國 立 交 通 大 學

應 用 數 學 系

碩 士 論 文

線性雙曲型偏微分方程之研究

Topics on linear hyperbolic equations

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摘　要

本論文主要研究線性的雙曲型偏微分方程(線性雙曲 PDEs)。首先，我們舉給幾個屬於此類型的實際例子。再來，使用幾種典型的方法來解線性雙曲 PDEs。同時並以不同形式來表示解，並且確定解的一致性。

當我們對 PDEs 使用積分轉換時(對於變數是整條實數線使用 Fourier 轉換；變數是半射線使用 Laplace 轉換)，再藉由逆積分轉換(inversion Fouier transform or inversion Laplce transform)來得到 PDEs 的解是必要的。但是執行逆積分轉換時，經常那些被積分會出現平方根。然而，平方根在複數平面上是多值的。為了能正確地進行逆轉換，我們利用適當的代數分析來建構多值函數的黎曼曲面，使其變成單值函數，以致於我們能正確地在分析上和數值上完成逆積分轉換。最後由一些例子說明整個架構。

Topics on linear hyperbolic equations

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Abstract

We study the linear hyperbolic partial differential equations (linear hyperbolic PDEs). First, we give some practical examples and show that they are governed by such type of the equations. Next, we apply several classical methods to solve the linear hyperbolic PDEs with the solutions being expressed in various forms. We then identify those solutions.

When we apply Fourier and Laplace transformations to the whole- and half-line PDEs,it is necessary to perform the inverse Fourier and Laplace transformations to derive the PDE solutions, and it is quite often that those integrals involve the square root operator which is multi-valued in the complex plane. In order to perform the inverse transformations correctly, we develop the Riemann surfaces from the complex plane with the proper algebraic structures to assure that the square root is now a single-valued function on the surfaces, and we are able to accomplish the inverse transformations analytically and numerically. Some examples are given to illustrate the entire scheme.

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Contents

1 Introduction

We begin our study of linear hyperbolic equations by showing classical examples. First, we present a simple transport for first order partial differential equations, and then we extend our discussion to system of first order equations in electronics. Later, we will show the wave equation for second order partial differential equations. Under several hypothesis of physical phenomenon, a vibrating string problem is changed into one dimensional wave equation. Conversely, we can give a proper approximation to physics by discussing the solution of this mathematical model. Hence, we analyze the partial differential equations to observe physical problems.

1.1 The Advection Equation

Definition 1.1. Let $u(x,t)$,$F(x,t)$ be $m \times 1$ vector and $A(x,t)$, $B(x,t)$ be $m \times m$ matrix .The system of first order equations

$$u_t(x,t) + A(x,t)u_x(x,t) + B(x,t)u(x,t) = F(x,t) \tag{1.1}$$

is said to be hyperbolic if $A(x,t)$ is real diagonalized.

Obviously, a single real equation

$$u_t(x,t) + cu_x(x,t) = F(x,t)$$

is a hyperbolic equation. Let the particles of pollutant be transported from left to right with a constant speed c in a river. Denote $u(x,t)$ the density of particles at the position x and time t in the river. Suppose this river is so narrow that no particles get scattered.

First, we consider there no particles get lost or added. At time t, the amount of the particles of pollutant in an interval $[a,b]$, where $0 < a < b$, is

$$M = \int_a^b u(x,s)dx.$$

Let $s, h > 0$. The particles of time $s + h$ are of time s transported to right with distance ch centimeters. Hence, the amount of the particles of pollutant at time $s + h$ is equal to the amount at time s, i.e.

$$M = \int_a^b u(x, s)dx = \int_{a+ch}^{b+ch} u(x, s + h)dx. \tag{1.2}$$

By the First Fundamental of Calculus, we differentiate the equation (1.2) for b, then it becomes

$$u(b, s) = u(b + ch, s + h). \tag{1.3}$$

Again we differentiate the equation (1.3) for h, hence

$$0 = cu_x(b + ch, s + h) + u_t(b + ch, s + h).$$

Let h=0, we derive a homogeneous transport equation.

$$u_t(b, s) + cu_x(b, s) = 0,$$

where b is arbitrary, hence we get the advection equation.

If the particles get lost or added in the river, then we obtain the nonhomogeneous advection equation

$$u_t + cu_x = F,$$

where F is the amount of particles which get loss or added per length at position x and time t.

1.2 The Wave Equation in One Dimension

Definition 1.2. The linear partial differential equation for second order

$$A(x, t)u_{xx} + B(x, t)u_{xt} + C(x, t)u_{tt} + D(x, t)u_x + E(x, t)u_t + F(x, t)u = 0$$

is said to be hyperbolic, parabolic, or elliptic at (x_0, t_0) if $B^2(x_0, t_0) - 4A(x_0, t_0)C(x_0, t_0)$ is positive, zero, or negative, respectively.It is hyperbolic, parabolic, or elliptic in the domain D if $B^2(x, t) - 4A(x, t)C(x, t)$ is positive, zero, or negative for all $(x, t) \in D$, respectively.

For the most part in this paper, we discuss the type of linear hyperbolic equations. The wave equation in one dimension is

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = F(x,t),$$

where $0 < x < l, t > 0$. It is hyperbolic in its region. Because $A(x,t) = -c^2, B(x,t) = 0, C(x,t) = 1, B^2(x,t) - 4A(x,t)C(x,t) = 0 - 4 \times (-c)^2 \times 1 = 4c^2 > 0$,for all $0 < x < l, t > 0$.

Now, we want to show how the motion of a string as a mathematical equation under several assumptions.

(1) The string with length l is flexible and elastic. It is so flexible such that it offers no resistance to bending. Hence, the tension is in the direction of tangent to the profile of the string. In an elastic uniformly string, the density is a constant (mass per unit length).

(2) There is no elongation of a single element of the string. By Hooke's law the tension is constant.

(3)The string has small transverse vibration.

(4)The weight of the string is small compared with the tension in the string.

Denote $u(x,t)$ the displacement from equilibrium position at time t and position x . Let ρ and T be a constant density and tension at time t and position x. For any two closed points x and $x + \Delta x$ in the string at time t as shown in Figure 1

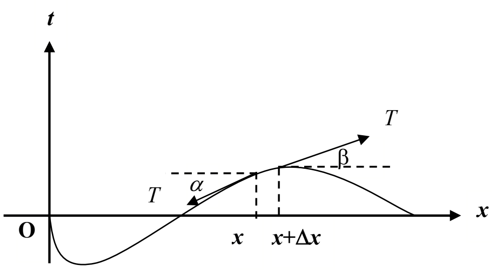


Figure 1: Two closed points in the string.

From Newton's Law $F = ma$, we get a equation for equivalent vertical force

$$T \sin\beta - T \sin\alpha = \rho \Delta x u_{tt}. \tag{1.4}$$

Since small transverse vibration of a string, $\sin\alpha \approx \tan\alpha$ and $\sin\beta \approx \tan\beta$. So, the equation (1.4) becomes

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} u_{tt}. \tag{1.5}$$

At time t, $\tan\alpha = (u_x)_x$ and $\tan\beta = (u_x)_{x+\Delta x}$, so equation (1.5) obtains

$$\frac{1}{\Delta x}[(u_x)_{x+\Delta x} - (u_x)_x] = \frac{\rho}{T} u_{tt}. \tag{1.6}$$

Let Δx be sufficiently small, then we derive the homogenous wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) = 0,$$

where $c^2 = \frac{T}{\rho}$.

Let there be an external force to a string. Hence, it appears an nonhomogeneous term.

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t),$$

where $c^2 = \frac{T}{\rho}$ and $f(x,t)$ the external force per unit length at position x and time t.

Remark 1.3. The value c is a wave speed. It is clear the unit of a tension T is $kg \cdot m/s^2$ and of a density ρ is kg/m. Then the unit of $c = \sqrt{\frac{T}{\rho}}$ is m/s . Here, the unit of c is indeed the unit of the speed.

Finally, we successfully transform a physical phenomenon into a mathematical equation under several properly hypotheses. By this mathematical equation, we can get a lot of information for the motion of the string vibrating. Thus, we will discuss the wave equation in detail in a later chapter.

1.3 The Telegraph Equation

The wave equation which we have discussed in section 1.2 can be replaced by the system of first order equations. Now, we want to present how the wave equation changes into the system of first order equations in electronics. Suppose a pair of transmission lines has a voltage $V(x,t)$ across them and a current $I(x,t)$ at position x and time t. The part of it is an interconnection of elements: capacitance, resistance, leakage resistance and inductance. Denote C the capacitance per unit length, R the resistance per unit length, G the leakage resistance per unit length and L the inductance per unit length.

Conductance G is the ability of an element to conduct electric current per unit length, and then conductance is the reciprocal of resistance

$$G = \frac{1}{R} = \frac{I}{V}.$$

Let μ, N, A be the permeability of core current, number of turns and cross section area of the inductor, respectively. When the current passes through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change

$$V = L\frac{dI}{dt},$$

where

$$L = \frac{N^2\mu A}{l}.$$

By Kirchhoff's Current Law and Kirchhoff's Voltage Law, we have system (1.7)

$$\begin{aligned} V(x,t) - RI(x,t)\Delta x - L\frac{\partial I(x,t)}{\partial t}\Delta x &= V(x+\Delta x, t), \\ I(x,t) - GV(x,t)\Delta x - C\frac{\partial V(x,t)}{\partial t}\Delta x &= I(x+\Delta x, t). \end{aligned} \tag{1.7}$$

Let Δx be small enough, then system (1.7) becomes the following system (1.8)

$$\begin{aligned} V_x(x,t) &= -RI(x,t) - LI_t(x,t), \\ I_x(x,t) &= -GV(x,t) - CV_t(x,t). \end{aligned} \tag{1.8}$$

Let

$$u = \begin{bmatrix} I \\ V \end{bmatrix},$$

then system (1.8) can be expressed the standard form of (1.1). Here,

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},$$

possesses two distinct real eigenvalues, A is real diagonalizable. Clearly, this system is hyperbolic. Differentiate the first and second of system (1.8) for x and t, respectively. Then we obtain

$$\begin{aligned} V_{xx}(x,t) &= -RI_x(x,t) - LI_{tx}(x,t), \\ I_{xt}(x,t) &= -GV_t(x,t) - CV_{tt}(x,t). \end{aligned} \tag{1.9}$$

Subtracting L times of second equation from first equation the system (1.9), hence we get the partial order equation for second order

$$CLV_{tt}(x,t) - V_{xx} = -(GL+CR)V_t(x,t) - GRV(x,t). \tag{1.10}$$

The equation

$$V_{tt}(x,t) - c^2 V_{xx} = -aV_t(x,t) - bV(x,t). \tag{1.11}$$

is called the telegraph equation, where $c = \frac{1}{CL}, a = GL + CR, b = GR$.

