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Finite Order Diagonal Automorphisms
on
Complex Simple Lie Algebras

研究生：劉 筱 凡

指導教授：蔡 孟 傑　教授

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研　究　生：劉筱凡　　Student：Hsiao-Fan Liu

指導教授：蔡孟傑　　Advisor：Dr. Meng-Kiat Chuah

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摘要

複單李代數上的一自同構若保持住一個 Cartan 子李代數且以係數乘積作用在相對應的根空間上，即稱為一對角的自同構。本篇論文中，我們研究複單李代數上的有限維對角的自同構。特別地，我們可藉由圖形來表示這些自同構，並且討論等價的圖形中彼此的組合性質。

關鍵字：對角的自同構，複單李代數。

中 華 民 國 九 十 六 年 六 月

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Student: Hsiao-Fan Liu Advisor: Meng-Kiat Chuah

Department of Applied Mathematics

National Chiao Tung University

Abstract

An automorphism on a complex simple Lie algebra is said to be diagonal if it preserves a Cartan subalgebra and acts as scalar multiples on the corresponding root spaces. In this thesis, we study finite order diagonal automorphisms on complex simple Lie algebras. In particular, we represent these automorphisms by some diagrams, and study the combinatorial properties of equivalent diagrams.

Keywords: diagonal automorphism, complex simple Lie algebra.

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Contents

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_i,$$

where $\Delta \subset \mathfrak{h}^*$ are the root. We say that an automorphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a diagonal automorphism if there exists a σ-stable Cartan subalgebra $\mathfrak{h}$ such as σ preserves all the root spaces $\mathfrak{g}_i$. The purpose of this thesis is to study the diagrammatic expression of finite order diagonal automorphisms, and to classify the equivalent diagrams when $\mathfrak{g}$ is a complex simple Lie algebra of type A.

Let σ be a diagonal automorphism with respect to $\mathfrak{h}$. So σ acts as scalar multiples on the root spaces $\mathfrak{g}_i$. If in addition σ is of finite order m, then the eigenvalues are some complex numbers c_i such that $c_i^m = 1$. In Section 2, we introduce the Vogan diagram of σ. It is a diagram which represents σ by indicating its eigenvalues on the vertices of the extended Dynkin diagram D^1 of $\mathfrak{g}$. This is the natural generalization of the Vogan diagrams of involutions on $\mathfrak{g}$ [7].

A Vogan diagram represents a finite order diagonal automorphism σ under a choice of simple roots. However, different diagrams may represent the same σ due to different simple roots. In this case we say that the diagrams are equivalent. In Section 3, we introduce an algorithm which combinatorially describes how two diagrams are equivalent. As an application, in Section 4, we give a precise method to determine equivalent diagrams for type A Lie algebras.

2 Vogan Diagrams

In this section, we introduce the Vogan diagrams of finite order diagonal automorphisms. Let $\mathfrak{g}$ be a complex simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$. Let Π denote a simple system, so $\Pi \subset \Delta \subset \mathfrak{h}^*$. The vertices of the extended Dynkin diagram are $\Pi \cup \{\varphi\}$, where φ is the lowest root. The vertices i of D^1 are equipped with canonical coefficients a_i, where a_i are positive integers without nontrivial common factor and $\sum_{\Pi\cup\{\varphi\}} a_i i = 0$.

Let σ be a diagonal automorphism on $\mathfrak{g}$ of order m which preserves $\mathfrak{h}$, and let $\mathbb{Z}_m$ denote the abelian group of $\mathbb{Z}$ modulo $m\mathbb{Z}$. Let $\epsilon = \exp(2\pi\sqrt{-1}/m) \in \mathbb{C}$ be the m-th primitive root of unity. The eigenvalues c_i of σ are exactly ϵ^{s_i}, where $s_i \in \mathbb{Z}_m$. It follows that if we assign s_i to vertex i of the Dynkin diagram D, then $\sigma X = \epsilon^{s_i} X$ for all $X \in \mathfrak{g}_i$. Moreover, Lie algebra homomorphism provides

$$\sigma[X,Y] = [\sigma X, \sigma Y] = [\epsilon^{s_i} X, \epsilon^{s_j} Y] = \epsilon^{s_i+s_j}[X,Y],$$

where $Y \in \mathfrak{g}_j$. This implies that the assignment to vertex φ is forced by Lie algebra homomorphism while the other vertices are assigned. In other words, we can assign such s_i to each simple root and hence there is an assignment $\{s_i\}_{\Pi\cup\varphi}$ on the vertices of D^1.

