \frac{\alpha}{\alpha+\beta}}$ and $(f^n)'(z_n) = \frac{p - p_-}{r_{n-1} - r_{n-2}}$. Because $\left(\frac{\alpha}{\alpha+\beta}, r_{n-2} \right) \subseteq \left(\frac{\alpha}{\alpha+\beta}, r_1 \right)$, $r_1 - \frac{\alpha}{\alpha+\beta} < p_-$ and $(r_{n-2}, r_{n-1}) \not\subseteq (p_-, p)$, we have $(f^n)'(y_n) \geq \frac{p_- - \frac{\alpha}{\alpha+\beta}}{r_1 - \frac{\alpha}{\alpha+\beta}} > 1$ and $(f^n)'(z_n) > 1$. Because $S_f < 0$, by (4.1) we have $S_{f^n} < 0$ and hence by Equation (4.2) applied to f^n , we obtain $(f^n)'(r_{n-2}) > \min\{(f^n)'(y_n), (f^n)'(z_n)\} > 1$. Because $r_0 = p$ we have $(f^2)'(r_0) > 1$. Thus, inductively we have that for $n \geq 3$,

$$(f^n)'(r_{n-1}) = f'(f^{n-1}(r_{n-1})) \cdot (f^{n-1})'(r_{n-1}) = f'(p_-) \cdot (f^{n-1})'(r_{n-1}) > 1.$$

By Equation (4.2) again, we get that $(f^n)'(x) > \min\{(f^n)'(r_{n-2}), (f^n)'(r_{n-1})\} > 1$ for all $x \in I_n = [r_{n-2}, r_{n-1})$. By using the same argument as above, we have that $(f^n)'(x) < -1$ for all $x \in \hat{I}_n = (\ell_{n-2}, \ell_{n-1}]$. The desired claim follows.

Second, we claim that for every $x \in [0, 1]$ whose orbit does not go through $\frac{\alpha}{\alpha+\beta}$, there exists a positive integer n_x such that

$$|(f^{n_x})'(x)| > 1. \quad (4.5)$$

For $x \in J$, claim (4.5) follows Equation (4.4) by taking $n_x = \tau(x)$. Next, we consider $x \in [0, p_-]$. Because $S_f < 0$ and $f' > 0$ on $[0, p_-]$, by Equation (4.2) we have that $f'(x) \geq \min\{f'(0), f'(p_-)\} > 1$ for all $x \in [0, p_-]$. Thus, Equation (4.5) holds for $x \in [0, p_-]$ by taking $n_x = 1$. Finally, consider $x \in [p, 1]$. Then $f(x) \in [0, p]$ and so the above result implies that $|(f^{m_x})'(f(x))| > 1$ for some integer $m_x \geq 1$. Because $S_f < 0$ and $f' < 0$ on $[p, 1]$, by Equation (4.2) we have $f'(x) \leq \max\{f'(p), f'(1)\} < -1$. Thus $|(f^{m_x+1})'(x)| = |(f^{m_x})'(f(x)) \cdot f'(x)| > 1$. Therefore, Equation (4.5) holds by taking $n_x = m_x + 1$. We have finished the proof of the claim.

Third, we claim that for any nonempty open set $U \subset [0, 1]$, there exists a positive integer n such that

$$f^n(U) \supset [0, 1]. \quad (4.6)$$

Let U be an interval in $[0, 1]$. Because $f(x) > x$ for $x \in (0, p_-]$ and $f([p, 1]) \subset (0, p]$, there are a positive integer n and a subinterval $U_0 \subset U$ such that $f^n(U_0) \subset J$. For convenience, we denote $R(x) = f^{\tau(x)}(x)$ for $x \in J$. The claim (4.4) says that R expands the lengths of