Let this transmission line have no energy lost, then $R = 0, G = 0$. Hence, above equation (1.11) becomes the homogenous wave equation

$$V_{tt}(x,t) - \frac{1}{CL} V_{xx} = 0.$$

Similarly, we can derive the homogenous wave equation of I

$$I_{tt}(x,t) - \frac{1}{CL} I_{xx} = 0.$$

2 Solutions of Linear Hyperbolic Equations

2.1 Solution of the Advection Equation

Example 2.1. *Using the method of characteristic to solving a I.V.P. of the advection equation.*

$$u_t(x,t) + 2tu_x(x,t) = 0, -\infty < x < \infty, t > 0,$$
$$u(x,0) = e^{-x^2}, -\infty < x < \infty.$$

The characteristic is

$$\frac{dx}{dt} = 2t,$$
$$x(0) = \xi.$$

Along this characteristic

$$x(t) = t^2 + \xi,$$

the solution $u(x,t)$ is a constant. Because

$$\begin{aligned}\frac{d}{dt}u(x(t),t) &= u_t + u_x\frac{dx}{dt}\\ &= u_t + 2tu_x\\ &= 0\end{aligned}$$

Hence, the solution

$$u(x(t),t) = u(x(0),0) = f(\xi) = e^{-(x-t)^2}.$$

2.2 D'Alembert's Solution to the Wave Equation

Consider a finite string problem with two fixed ends

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), 0 < x < l, t > 0, \\ u(x,0) &= f(x), 0 \le x \le l, \\ u_t(x,0) &= g(x), 0 \le x \le l, \\ u(0,t) &= 0, t \ge 0, \\ u(l,t) &= 0, t \ge 0. \end{aligned} \tag{2.1}$$

Chosen a new coordinate transformation (ξ, η)

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct. \end{aligned}$$

Hence, the wave equation (2.1) becomes

$$-4c^2 u_{\xi\eta} = 0.$$

So, the solution is

$$u(x,t) = p(\xi) + q(\eta), \tag{2.2}$$

where $p(\xi), q(\eta)$ are arbitrary functions of ξ, η, respectively.

First, we discuss a solution only depends on $p(\xi)$.

Fixed $x + ct = \xi$, then the solution is

$$u(x,t) = p(\xi),$$

Since,

$$\frac{dx}{dt} = -c,$$

Hence, along this characteristic $x + ct = \xi$, the wave is move to left with velocity c. Next, we consider a solution only depends on $q(\eta)$. First, we

Fixed $x - ct = \eta$, then the solution is constant of

$$u(x,t) = q(\eta),$$

Since

$$\frac{dx}{dt} = c.$$

Hence, along this characteristic $x - ct = \eta$, the wave is move to right with velocity c. Finally, we combine $p(\xi)$ and $q(\eta)$ together. One wave for $p(\xi)$ propagates to left along the line $x + ct = \xi$ with speed c; another $q(\eta)$ for propagates to right along the line $x - ct = \eta$ with speed c.

Applying initial conditions of (2.1) to a general solution (2.2), hence we get

$$p(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\bar{x})d\bar{x} + p(0), \xi \in [0, l],$$
$$q(\eta) = \frac{1}{2} f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\bar{x})d\bar{x} - p(0), \eta \in [0, l],$$

Thus, the D'Alembert's solution is

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x},$$

where $0 \leq x - ct \leq x + ct \leq l$. The above solution of the D'Alembert's solution form is only valid on the region shown as Figure 2.

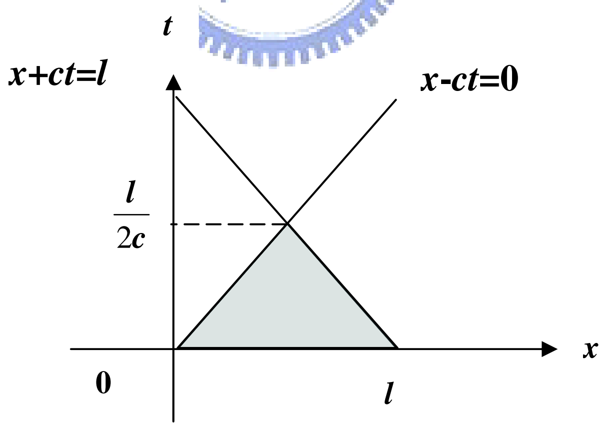


Figure 2: The region of D'Alembert's solution.

Consider an nonhomogeneous finite string problem with two fixed ends. After a corrdinate transformation (ξ, η) as above, then the wave equation with extra force term becomes

$$u_{\xi\eta}(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}) = -\frac{1}{4c^2}F(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}),$$

Integrating two sides of above equation for ξ from η to ξ, we get

$$u_\eta(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}) - \frac{1}{2}f'(\eta) - \frac{1}{2c}g(\eta) = -\frac{1}{4c^2}\int_\eta^\xi F(\frac{\bar{\xi}+\eta}{2}, \frac{\bar{\xi}-\eta}{2c})d\bar{\xi},$$

And we integrate above equation for η from η to ξ, then it yields the nonhomogeneous D'Alembert's solution

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} + \frac{1}{2c}\int_0^t \int_{x-c(t-\bar{t})}^{x+c(t-\bar{t})} F(\bar{x},\bar{t})d\bar{x}d\bar{t},$$

Applying the boundary condition, we try to extend the valid domain of the D'Alembert's solution to $D = \{(x,t) \mid -\infty < x < \infty, t \geq 0\}$ by extending functions f, g and F. The following table is that extends the domain of two functions f, g and F corresponding to different boundary conditions from $[0, l]$ to $(-\infty, \infty)$.

$x = 0$	extend functions f, g and F	$x = l$
$u(0,t) = 0$	odd at points $0, l$	$u(l,t) = 0$
$u(0,t) = 0$	odd at points 0,even at l	$u_x(l,t) = 0$
$u_x(0,t) = 0$	even at points 0,odd at l	$u(l,t) = 0$
$u_x(0,t) = 0$	even at points $0, l$	$u_x(l,t) = 0$

Table 1: Extend functions f, g and F.

Example 2.2. *Find the D'Alembert's solution for the following problem.*

$$u_{tt}(x,t) = u_{xx}(x,t), 0 < x < \pi, t > 0,$$

$$u(x,0) = \sin x, 0 \leq x \leq \pi,$$

$$u_t(x,0) = \cos x, 0 \leq x \leq \pi,$$

$$u(0,t) = 0, t \geq 0,$$

$$u(\pi,t) = 0, t \geq 0.$$

Using D'Alembert's solution formula, we get

$$u(x,t) = \frac{1}{2}[sin(x+ct) + sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cos(\overline{x})d\overline{x}, \tag{2.3}$$

where $0 \le x - ct \le x + ct \le \pi$. According to boundary conditions of this problem, f and g are odd at the point $x = 0$ and even at the point $x = \pi$. Then $f(x) = \sin x$, $g(x) = \cos x$, $x \in \mathbb{R}$. Hence,

$$u(x,t) = \sin x \cos t + \cos x \sin t, x \in \mathbb{R}, t > 0 \tag{2.4}$$

2.3 The method of Separation of Variables to the Wave Equation

In this section, we introduce a common method to solve the initial boundary value problem. The strategy of this method is separate independent variables for the function.

Example 2.3. *Using the method of separation of variables to solve the forced vibration of rectangular membrane problem.*

$$\begin{aligned}
u_{tt}(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t) + xy\sin t, 0 < x < \pi, 0 < y < \pi, t > 0,\\
u(x,y,0) &= 0, 0 \le x \le \pi, 0 \le y \le \pi,\\
u_t(x,y,0) &= 0, 0 \le x \le \pi, 0 \le y \le \pi,\\
u(0,y,t) &= 0, 0 \le y \le \pi, t \ge 0,\\
u(\pi,y,t) &= 0, 0 \le y \le \pi, t \ge 0,\\
u(x,0,t) &= 0, 0 \le x \le \pi, t \ge 0,\\
u(x,\pi,t) &= 0, 0 \le x \le \pi, t \ge 0.
\end{aligned} \tag{2.5}$$

Let

$$u(x,y,t) = U(x,y)T(t). \tag{2.6}$$

Substituting equation (2.6) into the wave equation of problem (2.5), then

$$UT" = c^2 \Delta UT.$$

Let

$$\frac{T"}{c^2T} = \frac{\Delta U}{U} = -\lambda.$$

Then

$$T'' + \lambda T = 0,$$

$$\Delta U + \lambda U = 0.$$

Let $\lambda = \alpha^2$, then we have

$$T = A\cos\alpha t + B\sin\alpha t,$$

where A and B are constant. Again separating the variables of $U(x,t)$, let $U(x,t) = X(x)Y(y)$. Then it yield two problems

$$X'' - \mu X = 0,$$

$$X(0) = 0,$$

$$X(\pi) = 0.$$

and

$$Y'' + (\lambda + \mu)Y = 0,$$

$$Y(0) = 0,$$

$$Y'(0) = 0.$$

Let $\mu = -\beta^2$ and $\gamma^2 = (\lambda + \mu) = \alpha^2 - \beta^2$. Hence, the solutions of above problems are

$$X_m(x) = \sin mx,$$

$$Y_n(y) = \sin ny,$$

where $\beta = m$ and $\gamma = n$. So,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{mn}\cos\alpha_{mn}t + b_{mn}\sin\alpha_{mn}t)\sin mx\sin ny,$$

where

$$a_{mn} = \frac{4}{ab}\int_0^{\pi}\int_0^{\pi} u(x,y,0)\sin mx\sin ny\,dxdy = 0,$$

$$b_{mn} = \frac{4}{\alpha_{mn}ab}\int_0^{\pi}\int_0^{\pi} u_t(x,y,0)\sin mx\sin ny\,dxdy = 0.$$

Assume the solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin\alpha_{mn}t\sin mx\sin ny,$$

and external forcing function

$$F(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}(t)\sin\alpha_{mn}t\sin mx\sin ny,$$

here

$$\begin{aligned} F_{mn}(t) &= \frac{4}{ab}\int_0^{\pi}\int_0^{\pi} F(x,y,t)\sin mx\sin nydxdy \\ &= \frac{4(m\pi\cos m\pi - \sin m\pi)(n\pi\cos n\pi - sinn\pi)\sin t}{\pi^2 m^2 n^2} \end{aligned}$$

Taking u and F into the wave equation in problem (2.5) Hence, we get the following equation

$$u''_{mn} + (m^2+n^2)u_{mn} = F_{mn},$$

where u is twice continuously differentiable with respect to t. Thus,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin\alpha_{mn}t\sin mx\sin ny,$$

where

$$\begin{aligned} u_{mn}(t) &= \frac{1}{\alpha_{mn}}\int_0^t F_{mn}\sin(\alpha_{mn})(t-\tau)d\tau \\ &= \frac{4(-1)^{m+n+1}}{mn\alpha_{mn}}\{\sin\alpha_{mn}t[\frac{\cos(1-\alpha_{mn})t-1}{2(1-\alpha_{mn})} + \frac{\cos(1+\alpha_{mn})t-1}{2(1+\alpha_{mn})}] \\ &+ \cos\alpha_{mn}t[\frac{\sin(1-\alpha_{mn})t-1}{2(1-\alpha_{mn})} + \frac{\sin 1+\alpha_{mn})t-1}{2(1+\alpha_{mn})}]\}, \end{aligned}$$

where $\alpha_{mn} = \sqrt{m^2+n^2}$

2.4 The Fourier Transforms to the Wave Equation

We often use the method of integral transform to solve the problem for initial value problems of the infinite or semi-infinite region. First, we introduce the method of Fourier Transform for a variable of all full real line. In general, a variable transformed is the spatial variable. And we will discuss the solution by Fourier Transform.

Definition 2.4. If $f(x)$ is absolutely integrable, then the Fourier Transform is

$$\mathscr{F}[f(x)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx.$$

Problems of partial differential equation can be reduced by problems of ordinary differential equation for the Fourier Transform $\hat{u}(x,t)$ of $u(x,t)$ by property (2.5). After solving the problem of $\hat{u}(x,t)$, there is an inversion theorem for Fourier Transform to help we transform $\hat{u}(x,t)$ back $u(x,t)$.