Given $s_i \in \mathbb{Z}_m$, we may write $s_i = [b_i]$ for some $b_i \in \mathbb{Z}$. We say that the assignment $\{s_i\}$ is *nontrivial* if the set $\{b_i\}_{\Pi\cup\varphi}$ has no nontrivial common factor. To ensure that the order of σ is m, $\{s_i\}$ is required to be nontrivial. In this respect, we associate the following diagram on $\mathfrak{g}$.

Definition 2.1. A Vogan diagram of order m on $\mathfrak{g}$ is a nontrivial assignment of $s_i \in \mathbb{Z}_m$ to each vertex i of D^1, such that $\sum_{\Pi\cup\{\varphi\}} a_i s_i$ is a positive multiple of m. In particular, if $\sum_{\Pi\cup\{\varphi\}} a_i s_i = m$, we call it a standard Vogan diagram .

Obviously, these s_i indicate the behavior of σ on $\mathfrak{g}_i$. Here, we may soon conclude that every Vogan diagram represents an automorphism on the corresponding complex simple Lie algebra of order m.

Recall that a Vogan diagram defined in [7] is a Dynkin diagram with an involution θ, such that the vertices fixed by θ are painted or not. For $m = 2$, σ is an involution on $\mathfrak{g}$ and the eigenvalues are ± 1. By ignoring φ, we obtain a Dynkin diagram with σ. We can let σ act on the roots i such that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{\sigma(i)}$. Then $\sigma(i) = i$ if and only if i is imaginary, and hence we obtain an automorphism σ on D. So the imaginary simple roots are the vertices of D which are fixed by σ. Namely, the vertices which are painted represent the eigenvalue -1 of σ. It is easy to see that $-1 = \epsilon^1$. Therefore, we can assign 1 to the painted vertices and 0 to the others. This leads an assignment to be a Vogan diagram of order 2 on $\mathfrak{g}$ and provides a natural way to define those Vogan diagrams of higher order m on $\mathfrak{g}$.

Throughout this paper, we shall always let V be the set of Vogan diagrams of order m on $\mathfrak{g}$ and V_s be the set of standard Vogan diagrams.

Theorem 2.2 (Kac[4][6]). *A Vogan diagram with $\{s_i\}$ represents an order m automorphism σ on $\mathfrak{g}$ by $\sigma X_i = \epsilon^{s_i} X_i$ on the root vector X_i of i. Conversely, every diagonal automorphism is represented by a Vogan diagram. Each Vogan diagram is equivalent to a standard Vogan diagram. Up to conjugation, the automorphisms obtained this way exhaust all m-th order diagonal automorphisms on $\mathfrak{g}$.*

This theorem follows from [4, Chap.X-5, Theorem 5.15]. Due to the choice of simple systems, two different Vogan diagrams may represent conjugate automorphisms. In such case, we say that these Vogan diagrams are *equivalent*. We summarize it as follows.

Definition 2.3. Two Vogan diagrams v and w are equivalent if and only if they represent two conjugate automorphisms.

We denote it by $v \sim w$. By Theorem 2.2, it allows us to use the diagrams to study finite order automorphisms on $\mathfrak{g}$. Since each automorphism of order m must give one assignment $\{s_i\}$ with $\sum_{\Pi\cup\{\varphi\}} a_i s_i = m$, we derive that every Vogan diagram $v \in V$ is equivalent to a standard Vogan diagram $v' \in V_s$.

In fact, Theorem 2.2 is generalized from the following theorem.

Theorem 2.4 (Borel-de Siebenthal [3]). *Every real form of a complex simple Lie algebra can be represented by a Vogan diagram with at most one painted vertex.*

For the case of $m = 2$, Theorem 2.2 says that each involution σ determines an assignment $\{s_i\}$ on D^1 where $\sum_{\Pi\cup\{\varphi\}} a_i s_i$ is even and s_i is either 1 or 0. Suppose there is only one black vertex of D. By Theorem 2.4, the vertex φ must be black. Namely, every Vogan diagram is equivalent to a diagram with $\sum_{\Pi\cup\{\varphi\}} a_i s_i = 2$.

