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A version of the implicit function theorem and its consequences for chaos of difference systems

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摘　　要

我們指一個差分系統的形式為$G^*(x_{t-\mu}, \cdots, x_{t-1}, x_t, x_{t+1}, \cdots, x_{t+\nu})=0$，此式$G^*$為一個$(\mu+1+\nu)$個變數映至$R^N$的函數，並且每個變數屬於$R^N$。我們考慮僅和$x_t$變數有關的差分系統稱之為靜態系統，並且引入在某些性質相當類似於靜態系統的差分系統稱之為半靜態系統。我們提供隱函數定理的一個變形版本。我們呈現再加一些條件下，一個靜態系統是混沌的。我們使用這個隱函數定理的變形去呈現對於正則靜態系統的些微C^1擾動下，混沌現象的穩定性。

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ABSTRACT

By a *difference system*, we mean a system of the form $G^*(x_{t-\mu}, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_{t+\nu}) = 0$, where each side of this equation is an $N \times 1$ column vector and $G^* : \mathrm{Dom}(G^*) \subset (\mathbb{R}^N)^{\mu+1+\nu} \to \mathbb{R}^N$ with $N, \mu, \nu \in \mathbb{N}$. We consider a *static system* as a difference system that depends only on x_t and a *quasi-static system* as a difference system that is in a certain sense relatively close to a static system. We provide a modified version of the implicit function theorem. We show that under additional conditions, a static system is chaotic. We use this version of the implicit function theorem to show the stability of chaos for regular static system under small C^1 perturbations.

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目　　錄

1 Introduction

Consider a difference system

$$G(x_{t-1},\, x_t,\, x_{t+1}) = 0. \tag{1}$$

Define an *orbit* as a sequence $\{x_t\}$ satisfying Eq. (1) for all $t \in \mathbb{Z}$. Suppose G reduces to a *static system* G^*, by which we mean that G^* is a function of x_t alone:

$$G^*(x_{t-1},\, x_t,\, x_{t+1}) = F(x_t).$$

In [1], we have the fact that if $F(x_t) = 0$ has multiple solutions at which the Jacobian matrix DF is nonsingular, then for G is in a certain sense relatively close to G^*, G displays chaotic dynamics.

The result will be based on simultaneous control of appropriate perturbation of static difference system. We provide a modified version of the implicit function theorem which inspired by the concept of Li and Malkin in [2].

In this paper, section 2 presents a modified version of the implicit function theorem and proof. Section 3 gives the definition of chaos and show that under additional conditions, a static system is chaotic. Section 4 defines quasi-static systems, establishes their properties, and presents stability of quasi-staticness and chaos.

2 A version of the implicit function theorem

Let $x \in \mathbb{R}^n$, $x = (x_1, x_2,..., x_n)$, $\|\cdot\|_2$ denotes the Euclidean norm, i.e. $\|x\|_2 = (x_1^2 + x_2^2+...+x_n^2)^{\frac{1}{2}}$, $\|\cdot\|_\infty$ denotes the sup norm, i.e. $\|x\|_\infty = \max\limits_{1\leq i\leq n} |x_i|$. In fact, for a $m \times n$ matrix B,

$$\|B\|_\infty = \max_{\xi\in\mathbb{R}^n} \frac{\|B\xi\|_\infty}{\|\xi\|_\infty} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |B_{ij}| \,.$$

Let $m, n \in \mathbb{N}$, for $H \subset \mathbb{R}^m \times \mathbb{R}^n$, and $\mathbf{C}^1(H, \mathbb{R}^n)$ denote the set of C^1 functions $F : Dom(F) \to \mathbb{R}^n$ such that $Dom(F) \supset H$. Let $E = \{F \mid_H : F \in C^1(H, \mathbb{R}^n)\}$. For $F_1, F_2 \in E$, define $\rho(F_1, F_2) \equiv$

$$\max\{\sup_{(y,\, z)\in H} \|F_1(y, z) - F_2(y, z)\|_2 \,, \sup_{(y,\, z)\in H} \|DF_1(y, z) - DF_2(y, z)\|_\infty\},$$

then (E, ρ) is a metric space. We will use the notation $U(x, r)$ and $U[x, r]$ for the open and closed ball, respectively, of radius r centred at the point $x \in X$, where X is a metric space. Let

$$N_\delta(F^*, H) = \{F \in E \mid \rho(F, F^*) < \delta, Dom(F) \subset Dom(F^*)\}.$$