intervals in J and hence there exists an integer $k > 0$ and a subinterval $V_0 \subset f^m(U_0)$ such that $R^k(V_0)$ contains a discontinuity point of R . Thus, there exists $m > 0$ such that $p \in f^m(V_0)$. Now it remains to prove that $f^{m+\ell}(V_0) = [0,1]$ for some $\ell > 0$. Because f maps $[p, 1]$ homeomorphically onto $[0, p]$, there exists a unique $d \in [p,1]$ such that $f(d) = \frac{\alpha}{\alpha+\beta}$. Then $f^{m+2\ell}(V_0) \supset [\frac{\alpha}{\alpha+\beta}, d]$ for some $\hat{\ell} > 0$. Indeed, because $|(f^2)'(y)| = |(f^{\tau(x)})'(x)| > 1$ for $x \in [r_1, p]$ and $f^2(x) = f(f(x)) \leq f(f(r_1)) = p_- < \frac{\alpha}{\alpha+\beta} < x$ for $x \in [\frac{\alpha}{\alpha+\beta}, r_1]$, we have that $f^2(x) < x$ for all $x \in [\frac{\alpha}{\alpha+\beta}, p)$ and hence $f^2(x) > x$ for all $x \in (p, d]$. Thus, there exists $\hat{\ell} > 0$ such that $f^{m+2\hat{\ell}}(V_0) \supset [\frac{\alpha}{\alpha+\beta}, d]$. Because $f^2([\frac{\alpha}{\alpha+\beta}, d]) = f([\frac{\alpha}{\alpha+\beta}, 1]) = [0, 1]$, $f^{m+2\hat{\ell}+2}(V_0) \supset f^2([\frac{\alpha}{\alpha+\beta}, d]) = [0, 1]$. The proof of the desired claim is complete.

Finally, we are in position to obtain the three properties of Devaney's chaos. Let U be any nonempty open interval in $[0,1]$. Then there exist a nonempty open interval V and a closed interval W such that $V \subset W \subset U$. By claim (4.6), there exists a positive integer n such that $f^n(V) \subset [0,1]$ and hence $f^n(W) \supset W$. By the fixed point theorem, f^n has a fixed point in W . Therefore, f has a periodic point in W and so in U . We have proved that the set of periodic points is dense in $[0,1]$. The claim (4.6) immediately implies that f is topologically transitive. For sensitive dependence of f , we take $\eta = \frac{1}{4}$. Let $x \in [0,1]$ and $\epsilon > 0$ be arbitrary. Take U to be the interval $(x, x + \frac{\epsilon}{2})$ or $(x - \frac{\epsilon}{2}, x)$ provided it is well defined. By claim (4.6), we have $f^n(U) \supset [0,1]$. Thus, there exists $y \in U$ such that $|f^n(x) - f^n(y)| > \frac{1}{4} = \eta$. The proof of the theorem is complete. \square

Now we consider the case when $\alpha > 1$ and $f(\frac{\alpha}{\alpha+\beta}) > \frac{\alpha}{\alpha+\beta}$. Let $p_- < p$ be as in Equation (3.2). Then $f'(0) = 0$ and hence there exists a unique point, namely q , in $(0, p_-)$ such that

$$f(q) = q. \tag{4.7}$$

Similar to Equation (4.3), we have that if $f^2(\frac{\alpha}{\alpha+\beta}) \leq q$, then there exist $p_- < \ell_1 < r_1 < p$ such that

$$f^2(\ell_1) = f^2(r_1) = p_-. \tag{4.8}$$

Moreover, if $f^2(\frac{\alpha}{\alpha+\beta}) < q$, the intermediate value theorem implies that there exist $p_- < \ell_1 < q_- < q_+ < r_1 < p$ such that

$$f^2(q_-) = f^2(q_+) = q. \tag{4.9}$$

Under the condition $f^2(\frac{\alpha}{\alpha+\beta}) = q$, Theorem 3.3 says that f^2 exhibits Li-Yorke chaos. In fact, the existence of Devaney's chaos is also possible.

