國立交通大學

應 用 數 學 系

碩 士 論 文

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Topics on Linear Parabolic Equations

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中 華 民 國 九 十 六 年 六 月

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A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in partial Fulfillment of the Requirements
for the Degree of Master
in
Applied Mathematics
June 1997
Hsinchu, Taiwan, Republic of China

中華民國九十六年六月

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摘　要

這篇論文中我們研究線性拋物偏微分方程。首先，我們舉出關於此種方程一些實際的例子。接下來，我們運用一般的方法去解決此方程，雖然同一例題用不同的解法，會得到不同的表示方式，但可以證明都是一樣的。

當我們應用 Fourier 和 Laplace 轉換去解決全線和半線的偏微分方程時，我們必須使用 inverse Fourier 和 Laplace 轉換去求出解析解，而這些被積分函數中有時會牽扯到平方根，但是在複數平面中，平方根是多值的。為了使我們的運算正確，所以我們從複數平面上的代數結構去發展黎曼空間，讓平方根是一個單值函數，我們可以利用數學軟體去完成 inverse 轉換。在此篇論文中，我們提出一些例題去說明及驗證此方法是可行的。

Topics on Linear Parabolic Equations

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Abstract

We study the linear parabolic partial differential equations(linear parabolic PDEs). First, we give some practical examples and show that they are governed by such type of the equations. Next, we apply several classical methods to solve the linear parabolic PDEs with the solutions being expressed in various forms. We then identify those solutions.

When we apply Fourier and Laplace transformations to the whole-and half-line PDEs, it is necessary to perform the inverse Fourier and Laplace transformations to derive the PDE solutions, and it is quite often that those integrals involve the square root operator which is multi-valued in the complex plane. In order to perform the inverse transformations correctly, we develop the Riemann surfaces from the complex plane with the proper algebraic structures to assure that the square root is now a single-valued function on the surfaces, and we are able to accomplish the inverse transformations analytically and numerically. Some examples are given to illustrate the entire scheme.

誌　　謝

首先我要感謝我的指導教授 李榮耀博士，因為他適時從旁的協助及建議，讓我的論文足以順利完成；同時也要感謝口試委員給我客觀的意見，讓我獲益良多。

感謝陪我一起討論、研究的兩個好朋友美如、文雯；因為他們不厭其煩的態度及適時的關心，讓我在遇到挫折時能藉此動力突破它。

最重要的當然還是要感謝我的家人，因為他們的支持和鼓勵，讓我在求學及研究的過程中沒有後顧之憂以完成我的論文。

Contents

1. Introduction

The parabolic equations occur commonly in applied science. Examples are models of many physical processes, financial models and Schrodinger equation. Before we introduce the linear parabolic PDE, we must classify the partial differential equation.

1.1 Classification

All linear, second-order partial differential equations can be classified as parabolic, hyperbolic or elliptic. Assuming $u_{xy} = u_{yx}$, the general second-order PDE in two independent variables has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + E(u_x, u_y, u, x, y) = 0,$$

where the coefficients *A*, *B*, *C* etc. may depend upon *x* and *y*, and $(x, y) \in$ domain D.

1. If $B^2 - 4AC > 0$, the equation is hyperbolic in D.
2. If $B^2 - 4AC = 0$, the equation is parabolic in D.
3. If $B^2 - 4AC < 0$, the equation is elliptic in D.

For example, wave equation $u_{tt} - c^2 u_{xx} = 0$ is a hyperbolic equation; heat equation $u_t - u_{xx} = 0$ is a parabolic equation; Laplace equation $u_{xx} + u_{yy} = 0$ is a elliptic equation.

In this chapter we introduce linear models of the parabolic PDE.

1.2 Linear models of the parabolic PDE

1.2.1 Heat and mass transfer

We consider the temperature $u(x, y, z, t)$ in a slab of material covering a three-dimensional domain D bounded by a closed surface S. The material at the point $(x, y, z,)$ has the property that the temperature u is attained by storing the energy in the form of random molecular motion. The total heat context of the solid is given by

$$Q(t) = \iiint_D cu\rho dV,$$

where dV is the volume element; for instance, $dV = dxdydz$ in Cartesian variables; the constant of proportionality c is the specific heat in ($cal/_{g^\circ c}$) and ρ is density ($g/_{cm^3}$).For an incompressible material, ρ is constant.

By Fourier's law, the rate of flow is proportional to the gradient of the temperature, ie.

$$\vec{v} = -k \ gradu ,$$

where the constant k is the thermal conductivity. The net inflow of heat through boundary S is

$$R(t) = \iint_S k \ gradu \cdot \vec{n} dA .$$

Therefore the integral conservation law of heat energy, the rate of change of energy in D must be equal to the flux of energy, so

$$\frac{d}{dt} \iiint_D c\rho u dV = \iint_S k \ gradu \cdot \vec{n} dA , \tag{1.1}$$

where c, ρ and k may depend on position.

For a medium where c, ρ and k are smooth, we apply Gauss' theorem (Divergence theorem) to express the right-hand side of (1.1), then we have

$$R(t) = \iiint_D div(k \ gradu) dV .$$

Since the boundaries of D are fixed in space, we may rewrite the left-hand side of (1.1) as (we suppose that u is continuous in D)

$$\frac{dQ}{dt} = \iiint_D c\rho u_t dV .$$

Thus, this gives

$$\iiint_D [c\rho u_t - \text{div}(\ \text{k} \ \ \text{grad u} \ )] dV = 0 .$$

We suppose that the integrand is continuous, and it must be zero. That is,

$$c\rho u_t - \text{div}(\ \text{k} \ \ \text{grad u} \ ) = 0.$$

For constant k , this reduces to

$$u_t - K \nabla^2 u = 0 ,$$

where $K = \dfrac{k}{c\rho}$ is called the thermal diffusivity of material and $\nabla^2 = div\, grad$ is the Laplacian given by $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ in Cartesian coordinates. This is known as the heat equation.

A more interesting example is that of molecular diffusion, in which two substances co-exist at each point and their relative properties vary in space and time. In the simplest case one of the substances, called matrix, is fixed and the other diffuses through it with a concentration (amount per unit volume) given by $c(x,t)$. Examples are a dye in a liquid, and smoke in the atmosphere; further examples are any mixture of substances such as a solute dissolved in a liquid or gas.

Then Fick's law relates the flux is proportional to the gradient of c by $\vec{q} = -D\,\nabla c$, where D is called diffusity. Conservation of mass implies that

$$\frac{\partial c}{\partial t} = -\nabla \cdot \vec{q} = D\,\nabla^2 c\,,$$

for constant diffusivity. If , however, the medium is moving with velocity v there is also mass transfer by convection. The total mass flux is

$$\vec{q} = c\vec{v} - D\,\nabla c\,,$$

so that c satisfies the convection-diffusion equation

$$\frac{\partial c}{\partial t} = D\,\nabla^2 c - \nabla \cdot (c\vec{v}) = D\,\nabla^2 c - [\vec{v} \cdot \nabla c + c(\nabla \cdot \vec{v})]\,,$$

for an incompressible liquid this reduces to

$$\frac{\partial c}{\partial t} = D\,\nabla^2 c - \vec{v} \cdot \nabla c\,.$$

This can be rewritten as

$$(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla)c = D\,\nabla^2 c\,.$$

Remark:

The form

$$\frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u = D\,\nabla^2 u + f$$

is also called a convection-diffusion or reaction-diffusion equations depending whether $f = 0$ or $\vec{v} = 0$ respectively.

1.2.2 Finance

Suppose we consider an option, which is contract giving its holder the right (but not the obligation) to buy (or sell) some asset, such as a number of stock-market shares, at some specified time, say T, when the exercise price, a previously agreed sum of money E, is paid for the asset. Suppose the asset is a share which is expected to gain in value in 0<t<T, but whose price is subject to unpredictable factor. If we hold an options, we can set up a "portfolio" of the option to protect ourselves against unpredictability. To do this we need to assess the value $V(s,t)$ of the option to buy a share at time T as a function of current time t and the asset value S. We suppose we have a cash balance M, and we hold a number Δ, which may vary in time, of the assets. Thus, the portfolio value is $P = M + S\Delta + V$. The cash balance accrues interest at a rate r; it also changes when we buy or sell assets, in a short time dt, we receive $rMdt$ in interest and spend $-Sd\Delta$ on assets. In the same time, t he asset price changes by dS and the option value by dV, so the overall change is

$$dP = rMdt - Sd\Delta + Sd\Delta + \Delta dS + dV = rMdt + \Delta dS + dV .$$

Now we suppose the instantaneous "rate-of-return" on the asset varies randomly.

$$\frac{dS}{S} = \mu dt + \sigma dx , \qquad (1.2)$$

where μ is a deterministic "growth rate" for the asset; dx is a small normal random variable of mean zero and variance dt, and σ is a parameter which measures how "volatile" the share price is.

By Taylor' Expansion series for dV,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \dots .$$

Let $dt \to 0$, dS is given by $\sigma^2 S^2 dx^2$ and then replace dx^2 by dt since dx has zero mean and variance dt.

$$dV = (\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S}\sigma S dx .$$

Thus,

$$dP = rMdt + \Delta\mu Sdt + \Delta\sigma Sdx + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu Sdt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}\sigma Sdx .$$

By observing, we can choose Δ to be $-\frac{\partial V}{\partial S}$ and we have

$$dP = rMdt + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt .$$

The final step is to use the idea of no arbitrage, it means that it is impossible to earn more than the risk-free interest rate r for a risk-free portfolio, so

$$dP = rPdt ,$$

and

$$\begin{aligned} rMdt + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt &= r(M + \Delta S + V)dt \\ &= rMdt + rVdt - r\frac{\partial V}{\partial S}Sdt. \end{aligned}$$

Hence, we derive the **Black-Scholes** equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r(V - S\frac{\partial V}{\partial S}) .$$

I. Methods of solving the linear parabolic PDE

In this chapter we introduce six methods to solve the heat conduction problem.

2.1 <u>Separation of Variables</u>

We first consider the heat conduction problem

$$\begin{aligned} &\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0, && 0<x<\pi,\ \ t>0 \\ &u(0,t)=u(\pi,t)=0, && t\ge 0 \\ &u(x,0)=f(x). && 0\le x\le \pi \end{aligned} \tag{2.1}$$

We look for a specific type of solution; namely, a product of a function of x only and a function of t only

$$u(x,t)=X(x)T(t).$$

We substitute the function u into the differential equation, and divide u. This gives

$$\frac{T'}{T}-k\frac{X''}{X}=0,$$

or

$$\frac{T'}{T}=k\frac{X''}{X}. \tag{2.2}$$

The left-hand side of this equation depends only upon t. The right-hand side is independent of t. We say that heat equation (in 0<x<l) is separable.

If we take the partial derivative with respect to t of both sides of the separated equation, we find that

$$\frac{d}{dt}\left[\frac{T'}{T}\right]=0.$$

It follows that

$$\frac{T'}{T}=-\lambda,$$

where λ is a constant. Then by (2.2) we have

$$k\frac{X''}{X}=-\lambda.$$

Thus $u(x,t)=X(x)T(t)$ is a solution of heat equation if and only if X and T satisfy the two ordinary equations

$$\begin{aligned} kX'' + \lambda X &= 0, \\ T' + \lambda T &= 0, \end{aligned} \tag{2.3}$$

for some constant λ.

We can solve the two ordinary differential equations (2.3) to obtain particular solutions of the partial differential equation (2.1). For each value of λ the equation $kX'' + \lambda X = 0$ has two linearly independent solutions. The families of solutions of (2.1) are given by

$$\begin{aligned} &e^{-\lambda t}\cos\sqrt{\frac{\lambda}{k}}x,\ \ e^{-\lambda t}\sin\sqrt{\frac{\lambda}{k}}x, && \text{for } \lambda > 0 \\ &1, x, && \text{for } \lambda = 0 \\ &e^{-\lambda t+\sqrt{\frac{-\lambda}{k}}x},\ \ e^{-\lambda t-\sqrt{\frac{-\lambda}{k}}x}. && \text{for } \lambda < 0 \end{aligned}$$

Now, consider the boundary conditions. Since we wish to have $u = 0$ for x=0 and x=1, we only consider those solutions of the first equation (2.3) which also satisfy these conditions. We must have

$$\begin{aligned} &kX'' + \lambda X = 0, && 0 < x < \pi \\ &X(0) = X(\pi) = 0. \end{aligned}$$

This homogeneous problem always has the trivial solution X=0, but we are interested in cases where this is not the only solution.

Case1. $\lambda > 0$,

$$X(x) = a\sin\sqrt{\frac{\lambda}{k}}x + b\cos\sqrt{\frac{\lambda}{k}}x.$$

X(0)= 0 tells us that

$$X(x) = a\sin\sqrt{\frac{\lambda}{k}}x,$$

and $X(\pi) = 0$ gives

$$a\ \sin\sqrt{\frac{\lambda}{k}}\pi = 0.$$

Then X need not be identically zero if and only if

$$\sin\sqrt{\frac{\lambda}{k}}\pi = 0\,,$$

ie.

$$\sqrt{\frac{\lambda}{k}}\pi = n\pi\ , \qquad \text{n} = 1, 2, 3\ ...$$

That is,

$$\lambda_n = n^2 k, \qquad \text{n} = 1, 2, 3\ ...$$

These value λ_n are called the eigenvalues of the problem, and the functions

$$X_n(x) = \sin nx, \qquad \text{n} = 1, 2, 3, \ldots$$

are the corresponding eigenfunctions.

Case2. $\lambda = 0$,

$$\text{X(x) = a + b x,}$$

We know

$$\text{X(0)} = \text{a} + \text{b}*0 = 0 \;\Rightarrow\; \text{a} = 0 \;\Rightarrow\; \text{X(x)} = \text{bx},$$

and

$$\text{X}(\pi) = \text{b}*\pi = 0 \;\Rightarrow\; \text{b} = 0.$$

This gives the trivial solution X= 0.

Case3. $\lambda < 0$,

$$X(x) = a\, e^{\sqrt{\frac{-\lambda}{k}}x} + b\, \mathrm{e}^{-\sqrt{\frac{-\lambda}{k}}x}\,.$$

We have

$$X(0) = a*1 + b*1 = 0\,,$$

and

$$X(\pi) = a\, e^{\sqrt{\frac{-\lambda}{k}}\pi} + b\, e^{-\sqrt{\frac{-\lambda}{k}}\pi} = 0\,,$$

but these can not find a or b.