Theorem 1 *Let $m, n \in \mathbb{N}$, $Y \subset \mathbb{R}^m$, $Z \subset \mathbb{R}^n$, $H \subset \mathbb{R}^m \times \mathbb{R}^n$, and $F^* \in E$. Suppose:*

(a1) Y, Z, and H are compact, and $Y \times Z \subset H$.

(a2) There is a unique function $f^* : Y \to Z$ such that for all $y \in Y$, $F^*(y, f^*(y)) = 0$.

(a3) $f^*(Y) \subset$ the interior of Z and for any $y \in Y$, $D_2F^*(y, f^*(y))$ is nonsingular.

Then

(c1) there exists a $\overline{\delta} > 0$ for any $F \in N_{\overline{\delta}}(F^*, H)$, there is a unique function $f^F : Y \to Z$ such that for any $y \in Y$, $F(y, f^F(y)) = 0$, and the unique function f^F is C^1.

(c2) $\sup_{F\in N_{\overline{\delta}}(F^*,H)}(\max\{\sup_{y\in Y}\left\|f^F(y)-f^*(y)\right\|,\ \sup_{y\in Y}\left\|Df^F(y)-Df^*(y)\right\|\})\ \to\ 0$ *as* $\overline{\delta}\downarrow 0$.

Proof of Theorem 1. Since f^* is continuous and Y is compact, we have $f^*(Y)$ is compact. From (a3), for any $y\in Y$ there exists a $\varepsilon_y>0$ such that $U(f^*(y),\,\varepsilon_y)\subset Z$. Because $f^*(Y)$ is compact, there exist $\varepsilon_{y_1},\ \varepsilon_{y_2},\ldots\varepsilon_{y_n};\ y_1,\ y_2,\ldots y_n$ such that $f^*(Y)\subset\cup_{i=1}^{n}U(f^*(y_i),\,\varepsilon_{y_i})$. Let $\eta_0=\min_{1\le i\le n}\varepsilon_{y_i}$. Denote $V_1=U(F^*,\,1)$, $W_{\eta_0}=\cup_{y\in Y}U(f^*(y),\,\eta_0)$. Then $W_{\eta_0}\subset Z$. For any $y\in Y$, we define a function $g_y:V_1\times U(f^*(y),\,\eta_0)\to\mathbb{R}^n$ by

$$g_y(F,\,z)=z-(D_2F^*(y,\,f^*(y)))^{-1}F(y,\,z)$$

then $g_y(F^*,\ f^*(y))=f^*(y)$ and $D_2g_y(F^*,\ f^*(y))=0$. We denote $T_y=D_2F^*(y,\,f^*(y))$. By assumption (a3) and the map $T:A\to A^{-1}$ is continuous, there exists a constant $M>0$ such that for any $y\in Y$,

$$\left\|(D_2F^*(y,\,f^*(y)))^{-1}\right\|<M.$$

Since D_2F^* is continuous on the compact set $Y\times f^*(Y)$, there exists a δ_1, $0<\delta_1<\min\{\frac{1}{4M},\,1,\,\eta_0\}$ such that for any $y\in Y$,

$$\begin{aligned}
&\|D_2F(y,\,z)-D_2F^*(y,\,f^*(y))\|\\
\le\ &\|D_2F(y,\,z)-D_2F^*(y,\,z)\|+\|D_2F^*(y,\,z)-D_2F^*(y,\,f^*(y))\|\\
<\ &\|DF(y,\,f^*(y))-DF^*(y,\,f^*(y))\|+\frac{1}{4M}\\
<\ &\rho(F,\,F^*)+\frac{1}{4M}<\frac{1}{2M}\le\frac{1}{2\left\|T_y^{-1}\right\|}
\end{aligned}$$

provided $F\in U[F^*,\,\delta_1]$, $z\in U[f^*(y),\,\delta_1]$.