Theorem 4.3: Let $f = f_{\lambda, \alpha, \beta}$ be given by Equation (3.1) with $\alpha > 1$ and $f^2(\frac{\alpha}{\alpha+\beta}) = q$, where q is in Equation (4.7), and let ℓ_1, r_1 as in Equation (4.8). If $\min \{|f'(q)|, |f'(p)|\} > 1$, $\max \{r_1 - \frac{\alpha}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} - \ell_1\} < p_- - q$ and the Schwarzian derivative $S_f(x) < 0$ for $x \in [q, f(\frac{\alpha}{\alpha+\beta})]$, then f exhibits Devaney's chaos on $[q, f(\frac{\alpha}{\alpha+\beta})]$.

The proof is similar to the one for Theorem 4.2. We omit it here.

If Devaney's chaotic motion presents, then the initial condition will be an important determinant to the future development for the economy. This result contradicts the traditional Solow growth model because two economies which only differ from each other

at initial conditions will behave very differently not only in the short run but also in the long run.

4.2. Smale’s chaos

Based on the pioneering article by Smale [30] in dynamical systems, one can define Smale’s chaos as follows (see also [31]).

Definition 4.4: Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a map and Λ be a subset of \mathbb{R} . We say that h exhibits Smale’s chaos on Λ if there exist an integer $N > 1$ and a function $\varphi : \Lambda \rightarrow \Sigma_N$, where $\Sigma_N = \{s_0s_1s_2 \cdots \mid s_i = 1, 2, \dots, \text{ or } N \text{ for all } i \geq 0\}$, such that φ is continuous and one to one from Λ onto Σ_N , its inverse φ^{-1} is continuous, and for any $x \in \Lambda$,

$$\varphi(h(x)) = \sigma(\varphi(x)),$$

where σ is the *shift map* on Σ_N defined by

$$\sigma(s_0s_1s_2 \cdots) = s_1s_2s_3 \cdots.$$

For the case when $f^2(\frac{\alpha}{\alpha+\beta}) < q$, our model can exhibit Smale’s chaos.

Theorem 4.5: Let $f = f_{\lambda,\alpha,\beta}$ be given by Equation (3.1) with $\alpha > 1$ and $f^2(\frac{\alpha}{\alpha+\beta}) < q$, where q is in Equation (4.7), and let $p_- < \ell_1 < q_- < q_+ < r_1$ as in Equations (4.8) and (4.9). Let

$$\Lambda = \{x \in [q, f(q_+)] : f^n(x) \in [q, f(q_+)] \text{ for all } n \geq 0\}.$$

If one of the following holds:

- (H1) $\min\{|f'(q_-)|, |f'(q_+)|\} > 1$;
- (H2) $\max\{r_1 - q_+, q_- - \ell_1\} < p_- - q$, $f'(p) < -1$ and the Schwarzian derivative $S_f(x) < 0$ for $x \in [q, f(q_+)]$,

then:

- (1) the set Λ is invariant under f and is a Cantor set (i.e. closed, bounded, totally connected, and perfect);
- (2) the map f has periodic points of all periods in Λ ;
- (3) the map f exhibits Smale’s chaos on Λ ; and
- (4) every orbit with an initial point in $[0,1] \setminus \Lambda$ converges to the origin.

Proof: Consider the case when (H1) holds. For item (1), first we show that the invariance $f(\Lambda) = \Lambda$. It follows immediately from the definition of Λ that $f(\Lambda) \subset \Lambda$. We prove $\Lambda \subset f(\Lambda)$ by contraction. Let $x \in \Lambda$. Because $f([q, q_-]) = [q, f(q_+)]$, there exists $y \in [q, q_-]$ such that $f(y) = x$. Suppose $y \notin \Lambda$. Then there is $m \geq 1$ such that $f^m(y) \notin [q, f(q_+)]$ and so $f^{m-1}(x) = f^{m-1}(f(y)) = f^m(y) \notin \Lambda$. This contradicts the fact that $f(\Lambda) \subset \Lambda$. Second, we show that Λ is compact. Because $\Lambda \subset [0,1]$, Λ is bounded. Let $J_0 = [0,1]$ and inductively

define $J_n = \{x \in [0,1] : f(x) \in J_{n-1}\}$ for $n \geq 1$. By the definition of Λ , we have $\Lambda = \bigcap_{n=0}^{\infty} J_n$. Because f is continuous, J_n is closed for all $n \geq 0$. Hence, Λ is closed and so is compact.