3 Equivalent Diagrams

In the previous section, we have constructed the Vogan diagram v of the automorphism σ with respect to a simple system. If we change the simple system, we obtain a different diagram w which also represents σ, and in this case we say that v and w are equivalent. In this section, we give a combinatorial description for equivalent diagrams.

Let $\mathrm{aut}(\Delta)$ denote the automorphism group of the root system Δ. It acts on the simple system, and so it also acts of the Vogan diagrams V. Since it acts transitively on all the ordered simple system, the orbits of its action on V are precisely the equivalent classes of Vogan diagrams. Namely, $v \sim w$ if and only if there exists some $f \in \mathrm{aut}(\Delta)$ such that $f(v) = w$. We shall give a combinatorial description for such $f \in \mathrm{aut}(\Delta)$.

Consider two possible ways to change the choice of simple system. Let $\mathscr{W}$ be the Weyl group generated by simple reflections and $\mathrm{aut}(\mathrm{D})$ be the set of all automorphisms on the Dynkin diagram D. Evidently, $\mathrm{aut}(\mathrm{D})$ is a subgroup of $\mathrm{aut}(\Delta)$. Moreover, $\mathscr{W}$ is a normal subgroup [5, Lemma 9.2]. Exactly, a possible change of simple system is provided as follows.

Theorem 3.1. *There is a semi-direct product*

$$\mathrm{aut}(\Delta) = \mathscr{W} \times \mathrm{aut}(\mathrm{D}).$$

This theorem follows from [4, Chap.X-3, Theorem 3.29] or [5, Chap.12.2].

Due to $\mathrm{aut}(\Delta) = \mathscr{W} \times \mathrm{aut}(\mathrm{D})$, we shall consider an operation which acts on Δ by reflection corresponding to the simple root i. As a result, it leads to an equivalent Vogan diagram. In what follows, we introduce and generalize F_i developed in [1] and [2]. Let v be a Vogan diagram of an order m automorphism, with assignment of $\{s_j\}$ to the vertices j. Given a vertex i, we define $F_i(v)$ to be another Vogan diagram with assignment of $\{t_j\}$ as follows.

$$
(3.1)\quad F_i(v):\begin{cases} t_i = -s_i, & \\ t_j = s_j + s_i & \text{if } j \text{ is an adjacent root of equal or shorter length,} \\ t_j = s_j + 2s_i & \text{if } j \text{ is a longer root joint to } i \text{ by a double edge,} \\ t_j = s_j + 3s_i & \text{if } j \text{ is a longer root joint to } i \text{ by a triple edge,} \\ t_j = s_j & \text{if } j \text{ and } i \text{ are not adjacent.} \end{cases}
$$

Indeed, F_i corresponds to the reflection defined by the simple root i. Consider the effect of the reflection defined by i. There is no effect to vertices j which are not adjacent to i, since it means that the simple roots j and i are mutually orthogonal. So, we obtain $t_j = s_j$. For adjacent vertices, we consider just locally on the plane, namely A_2, B_2 and G_2. There are five conditions for adjacent vertices. In each case, one can draw the roots of i and j on the plane and justify the condition of F_i visually.

Such F_i provides a way to judge whether two Vogan diagrams are equivalent, and they generate the Weyl group $\mathscr{W}$. Definition 2.3 therefore can be restated as follows.

Proposition 3.2. *Two Vogan diagrams v and w are equivalent if and only if there exists a sequence of Vogan diagrams v_a with*

$$
(3.2)\qquad v = v_0 \mapsto v_1 \mapsto \dots \mapsto v_k = w,
$$

such that each $v_a \to v_{a+1}$ is given by some F_i of (3.1) *or a diagram automorphism.*

Let $\mathrm{int}(\mathfrak{g})$ be the subgroup of $\mathrm{aut}(\mathfrak{g})$ generated by $\{\exp(\mathrm{ad}_X)\ ;\ X \in \mathfrak{g}\}$, where $\exp : \mathrm{end}(\mathfrak{g}) \longrightarrow \mathrm{aut}(\mathfrak{g})$ is the exponential map. The members of $\mathrm{int}(\mathfrak{g})$ are called *inner automorphisms*. The theorem of Kac [4, Chap.X-5, Theorem 5.16] says that each diagonal automorphism of finite order on $\mathfrak{g}$ is an inner automorphism. The inner automorphism corresponds only to F_i, without diagram automorphisms. Thus, we may impose a stricter notion on the Vogan diagrams. Namely, two equivalent Vogan diagrams v and w are said to be *inner equivalent* if and only if each $v_a \to v_{a+1}$ in (3.2) is given by some F_i and not a diagram automorphism.