Property 2.5. If $f(x)$ is absolutely integrable, approaches zero as $x \to \pm\infty$ and has a first derivative, then

$$\mathscr{F}[f'(x)](\omega) = -i\omega\mathscr{F}[f(x)](\omega)$$

Property 2.6. If $f(x)$ is absolutely integrable, then

$$\mathscr{F}[e^{icx}f(x)](\omega) = \mathscr{F}[f(x)](\omega + c).$$

Theorem 2.7. *If $f(x)$ is absolutely integrable, then the Fourier Transform is*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}[f(x)](\omega).$$

Example 2.8. *Using the Fourier Transform to solve a long string problem.*

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), -\infty < x < \infty, t > 0, \\ u(x,0) &= e^{-|x|}, -\infty < x < \infty, \\ u_t(x,0) &= 0, -\infty < x < \infty, \end{aligned} \tag{2.7}$$

Let the Fourier Transform of u for a variable x is

$$\mathscr{F}[u(x,t)](\omega) = \hat{u}(\omega,t) = \int_{-\infty}^{\infty} e^{i\omega x}u(x,t)dx.$$

Then problem (2.7) is reduced to the following problem (2.8) for second order ordinary differential equation corresponding initial value conditions by property 2.5.

$$\begin{aligned} \frac{d}{dt^2}\hat{u}(\omega,t) &= -c^2\omega^2\hat{u}(\omega,t), -\infty < \omega < \infty, t > 0, \\ \hat{u}(\omega,0) &= \frac{2}{1+\omega^2}, -\infty < \omega < \infty, \\ \frac{d}{d\omega}\hat{u}(\omega,0) &= 0, -\infty < \omega < \infty. \end{aligned} \tag{2.8}$$

Then

$$\hat{u}(\omega, t) = \frac{2}{1+\omega^2}\cos\omega ct$$
$$= \frac{2}{1+\omega^2}\frac{e^{i\omega ct} - e^{-i\omega ct}}{2}$$

By property 2.6, it yields the solution

$$u(x,t) = \frac{1}{2}[e^{-|x+ct|} + e^{-|x-ct|}].$$

2.5 The Laplace Transforms to the Wave Equation

In section (2.4), we have introduced a method of Fourier Transform in order to solving problem of infinite line region. Now, we present a method of Laplace Transform for solving a half-line extent. In general, a variable transformed is the time variable. The property of Laplace Transform is similar to Fourier Transform.

Definition 2.9. If $f(t)$ is absolutely integrable, then the Laplace Transform is

$$\mathscr{L}[f(t)](s) = F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{2.9}$$

The properties of Laplace Transform are similar to Fourier Transform.

Property 2.10. If $f(t)$ is absolutely integrable, approaches zero as $t \to \infty$ and has a first derivative for $t > 0$, then

$$\mathscr{L}[f'(t)](s) = s\mathscr{L}[f(t)](s) - f(0).$$

In general, if $f(t)$ and $f^{(m)}(t)$ are absolutely integrable, $m = 1, 2, n-1$, approaches zero as $t \to \infty$

$$\mathscr{L}[f^{(n)}(t)](s) = s^n\mathscr{L}[f(t)](s) - s^{n-1}f(0) - ... - f^{(n-1)}(0).$$

Property 2.11. If $\mathscr{L}^{-1}[F(s)] = f(t)$, then

$$\mathscr{L}^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a) & \text{if } t > a, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.12. *If $F(s)$ is the Laplace transform of a real function $f(t)$, with the complex transformed variables s, then the inversion integral is*

$$f(t) = \lim_{L\to\infty} \int_{\gamma-iL}^{\gamma+iL} \mathscr{L}[f(t)](s)e^{st}ds. \tag{2.10}$$

Example 2.13. *Using the Laplace Transform to solve one fixed end string problem.*

$$\begin{aligned} u_{tt}(x,t) &= 4u_{xx}(x,t), 0 < x < \infty, t > 0, \\ u(x,0) &= 0, 0 < x < \infty, \\ u_t(x,0) &= 0, 0 < x < \infty, \\ u(0,t) &= 3\sin t, t \geq 0. \end{aligned} \tag{2.11}$$

Let $U(x,s) = \mathscr{L}[u(x,t)], u(x,t)$ is bounded. Then problem (2.11) is reduced to the following problem (2.12) for second order ordinary differential equation by property 2.10.

$$\begin{aligned} s^2U(x,s) &= U_{xx}(x,s), 0 < x < \infty, t > 0, \\ U(x,0) &= 0, 0 < x < \infty, \\ U_t(x,0) &= 0, 0 < x < \infty, \\ U(0,s) &= \frac{3}{s^2+1}, s \geq 0. \end{aligned} \tag{2.12}$$

Then we have

$$U(x,s) = ae^{sx} + be^{-sx}, \tag{2.13}$$

where a and b are constant. Applying a boundary condition of problem (2.12) and $U(x,s)$ is bounded, then

$$U(x,s) = \frac{3}{s^2+1}e^{-sx} \tag{2.14}$$

By property 2.11, the solution is

$$u(x,t) = \begin{cases} 3\sin(t-x) & \text{if } t > x, \\ 0 & \text{otherwise.} \end{cases}$$

2.6 The method of characteristics to the Wave Equation

In this section, we will discuss how we approximate the solution for hyperbolic linear equation of second order by two characteristics. The following linear hyperbolic equation for second order is

$$A(x,t)u_{xx}x,t + B(x,t)u_{x,t}(x,t) + C(x,t)u_{tt}(x,t) + e(x,t,u_t,u_x) = 0, \tag{2.15}$$

where $B^2(x,t) - 4A(x,t)C(x,t) > 0$ for all (x,t) in its region. Let

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= S(x,t), \\ \frac{\partial u(x,t)}{\partial t} &= T(x,t). \end{aligned} \tag{2.16}$$

Then

$$\begin{aligned} dS &= u_{xx}dx + u_{xt}dt, \\ dT &= u_{tx}dx + u_{tt}dt. \end{aligned} \tag{2.17}$$

Substitute equations (2.17) into equation (2.15), then we obtain

$$A(x,t)(\frac{dS - u_{xt}dt}{dx}) + B(x,t)u_{xt}(x,t) + C(x,t)(\frac{dT - u_{xt}dx}{dt}) + e(x,t,u_t,u_x) = 0. \tag{2.18}$$

Multiplying two sides of above equation (2.18) by $\frac{dt}{dx}$, then we obtain

$$[A(\frac{dt}{dx})^2 - B(\frac{dt}{dx}) + C]u_{xt} - [A\frac{dS}{dx}(\frac{dt}{dx}) + C\frac{dT}{dx} + e(\frac{dt}{dx})] = 0. \tag{2.19}$$

Let a tangent slope at every point on the curve C satisfy a root of

$$A(\frac{dt}{dx})^2 - B(\frac{dt}{dx}) + C = 0. \tag{2.20}$$

Since

$$B^2(x,t) - 4A(x,t)C(x,t) > 0,$$

there are two curve. These are called characteristics. On two characteristics, the equation (2.19) is simplified to solve

$$A\frac{dS}{dx}(\frac{dt}{dx}) + C\frac{dT}{dx} + e(\frac{dt}{dx})] = 0.$$

The procession for solving this problem is shown as follows.

First, we have to know what the values m of the tangent slope at initial points P and Q on the curve C are. Let m^+ and m^- be the right characteristic and the left characteristic at one point, respectively. As shown in Figure 3. This value is easy to get from equation (2.20).

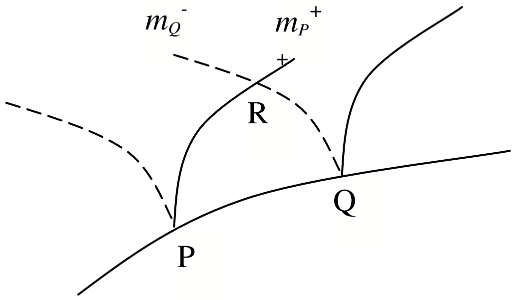


Figure 3: Two characteristics.

Second, we use lines to approximate these curves and we have to find the point R. An line equation of the right characteristic at the point P is

$$L_1 : t - t_P = m_p^+(x_R - x_P)$$

and a line equation of the left characteristic at the point Q is

$$L_2 : t - t_Q = m_Q^-(x_R - x_Q)$$

The intersection of above two lines L_1 and L_2 is the point R. Hence, we can find a pint R from two lines L_1 and L_2.

Third, we want to find the value of S_R and T_R. Denote m_{av} the average of the tangent slope of any two points in the line. From P to R of L_1, we derive the equation

$$A_P(x, t)(S_R - S_P)m_{av} + C_P(T_R - T_P) + e_P(t_R - t_P) = 0.$$

From Q to R of L_2, we obtain the equation

$$A_Q(x,t)(S_R - S_Q)m_{av} + C_Q(T_R - T_Q) + e_Q(t_R - t_Q) = 0,$$

where an index of variables A, C, S, T, G and t represents the values of A, C, S, T, G and t at that point. Finally, we have value of u at the point R from following equation (2.21)

$$u_R - u_P = \frac{1}{2}(S_P + S_R)(x_R - x_P) + \frac{1}{2}(T_P + T_R)(t_R - t_P) \tag{2.21}$$

Example 2.14. *Using the method of characteristics to calculate displacement u at time $t = 0.3$ of the point on the string$x = 0.4$.*

$$\begin{aligned}
u_{tt}(x,t) &= u_{xx}, 0 < x < l, t > 0,\\
u(x,0) &= \frac{1}{2}x(1-x), 0 \le x \le l,\\
u_t(x,0) &= 0, 0 \le x \le l,\\
u(0,t) &= u(l,t) = 0, t \ge 0.
\end{aligned} \tag{2.22}$$

$A(x,t) = -1, B(x,t) = 0, C(x,t) = 1$ and $e(x,t) = 0$ in the partial differential equation of problem (2.15). Since $m_P^+ = 1$ and $m_Q^- = -1$, the right characteristic through P and the left characteristic through Q are $L_1 : x - t = 0.1, L_2 : x + t = 0.7$, respectively. Here, $S(x,t) =$ and $T(x,t) = 0, 0 \le x \le l, t > 0$. Then $S_P = 0.4, S_Q = -0.2, T_P = 0, T_Q = 0$.

From P to R in L_1, we have

$$(-1)(S_R - 0.4) + T_R = 0.$$

From Q to R in L_2, we get

$$(S_R + 0.2) + T_R = 0.$$

Those equations imply $S_R = 0.1, T_R = -0.3$. Finally, the value of u at the point R from equation (2.21) is

$$0.045 + \frac{0.4 + 0.1}{2}(0.4 - 0.1) + \frac{0 - 0.3}{2}(0.3 - 0) = 0.075.$$

Thus, a displacement u at time $t = 0.3$ of the point on the string $x = 0.4$ is 0.075.

2.7 The method of Finite Difference to the Wave Equation

The methods of integral transforms (like as Fourier, Laplace transform) and separation of variables are only valid on special problems. And we obtain an infinite integral form or a sum of infinite series form of solutions. If we can not compute them exactly, we want to approximate them by numerical methods. Suppose a function $u(x,t)$ and its derivatives are continuous and finite. We give $v(x,t)$ a good approximation to $u(x,t)$.