We know

$$\sinh\sqrt{\frac{-\lambda}{k}}x=\frac{e^{\sqrt{\frac{-\lambda}{k}}x}-e^{-\sqrt{\frac{-\lambda}{k}}x}}{2}, \qquad \cosh\sqrt{\frac{-\lambda}{k}}x=\frac{e^{\sqrt{\frac{-\lambda}{k}}x}+e^{-\sqrt{\frac{-\lambda}{k}}x}}{2},$$

so that a general solution

$$X(x)=A\sinh\sqrt{\frac{-\lambda}{k}}x+B\cosh\sqrt{\frac{-\lambda}{k}}x.$$

Therefore

$$X(0)=A\sinh 0+B\cosh 0=A*0+B*1=0 \Rightarrow B=0 \Rightarrow X(x)=A\sinh\sqrt{\frac{-\lambda}{k}}x.$$

And

$$X(\pi)=A\ \sinh\sqrt{\frac{-\lambda}{k}}\pi=0=A\ \frac{e^{\sqrt{\frac{-\lambda}{k}}\pi}-e^{-\sqrt{\frac{-\lambda}{k}}\pi}}{2}=0 \ \Rightarrow\ A=0.$$

This also gives the trivial solution X= 0.

Having found a sequence of values of λ, we can look at the corresponding functions T(t). These are easily seen to be multiples of e^{-n^2kt}

We have constructed the particular solutions

$$u_n(x,t)=\sin nx\ \mathrm{e}^{-n^2kt}.$$

Which satisfy all the homogeneous conditions of the problem (2.1). The same is true of any finite linear combination. We attempt to represent the solution of (2.1) as an infinite series in the functions u_n :

$$u(x,t)=\sum_{n=1}^{\infty}a_n\ \sin nx\ e^{-n^2kt}. \qquad (2.4)$$

We need to determine the coefficients a_n such that $u(x,t)$ satisfies the initial condition $u(x,0)=f(x)$. Thus we require

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}a_n\ \sin nx.$$

But the right side is just the Fourier sine series of the function $f(x)$ on the interval $(0,l)$.

Therefore the coefficients a_n are the Fourier coefficients given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \sin nx\ dx, \qquad n = 1, 2, \ldots \tag{2.5}$$

Therefore we have obtained a solution to (2.1) given by the infinite series (2.4) where the coefficients a_n are given by (2.5).

Substituting the expression for the a_n into the solution formula (2.4) allows us to write the solution as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\int_0^{\pi} f(\xi)\ \sin n\xi d\xi\right)\ e^{-n^2kt} \sin nx \\ &= \int_0^{\pi} \left(\frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2kt}\ \sin n\xi \sin nx\right) f(\xi) d\xi \\ &= \int_0^{\pi} K(x,\xi,t) f(\xi) d\xi. \end{aligned}$$

Remark:

If is defined and integrable on the interval $[-\pi, \pi]$, then its Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx\ , \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx\ , \qquad n = 1, 2, \cdots$$

Example 2.1 :(Using Separation of Variables)

$$
\begin{aligned}
&u_t - u_{xx} = 0\ , && 0 < x < \pi,\ \ t > 0 \\
&u(0,t) = u(\pi,t) = 0, && t \ge 0 \\
&u(x,0) = x^2(\pi - x). && 0 \le x \le \pi
\end{aligned}
$$

By equation (2.4) we can know

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx\ ,$$

where

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)\sin nx\, dx \\
&= 2\,[(-1)^{n+1}\frac{\pi^2}{n} + (-1)^n \frac{2}{n^3} - \frac{2}{n^3}] - \frac{2}{\pi}[(-1)^{n+1}\frac{\pi^3}{n} + (-1)^n \frac{6\pi}{n^3}] \\
&= (-1)^n \frac{-8}{n^3} - \frac{4}{n^3} = -4[1 + 2(-1)^n]n^{-3}
\end{aligned}
$$

Then the solution is

$$u(x,t) = -4\sum_{n=1}^{\infty}[1 + 2(-1)^n]n^{-3}e^{-n^2 t} \sin nx\ .$$

2.2 Finite Fourier Transformation and nonhomogeneous problem

Recall the heat conduction problem (2.6). The solution of the problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n\ e^{-n^2 t} \sin(nx),$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx, \qquad \text{n} = 1,2,...$$

We shall now treat the corresponding nonhomogeneous problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(x), && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = u(\pi,t) = 0\,, && t \ge 0 \\ &u(x,0) = f(x)\ . && 0 \le x \le \pi \end{aligned} \tag{2.7}$$

by expanding the solution in a Fourier series in terms of the same set of functions.

To solve the above nonhomogeneous problem, we expand the solution in a Fourier sine series for each fixed t

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\sin(nx)\,.$$

The set of sine coefficients

$$b_n(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\sin(nx)dx\,,$$

which is called undetermined coefficient and is also called the finite sine transform of $u(x,t)$.

If u_{xx} is continuous, its finite sine transform is given by

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} u_{xx}(x,t)\sin(nx)dx &= \frac{2}{\pi}[u_x(x,t)\sin(nx) - u(x,t)\cos(nx)]_0^{\pi} - n^2\frac{2}{\pi}\int_0^{\pi} u(x,t)\sin(nx)dx \\ &= -n^2 b_n(t). \end{aligned}$$

Because of $u(0,t) = u(\pi,t) = 0$.

If u_t is continuous, we can interchange integration and differentiation to show that

$$\frac{2}{\pi}\int_0^{\pi} u_t(x,t)\sin(nx)dx = \frac{d}{dt}b_n(t)\,.$$

Taking the finite sine transform of both sides of (2.7) leads to the equation

$$b_n{}'(t) + n^2 b_n(t) = B_n(t)\,, \tag{2.8}$$

where

$$B_n(t) = \frac{2}{\pi}\int_0^{\pi} F(x,t)\sin(nx)dx\ . \tag{2.9}$$

The initial condition $u(x,0) = 0$ means that

$$b_n(0) = 0. \tag{2.10}$$

Taking sine transform has reduced the partial differential problem (2.7) to the ordinary differential problem (2.8), (2.9). We now solve this problem

$$b_n'(t) + n^2 b_n(t) = B_n(t),$$
$$b_n(0) = 0.$$

We have

$$b_n(t) = \int_0^t e^{-n^2(t-\tau)} B_n(\tau) d\tau .$$

If the problem (2.7) has a solution u with u_t and u_{xx} continuous, it must have the Fourier sine series

$$u(x,t) \sim \sum_{n=1}^{\infty} [\int_0^t e^{-n^2(t-\tau)} B_n(\tau) d\tau] \sin(nx)$$

Recall the problem (2.6), the solution u can be written as

$$u(x,t) = \int_0^{\pi} K(\xi, x, t) f(\xi) d\xi ,$$

where

$$K(\xi, x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin(n\xi) \sin(nx) .$$

For our nonhomogeneous problem, we would like a similar form as above. We use the definition (2.9) of $B_n(t)$, and formally interchange integration and summation. This gives

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} [\int_0^t e^{-n^2(t-\tau)} (\frac{2}{\pi} \int_0^{\pi} F(\xi,\tau) \sin(n\xi)\, d\xi)\, d\tau] \sin(nx) \\ &= \int_0^t \int_0^{\pi} [\frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2(t-\tau)} \sin(n\xi) \sin(nx)] F(\xi,\tau) d\xi\, d\tau \\ &= \int_0^t \int_0^{\pi} K(x,\xi,t-\tau) F(\xi,\tau) d\xi\, d\tau, \end{aligned}$$

and only need F and F_X continuous.

If instead of the homogeneous initial condition $u(x,0) = 0$, we have $u(x,0) = f(x)$ in

(2.7), we must simply replace (2.10) by

$$b_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

Then

$$b_n(t) = \int_0^t e^{-n^2(t-\tau)} B_n(\tau)d\tau + b_n(0)e^{-n^2 t},$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}[\int_0^t e^{-n^2(t-\tau)} B_n(\tau)d\tau]\sin(nx) + \sum_{n=1}^{\infty} b_n(0)e^{-n^2 t}\sin nx \\ &= \int_0^t\int_0^{\pi} K(x,\xi,t-\tau)F(\xi,\tau)d\xi d\tau + \int_0^{\pi} K(\xi,x,t)f(\xi)d\xi. \end{aligned}$$

Example 2.2 : (Using Finite Fourier transformation)

$$\begin{aligned} &u_t - u_{xx} = t\ \sin^3 x\,, && 0 < x < \pi,\ \ t > 0 \\ &u(0,t) = u(\pi,t) = 0, && t > 0 \\ &u(x,0) = 0. && 0 < x < \pi \end{aligned}$$

We can find that

$$\begin{aligned} B_n(\tau) &= \frac{2}{\pi}\int_0^{\pi} t\sin^3 x\sin nx\, dx = \frac{2t}{\pi}\int_0^{\pi}\sin^3 x\sin nx\, dx \\ &= \frac{t}{4\pi}\int_0^{\pi} 3\cos(n-1)x - 3\cos(n+1)x - \cos(n-3)x + \cos(n+3)x\, dx \\ &= \begin{cases} \frac{3}{4}\,, & \text{n} = 1 \\ -\frac{1}{4}t\,, & n = 3 \\ 0\,, & \text{otherwise} \end{cases} \end{aligned}$$

Then the solution is

$$u(x,t)=\sum_{n=1}^{\infty}[\int_0^t e^{-n^2(t-\tau)}B_n(\tau)d\tau]\sin(nx)$$
$$=\int_0^t \frac{3}{4}\tau\, e^{-(t-\tau)}\sin x\, d\tau-\int_0^t \frac{1}{4}\tau\, e^{-9(t-\tau)}\sin 3x\, d\tau$$
$$=\frac{3}{4}e^{-t}\sin x\int_0^t \tau\, e^{\tau}d\tau-\frac{1}{4}e^{-9t}\sin 3x\int_0^t \tau\, e^{9\tau}d\tau$$
$$=\frac{3}{4}(t-1+e^{-t})\sin x-\frac{1}{36}(t-\frac{1}{9}+\frac{1}{9}e^{-9t})\sin 3x.$$

2.3 Fourier transforms

First let us begin with functions of one variable. The Fourier transform of a function $f(x),\ x\in R,$ is defined by the equation

$$F[f](w)=\hat{f}(w)=\int_{-\infty}^{\infty} f(x)e^{iwx}dx\ .$$

If f is absolutely integrable, ie. $\int_{-\infty}^{\infty}|f(x)|dx<\infty$, then $\hat{f}$ can be shown to exist. Nice property of the Fourier transform is the simple form of the inversion formula, or inverse transform. It is

$$F^{-1}[\hat{f}](x)\equiv f(x)=\frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}\hat{f}(w)e^{-iwx}dw\,.$$

It dictates how to get back from the transform domain. Besides, we have some operational formulas for Fourier transform:

1. $F[f'](w)=-iwF[f](w)$
2. $F[ixf(x)](w)=\dfrac{d}{dw}F[f](w)$
3. $F[f(ax-b)](w)=\dfrac{1}{|a|}e^{iw\frac{b}{a}}\hat{f}(\dfrac{w}{a})$
4. $F[e^{icx}f(x)](w)=\hat{f}(w+c)$
5. $F[\cos cx\, f(x)](w)=\dfrac{1}{2}[\hat{f}(w+c)+\hat{f}(w-c)$

$$F[\sin cx\, f(x)](w) = \frac{1}{2i}[\hat{f}(w+c) - \hat{f}(w-c)$$

<u>The convolution theorem</u> for Fourier transform

If $f(x)$ and $g(x)$ are both absolutely integrable and square integrable, then

$$F[f * g(x)](w) = \hat{f}(w)\hat{g}(w).$$

<u>Proof</u>:

$$F[f * g(x)](w) = F[\int_{-\infty}^{\infty} f(x-y)g(y)dy] = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x-y)g(y)dy]\, e^{iwx}dx$$

$$= \int_{-\infty}^{\infty} g(y)e^{iwy}[\int_{-\infty}^{\infty} f(x-y)e^{iw(x-y)}dx]dy = \hat{f}(w)\int_{-\infty}^{\infty} g(y)e^{iwy}dy$$

$$= \hat{f}(w)\hat{g}(w). \qquad q.e.d$$

Some Fourier transforms can be calculated directly; many require complex contour integration. In the following example we try to solve the infinite-slab heat conduction problem.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = f(x),$$
$$u(x,t) \ \ bounded.$$

We suppose that $f(x)$ is absolutely integrable. We make the hypothesis that u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ are continuous in x and t, and absolutely integrable in x, uniformly in t. Then u and $\frac{\partial u}{\partial t}$ approach zero as $x \to \infty$.

If our hypotheses are valid, u has a Fourier transform, for fixed t,

$$\hat{u}(w,t) = \int_{-\infty}^{\infty} u(x,t)e^{iwx}dx,$$

and

$$F[u_t](w,t) = \int_{-\infty}^{\infty} u_t(x,t)e^{iwx}dx = \frac{d}{dt}\hat{u}(w,t),$$

$$F[u_{xx}](w,t) = \int_{-\infty}^{\infty} u_{xx}(x,t)e^{iwx}dx = -w^2\hat{u}(w,t).$$

Taking Fourier transforms with respect to x in the problem, we obtain the initial value problem

$$\frac{d\hat{u}}{dt} + w^2\hat{u} = 0 \qquad -\infty < w < \infty,\ t > 0$$
$$\hat{u}(w,0) = \hat{f}(w)$$

Then we solve the ordinary differential problem

$$\frac{1}{\hat{u}}d\hat{u} = -w^2 dt,$$
$$\int_0^t \frac{1}{\hat{u}}d\hat{u} = \int_0^t -w^2 d\tau,$$
$$\ln\hat{u}(w,t) - \ln\hat{u}(w,0) = -w^2 t,$$
$$\hat{u}(w,t) = \hat{u}(w,0)e^{-w^2 t} = \hat{f}(w)e^{-w^2 t},$$

whose solution is

$$\hat{u}(w,t) = \hat{f}(w)e^{-w^2 t}.$$

Then the solution formula is

$$u(x,t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{-iwx}\hat{f}(w)e^{-w^2 t}dw. \qquad (2.11)$$

Example 2.3 : (Using Fourier transformation to solve the infinite-slab heat conduction problem)

$$u_t - u_{xx} = 0, \qquad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = e^{-x^2},$$
$$u(x,t)\ bounded.$$

In this problem $f(x) = e^{-x^2}$, we can find

$$\hat{f}(w)=\int_{-\infty}^{\infty}e^{-x^2}e^{iwx}dx=\sqrt{\pi}e^{-\frac{w^2}{4}}.$$

Then the solution is

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\pi}e^{-\frac{w^2}{4}}e^{-w^2t}e^{iwx}dw$$

$$=(1+4t)^{-\frac{1}{2}}e^{\frac{-x^2}{1+4t}}.$$

2.4 Sine and Cosine transforms

If $f(x)$ is given for $0<x<\infty$, its sine transform is defined as

$$F_s(f)\equiv\int_0^{\infty}f(x)\sin wx\,dx.$$

If we extend $f(x)$ to $-\infty<x<\infty$ as an odd function, ie. $f(-x)=-f(x)$, we have

$$\hat{f}(w)=\lim_{L\to\infty}\int_{-L}^{L}f(x)e^{iwx}dx=\lim_{L\to\infty}\int_{-L}^{L}f(x)(\cos wx+i\sin wx)dx$$

$$=\lim_{L\to\infty}\int_{-L}^{L}if(x)\sin wxdx$$

$$=2i\lim_{L\to\infty}\int_{0}^{L}f(x)\sin wxdx.$$

Hence, the inverse theorem becomes

$$f(x)=\frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}e^{-iwx}\;2iF_s[f]\,dw.$$

The sine transform is clearly an odd function of w. Hence the integral on the right becomes

$4\int_0^L\sin wx\,F_s[f]\,dw$. Thus the inversion theorem is

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin wx\,F_s[f]dw,$$

or

$$f(x) = \frac{2}{\pi} F_s[F_s[f](w)](x),$$

for a function $f(x)$ defined for $0 < x < \infty$.