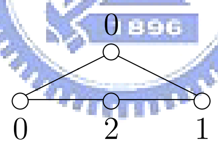
4 Equivalence Classes of Type A Diagrams

Recall that Kac's theorem says that a Vogan diagram is equivalent to a standard diagram, but it does not say which standard diagram. In this section, we provide a method to find the standard diagram explicitly. We also show the method to judge whether two Vogan diagrams are equivalent.

Let $V(A_n^1)$ denote the set of all Vogan diagrams of order m on A_n^1. Label the vertices of A_n^1 naturally by $0, 1, ..., n$, where 0 is the extra vertex and express a Vogan diagram on A_n^1 by

(4.1) $\quad (i_1, i_2, ..., i_k) \in V(A_n^1), \;\; 0 \leq i_1 \leq ... \leq i_k \leq n$ and k is a multiple of m.

Recall that each vertex i of a Vogan diagram is assigned by $s_i \in \mathbb{Z}_m$, and thus we allow an index to appear s (or $s + m\mathbb{Z}$) times in $(i_1, ..., i_k)$ if and only if the corresponding vertex is assigned by s. We also allow 0 to appear, and we may ignore it since the assignment to 0 is forced. So for example $(2, 2, 3)$ and $(1, 1, 1, 2, 2, 3)$ both refer to the following Vogan diagram of order 3 on A_3^1.



Define the *standard* Vogan diagrams by setting $k = m$ in (4.1), namely

$$(i_1, ..., i_m) \in V(A_n^1), \;\; 0 = i_1 \leq i_2 \leq ... \leq i_m \leq n.$$

From now on, we shall let $V_s(A_n^1)$ denote the set of all standard Vogan diagrams of order m on A_n^1. It is easily seen that $V_s(A_n^1) \subset V(A_n^1)$.

Recall that we want to find a standard Vogan diagram which is equivalent to a given diagram $v \in V(A_n^1)$. This will be done in Proposition 4.4, where we construct a map $\tau : V(A_n^1) \longrightarrow V_s(A_n^1)$ which satisfies $v \sim \tau(v)$. The next few lemmas study a function $\phi : V(A_n^1) \longrightarrow \mathbb{C}$ which will be used to construct τ.

Let $\mathbb{C}$ be the set of all complex numbers. Define

$$\phi : V(A_n^1) \longrightarrow \mathbb{C}, \qquad (4.2)$$
$$\phi(i_1, ..., i_k) = \sum_{p=0}^{k-1} \epsilon^p i_{k-p},$$

where ϵ is the m-th primitive root of unity as before.

By this definition of ϕ, we can check that

$$\phi(i_1, ..., i_r, ..., i_k) = \phi(i_{r+1}, ..., i_k) + \epsilon^{k-r}\phi(i_1, ..., i_r). \qquad (4.3)$$

For example,

$$\begin{aligned}
&\phi(2, 2, 3, 3, 6, 6, 6) \\
&= (1 + \epsilon + \epsilon^2)6 + \epsilon^3\phi(2, 2, 3, 3) \\
&= (1 + \epsilon + \epsilon^2)6 + \epsilon^3((1 + \epsilon)3 + \epsilon^2\phi(2, 2)) \\
&= (1 + \epsilon + \epsilon^2)6 + \epsilon^3((1 + \epsilon)3 + \epsilon^2(1 + \epsilon)2) \\
&= (1 + \epsilon + \epsilon^2)6 + (\epsilon^3 + \epsilon^4)3 + (\epsilon^5 + \epsilon^6)2.
\end{aligned}$$

Two important properties of ϵ are

$$\epsilon^m = 1 \ , \ 1 + \epsilon + ... + \epsilon^{m-1} = 0. \qquad (4.4)$$

The second equation is proved by $(\epsilon - 1)(1 + \epsilon + ... + \epsilon^{m-1}) = 0$, and hence we know that $1, \epsilon, \epsilon^2, ..., \epsilon^{m-2}$ are linearly independent. Observe that ϕ is well-defined because of the second equation in (4.4). Namely, if the indices i_r appear several times for a same diagram as allowed in (4.1), then the value of ϕ remains the same. Further, the second equation in (4.4) yields

$$1 + \epsilon + ... + \epsilon^{s-1} = -\epsilon^s(1 + \epsilon + ... + \epsilon^{m-s-1}), \text{ for } s \in \{0, 1, ..., m-1\}. \qquad (4.5)$$