And therefore

$$\begin{aligned}
\|(D_2g_y(F,\,z)\| &= \left\|I-T_y^{-1}D_2F(y,\,z)\right\|=\left\|T_y^{-1}T_y-T_y^{-1}D_2F(y,\,z)\right\|\\
&\le \left\|T_y^{-1}\right\|\|T_y-D_2F(y,\,z)\|\le\frac{1}{2}.
\end{aligned}$$

By Mean Value Theorem applied to $g_y(F,\,\cdot)$, For any $F \in U[F^*,\,\delta_1]$, $y \in Y$, and any two points $z_1,\,z_2 \in U[f^*(y),\,\delta_1]$,

$$\|g_y(F,\,z_1) - g_y(F,\,z_2)\| \le \frac{1}{2}\,\|z_1 - z_2\|\,.$$

Now, we choose a $\overline{\delta}$, $0 < \overline{\delta} < \min\{\frac{1}{2M}\delta_1,\,\delta_1\}$ such that for any $y \in Y$, and $F \in U[F^*,\,\overline{\delta}]$,

$$\begin{aligned}\|F(y,\,f^*(y))\| &= \|F(y,\,f^*(y)) - F^*(y,\,f^*(y))\| \\ &\le \rho(F,\,F^*) \le \overline{\delta} < \frac{1}{2M}\delta_1 \le \frac{1}{2\left\|T_y^{-1}\right\|}\delta_1,\end{aligned}$$

and therefore

$$\begin{aligned}\|g_y(F,\,f^*(y)) - f^*(y)\| &= \left\|T_y^{-1}\cdot F(y,\,f^*(y))\right\| \\ &\le \left\|T_y^{-1}\right\|\,\|F(y,\,f^*(y))\| < \frac{1}{2}\delta_1.\end{aligned}$$

Thus for any $y \in Y$, $F \in U[F^*,\,\overline{\delta}]$ and $z \in U[f^*(y),\,\delta_1]$ one has

$$\begin{aligned}\|g_y(F,\,z) - f^*(y)\| &\le \|g_y(F,\,z) - g_y(F,\,f^*(y))\| + \|g_y(F,\,f^*(y)) - f^*(y)\| \\ &< \frac{1}{2}\,\|z - f^*(y)\| + \frac{1}{2}\delta_1 \le \delta_1.\end{aligned}$$

This implies that for any $y \in Y$ and any (fixed) $F \in U[F^*,\,\overline{\delta}]$, the map $z \to g_y(F,\,z)$ is a contraction of the complete metric space $U[f^*(y),\,\delta_1]$ into itself. Hence by the contraction mapping principle, there exists a unique fixed point, say $\psi_y(F)$, and so $g_y(F,\,\psi_y(F)) = \psi_y(F)$ or, equivalently, $F(y,\,\psi_y(F)) = 0$.

Given a $F \in U[F^*,\,\overline{\delta}]$, for any $y \in Y$ there exists a unique $\psi_y(F)$ such that $F(y,\,\psi_y(F)) = 0$. We define the function $f^F : Y \to Z$ by $f^F(y) = \psi_y(F)$. Therefore, for any $F \in N_{\overline{\delta}}(F^*,\,H)$, there is a unique function $f^F : Y \to Z$ such that for all $y \in Y$, $F(y,\,f^F(y)) = 0$. It remains to show that f^F is C^1.

For all $F \in N_{\overline{\delta}}(F^*,\,H)$,

$$\left\|D_2F(y,\,f^F(y)) - D_2F^*(y,\,f^*(y))\right\| < \frac{1}{2M} < \frac{1}{M} < \left\|D_2F^*(y,\,f^*(y))^{-1}\right\|^{-1}.$$

Hence $D_2F(y,\, f^F(y))$ is nonsingular for all $y \in Y$. (see [3], p. 209)

Therefore for all $F \in N_{\overline{\delta}}(F^*,\, H)$, there is a unique function $f^F : Y \to Z$ such that for all $y \in Y$, $F(y,\, f^F(y)) = 0$, and $D_2F(y,\, f^F(y))$ is nonsingular. By implicit function theorem (see [4], p. 374), the function f^F is unique and C^1. We complete the proof of (c1).