A stretch string fixed two points problem (2.14) has done by the methods of characteristic. Now, we try to solve it by the method of finite difference for $l = 1$. Denote mesh parameter $h = \frac{l}{N}$ and $k = h$ such that $x = 0, h, 2h, \ldots, Nh = 1, t = 0, k, 2k, \ldots, N = 0.1$. Let a mash function $v(nh, mk)$ satisfies

$$\begin{aligned}
\Lambda_h[v] &= -\frac{v((n+1)h, mk) - 2v(nh, mk) + v((n-1)h, mk)}{h^2} \\
&\quad + \frac{v(nh, (m+1)k) - 2v(nh, mk) + v(nh, (m-1)k)}{k^2} = 0, \\
u(x,0) &= \frac{1}{2}x(1-x), 0 \le x \le 1, \\
u_t(x,0) &= 0, 0 \le x \le 1, \\
u(0,t) &= u(1,t) = 0, t \ge 0.
\end{aligned} \tag{2.23}$$

Then we get a recursion formula

$$v(nh, (m+1)k) = v((n+1)h, mk) - v((n-1)h, mk) - v(nh, (m-1)k), \tag{2.24}$$

$m \ge 1$. But there does not have the value v at time -1 of point nh in above recursion formula (2.24). Initial conditions of problem (2.23) provides some information

$$v(nh, -k) = v(nh, k). \tag{2.25}$$

Put (2.25) into a recursion formula (2.24), then it becomes

$$v(nh, k) = \frac{1}{2}[v((n+1)h, 0) - v((n-1)h, 0)]. \tag{2.26}$$

Thus, the following table and Figure 4 shown the discrete values

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0.045	0.08	0.105	0.12	0.125	0.12	0.105	0.08	0.045	0
0.1	0	0.04	0.075	0.1	0.115	0.1225	0.115	0.1	0.075	0.04	0
0.2	0	0.03	0.06	0.085	0.1025	0.105	0.1025	0.085	0.06	0.03	0
0.3	0	0.02	0.04	0.0625	0.075	0.0825	0.075	0.0625	0.04	0.02	0

Table 2:Discrete data to problem (2.14) with $l = 1$.

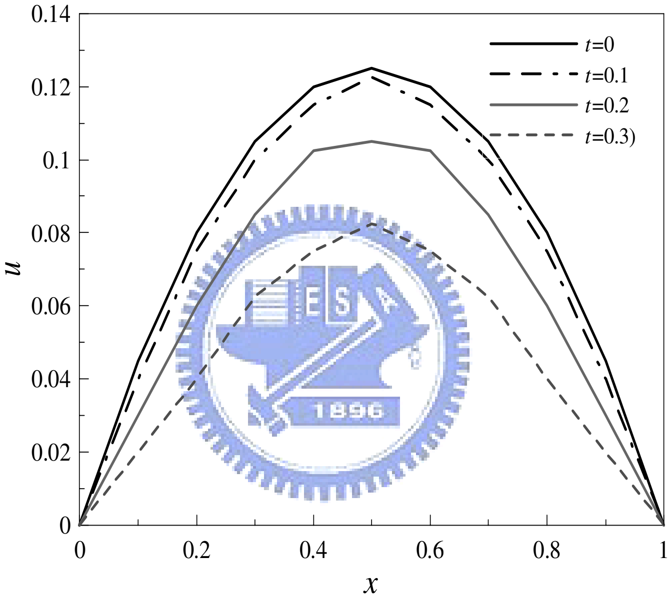


Figure 4: Solutions are solved by Finite Differences at t=0,0.1,0.2,0.3.

3 The Wave equation in different domains

3.1 Solution of the finite string problem

Consider a stretch string fixed two end points problem (2.1). The D'Alembert's solution of a finite string problem is

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x},$$

where $f(x)$ and $g(x)$ odd extend about points $x = 0$ and $x = l$. They extend the domain $D = \{(x,t) \mid 0 \le x \le l, t > 0\}$ to $\{(x,t) \mid x \in \mathbb{R}, t > 0\}$.

Standing wave solutions of a finite string is

$$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi c}{l})t + b_n \sin(n\pi cl)t] \sin(\frac{n\pi}{l}x),$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l})xdx, b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin(\frac{n\pi}{l})xdx.$$

Proposition 3.1. *The integral form of D'Alembert's solution is the same as the infinite series form from the method of separation of variables.*

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} \\
&= \frac{1}{2}[\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l})(x+ct) + \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l})(x-ct)] \\
&+ \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{n\pi c}{l}\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{l}\bar{x}d\bar{x}) \\
&= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}x)\cos(\frac{n\pi c}{l}t) + \frac{n\pi}{2l}\sum_{n=1}^{\infty}\int_{x-ct}^{x+ct} b_n \sin(\frac{n\pi}{l})\bar{x}d\bar{x} \\
&= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}x)\cos(\frac{n\pi c}{l}t) - \frac{1}{2}[\sum_{n=1}^{\infty} b_n \cos(\frac{n\pi}{l})(x+ct) - \sum_{n=1}^{\infty} b_n cos(\frac{n\pi}{l})(x-ct)] \\
&= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}x)\cos(\frac{n\pi c}{l})t + b_n \sin(\frac{n\pi}{l}x)\sin(n\pi cl)t \\
&= \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi c}{l})t + b_n \sin(n\pi cl)t]\sin(\frac{n\pi}{l}x).
\end{aligned}$$

The integral form of D'Alembert's solution is the same as the series form of separation of variables.

There are differences between D'Alembert's formula and Separation of variables. The method of separation of variables also solves the special problem of the wave equation with nonconstant c. This method is restricted to on a finite boundary condition. Like as rectangle,circle or cylinder,and so on. In fact, it does not work for all equations. For instance, a partial differential equation with the variable coefficients. It is hard to separate the equation such that one side of only involving in x and the other in y. Even if it with constant coefficients, it may be not apply to.For example, $u_{xx}+u_{xy}+u_{yy} = 0$ is not separable in rectangular coordinates. But it can be in polar coordinates.

There is the comparison between an analytic solution and numerical solutions in Example (2.14) with $l = 1$ in following Table.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Analytic solution	0	0.02	0.04	0.06	0.075	0.08	0.075	0.06	0.04	0.02	0
Finite Difference	0	0.02	0.04	0.0625	0.075	0.0825	0.075	0.0625	0.04	0.02	0
Characteristic	0	0.02	0.04	0.06	0.075	0.08	0.075	0.06	0.04	0.02	0

Table 3: Discrete solutions by different methods to Example (2.14) with $l = 1$ at $t = 0.3$.

From above Table 3, the solution by characteristics is an exact solution. Because characteristics are just lines in this example for constant speed c, there is no error in it. Hence, we can get a correct solution. The solution by the finite difference method is approximate to the accurate solution. It is a stable solution with an error of second order. Comparing to analytic solution, the numerical solution is the discrete data. We must evaluate the value one by one. If it is not easy to find the analytic solution, then it is a good choice to evaluate the discrete solution by the computer and give a good approximation for analytic solution.

3.2 Solution of the infinite problem

$$
\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t), -\infty < x < \infty, t > 0, \\
u(x,0) &= f(x), -\infty < x < \infty, \\
u_t(x,0) &= g(x), -\infty < x < \infty, \\
\lim_{x\to\infty} u(x,t) &= \lim_{x\to\infty} \frac{\partial u(x,t)}{\partial t} = 0, t > 0.
\end{aligned}
$$

Fourier Transform of this initial-fixed boundary value problem on x is

$$
\begin{aligned}
\frac{d^2}{dt^2}\hat{u}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) &= 0, -\infty < \omega < \infty, t > 0, \\
\hat{u}(\omega,0) &= F(\omega), -\infty < \omega < \infty, \\
\frac{d}{dt}u(\hat{\omega},0) &= G(\omega), -\infty < \omega < \infty.
\end{aligned}
\tag{3.1}
$$

Problem (3.2) becomes ordinary differential equation of ω for second order, then the solution is

$$
\begin{aligned}
\hat{u}(\omega,t) &= F(\omega)\cos\omega ct + \frac{G(\omega)}{ct}\sin\omega ct \\
&= F(\omega)\frac{e^{i\omega ct}+e^{-i\omega ct}}{2} + \frac{G(\omega)}{c\omega}\frac{e^{i\omega ct}-e^{-i\omega ct}}{2i}.
\end{aligned}
\tag{3.2}
$$

By Inversion of Fourier Transform Theorem, we have

$$
u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x},
$$

4 Riemann surface of genus N

The procession of solving the general initial value problem of linear hyperbolic equations by Fourier transform or Laplace transform may meet the integral of the square root function. Here, we present how to deal with this multi-valued functions such that the integral is meaning and correct evaluated by Mathematica.

4.1 Introduction

We know the square root function is a multi-valued function in the complex plane. Because

$$z = re^{i\theta} = e^{i(\theta+2\pi)},$$

Then

$$\sqrt{z} = \begin{cases} \sqrt{r}e^{i(\frac{\theta}{2})} \\ \sqrt{r}e^{i(\frac{\theta+2\pi}{2})} = -\sqrt{r}e^{i(\frac{\theta}{2})} \end{cases}$$

It is clear that a square root function is a two-valued function defined on the complex plane $\mathbb{C}$. So, it is not continuous on $\mathbb{C}$. Moreover, it is not analytic on $\mathbb{C}$. Hence, it is meaningless for evaluate the integral of multi-valued function. The only way to change multi-valued functions into single value function is redefined the domain. Denote two the square root functions

$$f_1(z) = |z|^{\frac{1}{2}} e^{\frac{iarg(z)}{2}}$$

and

$$f_2(z) = |z|^{\frac{1}{2}} e^{\frac{iarg(z)}{2}}.$$

Let $D_1 = \{\mathbb{C} \setminus (-\infty, 0] \mid arg(z) \in [-\pi, \pi)\}$ and $D_2 = \{\mathbb{C} \setminus (-\infty, 0] \mid arg(z) \in [\pi, 3\pi)\}$ be the domains of square root functions $f_1(z)$ and $f_2(z)$, respectively. Here, functions $f_1(z)$ and $f_2(z)$ are single valued function in each domain. Since $f_1(z)$ and $f_2(z)$ are both discontinuous on the negative real line, denote branch cuts $C_1 = \{(-\infty, 0] \mid arg(z) = \pi\}$ and $C_2 = \{(-\infty, 0] \mid arg(z) = 3\pi\}$.

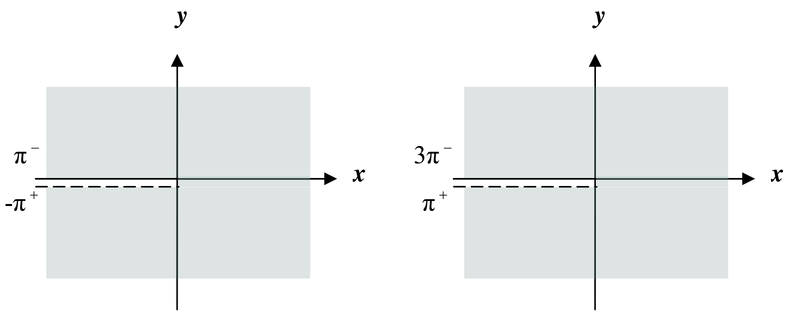


Figure 5: The domain D_1 and D_2.

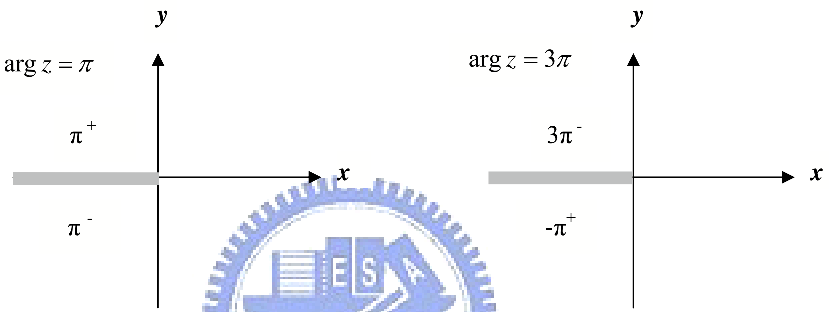


Figure 6: The branch cut C_1 and C_2.