For the following proposition and lemma, it is convenient for us to rewrite (4.1) to be

$$(0^{s_0}, 1^{s_1}, ..., n^{s_n}) \in V(A_n^1), \qquad (4.6)$$

to denote the Vogan diagram with value s_i at vertex i. For example, if $m = 4$,

$$(2,3,3,6,6,6) = (2^1, 3^2, 6^3) = (2^5, 3^2, 6^3) = (1^0, 2^5, 3^2, 6^3).$$

Here vertex 2 is assigned $s_2 = 1$ or $s_2 = 5$, and so on.

The function ϕ is useful, because it is invariant under $F_1, ..., F_{n-1}$, as shown by the following lemma. Let R_c denote counter clockwise rotation of assignment $\{s_i\}$ by c steps. For example, in the following diagram, the right diagram is obtained by applying R_2 to the left diagram.

Two Vogan diagrams on A_4^1 with $m = 3$.

Lemma 4.1. *Let $v \in V(A_n^1)$ be a Vogan diagram. Then*

(a) $\phi F_r(v) = \phi(v)$ *for all* $r = 1, 2, ..., n-1$.

(b) $\phi F_n(v) = \phi R_1(v)$.

(c) $\phi F_0(v) = \phi R_{-1}(v)$.

Proof. To prove this lemma, we use the notation in (4.6) to express v. For part (a), the identity (4.3) says that

$$\begin{aligned}
&\phi F_r(0^{s_0}, ..., n^{s_n}) \\
&= \phi(0^{s_0}, ..., (r-1)^{s_{r-1}+s_r}, r^{m-s_r}, (r+1)^{s_{r+1}+s_r}, ..., n^{s_n}) \\
&= \phi((r+2)^{s_{r+2}} + ... + n^{s_n}) \\
&\quad + \epsilon^{s_{r+2}+...+s_n} \phi((r-1)^{s_{r-1}+s_r}, r^{m-s_r}, (r+1)^{s_{r+1}+s_r}) \\
&\quad + \epsilon^{s_{r-1}+...+s_n} \phi(0^{s_0}, ..., (r-2)^{s_{r-2}}).
\end{aligned} \tag{4.7}$$

A direct computation shows that

$$\phi((r-1)^{s_{r-1}+s_r}, r^{m-s_r}, (r+1)^{s_{r+1}+s_r}) = \phi(r-1, r, r+1). \tag{4.8}$$

Therefore, when we substitute (4.8) into the middle term of the last expression of (4.7), we get

$$\begin{aligned}
&\phi F_r(0^{s_0}, ..., n^{s_n}) \\
&= \phi((r+2)^{s_{r+2}} + ... + n^{s_n}) \\
&\quad + \epsilon^{s_{r+2}+...+s_n}\phi(r-1, r, r+1) \\
&\quad + \epsilon^{s_{r-1}+...+s_n}\phi(0^{s_0}, ..., (r-2)^{s_{r-2}}) \\
&= \phi(0^{s_0}, ..., n^{s_n}).
\end{aligned}$$

This proves part (a) of the lemma.

To show part (b), the definition (4.2) yields

$$\begin{aligned}
&\phi R_1(v) \\
&= \phi(0^{s_n}, 1^{s_0}, ..., n^{s_{n-1}}) \\
&= (1+\epsilon+\cdots+\epsilon^{s_{n-1}-1})\cdot n + \epsilon^{s_{n-1}}(1+\epsilon+\cdots+\epsilon^{s_{n-2}-1})(n-1) \\
&\quad +... + \epsilon^{s_{n-1}+...+s_1}(1+\epsilon+\cdots+\epsilon^{s_0-1}).
\end{aligned} \tag{4.9}$$

Subtracting $(1+\epsilon+\cdots+\epsilon^{s_{n-1}+...+s_1-1})$ before adding it in (4.9) gives

$$\begin{aligned}
&\phi R_1(v) \\
&= \phi(1^{s_1}, 2^{s_2}, ..., (n-1)^{s_{n-1}}) \\
&\quad + (1+\epsilon+\cdots+\epsilon^{s_{n-1}+...+s_1+s_0-1}).
\end{aligned}$$