Similarly, we can define the cosine transform

$$F_c[f] \equiv \int_0^\infty f(x) \cos wx \, dx,$$

for a function $f(x)$ defined for $0 < x < \infty$.

If we extend $f(x)$ to $-\infty < x < \infty$ as an even function, ie. $f(-x) = f(x)$, we have

$$\hat{f}(w) = 2F_c[f].$$

The function $F_c[f]$ is even in w. Hence the inversion theorem becomes

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos wx \, F_c[f] \, dw,$$

or

$$f(x) = \frac{2}{\pi} F_c[F_c[f](w)](x),$$

for a function $f(x)$ defined for $0 < x < \infty$.

Sine and cosine transform are often useful in treating problems with boundary conditions only at x=0. And we can note that

$$F_s[f''] = f(0)w - w^2 F_s[f],$$
$$F_c[f''] = -f'(0) - w^2 F_c[f],$$

provided $f(x)$ and $f'(x) \to 0$ as $x \to \infty$. Thus sine transform is particularly useful when $f(0)$ is given, while the cosine transform is useful when $f'(0)$ is known.

Example 2.4 : (Using sine or cosine transformation to solve the heat conduction problem in a half-infinite slab)

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0\ , \qquad 0<x<\infty,\ t>0$$
$$u(0,t)=0, \tag{2.12}$$
$$u(x,0)=f(x),$$
$$u(x,t)\ bounded.$$

We suppose that $f(x)$ is absolutely integrable, and that u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ are continuous and absolutely integrable in x for each fixed t. we have

$$F_s[u](w,t)=\int_0^\infty u(x,t)\sin wx\,dx,$$

and

$$F_s[u_t](w,t)=\int_0^\infty u_t(x,t)\sin wx\,dx=\frac{d}{dt}F_s[u](w,t),$$
$$F_s[u_{xx}](w,t)=\int_0^\infty u_{xx}(x,t)\sin wx\,dx=u(0,t)w-w^2F_s[u](w)=-w^2F_s[u](w,t).$$

Taking the sine transform with respect to x in the problem, and putting $U(w,t)=F_s[u]$, we find

$$\frac{dU}{dt}+w^2U=0, \qquad 0<w<\infty,\ \mathrm{t}>0$$
$$U(w,0)=F_s[f].$$

Thus

$$U(w,t)=F_s[f]e^{-w^2t},$$

and

$$u(x,t)=\frac{2}{\pi}\lim_{L\to\infty}\int_0^L F_s[f](w)e^{-w^2t}\ \sin\ wx\,dw. \tag{2.13}$$

The problem coincides with the solution of the heat conduction in an infinite slab, provided we extend $f(x)$ to $-\infty<x<\infty$ as an odd function. The corresponding solution $u(x,t)$ of the problem (2.12) is then also odd at x=0, and hence $u(0,t)=0$

2.5 Laplace transforms

We consider a function $f(x)$ which vanishes for negative values of x:

$$f(x) = 0 \qquad for \;\; x < 0 .$$

Then if $e^{-s_1 x} f(x)$ is absolutely integrable, so is $e^{-s\,x} f(x)$ for $s \geq s_1$. It follows that the Fourier transform $\hat{f}(\xi)$ is analytic in a half-plane $\ln \xi > s_1$.

We define the Laplace transform

$$L[f](s) \equiv \int_0^\infty e^{-sx} f(x) dx ,$$

or

$$L[f](s) \equiv F[f](is) .$$

By integration by parts we find that

$$L[f'] = \int_0^\infty e^{-sx} f'(x) dx = sL[f] - f(0),$$
$$L[f''] = s^2 L[f] - sf(0) - f'(0).$$

The convolution theorem

$$L[f * g] = L[f] \cdot L[g]$$

follows from that for the Fourier transform.

By inversion theorem for the Fourier transform, we can find that the inverse formula for the Laplace transform is

$$f(x) = \frac{1}{2\pi i} \lim_{L \to \infty} \int_{s-iL}^{s+iL} L[f](\sigma) e^{\sigma x} \, d\sigma$$

where $s > s_1$ so that $L[f](\sigma)$ is analytic for $\mathrm{Re}\,\sigma \geq s_1$, and the path is vertical.

We consider the problem of heat conduction in an infinite slab, as mentioned in Section 2.3. In Section 2.3 we use Fourier transform to solve it.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = f(x)$$
$$u(x,t)\ bounded$$

Taking the Laplace transform with respect to t in the problem

Let

$$U(x,s) = \int_0^\infty e^{-ts} u(x,t)dt\,.$$

Then

$$\int_0^\infty e^{-ts} u_t(x,t)dt = sU(x,s) - f(x)\,.$$

We suppose that $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ are bounded and continuous, so that we obtain

$$\int_0^\infty e^{-ts} u_{xx}(x,t)dt = \frac{\partial}{\partial x^2} U(x,s)\,.$$

Thus, this gives

$$sU(x,s) - f(x) - \frac{\partial^2 U(x,s)}{\partial x^2} = 0\,, \qquad -\infty < x < \infty,\ s > 0 \tag{2.14}$$

For each fixed s this is an ordinary differential equation for $U(x,s)$ considered as a function of x. We now solve the equation (2.14) by means of the Fourier transform.

Let

$$\widehat{U}(w,s) = \int_{-\infty}^\infty U(x,s) e^{iwx} dx\,,$$

and

$$F[U_{xx}](w,s) = \int_{-\infty}^\infty U_{xx}(x,s) e^{iwx} dx = -w^2 \widehat{U}(w,s)\,.$$

Thus we have

$$s\widehat{U}(w,s) - \widehat{f}(w) + w^2 \widehat{U}(w,s) = 0\,, \qquad -\infty < w < \infty,\ s > 0 \tag{2.15}$$

The solution of the problem (2.15) is

$$\hat{U}(w,s)=\frac{\hat{f}(w)}{s+w^2}=\hat{f}(w)\cdot\frac{1}{\left(\sqrt{s}\right)^2+w^2}.$$

By the convolution theorem for the Fourier transform and

$$F^{-1}[\frac{1}{\left(\sqrt{s}\right)^2+w^2}](x,s)=\frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x|},$$

the solution of the problem (2.14) is given by

$$U(x,s)=\frac{1}{2\sqrt{s}}\int_{-\infty}^{\infty}e^{-\sqrt{s}|x-y|}f(y)dy.$$

The inverse Laplace transform of U(x, s) is

$$\begin{aligned} u(x,t)&=\frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL}(\frac{1}{2\sqrt{\sigma}}\int_{-\infty}^{\infty}e^{-\sqrt{\sigma}|x-y|}f(y)dy)e^{\sigma t}d\sigma \\ &=\frac{1}{2}\int_{-\infty}^{\infty}(\frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL}\frac{1}{\sqrt{\sigma}}e^{-\sqrt{\sigma}|x-y|}e^{\sigma t}d\sigma)f(y)dy. \end{aligned} \tag{2.16}$$

Therefore, to find $u(x,t)$, we need the inverse Laplace transform of $\frac{1}{\sqrt{s}}e^{-\sqrt{s}|x-y|}$. The function

$$g(\sigma)=\frac{1}{\sqrt{\sigma}}e^{-\sqrt{\sigma}|x-y|},$$

is multiple-valued, and we want to choose a particular branch cut of it. We choose that branch cut of $\frac{1}{\sqrt{\sigma}}$ along the negative real axis: $-\pi\le\arg\sigma<\pi$

Now we want to solve

$$L^{-1}[\frac{e^{-\sqrt{s}|x-y|}}{\sqrt{s}}]=\frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL}e^{\sigma t}g(\sigma)d\sigma,\qquad s>0$$

We apply Cauchy's theorem to the integral of $e^{\sigma t}g(\sigma)$ over the contour C, as shown in Figure (2.1)

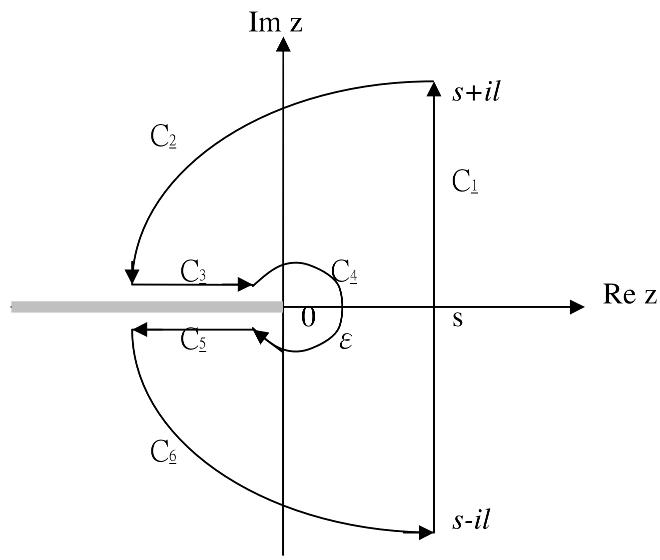


Figure 2.1 Contour C

Since $g(\sigma)$ is analytic inside this contour,

$$\oint_C e^{\sigma t} g(\sigma) = \int_{s-iL}^{s+iL} e^{\sigma t} g(\sigma) d\sigma + \sum_{n=2}^{6} \int_{C_n} e^{\sigma t} g(\sigma) d\sigma = 0 .$$

We let $L \to \infty$ and $\varepsilon \to 0$, we can find that

<u>Contour C_2</u>:

$\sigma = s + Le^{i\theta}$, θ *from* $\frac{\pi}{2}$ *to* π, $d\sigma = Lie^{i\theta} d\theta$.

$\int_{C_2} e^{\sigma t} g(\sigma) d\sigma = \int_{\frac{\pi}{2}}^{\pi} e^{(s+Le^{i\theta})t} g(s + Le^{i\theta}) L \cdot i \cdot e^{i\theta} d\theta$.

Since

$$\left| g(s + Le^{i\theta}) \right| = \left| \frac{1}{\sqrt{s + Le^{i\theta}}} \right| = \frac{1}{\left| \sqrt{s + Le^{i\theta}} \right|} = \frac{1}{\left| s + Le^{i\theta} \right|^{1/2}} \le \frac{1}{\sqrt{s - L}}$$

approach zero as $L \to \infty$. By Jordan's lemma the integrals over this contour C_2 approach zero.

Similarly to contour C_6, $\sigma = s + Le^{i\theta}$, θ *from* π *to* $\frac{3\pi}{2}$, $d\sigma = Lie^{i\theta}d\theta$, the integrals over the contour C_6 approach zero.

Contour C_4:

$\sigma = \varepsilon e^{i\theta}$, θ *from* $-\pi$ *to* π, $d\sigma = \varepsilon ie^{i\theta}d\theta$.

$$\int_{C_4} e^{\sigma t} g(\sigma)d\sigma = \int_{-\pi}^{\pi} e^{\varepsilon e^{i\theta} t} g(\varepsilon e^{i\theta})\, \varepsilon \cdot i \cdot e^{i\theta} d\theta .$$

Since

$$\left| g(\varepsilon e^{i\theta}) \right| = \left| \frac{1}{\sqrt{\varepsilon e^{i\theta}}} \right| = \frac{1}{\left| \sqrt{\varepsilon e^{i\theta}} \right|} = \frac{1}{\left| \varepsilon e^{i\theta} \right|^{1/2}} \le \frac{1}{\sqrt{\varepsilon}} ,$$

and

$$\left| e^{\varepsilon e^{i\theta} t} \right| = \left| e^{\varepsilon(\cos\theta + i\sin\theta)t} \right| = \left| e^{\varepsilon\cos\theta \cdot t} \cdot e^{i\varepsilon\sin\theta \cdot t} \right| = \left| e^{\varepsilon t \cdot \cos\theta} \right| \le e^{\varepsilon t} ,$$

thus

$$\left| \int_{C_4} e^{\sigma} g(\sigma)d\sigma \right| \le 2\pi\varepsilon e^{\sigma} \varepsilon^{-1/2} = 2\pi e^{\sigma} \varepsilon^{1/2}$$

approach zero as $\varepsilon \to 0$

Contour C_3 and C_5:

Recall

$$g(\sigma) = \frac{1}{\sqrt{\sigma}} e^{-\sqrt{\sigma}|x-y|} = |\sigma|^{1/2} e^{-i\frac{\arg\sigma}{2}} \cdot e^{-|x-y||\sigma|^{1/2} e^{i\frac{\arg\sigma}{2}}} .$$

Then putting $\sigma = -\mu^2$, we have

$$g(\sigma) = \frac{1}{i\mu} e^{-i\mu|x-y|}, \qquad for\ \arg\sigma = \pi$$

$$g(\sigma) = \frac{1}{-i\mu} e^{i\mu|x-y|}, \qquad for\ \arg\sigma = -\pi$$

$$\int_{C_3} e^{\sigma t} g(\sigma)d\sigma + \int_{C_5} e^{\sigma t} g(\sigma)d\sigma = \int_{\infty}^{0} \frac{e^{-\mu^2 t - i\mu|x-y|}}{i\mu}(-2\mu d\mu) + \int_{0}^{\infty} \frac{e^{-\mu^2 t + i\mu|x-y|}}{-i\mu}(-2\mu)d\mu$$

$$= \frac{4}{i}\int_{0}^{\infty} e^{-\mu^2 t} \cos \mu(x-y)\, d\mu.$$

Thus we find

$$\lim_{L\to\infty} \int_{s-iL}^{s+iL} e^{\sigma t} g(\sigma)d\sigma = \frac{-4}{i}\int_{0}^{\infty} e^{-\mu^2 t} \cos \mu(x-y)d\mu\,.$$

Hence by the inversion formula

$$L^{-1}[\frac{e^{-\sqrt{s}|x-y|}}{\sqrt{s}}] = \frac{2}{\pi}\int_{0}^{\infty} e^{-\mu^2 t} \cos \mu(x-y)\, d\mu\,.$$

Recalling the Fourier transform of e^{-ax^2} is $\sqrt{\frac{\pi}{a}} e^{-w^2/4a}$, we see that

$$\frac{2}{\pi}\int_{0}^{\infty} e^{-\mu^2 t} \cos \mu(x-y)\, d\mu = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\mu^2 t} \cos \mu(x-y)\, d\mu$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\mu^2 t + i\mu|x-y|}\, d\mu = \frac{1}{\pi}\sqrt{\frac{\pi}{t}} e^{-(x-y)^2/4t}$$

$$= \frac{1}{\sqrt{\pi t}} e^{-(x-y)^2/4t}.$$

We return to (2.16), we find the solution formula

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)dy\,. \tag{2.17}$$

2.6 <u>Finite Difference method</u>

Suppose that $u = u(x,t)$ is a function of two independent variables. The first partial derivatives of u are defined as limits of difference quotients.

$$u_x = \frac{\partial u}{\partial x}(x,t) = \lim_{h\to 0} \frac{u(x+h,t) - u(x,t)}{h},$$

and

$$u_t = \frac{\partial u}{\partial t}(x,t) = \lim_{k\to 0} \frac{u(x,t+k) - u(x,t)}{k}.$$

We can use the respective difference quotients to approximate these partial derivatives,

$$u_x(x,t) \cong \frac{u(x+h,t)-u(x,t)}{h}, \tag{2.18}$$

$$u_t(x,t) \cong \frac{u(x,t+k)-u(x,t)}{k}. \tag{2.19}$$

To analyze the truncation errors (TE) associated with the approximation (2.18) and (2.19), we require Taylor's theorem in two variables.