Next, we prove the (c2). Let $y \in Y$ and $F \in N_{\overline{\delta}}(F^*,\, H)$. Then

$$\begin{aligned}
& \left\|\psi_y(F) - \psi_y(F^*)\right\| \\
= \;& \left\|g_y(F,\, \psi_y(F)) - g_y(F^*,\, \psi_y(F^*))\right\| \\
\leq \;& \left\|g_y(F,\, \psi_y(F)) - g_y(F,\, \psi_y(F^*))\right\| + \left\|g_y(F,\, \psi_y(F^*)) - g_y(F^*,\, \psi_y(F^*))\right\| \\
\leq \;& \frac{1}{2}\left\|\psi_y(F) - \psi_y(F^*)\right\| + \left\|g_y(F,\, \psi_y(F^*)) - g_y(F^*,\, \psi_y(F^*))\right\|
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|\psi_y(F) - \psi_y(F^*)\right\| &\leq 2\left\|g_y(F,\, \psi_y(F^*)) - g_y(F^*,\, \psi_y(F^*))\right\| \\
&= 2\left\|T_y^{-1}(F(y,\, \psi_y(F^*) - F^*(y,\, \psi_y(F^*))\right\| \\
&\leq 2M\left\|F(y,\, \psi_y(F^*) - F^*(y,\, \psi_y(F^*)\right\| \\
&\leq 2M \sup_{(y,\, z) \in H} \|F(y,\, z) - F^*(y,\, z)\| \\
&\leq 2M\overline{\delta} \to 0 \text{ as } \overline{\delta} \to 0.
\end{aligned}$$

That is,

$$\sup_{F \in N_{\overline{\delta}}(F^*, H)} \sup_{y \in Y} \left\|f^F(y) - f^*(y)\right\| \to 0 \text{ as } \overline{\delta} \to 0. \tag{2}$$

Now, we remain to prove

$$\sup_{F \in N_{\overline{\delta}}(F^*, H)} \sup_{y \in Y} \left\|Df^F(y) - Df^*(y)\right\| \to 0 \text{ as } \overline{\delta} \to 0.$$

Since D_1F^* is continuous on the compact set $Y \times Z$, there exists a $N_1 > 0$ such that $\|D_1F^*(y, z)\| \leq N_1$ for all $(y, z) \in Y \times Z$. Let $y \in Y$ and $F \in N_{\overline{\delta}}(F^*, H)$. Then

$$\begin{aligned} \|D_1F(y, z)\| - \|D_1F^*(y, z)\| &\leq \|D_1F(y, z) - D_1F^*(y, z)\| \\ &\leq \|DF(y, z) - DF^*(y, z)\| \\ &\leq \sup_{(y, z)\in H} \|DF(y, z) - DF^*(y, z)\| \leq \overline{\delta} < 1. \end{aligned}$$

Thus

$$\|D_1F(y, z)\| \leq 1 + \|D_1F^*(y, z)\| \leq 1 + N_1 \tag{3}$$

for any $y \in Y$ and $F \in N_{\overline{\delta}}(F^*, H)$. Similarly, there exists a $N_2 > 0$ such that $\|D_2F^*(y, z)\| \leq N_2$ for all $(y, z) \in Y \times Z$ and

$$\|D_2F(y, z)\| \leq 1 + \|D_2F^*(y, z)\| \leq 1 + N_2$$

for any $y \in Y$ and $F \in N_{\overline{\delta}}(F^*, H)$.