Note that

$$\begin{aligned}
&\phi F_n(v) \\
&= \phi(0^{s_0+s_n}, 1^{s_1}, ..., (n-1)^{s_{n-1}+s_n}, n^{m-s_n}) \\
&= \phi((n-1)^{s_n}, n^{m-s_n}) + \epsilon^{(m-s_n)+s_n}\phi(1^{s_1}, ..., (n-1)^{s_{n-1}}) \\
&= \phi(1^{s_1}, 2^{s_2}, ..., (n-1)^{s_{n-1}}) + (1+\epsilon+\cdots+\epsilon^{m-s_n-1}).
\end{aligned}$$

The last expression is proved by using (4.5).

Since $\sum_{i=0}^{n} s_i$ is a multiple of m, it follows that

$$1+\epsilon+\cdots+\epsilon^{s_{n-1}+...+s_1+s_0-1} = 1+\epsilon+\cdots+\epsilon^{m-s_n-1}.$$

This verifies part (b) of the lemma and (c) follows from the same argument in (b), completing the proof.

□

Lemma 4.2. *$F_n F_{n-1} \cdots F_1 = R_{-1}$ and $F_1 F_2 \cdots F_n = R_1$.*

Proof. Given a Vogan diagram v, let v_i be its value at vertex i. We first claim that

(4a) $$(F_i F_{i-1} \cdots F_1(v))_i = -v_1 - v_2 - ... - v_i \text{ and}$$

(4b) $$(F_i F_{i-1} \cdots F_1(v))_{i+1} = v_1 + v_2 + ... + v_{i+1} \text{ for all } i = 1, 2, ..., n.$$

We prove (4a) and (4b) by induction on i. It is clear that $(F_1(v))_1 = -v_1$ and $(F_1(v))_2 = v_1 + v_2$, so (4a) and (4b) hold for $i = 1$.

Now suppose that (4a) and (4b) hold for i, and we want to show that they therefore hold for $i+1$ as well. It is obvious that

(4.10) $$(F_i F_{i-1} \cdots F_1(v))_{i+2} = v_{i+2}.$$

Then

(4.11) $$\begin{aligned}(F_{i+1} \cdots F_1(v))_{i+1} &= -(F_i \cdots F_1(v))_{i+1} \\ &= -v_1 - v_2 - ... - v_{i+1} \quad \text{by (4b).}\end{aligned}$$

Also,

(4.12) $$\begin{aligned}&(F_{i+1} \cdots F_1(v))_{i+2} \\ &= (F_i \cdots F_1(v))_{i+2} + (F_i \cdots F_1(v))_{i+1} \\ &= (v_1 + v_2 + ... + v_{i+1}) + v_{i+2} \qquad \text{by (4b) and (4.10).}\end{aligned}$$

By (4.11) and (4.12), we have shown that (4a) and (4b) are also true for $i+1$. This completes the induction, and so (4a) and (4b) hold for all $i = 1, 2, ..., n$.

To prove the lemma, we want to show that

(4c) $$(F_n \cdots F_1(v))_i = v_{i+1}.$$

By (4a) and (4b),

$$\begin{aligned}(F_{i+1}\cdots F_1(v))_i &= (F_i\cdots F_1(v))_i + (F_i\cdots F_1(v))_{i+1}\\ &= (-v_1 - v_2 - \ldots - v_i) + (v_1 + v_2 + \ldots + v_{i+1})\\ &= v_{i+1}.\end{aligned}$$

Since $F_{i+2}, F_{i+3}, \cdots, F_n$ has no effect on vertex i, it leads to (4c). This proves the lemma.

By similar arguments, we obtain $F_1F_2\cdots F_n = R_1$.