Assume that u is twice continuously differentiable and that h is positive. From Taylor's theorem, we have

$$u(x+h,t) = u(x,t) + u_x(x,t)h + u_{xx}(\xi,t)\frac{h^2}{2} \qquad x < \xi < x+h,$$

which can be rearranged to read

$$u_x(x,t) = \frac{u(x+h,t)-u(x,t)}{h} - \frac{h}{2}u_{xx}(\xi,t). \tag{2.20}$$

In (2.20) we find the partial derivative $u_x(x,t)$ expressed as a difference quotient, plus a truncation error, $-\frac{h}{2}u_{xx}(\xi,t)$. If u is twice continuously differentiable, so that $u_{xx}(x,t)$ is bounded on the interval $[x, x+h]$, then the truncation error can be made small by choosing h small.

We also require finite-difference formulas for the higher order derivatives of u. To obtain a difference formula for u_{xx}, we use Taylor's theorem to write

$$u(x+h,t) = u(x,t) + u_x(x,t)h + \frac{\partial^2 u}{\partial x^2}(x,t)\frac{h^2}{2} + \frac{\partial^3 u}{\partial x^3}(x,t)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(\xi,t)\frac{h^4}{24}. \tag{2.21}$$

and

$$u(x-h,t) = u(x,t) - u_x(x,t)h + \frac{\partial^2 u}{\partial x^2}(x,t)\frac{h^2}{2} - \frac{\partial^3 u}{\partial x^3}(x,t)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(\xi,t)\frac{h^4}{24}. \tag{2.22}$$

where $x < \xi_1 < x+h$ and $x-h < \xi_2 < x$. Adding equations (2.21) and (2.22) and then solving for u_{xx} gives a centered difference formula for $u_{xx}(x,t)$:

$$\begin{aligned} u_{xx}(x,t) &= \frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} + TE, \\ TE &= -\frac{h^2}{24}[\frac{\partial^4 u}{\partial x^4}(\xi_1,t) + \frac{\partial^4 u}{\partial x^4}(\xi_2,t)] = -\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi,t), \qquad x-h < \xi < x+h. \end{aligned} \tag{2.23}$$

where TE be simplified by the intermediate-value theorem.

Example 2.5 : (Use Finite Difference method)

$$u_t - u_{xx} + (\sin xt)u = 0\,, \quad for\ 0 < x < 1, t > 0$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = f(x).$$

Let $h = \frac{1}{N}$ for some integer N. Denote $x = 0, h, 2h, \cdots, Nh = 1$ and $t = 0, k, 2k, \cdots$. From (2.20) and (2.23),

$$\begin{aligned}\Lambda_h[u] &\equiv \frac{u(x,t+k)-u(x,t)}{k} - \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} + (\sin xt)u(x,t) \\ &= \frac{k}{2}u_{tt}(x,t+\theta_1 k) - \frac{h^2}{12}u_{xxxx}(x+\theta_2 h,t),\end{aligned} \tag{2.24}$$

where $0 < \theta_1 < 1$ and $-1 < \theta_2 < 1$. The right hand side is certainly small if h and k are small.

Let $v(nh, mk)$ be a function defined only at these points $x = nh,\ t = mk$. Let it satisfy the equation

$$\Lambda_h[v] = 0\,, \tag{2.25}$$

at those points where $0 < nh < 1$.

We replace the boundary conditions and the initial conditions by

$$v(0,mk) = v(1,mk) = 0,$$
$$v(nh,0) = f(nh).$$

We can solve the difference equation (2.25) for $v(nh, mk)$ in terms of the values of v at the time mk,

$$\begin{aligned}v(nh,mk) &= \frac{k}{h^2}[v((n+1)h,mk) + v((n-1)h,mk)] \\ &\quad + [1 - \frac{2k}{h^2} - k\sin nhmk]v(nh,mk).\end{aligned} \tag{2.26}$$

Thus we can compute $v(nh,k)$ by the given initial values, v at time $2k$ in terms of its values at k, and so on. It gives $v(nh,mk)$ for any n and m.

We hope that $v(nh,mk)$ is a good approximation to $u(nh,mk)$. Define the error w be

$$w(nh,mk) = u(nh,mk) - v(nh,mk)\,.$$

By (2.24) and (2.25),

$$\Lambda_h[w] = \frac{k}{2} u_{tt}(nh, mk + \theta_1 k) - \frac{h^2}{12} u_{xxxx}(nh + \theta_2 h, mk) .$$

If the constants A and B are bounds for $\frac{1}{12}|u_{xxxx}|$ and $\frac{1}{2}|u_{tt}|$, respectively. We see that

$$|\Lambda_h[w]| \le Ah^2 + Bk , \tag{2.27}$$

and we have

$$w(0, mk) = w(1, mk) = 0,$$
$$w(nh, 0) = 0.$$

The inequality (2.27) can be written

$$\left| w(nh, (m+1)k) - \{ \frac{k}{h^2} [w((n+1)h, mk) + w((n-1)h, mk) + [1 - \frac{2k}{h^2} - k \sin nhmk] w(nh, mk) \} \right| \quad \text{F}$$
$$\le (Ah^2 + Bk)k.$$

or each time $t = mk$ we define

$$M_m = \max_{0<n<N} |w(nh, mk)| .$$

We suppose that

$$k \le \frac{h^2}{2 + h^2} \quad , \tag{2.28}$$

so that the coefficient of $v(nh, mk)$ in (2.26) is nonnegative. The inequality becomes

$$|w(nh, (m+1)k)| \le \frac{2k}{h^2} M_m + [1 - \frac{2k}{h^2} - k \sin nhmk] M_m + (Ah^2 + Bk)k$$
$$= [1 - k \sin nhmk] M_m + (Ah^2 + Bk)k.$$

Therefore

$$M_{m+1} \le (1+k) M_m + (Ah^2 + Bk)k .$$

We multiply the inequality by $(1+k)^{-(m+1)}$ and transpose:

$$(1+k)^{-(m+1)} M_{m+1} - (1+k)^{-m} M_m \le (Ah^2 + Bk)k(1+k)^{-(m+1)} .$$

We sum both from $m = 0$ to $\frac{T}{k} - 1$ (we suppose $\frac{T}{k}$ is an integer.)

$$(1+k)^{-T/k} M_{T/k} - M_0 \le (Ah^2 + Bk)k \frac{(1+k)^{-1} - (1+k)^{-T/k - 1}}{1 - (1+k)^{-1}} .$$

Since $w(nh, 0) = 0$, we see that $M_0 = 0$. Hence

$$\left|w(nh,T)\right| \le (Ah^2+Bk)[(1+k)^{T/k}-1] \le e^T(Ah^2+Bk).$$

Thus for a fixed T $\left|u(nh,T)-V(nh,T)\right| \to 0$ uniformly in x as $h \to 0$ and $k \to 0$, provided that the inequality (2.28) is satisfied. The inequality of the type (2.28) is called a stability condition for the problem.

Example 2.6 : (Use Finite Difference method)

$$\begin{aligned} &u_t - u_{xx} = 0, && 0<x<1, t>0 \\ &u(0,t)=u(1,t)=0, && t>0 \\ &u(x,0)=100\sin \pi x. && 0<x<1 \end{aligned}$$

Using separation of variables as in Section 2.1, the exact solution is

$$u(x,t)=100^{-\pi^2 t}\sin \pi x.$$

By equation (2.20) and (2.23)

$$u_t - u_{xx} \cong \frac{u(x,t+k)-u(x,t)}{k} - \frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} = 0.$$

Let $h=\frac{1}{N}$ for some integer N. Denote $x=0,h,2h,\cdots,Nh=1$ and $t=0,k,2k,\cdots$.

Then we have that

$$u(nh,(m+1)k)=\frac{k}{h^2}[u((n+1)h,mk)+u((n-1)h,mk)]+[1-\frac{2k}{h^2}]u(nh,mk).$$

In this problem we must suppose that

$$1-\frac{2k}{h^2} \ge 0.$$

Let $r=\frac{k}{h^2}$. In general the solution is stable if and only if $r \le \frac{1}{2}$.

(1) $r=\frac{1}{2}$ Choosing $k=0.005,\ h=0.1,\ N=10$ gives the numerical solution shown in Table 2.1.

T=0.5	Numerical	Exact
x=0.0	0.000000	0.000000
x=0.1	0.204463	0.222241
x=0.2	0.388912	0.422728
x=0.3	0.535291	0.581836
x=0.4	0.629273	0.683989
x=0.5	0.661656	0.719188
x=0.6	0.629273	0.683989
x=0.7	0.535291	0.581836
x=0.8	0.388912	0.4227528
x=0.9	0.204463	0.222241
x=1.0	0.000000	0.000000

Table 2.1 Comparison of the numerical solution and exact solution k=0.005, h=0.1

(2) $r = \frac{1}{6}$ Choosing $k = 0.001667$, $h = 0.1$, $N = 10$ gives the numerical solution shown in Table 2.2.

T=0.5	Numerical	Exact
x=0.0	0.000000	0.000000
x=0.1	0.222040	0.222241
x=0.2	0.422346	0.422728
x=0.3	0.581309	0.581836
x=0.4	0.683370	0.683989
x=0.5	0.718538	0.719188
x=0.6	0.682270	0.683989
x=0.7	0.581309	0.581836
x=0.8	0.422346	0.4227528
x=0.9	0.222040	0.222241
x=1.0	0.000000	0.000000

Table 2.1 Comparison of the numerical solution and exact solution k=0.001667, h=0.1

II. Comparison with various solving methods

3.1 The limit of Separation of variables

In chapter 2 we introduce the Separation of variables to solve the problem. But in the processes of solving the problem we can find the limit of Separation of variables.

1. The differential operator L must be separable.

   Example:

$$\begin{aligned} &u_{tt} + 2u_{xt} + u_{xx} = 0, \qquad 0 < x < \pi,\ t > 0 \\ &u(0,t) = u(\pi,t) = 0, \\ &u(x,0) = f(x). \end{aligned} \tag{3.1}$$

   We can not use the Separation of variables to solve the problem (3.1). Because if

$$u(x,t) = X(x)T(t).$$

   We substitute u into the differential equation, and divide u, this gives

$$\frac{T''}{T} + 2\frac{X'T'}{XT} + \frac{X''}{X} = 0. \tag{3.2}$$

   By (3.2) we can know the equation in (3.1) is not sepsrable.

   Then if the differential equation contains u_{xt}, then the problem can not be solved by Separation of variables.

2. All boundary conditions must be on lines x=constant. That is, the range of x must be bounded.

3. The linear operators defining the boundary conditions at x=constant must involve no partial derivatives of u with respect to t, and their coefficients must be independent of t.

3.2 Sine- and cosine-transform v.s Fourier transform

In general we use sine- or cosine- transform to solve the half-infinite slab heat conduction problem. But we can also use Fourier transform to solve this problem if we extend $f(x)$ to $-\infty < x < \infty$ as an odd or even function. Recalling the problem (2.12), we extend $f(x)$ to

$-\infty < x < \infty$ as an odd function, then the problem becomes

$$\begin{aligned} &u_t - u_{xx} = 0, \qquad -\infty < x < \infty \\ &u(x,0) = f(x), \\ &f(x) \text{ is an odd function.} \end{aligned} \tag{3.3}$$

Because $f(x)$ is odd at x=0, we have

$$\hat{f}(w) = 2iF_s[f](w).$$

By the solution (2.11) and (3.3)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{-w^2 t}e^{iwx}dw \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} iF_s[f](w)e^{-w^2 t}(\cos wx - i\sin wx)dw. \end{aligned}$$

Since $F_s[f](w)$ and $\sin wx$ are an odd function of w and $\cos wx$ is an even function of w, then

$$u(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} F_s[f](w)e^{-w^2 t}\sin wx dw$$

is the same as the solution (2.13).

3.3 <u>Fourier Transform and Laplace Transform</u>

In chapter 2, we use the Fourier transform and Laplace transform to solve the infinite-slab heat conduction problem and we gain two solutions (2.11) and (2.17). We must identify that tow solutions are the same.

With the Fourier transform

$$u(x,t) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{-iwx}\hat{f}(w)e^{-w^2 t}dw,$$

where

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{iwx}dx.$$

Then we have

$$u(x,t)=\frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}e^{-iwx}e^{-w^2t}\left(\int_{-\infty}^{\infty}f(y)e^{iwy}dy\right)dw$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)[\lim_{L\to\infty}\int_{-L}^{L}e^{-w^2t}e^{iw(y-x)}dw]dy.$$

Recalling the Fourier transform of e^{-ax^2} is $\sqrt{\frac{\pi}{a}}e^{-w^2/4a}$.

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\sqrt{\frac{\pi}{t}}\,e^{-(x-y)^2/4t}dy$$
$$=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}f(y)e^{-(x-y)^2/4t}dy,$$

is the same as the solution (2.17)

For Fourier transform we need to integrate the function from -∞ to ∞, then we usually take Fourier transform into PDE with respect to x for fixed t because of $x\in R$.

Similar to Laplace transform we need to integrate the function from 0 to ∞, then we take Laplace transform into PDE with respect to t for fixed x because of $t>0$.

III. Develop the function $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ to solve linear parabolic PDE

We know that there are some differential equations whose solution space is in the Riemann surface. In this chapter, we want to compute the integrals $\int_\gamma \frac{1}{f(z)}dz$, where γ is in the Riemann surface of algebraic curve $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$. We will develop an algorithm such that we can compute the integrals $\int_\gamma \frac{1}{\sqrt{\prod_{j=1}^{n}(z-z_j)}}dz$ by Mathematica®5.