Let functions defined on above two cuts be

$$f_3(z) = i|z|^{\frac{1}{2}}, z \in C_1.$$

and

$$f_4(z) = -i|z|^{\frac{1}{2}}, z \in C_2.$$

And functions $f_3(z)$ and $f_4(z)$ are single valued function on each cuts. These domains are glued together is a Riemann surface domain. Hence, we construct a simple value function on Riemann surface as following:

$$\sqrt{z} = \begin{cases} f_1(z), z \in D_1 \\ f_2(z), z \in D_2 \\ f_3(z), z \in C_1 \\ f_4(z), z \in C_2 \end{cases}$$

Remark 4.1. The square root function is analytic in the domain $D_1 \cup D_2 \cup C_1 \cup C_2$. Since functions $f_1(z)$, $f_2(z)$, $f_3(z)$ and $f_4(z)$ are analytic in each domains. And any path pass from D_1 to D_2 through the cut C_1 is also continuous. Because $z \in D_1$ and $argz$ tends to π, then $f_1(z)$tends to $i|z|^{\frac{1}{2}}$. It is equal to the function $f_2(z)$, where $argz = \pi$. It is similar to any path from D_2 to D_1 through the cut C_2 is still continuous. Hence, the function is analytic on Riemann surface.

Remark 4.2. $f_1(z) = -f_2(z)$ and $f_3(z) = -f_4(z)$.
Since $z \in D_2$, $arg(z) \in [\pi, 3\pi)$.

$$\begin{aligned} f_2(z) &= |z|^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}} \\ &= |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} e^{i\pi} \\ &= -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} \\ &= -f_1(z), \end{aligned}$$

where $\theta \in [-\pi, \pi)$.

We will develop an algorithm such that we can evaluate the integrals of a square root function by Mathematica.

4.2 The algebraic structure on Riemann surface

Since $f(z) = \sqrt{z}$ is a two-valued function, we need construct branch cuts to cut our plane. For every point in complex plane, we have $arg(z - 0) \in [-\pi, \pi)$. In following Figure 7, + edge is the initial edge and − edge is terminal edge. The argument of + edge is $-\pi$ and of − edge is π.

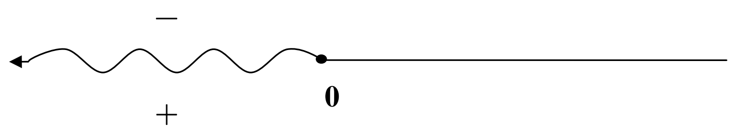


Figure 7: The cut plan of $\sqrt{z}$.

We define $(z, f(z))$ belong to sheet I if $arg(z-0) \in [-\pi, \pi)$; $(z, f(z))$ belong to sheet II if $arg(z-0) \in [\pi, 3\pi)$.

Remark 4.3. $z \in I^+$:

$$\begin{aligned} f(z) &= |z|^{\frac{1}{2}} e^{i\frac{-\pi}{2}} \\ &= -i|z|^{\frac{1}{2}} \end{aligned}$$

$z \in I^-$:

$$\begin{aligned} f(z) &= |z|^{\frac{1}{2}} e^{i\frac{2\pi}{2}} \\ &= i|z|^{\frac{1}{2}} \end{aligned}$$

Hence, $f(z)|_{I^-} = -f(z)|_{I^+}$. Since - edge in I is + edge in II, it means $f(z)|_{II} = -f(z)|_I$.

After presenting the algebraic structure of $f(z) = \sqrt{z}$ on Riemann surface, we have to show the cut structure for the general horizontal and vertical cuts. Let $p(z) = \sqrt{(z-z_1)(z-z_2)}$ For

Case1.$z_1, z_2 \in \mathbb{R}$(Horizontal cut)

z_1, z_2 are branch points of $p(z)$. It is discontinuous on this interval $[z_1, z_2]$. Hence, there is a branch cut between two points.

For example, $p(z) = \sqrt{z-1}\sqrt{z-2}$, chosen a point $0 \in (-\infty, 1)$.

(i) For 0 in + edge: Then $arg(z-1) = arg(z-2) = -\pi$

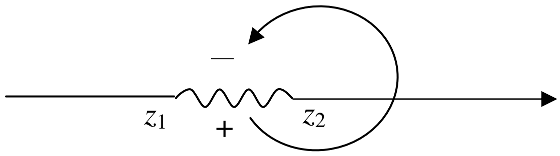


Figure 8: The algebraic structure of $p(z)$ for two branch points in horizontal.

(ii) For 0 in - edge: Then $arg(z-1) = arg(z-2) = \pi$

$$arg(0-1) = arg(-1) = \begin{cases} -\pi \\ \pi \end{cases}$$

$$arg(0-2) = arg(-2) = \begin{cases} -\pi \\ \pi \end{cases}$$

Taken the principal argument of a negative number is $-\pi$, hence

$$\sqrt{-1} \cdot \sqrt{-2} = |1|^{\frac{1}{2}} e^{i(\frac{-\pi}{2})} \cdot |2|^{\frac{1}{2}} e^{i(\frac{-\pi}{2})} = |2|^{\frac{1}{2}} e^{i(-\pi)} = -|2|^{\frac{1}{2}} \tag{4.1}$$

On the other hand, taken the principal argument of a negative number is π , then

$$\sqrt{-1} \cdot \sqrt{-2} = |1|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} \cdot |2|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} = |2|^{\frac{1}{2}} e^{i(\pi)} = -|2|^{\frac{1}{2}} \tag{4.2}$$

Here, the value in equation (4.1) is equal to equation (4.2). So, $p(z)$ is continuous on this interval $(-\infty, 1)$. Pick up one point $\frac{3}{2} \in (1, 2)$.

(i) For $\frac{3}{2}$ in + edge: Then $arg(z-1) = arg(z-2) = -\pi$

(ii) For $\frac{3}{2}$ in − edge: Then $arg(z-1) = arg(z-2) = \pi$

$$arg(\frac{3}{2} - 1) = arg(\frac{1}{2}) = 0$$

$$arg(\frac{3}{2} - 2) = arg(\frac{-1}{2}) = \begin{cases} -\pi \\ \pi \end{cases}$$

By the same way as above, chosen the principal argument of a negative number is $-\pi$, hence

$$\sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = |\frac{1}{2}|^{\frac{1}{2}} e^{i(0)} \cdot |\frac{1}{2}|^{\frac{1}{2}} e^{i(\frac{-\pi}{2})} = |\frac{1}{2}| e^{i(\frac{-\pi}{2})} = -|\frac{1}{2}| i \tag{4.3}$$

Chosen taken the principal argument of a negative number is π, then

$$\sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = |\frac{1}{2}|^{\frac{1}{2}} e^{i(0)} \cdot |\frac{1}{2}|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} = \frac{1}{2} e^{i(\frac{\pi}{2})} = |\frac{1}{2}| e^{i(\frac{\pi}{2})} = |\frac{1}{2}| i \tag{4.4}$$

Since the value in equation (4.3) is unequal to equation (4.4). Then the function is discontinuous on this interval $(1, 2)$. So, there is a branch cut in it. Of course, a square root function is continuous for positive real number. Hence, there is a branch cut between two points $z_1, z_2 \in \mathbb{N}$.

Case2.$z_1, z_2 \in i\mathbb{R}$(vertical cut)

The branch cut for the vertical cut structure is the same as horizontal. Denote the interval $[z_1, z_2]$ be the branch cut. The algebraic structure of $p(z)$ for two branch points in vertical as shown in Figure 9 We define the sheet is $I = \{z \mid arg(z - z_j) \in [-\frac{3}{2}\pi, \frac{1}{2}\pi)\}$ and another sheet

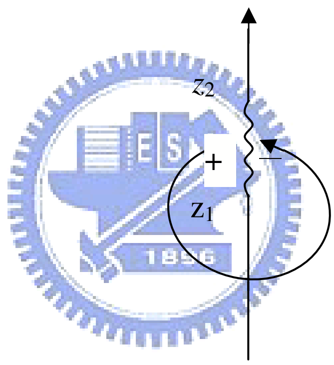


Figure 9: The algebraic structure of $p(z)$ for two branch points in vertical.

is $II = \{z \mid Arg(z - z_j) \in [\frac{1}{2}\pi, \frac{5}{2}\pi)\}$, where z_j is the branch point of $p(z)$, for each positive integer $j = 1, 2$. $+$ edge is the initial edge and $-$ edge is terminal edge. The argument of $+$ edge is $-\frac{3}{2}\pi$ and of $-$ edge is $\frac{1}{2}\pi$.

Remark 4.4. As we know, a curve crosses the cut from one sheet to another sheet. Hence, if a curve goes through $2N - 1$ cuts for $N \in \mathbb{N}$, then a curve will cross to another sheet. So, there is a branch cut at that line segment.

Remark 4.5. By the same way as horizontal case, thus the value in sheet II is minus of in sheet I. If $z \in I^{+}$(+ edge of sheet I) and $z \in [z_1, z_2]$, then $arg(z - z_1) = -\frac{3}{2}\pi$ and $arg(z - z_1) = \frac{1}{2}\pi$. So,

$$\begin{aligned}
p(z) &= \sqrt{(z - z_1)(z - z_2)} \\
&= \prod_{j=1}^{2} \sqrt{(z - z_j)} \\
&= |z - z_1|^{\frac{1}{2}} e^{i(-\frac{3\pi}{4})} \cdot |z - z_2|^{\frac{1}{2}} e^{i(-\frac{\pi}{4})} \\
&= -\prod_{j=1}^{2} |z - z_j|^{\frac{1}{2}}
\end{aligned}$$

If $z \in I^{-}$(− edge of sheet I), then $arg(z - z_1) = \frac{1}{2}\pi$ and $arg(z - z_1) = -\frac{1}{2}\pi$. So,

$$\begin{aligned}
p(z) &= \sqrt{(z - z_1)(z - z_2)} \\
&= \prod_{j=1}^{2} \sqrt{(z - z_j)} \\
&= |z - z_1|^{\frac{1}{2}} e^{i(\frac{\pi}{4})} \cdot |z - z_2|^{\frac{1}{2}} e^{i(-\frac{\pi}{4})} \\
&= \prod_{j=1}^{2} |z - z_j|^{\frac{1}{2}}
\end{aligned}$$

So, $f(z)|_{I^-} = -f(z)|_{I^+}$. It means $f(z)|_{II} = -f(z)|_{I}$.

4.3 The geometric structure on Riemann surface

After presenting the algebraic structure, we want to know how does it look like in geometric. For the horizontal branch cut case, by above definition, sheet I and sheet II is denoted by $I = \{z \mid arg(z - z_j) \in [-\pi, \pi)\}$ and $II = \{z \mid arg(z - z_j) \in [\pi, 3\pi)\}$,respectively.We glue the + edge of sheet I to the − edge of sheet II. When the integral path crossing the branch cut, then it pass from one sheet to the other sheet. If we want to evaluate the value in sheet II, then we only evaluate the negative value of the value in sheet I. Now, we discuss the geometric structure on Riemann surface of $f(z) = \sqrt{z}$. Using stereo projection, we have a one-to-one correspondence between the complex plane and surface. Regard ∞ in the complex plane is a point and is projected to a north pole of the surface. Hence, a geometric structure of f as Figure 10 shown. And the geometric structure of vertical cuts on Riemann surface of $p(z)$ is the same as horizontal cuts.