□

Lemma 4.3. *Let $v \in V(A_n^1)$. Then there exists $v' \in V_s(A_n^1)$ such that $v' \sim v$ and $\phi(v') = \phi(v)$.*

Proof. Let $v \in V(A_n^1)$. We claim that there exists $i_1, ..., i_N \in \{1, ..., n\}$ such that

$$F_{i_N}F_{i_{N-1}}\cdots F_{i_1}(v) \in V_s(A_n^1). \tag{4.13}$$

By Theorem 2.2, there exists $w \in V_s(A_n^1)$ such that $v \sim w$. Since the simple reflections $F_i \in \mathscr{W}$ and aut(D) generate aut(Δ), there exists a sequence $\{v_a\}$ such that

$$v = v_0 \mapsto v_1 \mapsto \ldots \mapsto v_{r-1} \mapsto v_r = w, \tag{4.14}$$

where each $v_a \mapsto v_{a+1}$ is given by some F_i (where $i \in \{1, ..., n\}$) or the diagram reflection $\gamma \in$ aut(D). Since $\mathscr{W}$ is a normal subgroup of aut(Δ), for each F_i, there exists some F_j such that $F_i\gamma = \gamma F_j$. Therefore, using another sequence in (4.14) if necessary, we may move the $\gamma' s$ so that they appear only at the end of the sequence. Further, since $\gamma^2 = 1$, we may assume that γ appears at most once in (4.14). We then let either v_{r-1} or v_r be $F_{i_N}F_{i_{N-1}}\cdots F_{i_1}(v)$ in (4.13). This proves (4.13) as claimed.

We now prove the lemma for $v \in V(A_n^1)$ by induction on N of (4.13). When $N = 1$, it obviously follows from Lemma 4.1. Indeed, if $i_1 < n$, then we choose $v' = F_{i_1}(v)$

and hence $\phi(v') = \phi(v)$. If otherwise, let $w = F_n(v)$ and choose $v' = R_1(w)$ since

$$\begin{aligned} \phi(v) &= \phi F_n(w) \\ &= \phi R_1(w) \quad \text{by Lemma 4.1(b).} \\ &= \phi(v'). \end{aligned} \tag{4.15}$$

Suppose that the assertion is true for $N-1$. For the case of N, let $w = F_{i_N} F_{i_{N-1}} \cdots F_{i_1}(v)$. The case of $i_N < n$ is obvious by Lemma 4.1 and the assumption. Now, we shall consider the case $w = F_n F_{i_{N-1}} \cdots F_{i_1}(v)$ only.

If $i_{N-1} = n$, then $w = F_{i_{N-2}} \cdots F_{i_1}(v)$ and hence there is nothing to prove. Conversely, if i_{N-1} is not adjacent to n, namely $0 < i_{N-1} < n-1$, then we can interchange $F_{i_{N-1}}$ and F_n to obtain $w = F_{i_{N-1}} F_n \cdots F_{i_1}(v)$. Then

$$\begin{aligned} \phi(w) &= \phi F_{i_{N-1}} F_n F_{i_{N-2}} \cdots F_{i_1}(v) \\ &= \phi F_n F_{i_{N-2}} \cdots F_{i_1}(v). \end{aligned}$$

Using the assumption again, the result is followed.

Consequently, it suffices to deal with

$$w = F_n F_{n-1} F_{n-2} \cdots F_{n-j}(v), \quad j \in \{1, ..., n-1\}. \tag{4.16}$$

Note that $F_n(w) = F_{n-1} F_{n-2} \cdots F_{n-j}(v)$ and hence

$$\phi F_n(w) = \phi(v) \quad \text{by Lemma 4.1(a).}$$

Therefore, the similar arguments in (4.15) imply that there is $v' = R_1(w) \in V_s(A_n^1)$ such that $\phi(v') = \phi(v)$. By induction, this proof is completed .

□

We are now ready to construct the map $\tau : V(A_n^1) \longrightarrow V_s(A_n^1)$ which satisfies $v \sim \tau(v)$. Define $P \subset \mathbb{C}$ by

$$P = \{b_0 + b_1\epsilon + ... + b_{m-2}\epsilon^{m-2} \; ; \; b_i \in \mathbb{Z} \text{ and } n \geq b_0 \geq b_1 \geq ... \geq b_{m-2} \geq 0\}.$$

Note that the image of ϕ is contained in P and we therefore define the useful τ as follows. Let $v \in V(A_n^1)$ and write $\phi(v) = b_0 + b_1\epsilon + ... + b_{m-2}\epsilon^{m-2} \in P$. Define

$$\begin{aligned} \tau : V(A_n^1) &\longrightarrow V_s(A_n^1), \\ \tau(v) &= (0, b_{m-2}, ..., b_1, b_0) \in V_s(A_n^1). \end{aligned} \tag{4.17}$$

Since τ is defined from ϕ, τ is well-defined clearly.