Before computing integrals, it is necessary to discuss the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$.

4.1. Fundamental introduction

For simplicity, we take $f(z)=\sqrt{z}$ to define a single-value and analytic function on the Riemann surface.

Now we let $z\in C$, and use polar form for z. That is,

$$z = re^{i\theta} \quad , \tag{4.1}$$

$$= re^{i(\theta+2\pi)} \quad . \tag{4.2}$$

Then by (4.1)

$$\sqrt{z} = r^{\frac{1}{2}}e^{i\frac{\theta}{2}},$$

and by (4.2)

$$\sqrt{z} = r^{\frac{1}{2}}e^{i(\frac{\theta+2\pi}{2})} = r^{\frac{1}{2}}e^{i(\frac{\theta}{2}+\pi)} = -r^{\frac{1}{2}}e^{i\frac{\theta}{2}}.$$

Therefore $f(z)=\sqrt{z}$ is a multi-valued function at each $z\in C$ and is not analytic on C.

How to make $f(z)=\sqrt{z}$ to be a single-valued and analytic at every point on C?

Consider two cuts from 0 to $-\infty$ ($i.e.$ the negative real axis) and

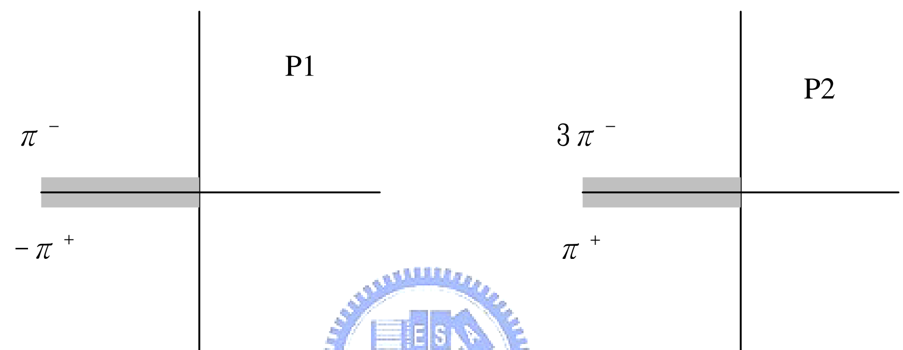
Let

$$P_1=\{C\setminus(-\infty,0]\ \mid\ \theta_1=\arg z\in[-\pi^+,\pi^-)\ \}$$

and

$$P_2=\{C\setminus(-\infty,0]\ \mid\ \theta_2=\arg z\in[\pi^+,3\pi^-)\ \}$$

as Fig. 4-1 shows.



(a) (b)

Fig. 4-1 P1, P2 plane

Define

$$f_1(z)=\sqrt{z}\ ,\quad z\in P_1\,.$$

$$f_2(z)=\sqrt{z}\ ,\quad z\in P_2\,.$$

Then

$f_1(z)=\sqrt{z}=|z|^{\frac{1}{2}}e^{i\frac{\theta_1}{2}}$ is single-valued at each $z\in P_1$ and analytic on P_1,

$f_2(z)=\sqrt{z}=|z|^{\frac{1}{2}}e^{i\frac{\theta_2}{2}}=|z|^{\frac{1}{2}}e^{i\frac{\theta_1+2\pi}{2}}=|z|^{\frac{1}{2}}e^{i\frac{\theta_1}{2}}e^{i\pi}=-|z|^{\frac{1}{2}}e^{i\frac{\theta_1}{2}}=-f_1(z)$ is also single-valued at each $z\in P_2$ and analytic on P_2.

Let

$$D_1 = \{ \ (-\infty, 0] \ | \ \arg z = \pi \ \},$$

as shown in Fig. 4-2.

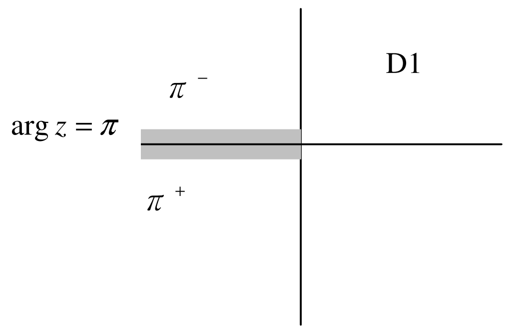


Fig. 4-2 $D_1 = \{ \ (-\infty, 0] \ | \ \arg z = \pi \ \}$

If $z \in P_1$ and $\arg z$ tends to π^- , then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$,

If $z \in P_2$ and $\arg z$ tends to π^+ , then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$,

So $\sqrt{z}$ is continuous cross the cut $(-\infty, 0]$ for $z \in D_1$.

We define

$$f_3(z) = \sqrt{z} \ , \ z \in D_1,$$

then

$$f_3(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}} \text{ for } z \in D_1 \text{ and analytic on } D_1.$$

Let

$$D_2 = \{ \ (-\infty, 0] \ | \ \arg z = 3\pi \ \},$$

as shown in Fig. 4-3.

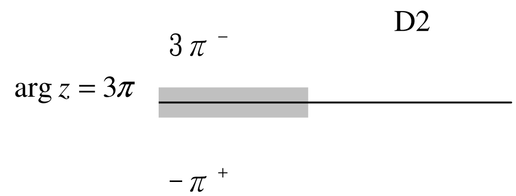


Fig. 4-3 $D_2 = \{ \ (-\infty,0] \ | \ \arg z = 3\pi \ \}$

If $z \in P_2$ and $\arg z$ tends to $3\pi^-$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{3\pi}{2}} = -i|z|^{\frac{1}{2}}$,

If $z \in P_1$ and $\arg z$ tends to $-\pi^+$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} = -i|z|^{\frac{1}{2}}$,

So $\sqrt{z}$ is continuous cross the cut $(-\infty,0]$ for $z \in D_2$.

We define

$$f_4(z) = \sqrt{z}\ , \quad z \in D_2\ ,$$

then

$$f_4(z) = -i|z|^{\frac{1}{2}} = -f_3(z) \ \text{ for } \ z \in D_2 \ \text{ and analytic on } \ D_2 .$$

According the discuss above, we can construct a single-valued function for $\sqrt{z}$.

We have the conclusion as the following:

Let $R_2 = P_1 \cup P_2 \cup (-\infty,0]$ and a function $F : R_2 \to C$, define

$$F(z) = \begin{cases} f_1(z) & , \quad z \in P_1 \\ f_2(z) & , \quad z \in P_2 \\ f_3(z) & , \quad z \in D_1 \\ f_4(z) & , \quad z \in D_2 \end{cases}$$

then $F(z)$ is single-valued and analytic at every point $z \in R_2$.

Note that $f_1(z) = -f_2(z)$ and $f_3(z) = -f_4(z)$.

Moreover, $F(z)$ is defined on a Riemann surface R_2 which is a generalization of the complex plane to a surface of more than one sheet such that a multi-valued function has only one value corresponding to each point on the surface.

4.2. Riemann surface of the algebraic curve $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in R$

Consider $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$, $z_j \in R$ and $z_1 > z_2 > z_3 > \ldots > z_n$ with n distance branch points.

4.2.1 The cut structure of $f(z)$

Since $f(z)$ is a two-valued function, we need branch cuts to define a single-valued and analytic function. But how can we construct branch cuts？

In this paper we by face the left direction to do cut explained. For convenience, let $n=2$ and $n=3$ to see what is going on？

First we check if there is any cut, for $n=2$ and $z_2=1$, $z_1=2$, as shown in Fig. 4-4.

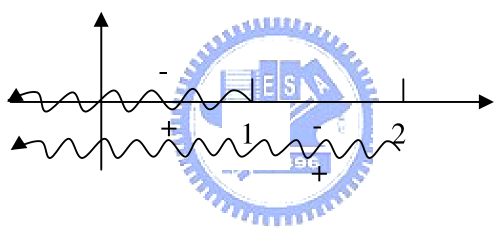


Fig. 4-4 The branch points are $z_2=1$ and $z_1=2$

Consider $-1 \in (-\infty,1)$, then we have

$$\arg(-1-1)=\arg(-2)=\begin{cases}-\pi \\ \pi\end{cases}.$$

$$\arg(-1-2)=\arg(-3)=\begin{cases}-\pi \\ \pi\end{cases}.$$

Taking $-\pi$: $\sqrt{-2}\cdot\sqrt{-3}=|2|^{\frac{1}{2}}|3|^{\frac{1}{2}}e^{i(\frac{-2\pi}{2})}=-|6|^{\frac{1}{2}}$. (4.3)

Taking π : $\sqrt{-2}\cdot\sqrt{-3}=|2|^{\frac{1}{2}}|3|^{\frac{1}{2}}e^{i(\frac{2\pi}{2})}=-|6|^{\frac{1}{2}}$. (4.4)

Since (4.3) = (4.4), there is no cut in $(-\infty,1)$

Consider $\frac{3}{2} \in (1,2)$, then we have

$$\arg(\frac{3}{2}-1) = \arg(\frac{1}{2}) = 0,$$

$$\arg(\frac{3}{2}-2) = \arg(-\frac{1}{2}) = \begin{cases} -\pi \\ \pi \end{cases}.$$

Taking $-\pi$: $\sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} e^{i(\frac{-\pi}{2})} = i\left|\frac{1}{4}\right|^{\frac{1}{2}}$. (4.5)

Taking π : $\sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} = -i\left|\frac{1}{4}\right|^{\frac{1}{2}}$. (4.6)

Since (4.5) $\neq$ (4.6), there is a cut in $(1,2)$

Hence we have the branch cut in [1,2], as shown in Fig. 4-5.

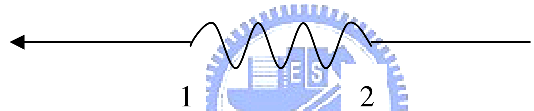


Fig. 4-5 The cut structure for $n=2$ branch points in horizontal

But we can use the simpler way to get branch cut. Recall Fig. 4-4. When crossing the cut even times in each line section, it will not change sign. When crossing the cut odd times in each line section will change sign, this implies the line section will form a branch cut. Hence we have the branch cut in $[z_2, z_1]$. The cut structure is shown in Fig.4-6.

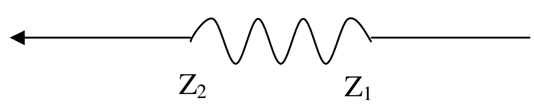
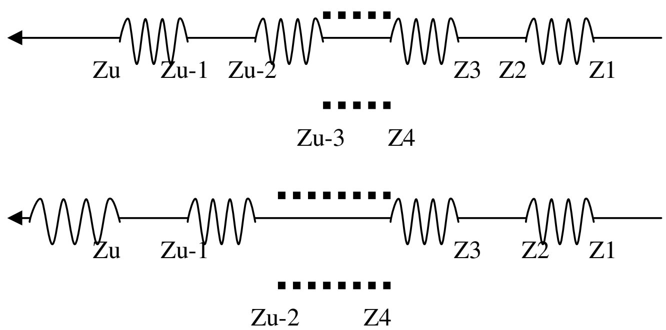


Fig.4-6 The cut structure for four branch points in horizontal

Now given n branch points, If n is even, then the branch cuts are $[z_n, z_{n-1}]$ 、

$[z_{n-2}, z_{n-3}]$ …… and $[z_2, z_1]$. If n is odd，then the branch cuts are $(-\infty, z_n]$、$[z_{n-1}, z_{n-2}]$…… and $[z_2, z_1]$. Show as Fig.4-7.



with $n \in even$

with $n \in odd$

Fig.4-7 branch cuts with $n \in even$ and $n \in odd$

4.2.2 <u>The algebraic and geometric structure for Riemann surface of horizontal cut</u>

For simplicity, we use $n = 3$ to discuss the structure for Riemann surface of $f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)}$ in horizontal cut.

(Ⅰ) Algebraic structure

As Fig.4-8 shows, $(-\infty, z_3]$、$[z_2, z_1]$ represent the cuts in this Riemann surface and 〝+〞, 〝−〞 are defined as following(the initial edge with +, the terminal edge with –) :

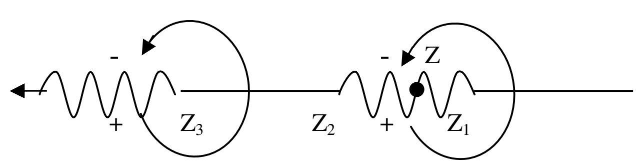


Fig.4-8 The algebraic structure for three branch points in horizontal

Case 1 : If $z \in I^{+}$ (+ edge of sheet I)

As the Fig.4-8 shows, $z \in [z_2, z_1]$

Since $z - z_j > 0 \;\Rightarrow\; \arg(z - z_j) = 0$ for $j = 2,3$.

$z - z_j < 0 \;\Rightarrow\; \arg(z - z_j) = -\pi$ for $j = 1$.

Then

$$f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)} = \prod_{j=1}^{3}\sqrt{z - z_j}$$

$$= |z - z_1|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} \cdot \prod_{j=2}^{3} |z - z_j|^{\frac{1}{2}} e^{i\cdot 0}$$

$$= e^{i(-\frac{\pi}{2})} \cdot \prod_{j=1}^{3} |z - z_j|^{\frac{1}{2}} = (-i) \cdot \prod_{j=1}^{3} |z - z_j|^{\frac{1}{2}}.$$

Case 2 : If $z \in I^{-}$ (− edge of sheet I)

As the Fig.4-11 shows, $z \in [z_2, z_1]$

Since $z - z_j > 0 \;\Rightarrow\; \arg(z - z_j) = 0$ for $j = 2,3$.

$z - z_j < 0 \;\Rightarrow\; \arg(z - z_j) = \pi$ for $j = 1$.

Then

$$f(z) = \sqrt{\prod_{j=1}^{3}(z - z_j)} = \prod_{j=1}^{3}\sqrt{z - z_j}$$

$$= |z - z_1|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} \cdot \prod_{j=2}^{3} |z - z_j|^{\frac{1}{2}} e^{i\cdot 0}$$

$$= e^{i(\frac{\pi}{2})} \cdot \prod_{j=1}^{3} |z - z_j|^{\frac{1}{2}} = (i) \cdot \prod_{j=1}^{3} |z - z_j|^{\frac{1}{2}}.$$

Note that $f(z)\,\big|_{I^{-}} = -f(z)\,\big|_{I^{+}}$, this result is the same with what we discuss before.

$$\Rightarrow \quad f(z)\,\big|_{II} = -f(z)\,\big|_{I}$$

(Ⅱ) Geometric structure

After knowing the algebraic structure, we will discuss about how to construct a geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$. According to algebraic structure for Riemann surface, we know that if n is even, then the branch cuts are $[z_n, z_{n-1}]$ 、$[z_{n-2}, z_{n-3}]$…… and $[z_2, z_1]$. It implies we have $\frac{n}{2}-1$ holes. If n is odd, then the branch cuts are $(-\infty, z_n]$ 、$[z_{n-1}, z_{n-2}]$…… and $[z_2, z_1]$. It implies we have $\frac{n-1}{2}$ holes. And we obtain one sheet with two edges in each cut by taken of counterclockwise which labeled the edge of lower- cut with + and the edge of upper- cut with –. Since there are two surface, one is, say sheet Ⅰ with $\arg f(z)\in[-\pi,\pi)$; another is, say sheet Ⅱ with $\arg f(z)\in[\pi,3\pi)$.