Next, we claim that there exists $\lambda > 1$ such that for any $x \in \Lambda \cap [q, f(q_+)]$,

$$|f'(x)| \geq \lambda. \tag{4.10}$$

Because $0, q, p$ are fixed points for f , the mean value theorem implies that there are $0 < a < q < b < p$ such that $f'(a) = f'(b) = 1$. Because f'' has at most one root on $(0,1)$, (H1) implies that $f'(x) > 1$ for all $x \in [q, q_-]$ and $f'(x) < -1$ for all $x \in [q_+, f(q_+)]$. The desired claim follows from the continuity of f' .

Now we prove that Λ is totally disconnected. Suppose, on the contrary, that $\Lambda \supset [y, z]$ for some $y < z$. Because Λ is invariant for $f, f^n([y, z]) \subset f^n(\Lambda) \subset \Lambda \subset [q, f(q_+)] \subset [0, 1]$. This leads to a contradiction. Indeed, the mean value theorem and the claim (4.10) together imply that

$$|f^n(y) - f^n(z)| = |(f^n)'(x)| |y - z| \geq \lambda^n |y - z| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

To prove that Λ is perfect, first notice that J_n consists of 2^n disjoint closed intervals. We order the 2^n components of J_n from left to right on the real line and denote the i th component by $J_{n,i}$. Also notice that the endpoints of each $J_{n,i}$ are contained in Λ . Let $x \in \Lambda$ and for $n \geq 0$, let $J_{n,i(n,x)}$ be the component of J_n that contains x . Then $J_{n+1, i(n+1,x)} \subset J_{n, i(n,x)}$ for all $n \geq 0$ and $x \in \bigcap_{n=0}^{\infty} J_{n,i(n,x)}$. Because Λ is totally disconnected, the length of $J_{n,i(n,x)}$ converges to 0 as n goes to ∞ . Therefore, there are endpoints from $J_{n,i(n,x)}$'s arbitrarily close to x . This proves that Λ is perfect. We have finished the proof of item (1).

For items (2) and (3), let $I_1 = [q, q_-]$ and $I_2 = [q_+, f(q_+)]$. Then $f(I_1) \supset I_1 \cup I_2$ and $f(I_2) \supset I_1 \cup I_2$. By using the same argument as in the proof of Theorem 4.2, the result in item (2) follows. Let σ be the shift map on Σ_2 . Define $\varphi : \Lambda \rightarrow \Sigma_2$ by $\varphi(x) = s_0 s_1 s_2 \dots$, where $s_i = 1$ if $f^i(x) \in I_1$ and $s_i = 2$ if $f^i(x) \in I_2$. Let $x \in \Lambda$, $\varphi(x) = s_0 s_1 s_2 \dots$, and $\varphi(f(x)) = t_0 t_1 t_2 \dots$. Then for any $i \geq 0$, $f^i(f(x)) = f^{i+1}(x) \in I_{s_{i+1}}$ and $f^i(f(x)) \in I_{t_i}$. Because $I_1 \cap I_2 = \emptyset, s_{i+1} = t_i$. Thus, $\varphi(f(x)) = \sigma(\varphi(x))$. Based on claim (4.10), it becomes a routine process to prove that φ is continuous, one to one, and onto, and φ^{-1} is continuous. For details, refer to the proof of Theorem 5.1 in [31, Chapter II]. Hence, the statement of item (3) is true.

For item (4), let $x \in [0, 1] \setminus \Lambda$. Then there exists $m \geq 0$ such that $f^m(x) \in [0, q) \cup (f(q_+), 1]$ and hence $f^{m+1}(x) \in [0, q)$. Because $f(y) < y$ for all $y \in (0, q)$ and 0 is the unique fixed point in $[0, q)$ for f . The continuity of f implies that $f^n(x)$ tends to 0 as n goes to the infinity.