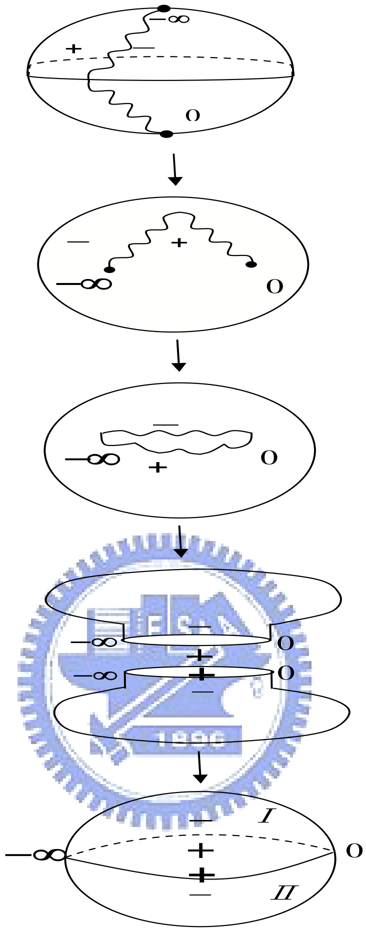


Figure 10: The geometric structure on Riemann surface of $p(z)$ in horizontal cuts.

Consider a general function $f(z) = \sqrt{\prod_{j=1}^{n}(z - z_j)}$. Since $z_1, z_2, ..., z_n$ are the n branch points, then there are $\lceil \frac{n}{2} \rceil - 1$ holes on Riemann surface, where $\lceil \frac{n}{2} \rceil = \begin{cases} \frac{n}{2} & n : even \\ \frac{n+1}{2} & n : odd \end{cases}$

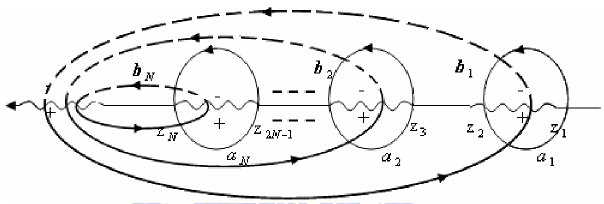
4.4 The integrals over a, b cycles on Riemann surface

After analyzing the algebraic structure and geometric structure of $p(z) = \sqrt{(z-z_1)(z-z_2)}$ on Riemann surface, we want to evaluate the integral of $\frac{1}{p(z)}$. Since every simple closed curves can be written as a linear combination of a, b cycles. Hence, we discuss the integral of canonical cycles a, b. Consider a general function $f(z) = \sqrt{\prod_{j=1}^{n}(z-z_j)}$. We want to evaluate $\oint_a \frac{1}{f(z)}$ and $\oint_b \frac{1}{f(z)}$ by another equivalent paths of a, b cycles such that the integrals are easier computed.

For horizontal branch cut:

The following Figure 11 is a, b cycles on Riemann surface.

For n is odd:



For n is even:

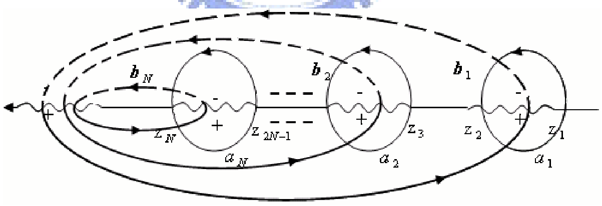


Figure 11: The a, b cycles.

Now, we want to evaluate the integral of $f(z) = \sqrt{(z+2)(z+1)(z-1)(z-2)(z-3)(z-4)}$ over a, b cycles as Figure 12. The algebraic and geometric structure:

For a_1, b_1 cycle:

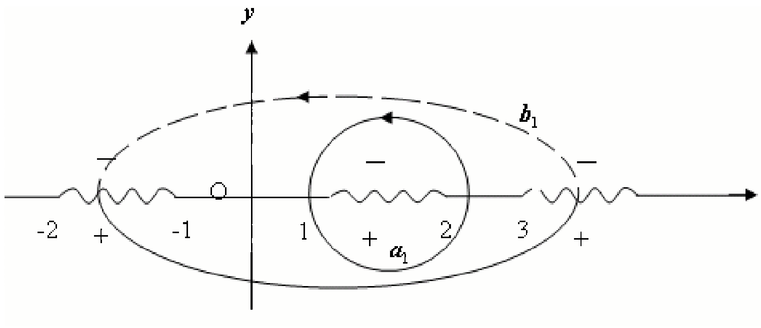


Figure 12: a_1, b_1 cycle.

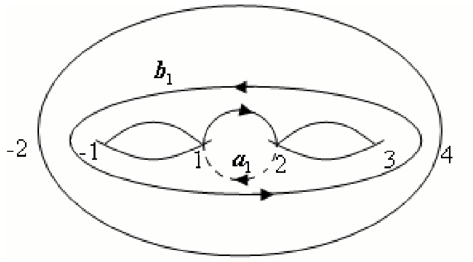


Figure 13: Geometric structure.

We want to evaluate the integral by Mathematica, but there is something wrong. The argument evaluated by Mathematica is $(-\pi, \pi]$, and the argument in our theory is $[-\pi, \pi)$. In Mathematica, it regards the argument $-\pi$ in our theory as π. Hence, we have to modify the value evaluated from Mathematica.

For $f(z) = \sqrt{z}$

In Theory:

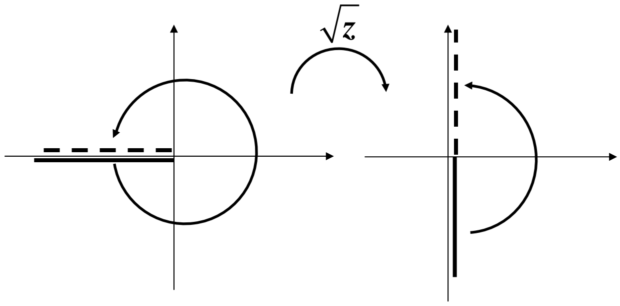


Figure 14: The argument in theory.

In Mathematica:

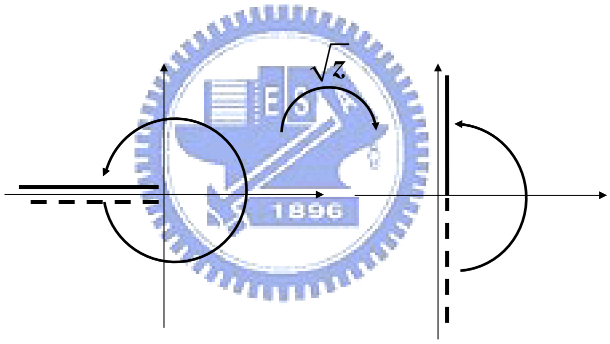


Figure 15: The argument in Mathematica.

Therefore,

$$f(z) = \sqrt{z} = \begin{cases} -M & arg(z) = -\pi \\ M & otherwise \end{cases}$$

By the same, we only modify the value for any $f(z) = \sqrt{z - z_j}$ when $arg(z - z_j) = -\pi$.

For a_1 cycle:

The integral over a_1 cycle is same over the circle of radius 1 at center $\frac{3}{2}$ and the circle in the sheet I, by Cauchy integral Theorem.

Let $z = \frac{3}{2} + e^{i\theta}$,then $dz = ie^{i\theta}d\theta$. Since $arg(z - z_j) \in [-\pi, \pi)$ for $1 \leq j \leq 6$. So, $\oint_{a_1} \frac{1}{f(z)}dz$ is same as the value evaluated by Mathematica. Hence,

$$\oint_{a_1} \frac{1}{f(z)}dz$$

$$= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{\sqrt{\frac{3}{2} + e^{i\theta} + 1}} \sqrt{\frac{3}{2} + e^{i\theta} + 2}\sqrt{\frac{3}{2} + e^{i\theta} - 1}\sqrt{\frac{3}{2} + e^{i\theta} - 2}\sqrt{\frac{3}{2} + e^{i\theta} - 3}\sqrt{\frac{3}{2} + e^{i\theta} - 4} \cdot d\theta$$

$$= -1.13022i$$

For the equivalent path a_1^*:

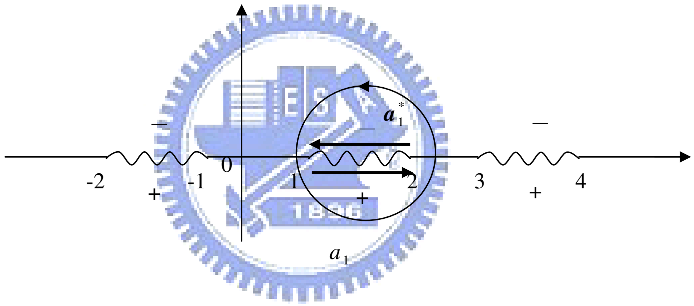


Figure 16: a_1 cycle.

The argument of Mathamatica evaluated is $(-\pi, \pi]$, then the argument of $-$ edge is regarded as π. Hence we must modify the integral on $-$ edge by multiple (-1).

	1 to 2		2 to 1	
Integral path	angle	value	angle	value
$z-4$	$-\pi$	$-M$	π	$+M$
$z-3$	$-\pi$	$-M$	π	$+M$
$z-2$	$-\pi$	$-M$	π	$+M$
$z-1$	0	$+M$	0	$+M$
$z+1$	0	$+M$	0	$+M$
$z+2$	0	$+M$	0	$+M$
Sheet	I	$+M$	I	$+M$
Total		$-M$		$+M$

Table 4: Angles and values for $z - z_j$ along integral path a_1^*.