By definition, the – edge of sheet Ⅰ is joined to the + edge of sheet Ⅱ, and the + edge of sheet Ⅰ is joined to the – edge of sheet Ⅱ. Whenever crossing the cut, we pass from one sheet to the other sheet and the value is continuous which from our construction.

Note that $f(z)\mid_{II} = -f(z)\mid_{I}$ and for $f(z)$, supra-half-ball represents sheet Ⅰ, and infra-half-ball represents sheet Ⅱ.

We take $n=3$ to discuss the geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ in horizontal cuts, as shown in Fig.4-9.

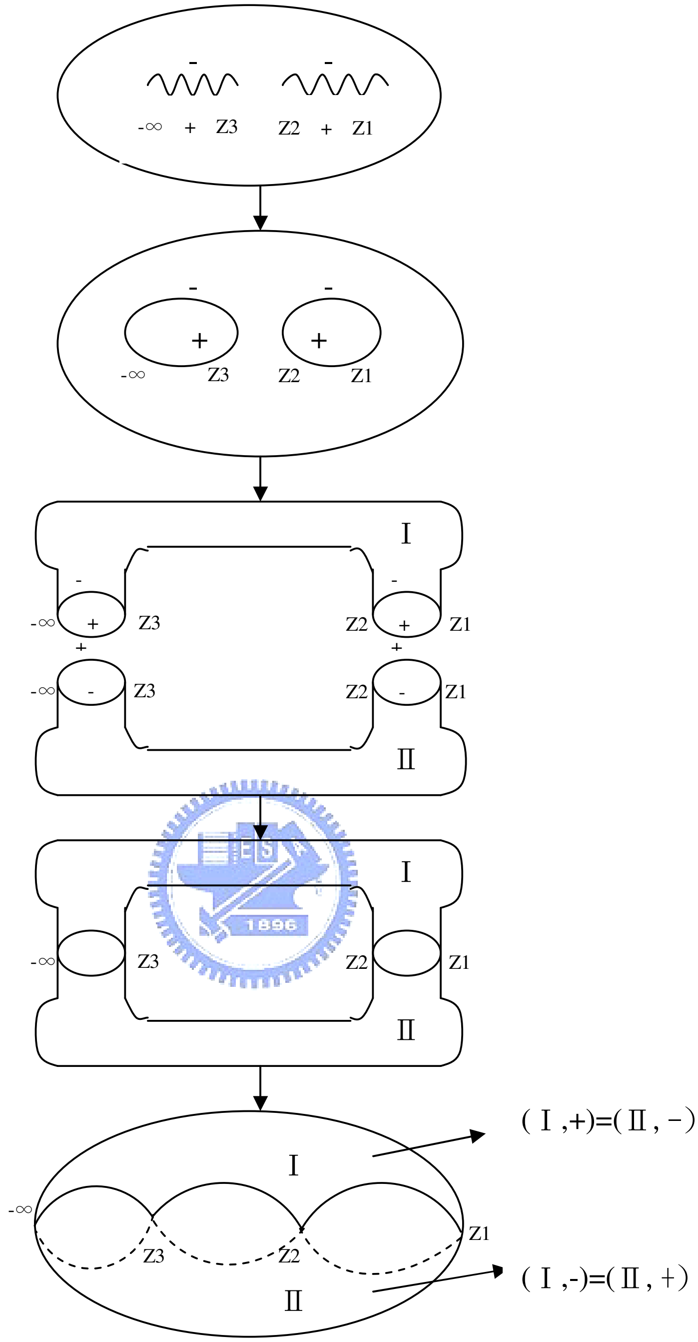


Fig.4-9 The geometric structure for Riemann surface with $n = 3$ in horizontal cut

(Ⅲ) Algebraic structure *v.s* Geometric structure

We also use $n = 3$ to discuss. Before talking about the relation between algebraic structure and geometric structure, we need to denote something as the following :

(a) If the curve is drawn by solid line :

In algebraic structure, it means the curve is in sheet Ⅰ;

In geometric structure, it means the curve is in the overhead Riemann surface.

(b) If the curve is drawn by dash line :

In algebraic structure, it means the curve is in sheet Ⅱ;

In geometric structure, it means the curve is in the ventral Riemann surface.

We give some example to show that the curve in algebraic structure and its corresponding in geometric structure in Fig.4-10 to Fig.4-12.

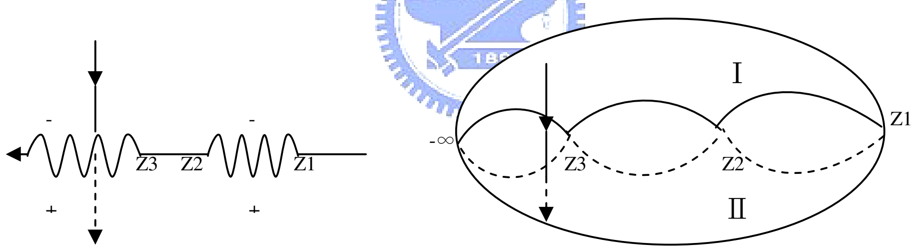


Fig.4-10 The rule in algebraic structure and geometric structure

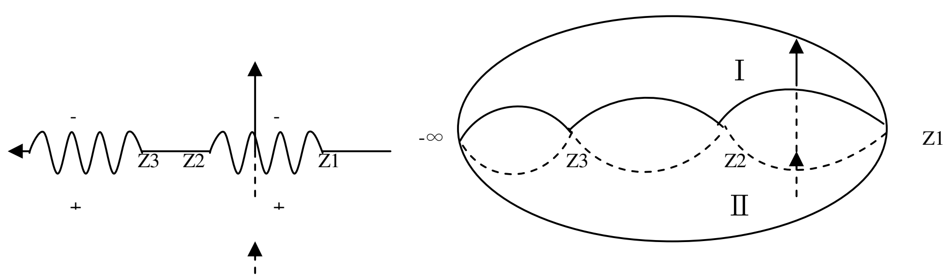


Fig.4-11 The rule in algebraic structure and geometric structure

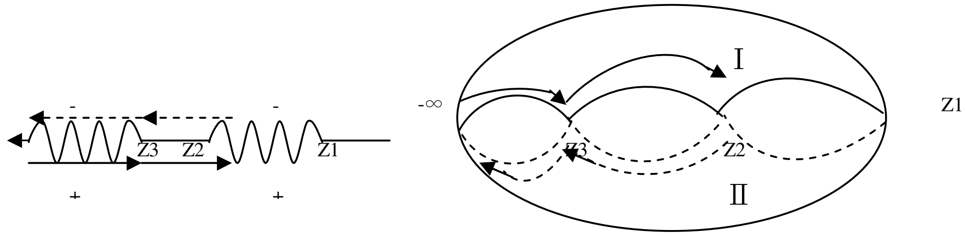


Fig.4-12 The rule in algebraic structure and geometric structure

4.3. Riemann surface of the algebraic curve $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in C$

In this section, we discuss the vertical cut structure. We define that $(z, f(z))$ belong to sheet I if and only if $\arg\prod_{j=1}^{n}(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2})$, *i.e.* $\arg(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2})$ for each j . And $f(z)\mid_{II} = -f(z)\mid_{I}$.

4.3.1 The vertical cut structure

We consider $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in C$ for $j=1,2,3,\cdots,n$ and we by face the up direction to do cut explained. The method of analyzing the vertical cut structure is the same as horizontal cut structure.

Then we can use the simpler way to get branch cut. We take $n=4$ with $z_1=i$ 、 $z_2=2i$ 、 $z_3=3i$ and $z_4=4i$, that is, $z_1<z_2<z_3<\ldots<z_n$, as shown in Fig.4-13.

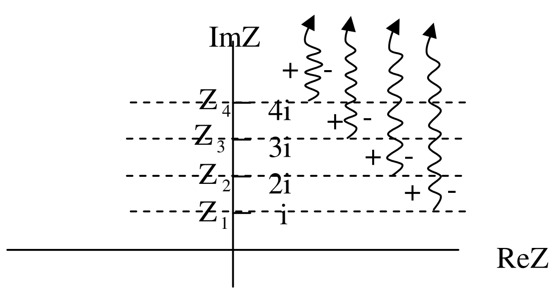


Fig.4-13 The cut appears at $z < z_j$ for each z_j

When crossing the cut even times in each line section, it will not change sign. When crossing the cut odd times in each line section will change sign，this implies the line section will form a branch cut. Hence we have the branch cuts in $[z_4, z_3]$ and $[z_2, z_1]$. The cut structure is showed in Fig.4-14.

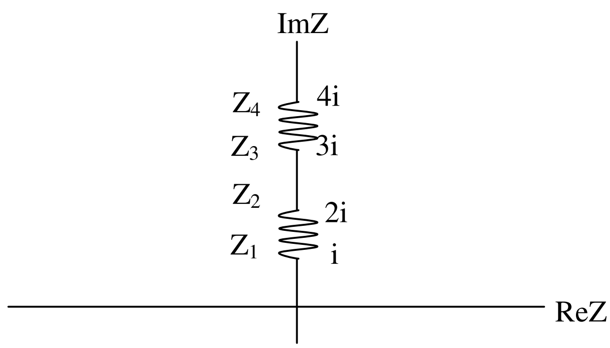


Fig.4-14 The cut structure for four branch points in vertical

4.3.2 <u>The algebraic and geometric structure for Riemann surface of vertical cut</u>

For simplicity, we use $n = 4$ to discuss the structure for Riemann surface of $f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)}$ in vertical cut. In the cut structure, we still depend on the countclockwise to take 〝+〞、〝−〞 sign. The definition of solid-line and dash-line are the same as horizontal cut case.

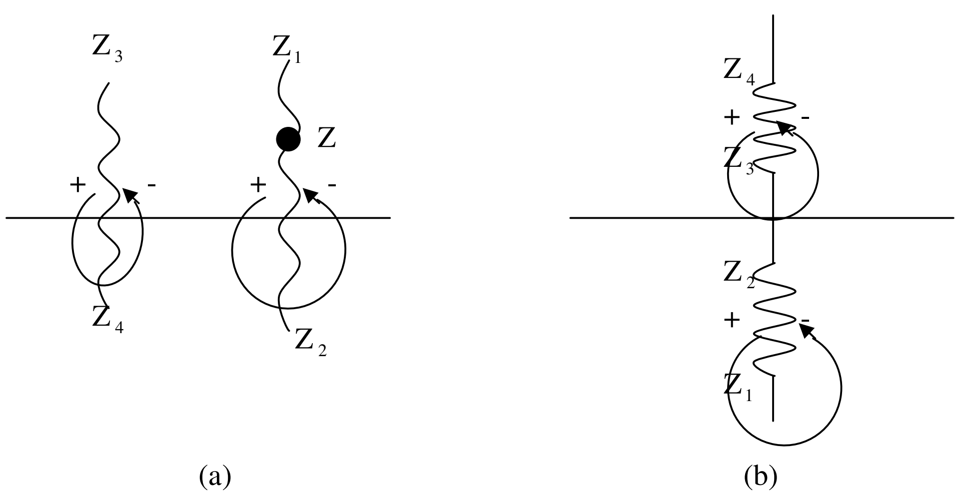


Fig.4-15 The algebraic structure for four branch points in vertical

(I) Algebraic structure

As Fig.4-15 shows, $[z_4, z_3]$ and $[z_2, z_1]$ represent the cuts in Riemann surface.

Case 1 : If $z \in I^+$ (+ edge of sheet I)

As the Fig.4-18 (a) shows, $z \in [z_2, z_1]$

Since $\arg(z - z_1) = -\frac{\pi}{2}$ and $\arg(z - z_2) = -\frac{3\pi}{2}$. $\arg(z - z_j) \in (-\pi, \frac{\pi}{2})$ for $j = 3,4$.

Then

$$f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)} = \prod_{j=1}^{4}\sqrt{z - z_j}$$

$$= |z - z_2|^{\frac{1}{2}} e^{i(-\frac{3\pi}{4})} \cdot \prod_{j=1,3,4} |z - z_j|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}}$$

$$= (-\frac{\sqrt{2}}{2} i)|z - z_2|^{\frac{1}{2}} \cdot \prod_{j=1,3,4} |z - z_j|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}}.$$

Case 2 : If $z \in I^-$ (– edge of sheet I)

As the Fig.4-18 (a) shows, $z \in [z_2, z_1]$.

Since $\arg(z - z_1) = -\frac{\pi}{2}$ and $\arg(z - z_2) = \frac{\pi}{2}$. $\arg(z - z_j) \in (-\pi, \frac{\pi}{2})$ for $j = 3,4$.

Then

$$f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)} = \prod_{j=1}^{4}\sqrt{z - z_j}$$

$$= |z - z_2|^{\frac{1}{2}} e^{i(\frac{\pi}{4})} \cdot \prod_{j=1,3,4} |z - z_j|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}}$$

$$= (\frac{\sqrt{2}}{2} i)|z - z_2|^{\frac{1}{2}} \cdot \prod_{j=1,3,4} |z - z_j|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}}.$$

Note that $f(z)\mid_{I^-} = -f(z)\mid_{I^+}$, this result is the same with what we discuss before.

$$\Rightarrow f(z)\mid_{II} = -f(z)\mid_{I}.$$

(Ⅱ) Geometric structure

The construct a geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ is the same as horizontal cuts.

By above example and illustration, we discusses the geometric structure for Riemann surface in vertical cuts. Show as Fig.4-16 (page 52).

4.4. The integrals over a, b cycles

We want to evaluate $\oint_a \frac{1}{f(z)}dz$ and $\oint_b \frac{1}{f(z)}dz$ for n branch points where a, b represent the a, b cycles over the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in C$, and develop an algorithm such that the integrals can be easily computed.