For arbitrary $v \in V(A_n^1)$, the following proposition gives us the method to express one form of standard Vogan diagrams which are equivalent to v.

Proposition 4.4. *Let $v \in V(A_n^1)$. Then v and $\tau(v)$ are equivalent.*

Proof. By Lemma 4.3, we can choose a standard Vogan diagram $v' \in V_s(A_n^1)$ such that $\phi(v) = \phi(v')$. Write $v' = (i_1, i_2, ..., i_m)$, and we obtain

$$\phi(v) = (i_m - i_1) + (i_{m-1} - i_1)\epsilon + ... + (i_2 - i_1)\epsilon^{m-2}.$$

By (4.17), we have that

$$\tau(v) = (0, i_2 - i_1, ..., i_{m-1} - i_1, i_m - i_1).$$

It is clear that $v' \sim \tau(v)$, which implies $v \sim \tau(v)$. This completes the proof. $\square$

By the above proposition, we are able to find a standard Vogan diagram $\tau(v)$ which is equivalent to a given diagram v. Together with the following proposition of Kac, we are also able to judge whether two given diagrams are equivalent.

Proposition 4.5 (Kac). *Two standard Vogan diagrams $v, w \in V_s(A_n^1)$ are equivalent if and only if there exists a diagram automorphism which maps v to w.*

Proof. Recall that a standard Vogan diagram which represents an automorphism is an assignment $\{s_i\}$ with $\sum s_i = m$. Therefore, v and $w \in V_s(A_n^1)$ are equivalent if and only if their corresponding automorphisms, σ and σ', are conjugate. The theorem of Kac [4, Chap.X-5, Theorem 5.16] says that two automorphisms on $\mathfrak{g}$ are conjugate if and only if $\{s_i\}$ can be transformed to $\{s_i'\}$ by a diagram automorphism. The proposition follows. $\square$

For example, consider the standard Vogan diagram $(0,1,1)$ of A_3^1. We obtain

$$(0,1,1) \mapsto (1,2,2) \mapsto (2,2,3). \tag{4.18}$$

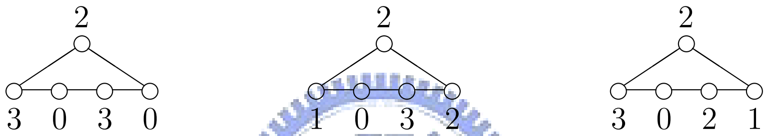
Obviously, the first step is achieved by rotation, while the second step by reflection. By Proposition 4.5, it implies immediately that $(0,1,1)$ and $(2,2,3)$ are equivalent.

We summarize our results in the following theorem.

Theorem 4.6. *Let v and $w \in V(A_n^1)$ be Vogan diagrams. Then v and w are equivalent if and only if $\tau(v)$ and $\tau(w)$ are related by a diagram automorphism.*

Proof. By Proposition 4.4 and Proposition 4.5, we derive this theorem. □

Example 4.7. Consider the following three Vogan diagrams v_1, v_2, v_3 with $m = 4$.

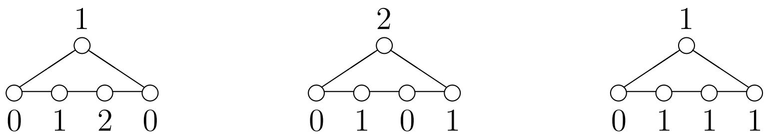


v_1 v_2 v_3

Write

$$v_1 = (0,0,1,1,1,3,3,3),\ v_2 = (0,0,1,3,3,3,4,4),\ \text{and } v_3 = (0,0,1,1,1,3,3,4).$$

By Theorem 4.6, we obtain standard diagrams $\tau(v_1), \tau(v_2)$ and $\tau(v_3)$ equivalent to v_1, v_2 and v_3, respectively. That is, $\tau(v_1) = (0,2,3,3),\ \tau(v_2) = (0,0,2,4),$ and $\tau(v_3) = (0,2,3,4)$ are as follows.



$\tau(v_1)$ $\tau(v_2)$ $\tau(v_3)$

Note that by diagram automorphism, $\tau(v_1) \sim \tau(v_2)$, but $\tau(v_2) \nsim \tau(v_3)$. Therefore, v_1 is equivalent to v_2, while v_3 is not equivalent to both v_1 and v_2.

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