Hence,

$$\oint_{a_1^*} \frac{1}{f(z)}dz = \int_1^2 \frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$
$$= -1.13022i$$

Therefore, the integral over a_1^* cycle is equal to over a_1 cycle

$$\oint_{a_1^*} \frac{1}{f(z)}dz = \oint_{a_1^*} \frac{1}{f(z)}dz = -1.13022i$$

For b_1 cycle:

The integral over b_1 cycle is same over the circle of radius $\frac{5}{2}$ at center 1 and the circle in the sheet I, by Cauchy integral Theorem. A dotted line is in sheet II. Let $z = 1 + \frac{5}{2}e^{i\theta}$,then $dz = \frac{5}{2}ie^{i\theta}d\theta$.Since $arg(z - z_j) \in [-\pi, \pi)$ for $1 \leq j \leq 6$. So, $\oint_{b_1} \frac{1}{f(z)}$ is evaluated correct by Mathematica. Hence,

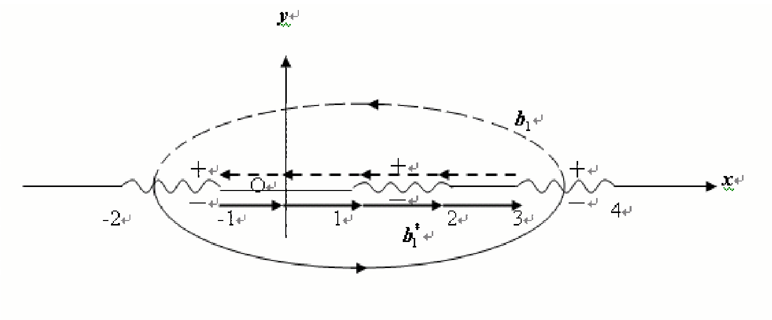


Figure 17: b_1 cycle.

$$\oint_{b_1} \frac{1}{f(z)} dz$$

$$= \int_{-\pi}^{\pi} \frac{\frac{5}{2} i e^{i\theta}}{\sqrt{\frac{5}{2} + e^{i\theta} + 1}\sqrt{\frac{5}{2} + e^{i\theta} + 2}\sqrt{\frac{5}{2} + e^{i\theta} - 1}\sqrt{\frac{5}{2} + e^{i\theta} - 2}\sqrt{\frac{5}{2} + e^{i\theta} - 3}\sqrt{\frac{5}{2} + e^{i\theta} - 4}} d\theta$$

$$= -0.0760776$$

For the equivalent path b_1^*:

	1 to 2		2 to 1	
Integral path	angle	value	angle	value
$z-4$	$-\pi$	$-M$	π	$+M$
$z-3$	$-\pi$	$-M$	π	$+M$
$z-2$	$-\pi$	$-M$	π	$+M$
$z-1$	0	$+M$	0	$+M$
$z+1$	0	$+M$	0	$+M$
$z+2$	0	$+M$	0	$+M$
Sheet	I	$+M$	II	$-M$
Total		$-M$		$-M$

Table 5: Angles and values for $z - z_j$ along integral path b_1^*.

$$\oint_{b_1^*} \frac{1}{f(z)} dz$$

$$= 2[\int_3^{-1} \frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}dz}]$$

$$= -0.0760776$$

For vertical branch cut : The argument evaluated by Mathematica is $(-\pi, \pi]$, but the argument in our theory is $[-\frac{3}{2}\pi, -\frac{1}{2}\pi)$. In Mathematica, it regards the argument belong $[-\frac{3}{2}\pi, -\pi]$ in our theory as $[-\frac{1}{2}\pi, \pi]$. Hence, we have to modify the value evaluated from Mathematica.

For $f(z) = \sqrt{z-i}$

In Theory:

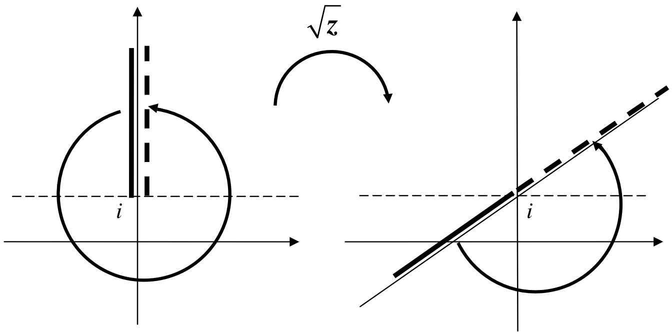


Figure 18: The argument in theory.

In Mathematica:

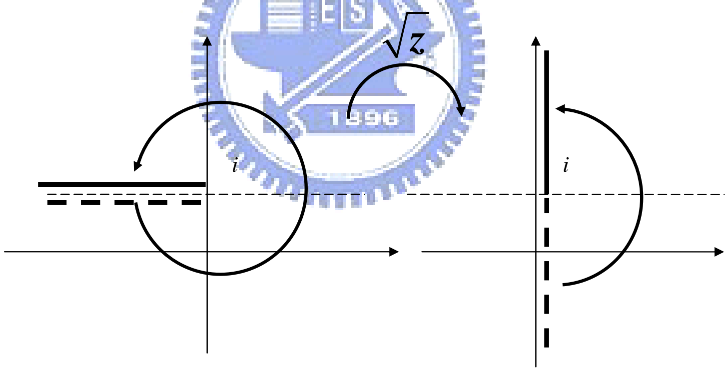


Figure 19: The argument in Mathematica.

Therefore,

$$f(z) = \sqrt{z} = \begin{cases} -M & arg(z-i) \in [-\frac{3}{2}\pi, -\frac{1}{2}\pi] \\ M & otherwise \end{cases}$$

By the same, we only modify the value for any $f(z) = \sqrt{z - z_j}$ when $arg(z - z_j) = [-\frac{3}{2}\pi, -\frac{1}{2}\pi]$.

For n is even:

The following Figure 20 is a, b cycles on Riemann surface.

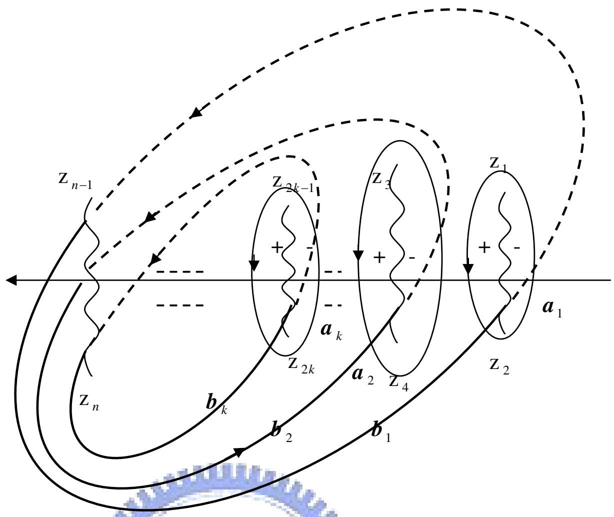


Figure 20: a, b cycle.

Now, we want to evaluate the integral of

$$f(z) = \frac{1}{\sqrt{\prod_{j=1}^{6}(z - z_j)}}$$

where $z_1 = 1 + 2i, z_2 = 1, z_3 = 3i, z_4 = i, z_5 = -1 + 3i$ and $z_6 = -1 + i$ over a, b cycles as in Figure 21.

Since the argument of Mathematica is $(-\pi, \pi]$, we must modify the value on the argument $[\frac{3}{2}\pi, -\pi]$ in vertical cut. We regard the branch point as the origin of the rectangular coordinate system in the plane. Hence, we modify the second sign of every rectangular coordinate system of center for each branch point.

For a_1 cycle:

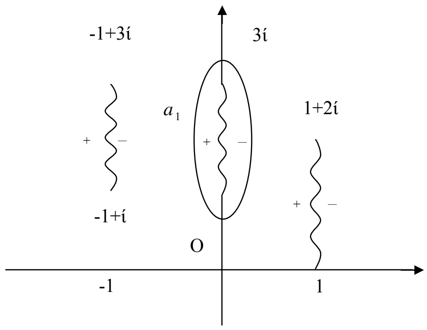


Figure 21: a_1 cycle.

The integral over a_1 cycle is the same as over the enclosed rectangle in Figure 22.

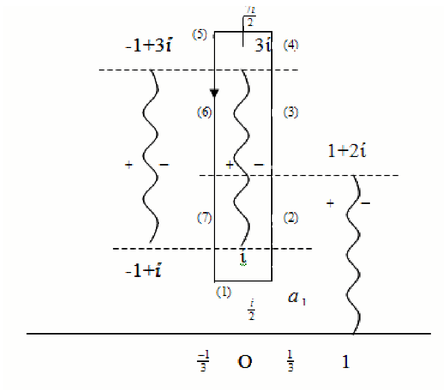


Figure 22: a_1 cycle.

Integral path	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$z-(1+2i)$	$+M$	$+M$	$-M$	$-M$	$-M$	$-M$	$+M$
$z-1$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$
$z-3i$	$+M$	$+M$	$+M$	$+M$	$-M$	$+M$	$+M$
$z-i$	$+M$	$+M$	$+M$	$+M$	$-M$	$-M$	$-M$
$z-(-1+3i)$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$
$z-(-1+i)$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$
Sheet	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$
Total	$-M$	$-M$	$+M$	$+M$	$+M$	$-M$	$+M$

Table 6: Angles and values for $z-z_j$ along integral path a_1.

$$\oint_{a_1} \frac{1}{f(z)}dz = -\int_{-\frac{1}{3}+i}^{\frac{1}{3}+2i} \frac{1}{f(z)}dz + \int_{\frac{1}{3}+2i}^{-\frac{1}{3}+3i} \frac{1}{f(z)}dz - \int_{-\frac{1}{3}+3i}^{-\frac{1}{3}+2i} \frac{1}{f(z)}dz + \int_{-\frac{1}{3}+2i}^{-\frac{1}{3}+i} \frac{1}{f(z)}dz$$
$$= 1.38321 - 2.33762i$$

For the equivalent path a_1^* in Figure 23.

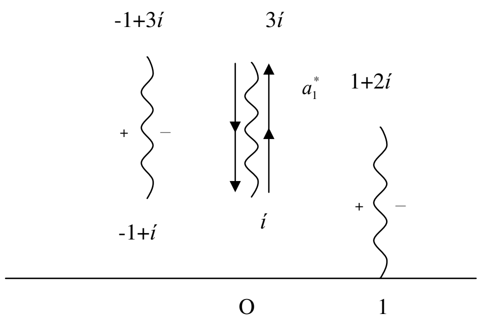


Figure 23: a_1^* cycle.

Integral path	$3i$ to $2i$	$2i$ to i	i to $2i$	$2i$ to $3i$
$z-(1+2i)$	$-M$	$+M$	$-M$	$+M$
$z-1$	$-M$	$-M$	$-M$	$-M$
$z-3i$	$+M$	$+M$	$+M$	$+M$
$z-i$	$-M$	$-M$	$+M$	$+M$
$z-(-1+3i)$	$+M$	$+M$	$+M$	$+M$
$z-(-1+i)$	$+M$	$+M$	$+M$	$+M$
Sheet	I	I	I	I
Total	$-M$	$+M$	$+M$	$-M$

Table 7: Angles and values for $z-z_j$ along integral path a_1^*.

$$\begin{aligned}\oint_{a_1^*}\frac{1}{f(z)}dz &= -\int_{3i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{i}\frac{1}{f(z)}dz-\int_{i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{3i}\frac{1}{f(z)}dz\\ &= 1.38321-2.33762i\end{aligned}$$

For b_1 cycle:

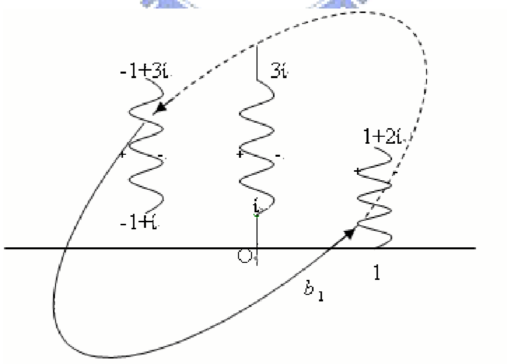


Figure 24: b_1 cycle.

Regard b_1 cycle as the following polygon.

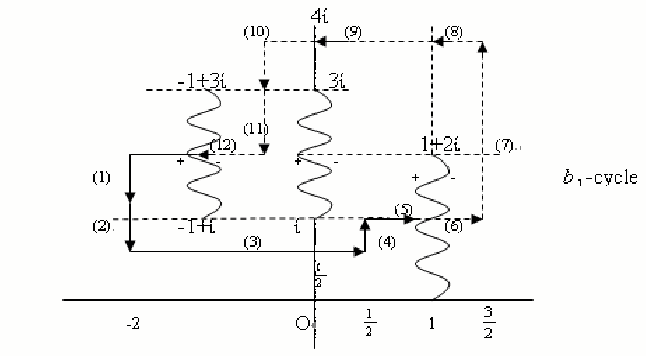


Figure 25: b_1 cycle.