4.4.1 The a, b cycles over the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$

(A) In horizontal cut :

Let $z_1, z_2, \cdots, z_n$ be the n branch points in $x-axis$ with $z_j \in C$, then

$f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ forms a $N-holes$ Riemann surface where $N \in Z^+ \cup \{0\}$ and

$$\begin{cases} N=\dfrac{n-1}{2} & for \quad n \quad odd \\ N=\dfrac{n-2}{2} & foe \quad n \quad even \end{cases}$$

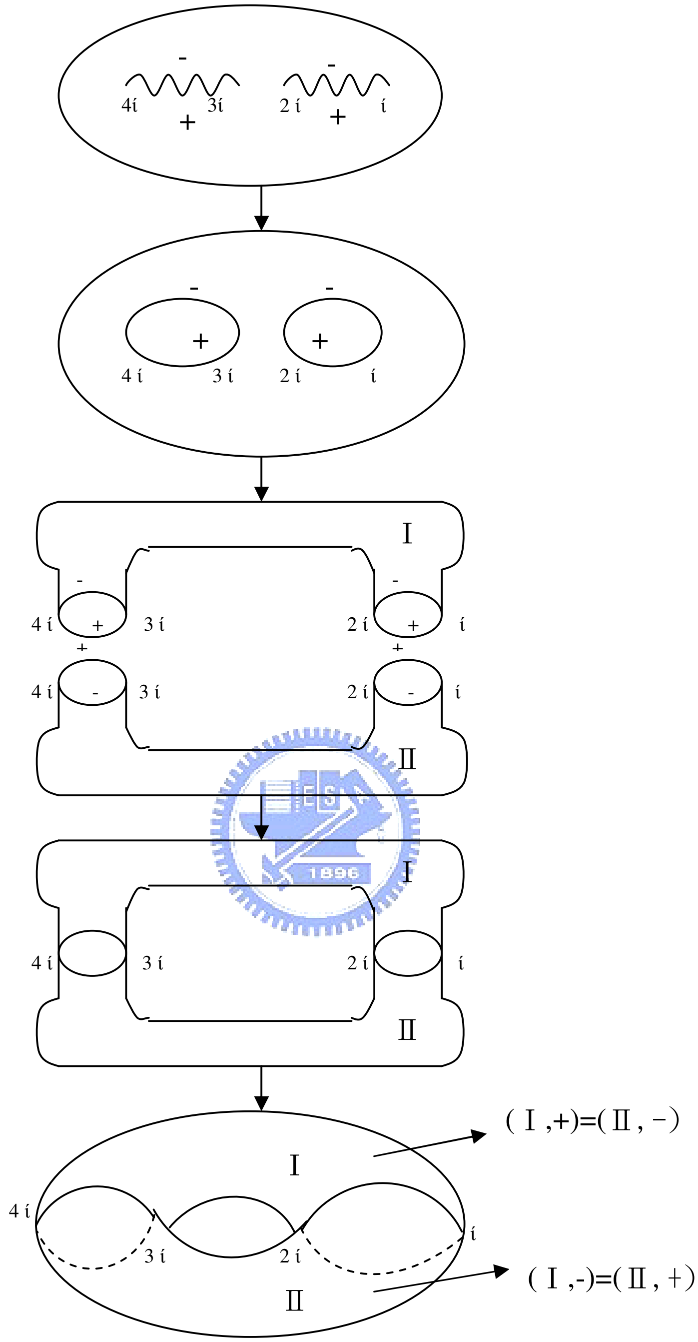


Fig.4-16 The geometric structure for Riemann surface with $n = 4$ in vertical cuts

So there are N a, b cycles. The Fig.4-17 represents the a, b cycles in the Riemann surface for n is even and the Fig.4-18 is the case for n is odd.

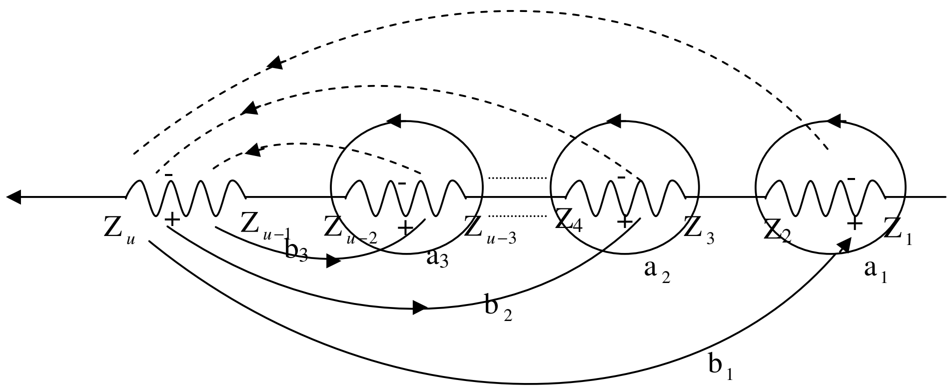


Fig.4-17 a, b cycles for horizontal cuts of even branch points

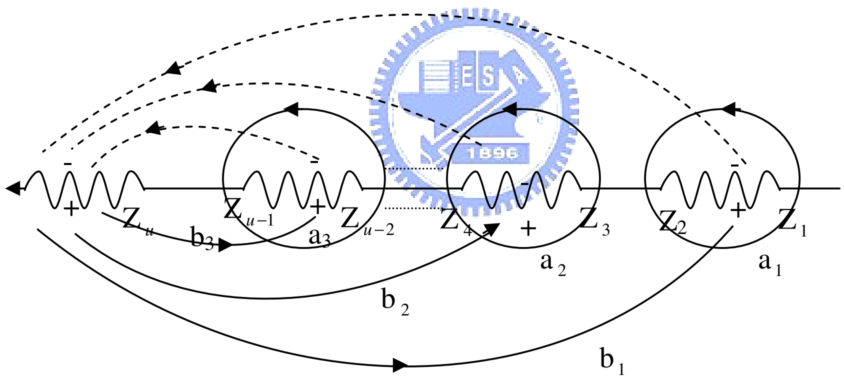


Fig.4-18 a, b cycles for horizontal cuts of odd branch points

(B) In vertical cut :

Let $z_1, z_2, \cdots, z_n \in C$ be the n branch points where n is even and $z_{2k} = \bar{z}_{2k-1}$, $k = 1,2,\cdots,\frac{n}{2}$. There are $\frac{n-2}{2}$ a, b cycles in the Riemann surface showed in Fig.4-19. For a_k cycle, it encloses the cut $\overline{z_{2k-1}z_{2k}}$, b_k cycle is passed through the cut $\overline{z_{2k-1}z_{2k}}$ from one sheet to the other.

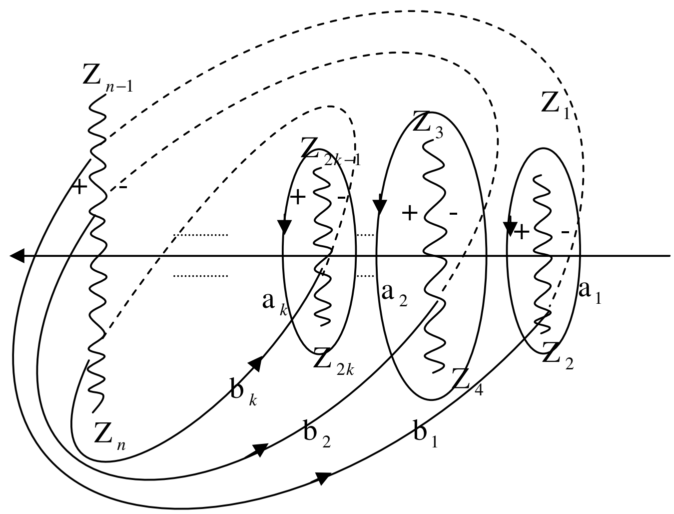


Fig.4-19 a, b cycles for vertical cuts

Let $z_1, z_2, \cdots\cdots, z_n \in C$ be the n branch points where n is even and $z_{2k} = \bar{z}_{2k-1}$, $k = 1,2,\cdots,\frac{n}{2}$. There are $\frac{n-2}{2}$ a, b cycles in the Riemann surface shown in Fig.4-20.

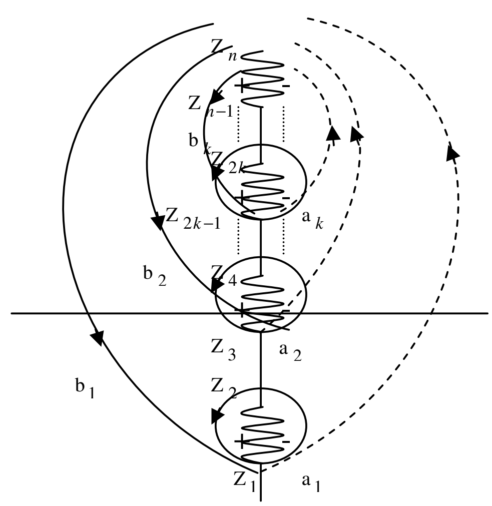


Fig.4-20 a, b cycles for vertical cuts

4.4.2 About 〝Mathematica〞 and how to modify

All programs in this paper are run by Mathematica®5. But we can not compute directly, before computing we need to give some adjustments. Since Mathematica®5 reads argument of any complex number in $(-\pi,\pi]$ only, then it just gives right answer in sheet Ⅰ in horizontal cuts (expect at the argument $-\pi$).

Consider the branch points z_j, $j=1,2,\ldots,n$. In horizontal cut structure we define $\arg(z-z_j)\in[-\pi,\pi)$, for $z\in C$. In vertical cut structure we define $\arg(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2})$, for $z\in C$. But in Mathematica®5, it defines $\arg(z-z_j)\in(-\pi,\pi]$, for $z\in C$. Then before computing the integral we must modify the function so that we can get the correct value.

In horizontal cut structure the value of $\sqrt{z-z_j}$ in our Theory and in Mathematica®5 are different at a point z with $\arg(z-z_j)=-\pi$, so we must modify the function $\sqrt{z-z_j}$ at $\arg(z-z_j)=-\pi$. But if point z with $\arg(z-z_j)=-\pi$ is only a point on the contour r, then it can not influence the value of $\int_r\sqrt{z-z_j}dz$ so that we need not modify the function $\sqrt{z-z_j}$.

In vertical cut structure the value of $\sqrt{z-z_j}$ in our Theory and in Mathematica®5 are different at some points z with $\arg(z-z_j)\in[-\frac{3\pi}{2},-\pi)$ so that we must modify the function $\sqrt{z-z_j}$ at $\arg(z-z_j)\in[-\frac{3\pi}{2},-\pi)$.

Besides, the askew cut structure is the same as horizontal cut and vertical cut structure. So we define $\arg(z-z_j)\in[\frac{\pi}{4},\frac{9\pi}{4})$, for $z\in C$. It implies that we must modify the function $\sqrt{z-z_j}$ at $\arg(z-z_j)\in(\pi,\frac{9\pi}{4})$.

4.4.3 Evaluation of $\oint_a \frac{1}{f(z)}dz$ and $\oint_b \frac{1}{f(z)}dz$

In this section we give three examples about the horizontal cut, vertical cut and askew cut. In the three examples we try to modify the function $f(z)$.

Example 4.1 :

Let $n=6$, and $z_1=4$、$z_2=3$、$z_3=2$、$z_4=1$、$z_5=-1$ and $z_6=-2$ are six branch points , as shown in Fig.4-21.

If $f(z)=\prod_{j=1}^{6}(z-z_j)^{\frac{1}{2}}$, then $\oint_r \frac{1}{f(z)}dz=?$ where $r=a,b$ cycles.

We use Mathematica®5 to compute the integral.

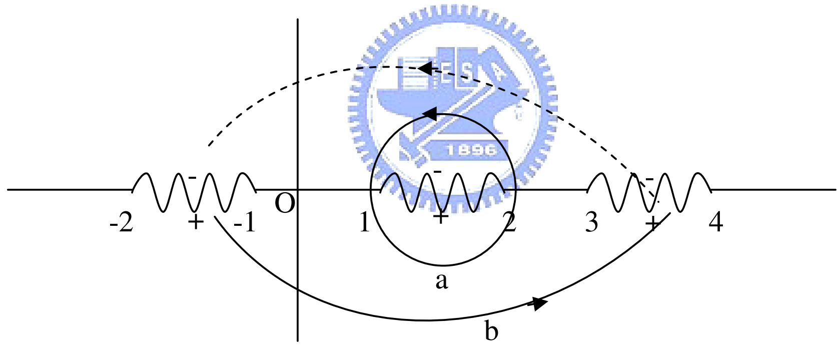


Fig.4-21 a,b cycles for six branch points in horizontal cut

(i) For a cycle :

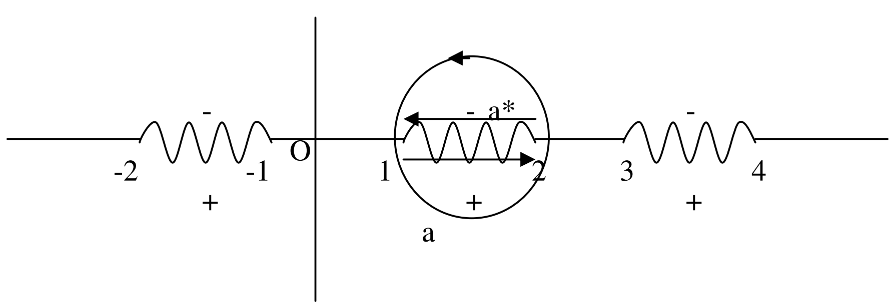


Fig.4-22 a,a^* cycles for six branch points in horizontal cut

We know that $\oint_a f(z)dz = \int_{a^*} f(z)dz$, we only compute the integral along a^* path.

For the equivalent path a^* :

Since $\arg(z-z_j) = -\pi$ is not the valid range in Mathematica®5, we must modify $f(z)$, as shown in Table 4.1. (M means the value of $f(z)$ in Mathematica®5.)

Branch points	*Value of* $f(z)$	
	(1,0) to (2,0)	*(2,0) to (1,0)*
z_1	$-M$	$+M$
z_2	$-M$	$+M$
z_3	$-M$	$+M$
z_4	$+M$	$+M$
z_5	$+M$	$+M$
z_6	$+M$	$+M$
Sheet I *or sheet* II	*(Sheet* I *)* $+M$	*(Sheet* I *)* $+M$
Total	$-M$	$+M$

Table 4.1 we must modify the valve of $f(z)$ for a^* cycle in Mathematica®5.