Consider when hypothesis (H2) holds. By using the same argument as in the proof of Theorem 4.2, we have the result, similar to claim (4.5), that for every $x \in \Lambda$, there exists an integer $n_x \geq 1$ such that $|(f^{n_x})'(x)| > 1$. Based on this, the rest of the proof is very similar to the one given above for hypothesis (H1). We leave the details to the readers. \square

Theorem 4.5 implies that the dynamical system (2.6) will display cycles of all periods under certain conditions, and hence Smale's chaos is possible. From Theorems 3.2, 3.3, 4.2, 4.3 and 4.5, the chaotic motion depends crucially on parameter values. Let us consider the model (3.1) with $\alpha = 2$ and $\beta = 1$, and λ varying, that is $f_\lambda(x) = \lambda x^2(1 - x)$.

If $0 < \lambda < \frac{27}{4}$, then $f([0,1]) \subset [0,1]$. Moreover, if $\lambda > 4$, then 0 and $\frac{1+\sqrt{1-\lambda}}{2}$ are fixed points for f . Also a simple calculation implies that its Schwarzian derivative

$$S_f(x) = \frac{-6(1 - 4x + 6x^2)}{(2x - 3x^2)^2},$$

which is negative for all $x \in (0,1)$ except the critical point $\frac{2}{3}$. See also Figure 1 for its bifurcation diagram. Corresponding Theorems 3.2, 3.3, 4.2, 4.3 and 4.5, we have the following results.

Example 4.6: Let $f_\lambda : [0, 1] \rightarrow \mathbb{R}$ be given by $f_\lambda(x) = \lambda x^2(1 - x)$, where $4 \leq \lambda \leq 6.75$ is a parameter. Then one has the following properties:

- (1) If $\lambda = 6.75$, then f_λ has periodic orbits of all periods and f_λ exhibits Li–Yorke chaos.
- (2) If $6 < \lambda < 6.75$, then f_λ has periodic orbits of all even periods and f_λ^2 exhibits Li–Yorke chaos.
- (3) There exists $\underline{\lambda} \approx 6.545$ such that f_λ exhibits Devaney’s chaos.
- (4) If $6.6 \leq \lambda \leq 6.75$, then there is a Cantor subset Λ of $[0,1]$ such that f_λ has periodic points of all periods in Λ and exhibits Smale’s chaos on Λ , and every orbit with an initial point in $[0,1] \setminus \Lambda$ converges to the origin. Note that since Λ has Cantor structure, Figure 1 appears that for $6.6 \leq \lambda < 6.75$, almost all orbits converge to the origin.

5. Conclusion

In this article, we have shown the existence of chaotic behaviour in an overlapping generations model with human capital accumulation. In addition to presenting the chaos in Li and Yorke condition, we also illustrate the chaotic trajectories in the sense of Devaney and Smale. Unlike other studies, we assume that excessive knowledge/information will reduce the productivity to accumulate human capital. Because traditional approach of human capital accumulation does not take the negative externality of human capital into account, it excludes the possibility of complex dynamics. Hence, our study highlights the important role of human capital externality and indicates that a more accurate estimation of the human capital accumulation function will be needed in the future study. Furthermore, the policy implications of our results considering the negative externality of human capital will be very different from those obtained from traditional studies.⁸ Another work for the future study is to extend the model to a higher-dimensional dynamical system. This can be achieved by including persistent habits in the model or by changing the formation of human capital accumulation.

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Notes

1. Day [1] argued that an excessive amount of capital will reduce overall productivity because of the ‘pollution effect’.
2. Average human capital was used by De la Croix and Doepke [10] to represent the quality of teachers.
3. The concept of ‘information pollution’ was introduced by Nielsen [13]; see also his interview by the BBC in [14].
4. For example, Eysenbach *et al.* [15] demonstrated the importance of the quality of medical information on the internet because misinformation can damage one’s human capital or even life.
5. We assume that there are only private schools in the economy. For studies considering educational systems, see [23–25].
6. Caucutt [26] has developed a model where schools could engage in price discrimination amongst students.
7. See [6], [10], [11], [23], [27], [28].
8. A good discussion about implications of policy decision when chaos presents can be found in Bullard and Butler [32].

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