Integral path	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
$z-(1+2i)$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$-M$	$-M$	$-M$	$+M$
$z-1$	$-M$	$-M$	$-M$	$-M$	$-M$	$+M$	$+M$	$+M$	$-M$	$-M$	$-M$	$-M$
$z-3i$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$-M$	$+M$	$+M$
$z-i$	$-M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$-M$	$-M$
$z-(-1+3i)$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$-M$	$-M$	$-M$
$z-(-1+i)$	$-M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$	$+M$
Sheet	I	I	I	I	I	II	II	II	II	II	II	II
Total	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$+M$	$-M$	$-M$	$+M$	$-M$

Table 8: Angles and values for $z - z_j$ along integral path b_1.

$$\oint_{b_1} \frac{1}{f(z)} dz = -\int_{-1+2i}^{-\frac{1}{2}+3i} \frac{1}{f(z)} dz + \int_{-\frac{1}{2}+3i}^{-\frac{1}{2}+2i} \frac{1}{f(z)} dz - \int_{-\frac{1}{2}+2i}^{-1+2i} \frac{1}{f(z)} dz$$
$$= 0.590344 - 1.16143i$$

For the equivalent path b_1^* in Figure 26

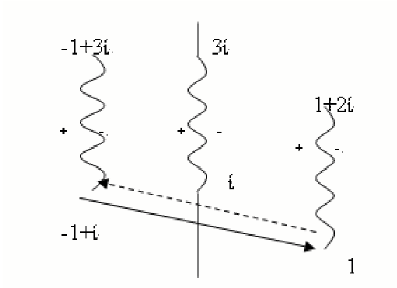


Figure 26: a_1 cycle.

	$-1+i$ to 1	1 to $-1+i$
Integral path	value	value
$z-(1+2i)$	$+M$	$+M$
$z-1$	$-M$	$-M$
$z-3i$	$+M$	$+M$
$z-i$	$+M$	$+M$
$z-(-1+3i)$	$+M$	$+M$
$z-(-1+i)$	$+M$	$+M$
Sheet	I	II
Total	$-M$	$+M$

Table 9: Angles and values for $z-z_j$ along integral path b_1^*.

$$\oint_{b_1^*} \frac{1}{f(z)}dz = 2\int_1^{-1+i} \frac{1}{f(z)}dz$$
$$= 0.590344 - 1.16143i$$

4.5 Solutions to Liner Hyperbolic Equations by Mathematica

Now, we want to solve the infinite problem (4.5) with the source term only involving time-independent.

Example 4.6. *Using Laplace transform, and then Fourier transform to solve the following I.V.P.*

$$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t) + sin2\sqrt{t}, -\infty < x < \infty, t > 0, \\ &u(x,0) = 0, -\infty < x < \infty, \\ &u_t(x,0) = 0, -\infty < x < \infty. \end{aligned} \tag{4.5}$$

First, using the method of Laplace Transform with respect to t, we have

$$s^2U(x,s) - U_{xx}(x,s) = \frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}, s > 0.$$

Note. $\mathscr{L}[\sin 2\sqrt{t}] = \frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}$. Since

$$\sin 2\sqrt{t} = 2\sqrt{t} - \frac{(2\sqrt{t})^3}{3!} + \frac{(2\sqrt{t})^5}{5!} - \frac{(2\sqrt{t})^7}{7!} + ...$$

Then

$$\begin{aligned} \mathscr{L}[\sin 2\sqrt{t}] &= \frac{2\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}} - \frac{8\Gamma(\frac{5}{2})}{3!s^{\frac{5}{2}}} + \frac{32\Gamma(\frac{7}{2})}{5!s^{\frac{7}{2}}} - \frac{128\Gamma(\frac{9}{2})}{7!s^{\frac{9}{2}}} + ... \\ &= \frac{\sqrt{\pi}}{s^{\frac{3}{2}}}[1 - \frac{1}{s} + \frac{1}{2s^2} - \frac{1}{3s^3} + ...] \\ &= \frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}. \end{aligned}$$

Using the method of Fourier Transform with respect to x, it becomes

$$s^2\hat{U}(\omega,s) + \omega^2\hat{U}(x,s) = \frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}2\pi\delta(\omega).$$

This implies

$$\hat{U}(\omega,s) = \frac{1}{s}\frac{s}{s^2+\omega^2}\frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}2\pi\delta(\omega).$$

By Convolution Theorem, we derives

$$\begin{aligned} U(x,t) &= \frac{1}{s}\frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}\{\mathscr{F}^{-1}[\frac{s}{s^2+\omega^2}]\mathscr{F}^{-1}[2\pi\delta(\omega)]\} \\ &= \frac{1}{s}\frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{s}}[\int_{\infty}^{\infty}\frac{1}{2}e^{-s|x-y|}\cdot 1dy] \\ &= \frac{\sqrt{\pi}}{s^3\sqrt{s}}e^{-\frac{1}{s}}. \end{aligned}$$

By Inversion Theorem of Laplace Transform, we have

$$u(x,t) = \frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL} U(x,\tau)e^{\tau t}d\tau, s > 0.$$

Let $G(x,\tau) = U(x,\tau)e^{\tau t}$ and we apply Cauchy's theorem to the integral of $G(x,s)$ over the contour shown as following Figure 27. Since $G(x,s)$ is analytic inside this contour C,

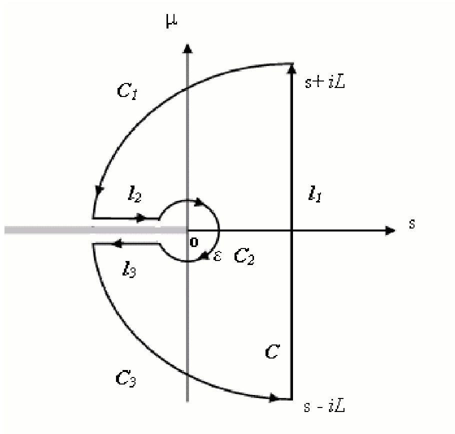


Figure 27: The integral contour C of $G(x,\tau)$.

(1)Along the path C_1 of contour C: Let $\tau = s + Le^{i\theta}, \frac{\pi}{2} \le \theta \le \pi, d\tau = iLe^{i\theta}d\theta$

$$\int_{C_1} G(x,\tau)d\tau = \int_{\frac{\pi}{2}}^{\pi} e^{(s+Le^{i\theta})t}U(x, s+Le^{i\theta})\cdot iLe^{i\theta}d\theta$$

Since

$$|U(x, s+Le^{i\theta})| = |\frac{\sqrt{\pi}}{(s+Le^{i\theta})^3\sqrt{s+Le^{i\theta}}}e^{-\frac{1}{s+Le^{i\theta}}}| \le |\frac{\sqrt{\pi}}{(L-s)^{\frac{7}{2}}}|$$

approach zero as $L \to \infty$. By Jordan's lemma the integrals over this contour C_1 approach zero.

(2)Along the path C_2 of contour C: Let $\tau = \varepsilon e^{i\theta}$, where $-\pi \leq \theta \leq \pi, d\tau = i\varepsilon e^{i\theta}$. Since

$$|U(x, \varepsilon e^{i\theta})| = |\frac{\sqrt{\pi}}{(\varepsilon e^{i\theta})}|^{\frac{7}{2}}|e^{-\frac{1}{\varepsilon e^{i\theta}}}| \leq \frac{\sqrt{\pi}}{\varepsilon}e^{-\frac{1}{\varepsilon}}, \tag{4.6}$$

By L'Hospital Rule twice times, we evaluate the value of the right hand side for above inequality (4.6) as $\varepsilon \to \infty$ is

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{-\frac{7}{2}}}{e^{\frac{1}{\varepsilon}}} = \lim_{\varepsilon \to 0} \frac{\frac{7}{2}\varepsilon^{-\frac{1}{2}}}{e^{\frac{1}{\varepsilon}}} = \lim_{\varepsilon \to 0} \frac{\frac{7}{4}\varepsilon^{\frac{5}{2}}}{e^{\frac{1}{\varepsilon}}} = 0.$$

By Jordan's lemma, the integral over this contour C_2 approach zero as $\varepsilon \to 0$.

(3)Along the path C_3 of contour C:

Similar as contour C_1, let $\tau = s + Le^{i\theta}, \pi \leq \theta \leq \frac{3\pi}{2}, d\tau = iLe^{i\theta}d\theta$

Then integrals over the contour C_3 approach zero.Hence, we have

$$u(x,t) = \frac{-1}{2\pi i}[\int_{\infty}^{0} G(x,\tau)d\tau + \int_{0}^{-\infty} G(x,\tau)d\tau]$$

	$-\infty$ to 0		0 to $-\infty$	
Integral path	angle	value	angle	value
s	π	$+M$	$-\pi$	$-M$
Sheet	I	$+M$	I	$+M$
Total		$+M$		$-M$

Table 10: Angles and values for s along the equivalent integral path.

We must modify the value on the integral path from a point 0 to $-\infty$ by Mathematica, so

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}[-M\{\int_{\infty}^{0} G(x,\tau)d\tau\} + M\{\int_{0}^{-\infty} G(x,\tau)d\tau\}] \\ &= \frac{1}{2\pi i}[2M\{\int_{0}^{-\infty} G(x,\tau)d\tau\}], \end{aligned}$$

where $M\{\int_C G(x,\tau)d\tau\}$ is represented by the integral value G evaluated by Mathematica on a contour C. At some time, the value of $u(x,t)$ at every position is the same, because the source term only involving the time-variable t.

Fixed $x = 1$

At the time $t = 1$, the value

$$u(x,t) = 0.396896.$$

At the time $t = 2$, the value

$$u(x,t) = 1.63306.$$

This is the same as the value evaluated by D'Alembert's solution for $t = 1, 2$. Here, D'Alembert's solution of problem (4.5) is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \sin 2\sqrt{\bar{t}}\, d\bar{x}\, d\bar{t} \\ &= -\frac{3}{2}\sqrt{t}\cos 2\sqrt{t} + \frac{1}{4}(3-4t)\sin 2\sqrt{t}. \end{aligned}$$

For fixed $x = 1$, the displacement in a long string as shown in the Figure 28.

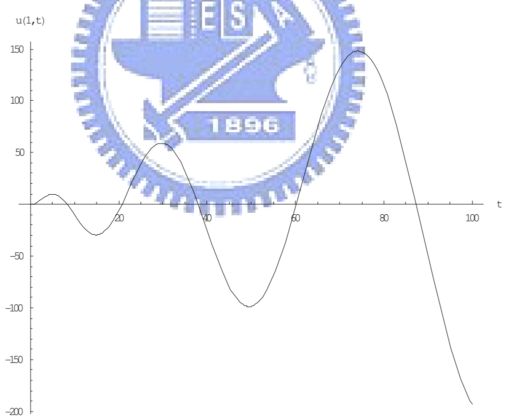


Figure 28: The graph of solutions to problem (4.5) at position $x = 1$.

References

[1] Weinberger H.F., A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Chelsea,1965.

[2] Roger Knobel, An introduction to the Mathematical Theory of Waves, 2000.

[3] G.D.SMITH, Numerical Solution of Partial Differential Equations,1964.

[4] Walter A. Strauss, Partial Differential Equations, 1992.

[5] Tyn Myint-U, Partial Differential Equations of Mathematical Physics,1973.

[6] Yu-Wei Hsiao, NCTU, Master thesis, Integral Evaluations on Three-sheeted Riemann surfaces of Genus N of type II, Taiwan, 2001.