By Mathematica®5,

$$-\int_1^2 \frac{1}{f(z)}dz + \int_2^1 \frac{1}{f(z)}dz = -2\int_1^2 \frac{1}{f(z)}dz = 3.3819\times10^{-49} - 1.13022i .$$

Therefore the integral over a_1 cycle is

$$\oint_a \frac{1}{f(z)}dz = \int_{a^*} \frac{1}{f(z)}dz = 3.3819\times10^{-49} - 1.13022i .$$

(ii) For b cycle :

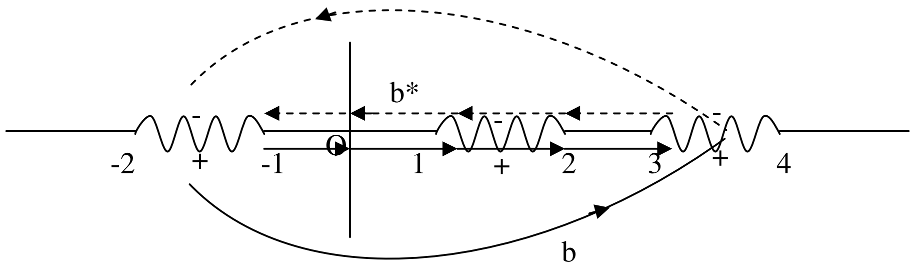


Fig.4-23 b, b^* cycles for six branch points in horizontal cut

For the equivalent path b^* :

Since the interval $(-1,1)$ and $(2,3)$ have no cut, so solid line in sheet Ⅰ implies + sign and dash line in sheet Ⅱ implies – sign . since $\arg(z - z_j) = -\pi$ is not the valid range in Mathematica®5, we get the Table 4.2.

Branch points	*Value of* $f(z)$	
	(1,0) to (2,0)	*(2,0) to (1,0)*
z_1	$-M$	$+M$
z_2	$-M$	$+M$
z_3	$-M$	$+M$
z_4	$+M$	$+M$
z_5	$+M$	$+M$
z_6	$+M$	$+M$
Sheet Ⅰ or sheet Ⅱ	*(Sheet Ⅰ)* $+M$	*(Sheet Ⅱ)* $-M$
Total	$-M$	$-M$

Table 4.2 we must modify the valve of $f(z)$ for b^* cycle in Mathematica®5.

By Mathematica®5,

$$\int_{-1}^{1}\frac{1}{f(z)}dz+\int_{2}^{3}\frac{1}{f(z)}dz-\int_{1}^{-1}\frac{1}{f(z)}dz-\int_{3}^{2}\frac{1}{f(z)}dz-\int_{1}^{2}\frac{1}{f(z)}dz-\int_{2}^{1}\frac{1}{f(z)}dz$$

$= -0.0760776+3.77621\times10^{-49}i$.

Therefore the integral over b cycle is

$$\oint_{b}\frac{1}{f(z)}dz=\int_{b^*}\frac{1}{f(z)}dz=-0.0760776+3.77621\times10^{-49}i\,.$$

Example 4.2 :

Let $n=6$，and $z_1=1+2i$、$z_2=1$、$z_3=3i$、$z_4=i$、$z_5=-1+3i$ and $z_6=-1+i$ are six branch points, as shown in Fig.4.24

If $f(z)=\prod_{j=1}^{6}(z-z_j)^{\frac{1}{2}}$, then $\oint_r\frac{1}{f(z)}dz=?$ where $r=a,b$ cycles.

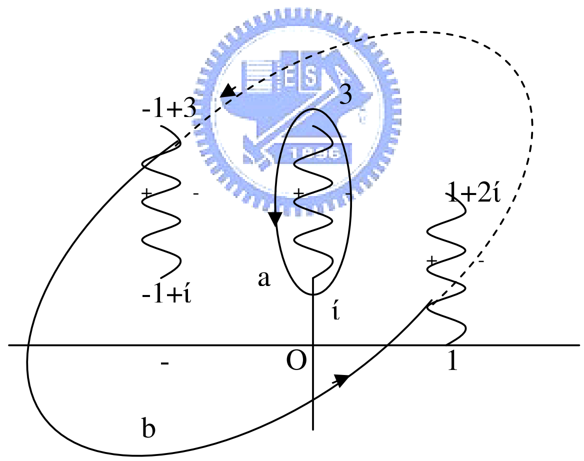


Fig.4.24 a,b cycles for six branch points in vertical cut

We use Mathematica®5 to compute the integral. Note that in vertical cut structure we must modify the function $\sqrt{z-z_j}$ at $\arg(z-z_j)\in[-\frac{3\pi}{2},-\pi)$.

(i) For a cycle:

For the equivalent path a^* : shown in Fig. 4.25

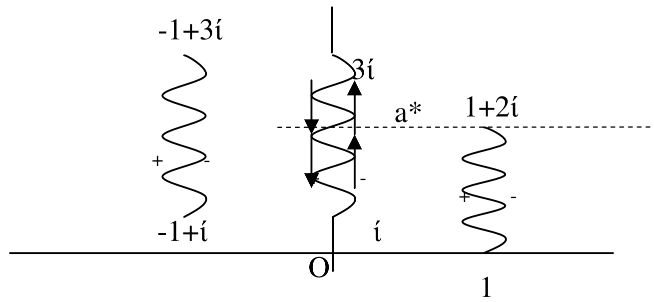


Fig.4.25 a* cycle for example 4.2

We can get the Table 4.4. For example, $\arg(z-z_1)\in[-\frac{3\pi}{2},-\pi)$ for z along the path (0,3i) to (0.2i) so that we must to modify $\sqrt{z-z_1}$.

Branch points	*Value of* $f(z)$			
	(0,3i) to (0.2i)	*(0,2i) to (0,i)*	*(0,2i) to (0,3i)*	*(0,i) to (0,2i)*
z_1	$-M$	$+M$	$-M$	$+M$
z_2	$-M$	$-M$	$-M$	$-M$
z_3	$+M$	$+M$	$+M$	$+M$
z_4	$-M$	$-M$	$+M$	$+M$
z_5	$+M$	$+M$	$+M$	$+M$
z_6	$+M$	$+M$	$+M$	$+M$
Sheet Ⅰ or sheet Ⅱ	$+M$	$+M$	$+M$	$+M$
Total	$-M$	$+M$	$+M$	$-M$

Table 4.4 we must modify the valve of $f(z)$ for a^* cycle in Mathematica®5.

By Mathematica®5,

$$\int_{a^*}\frac{1}{f(z)}dz=-\int_{3i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{i}\frac{1}{f(z)}dz-\int_{i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{3i}\frac{1}{f(z)}dz$$
$$=1.38321-2.33762i$$

Therefore the integral over a cycle is

$$\oint_a\frac{1}{f(z)}dz=\int_{a^*}\frac{1}{f(z)}dz=1.38321-2.33762i\,.$$

(ii) For b cycle:

For the equivalent path b^* :as shown in Fig.4.26

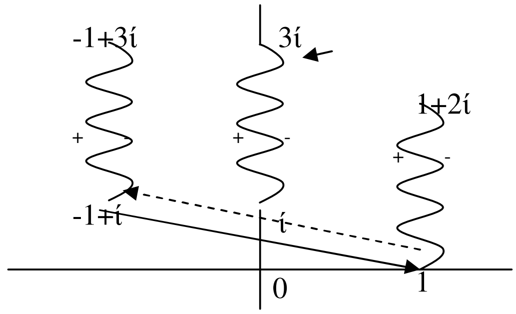


Fig.4.26 b^* cycle for example 4.2

We get the Table 4.6. For example, $\arg(z-z_1)\in[-\frac{3\pi}{2},-\pi)$ for z along -1+í to 1 so that we must to modify $\sqrt{z-z_1}$.

Branch points	*Value of* $f(z)$	
	-1+i to 1	*1 to -1+i*
z_1	$+M$	$+M$
z_2	$-M$	$-M$
z_3	$+M$	$+M$
z_4	$+M$	$+M$
z_5	$+M$	$+M$
z_6	$+M$	$+M$
Sheet I or sheet II	$+M$	$-M$
Total	$-M$	$+M$

Table 4.6 we must modify the valve of $f(z)$ for b^* cycle in Mathematica®5.

By Mathematica®5 ,

$$\int_{b^*}\frac{1}{f(z)}dz=-\int_{-1+i}^{1}\frac{1}{f(z)}dz+\int_{1}^{-1+i}\frac{1}{f(z)}dz=2\int_{1}^{-1+i}\frac{1}{f(z)}dz=0.590344-1.16143i\ .$$

Therefore the integral over b cycle is

$$\oint_b \frac{1}{f(z)}dz = \int_{b^*} \frac{1}{f(z)}dz = 0.590344 - 1.16143i .$$

Example 3 :

Let $n = 4$, $z_1 = 1$、$z_2 = i$、$z_3 = -i$ and $z_4 = -1$ are four branch points form a askew cut as shown in Fig.4.27.

If $f(z) = \prod_{j=1}^{4}(z - z_j)^{\frac{1}{2}}$, then $\oint_r \frac{1}{f(z)}dz = ?$ where $r = a, b$ cycles

Note that in askew cut structure we must modify the function $\sqrt{z - z_j}$ at $\arg(z - z_j) \in (\pi, \frac{9\pi}{4})$.

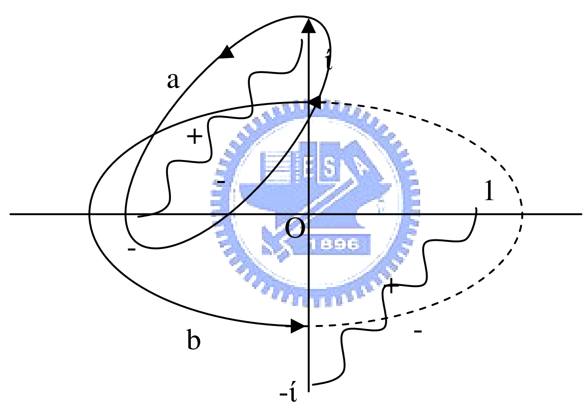


Fig.4.27 a, b cycles for four branch points in askew cut

(i) For a cycle :

For the equivalent path a^* : as shown in Fig.4.28

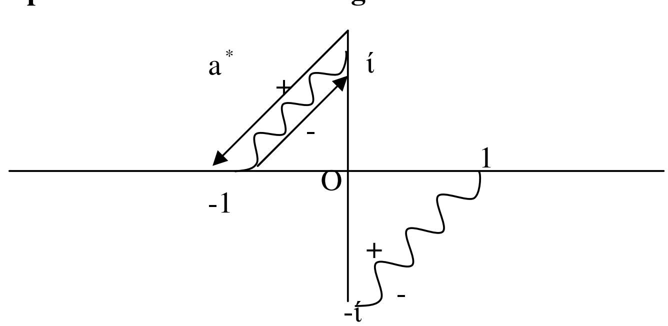


Fig.4.28 a^* cycle for example 4.3

We get the Table 4.8. For example, $\arg(z - z_1) \notin (\pi, \frac{9\pi}{4})$ for z along -1 to i and i to -1 so that we must not modify $\sqrt{z - z_1}$.

Branch points	*Value of* $f(z)$	
	-1 to i	*i to -1*
z_1 *=1*	$+M$	$+M$
z_2 *=i*	$-M$	$-M$
z_3 *=-i*	$+M$	$+M$
z_4 *=-1*	$-M$	$+M$
Sheet I or sheet II	$+M$	$+M$
Toatl	$+M$	$-M$

Table 4.8 we must modify the valve of $f(z)$ for a^* cycle in Mathematica®5.

By Mathematica®5,

$$\oint_{a^*} \frac{1}{f(z)}dz = \int_{-1}^{i} \frac{1}{f(z)}dz - \int_{i}^{-1} \frac{1}{f(z)}dz = 2\int_{-1}^{i} \frac{1}{f(z)}dz = 2.62206 - 2.62206i \, .$$

Therefore the integral over a cycle is

$$\oint_{a} \frac{1}{f(z)}dz = \int_{a^*} \frac{1}{f(z)}dz = 2.62206 - 2.62206i \, .$$

(ii) For b cycle :

For the equivalent path b^* : as shown in Fig. 4.29

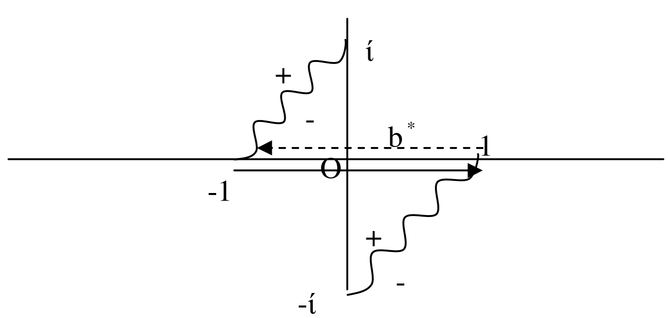


Fig.4.29 b^* cycle for example 4.3

We get the Table 4.10. For example, $\arg(z - z_1) \in (\pi, \frac{9\pi}{4})$ for z along the path -1 to 1 so that we must modify $\sqrt{z - z_1}$.

Branch points	Value of $f(z)$	
	-1 to 1	*1 to -1*
z_1 =*1*	$-M$	$+M$
z_2 =*i*	$-M$	$-M$
z_3 =*-i*	$+M$	$+M$
z_4 =*-1*	$-M$	$-M$
Sheet Ⅰ or sheet Ⅱ	$+M$	$-M$
Toatl	$-M$	$-M$

Table 4.10 we must modify the valve of $f(z)$ for b^* cycle in Mathematica®5.

By Mathematica®5,

$$\oint_{b^*} \frac{1}{f(z)} dz = -\int_{-1}^{1} \frac{1}{f(z)} dz - \int_{1}^{-1} \frac{1}{f(z)} dz = 0 .$$

Therefore the integral over b cycle is

$$\oint_{b} \frac{1}{f(z)} dz = \oint_{b^*} \frac{1}{f(z)} dz = 0 .$$

4.5 Application for Riemann integral

Recalling the heat conduction problem in section 2.5 we use Laplace and Fourier transformation to solve it. But we want to solve the integral

$$u(x,t) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} (\lim_{L\to\infty} \int_{s-iL}^{s+iL} \frac{1}{\sqrt{\sigma}} e^{-\sqrt{\sigma}|x-y|} e^{\sigma t} d\sigma) f(y) dy .$$

Since the path is from $s - il$ to $s + il$, we must not modify the integrate in Mathematica®5. Let $f(y) = y^2$, and by Mathematica®5 we can get

$$u(x,1)=\frac{1}{4\pi i}\int_{-\infty}^{\infty}(\lim_{L\to\infty}\int_{s-iL}^{s+iL}\frac{1}{\sqrt{\sigma}}e^{-\sqrt{\sigma}|x-y|}e^{\sigma}d\sigma)\,ydy = x. \tag{4.11}$$

We also compute the integrate in (2.17), and we can get that

$$u(x,1)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(x-y)^2/4}\,ydy = x,$$

is the same as (4.11). As shown in Table. 4.11, $\tilde{u}$ is the value of u which is computed by Mathematica®5.

t	1	2	3	4	5	6	7	8	9	10
$\tilde{u}$	$2+x^2$	$4+x^2$	$6+x^2$	$8+x^2$	$10+x^2$	$12+x^2$	$14+x^2$	$16+x^2$	$18+x^2$	$20+x^2$
u	$2+x^2$	$4+x^2$	$6+x^2$	$8+x^2$	$10+x^2$	$12+x^2$	$14+x^2$	$16+x^2$	$18+x^2$	$20+x^2$

Table 4.11 compare u with $\tilde{u}$

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