Let $y \in Y$ and $F \in N_{\overline{\delta}}(F^*, H)$. Then

$$\begin{aligned} &\left\|[D_2F(y, f^F(y))]^{-1}\right\| - \left\|[D_2F^*(y, f^*(y))]^{-1}\right\| \\ \leq\ &\left\|[D_2F(y, f^F(y))]^{-1} - [D_2F^*(y, f^*(y))]^{-1}\right\| \\ \leq\ &\left\|[D_2F(y, f^F(y))]^{-1}\right\| \left\|D_2F(y, f^F(y)) - D_2F^*(y, f^*(y))\right\| \left\|[D_2F^*(y, f^*(y))]^{-1}\right\| \\ \leq\ &\left\|[D_2F(y, f^F(y))]^{-1}\right\| \cdot \frac{1}{2M} \cdot M \end{aligned}$$

Thus

$$\left\|[D_2F(y, f^F(y))]^{-1}\right\| \leq 2\left\|[D_2F^*(y, f^*(y))]^{-1}\right\| \leq 2M$$

for any $y \in Y$ and $F \in N_{\overline{\delta}}(F^*, H)$.

Since D_2F^* is continuous on the compact set $Y \times Z$, for this $\overline{\delta} > 0$ there exists a

$0 < \delta_2 < \overline{\delta}$ such that for any $y \in Y$

$$\begin{aligned}
& \left\|[D_2F(y,\, f^F(y))]^{-1} - [D_2F^*(y,\, f^*(y))]^{-1}\right\| \\
\leq\ & \left\|[D_2F(y,\, f^F(y))]^{-1}\right\| \left\|D_2F(y,\, f^F(y)) - D_2F^*(y,\, f^*(y))\right\| \left\|[D_2F^*(y,\, f^*(y))]^{-1}\right\| \\
\leq\ & 2M \cdot \left\|D_2F(y,\, f^F(y)) - D_2F^*(y,\, f^*(y))\right\| \cdot M \\
\leq\ & 2M^2(\left\|D_2F(y,\, f^F(y)) - D_2F^*(y,\, f^F(y))\right\| + \left\|D_2F^*(y,\, f^F(y)) - D_2F^*(y,\, f^*(y))\right\|) \\
\leq\ & 2M^2(\sup_{(y,\, z)\in H} \|DF(y,\, z) - DF^*(y,\, z)\| + \overline{\delta}) \\
\leq\ & 2M^2(\delta_2 + \overline{\delta})
\end{aligned}$$

provided $F \in N_{\delta_2}(F^*,\, H),\ f^F(y) \in U[f^*(y),\, \delta_2]$.

That is,

$$\sup_{F\in N_{\delta_2}(F^*,H)} \sup_{y\in Y} \left\|[D_2F(y,\, f^F(y))]^{-1} - [D_2F^*(y,\, f^*(y))]^{-1}\right\| \to 0 \text{ as } \overline{\delta} \to 0. \quad (4)$$

Similarly, since D_1F^* is continuous on the compact set $Y \times Z$, for this $\delta_2 > 0$ there exists a $0 < \delta_3 < \delta_2$ such that for any $y \in Y$

$$\begin{aligned}
& \left\|D_1F(y,\, f^F(y)) - D_1F^*(y,\, f^*(y))\right\| \\
\leq\ & \left\|D_1F(y,\, f^F(y)) - D_1F^*(y,\, f^F(y))\right\| + \left\|D_1F^*(y,\, f^F(y)) - D_1F^*(y,\, f^*(y))\right\| \\
\leq\ & \sup_{(y,\, z)\in H} \|DF(y,\, z) - DF^*(y,\, z)\| + \delta_2 \\
\leq\ & \delta_3 + \delta_2
\end{aligned}$$

provided $F \in N_{\delta_3}(F^*,\, H),\ f^F(y) \in U[f^*(y),\, \delta_3]$.

That is,

$$\sup_{F\in N_{\delta_3}(F^*,H)} \sup_{y\in Y} \left\|D_1F(y,\, f^F(y)) - D_1F^*(y,\, f^*(y))\right\| \to 0 \text{ as } \delta_2 \to 0. \quad (5)$$

Let $y \in Y$ and $F \in N_{\delta_3}(F^*,\, H)$.

$$D_1F(y,\, f^F(y))I + D_2F(y,\, f^F(y))Df^F(y) = 0$$

By Eqs. (3), (4) and (5), we have

$$\begin{aligned}
& \left\|Df^F(y) - Df^*(y)\right\| \\
\leq\ & \left\|[D_2F(y,\, f^F(y))]^{-1}D_1F(y,\, f^F(y)) - [D_2F^*(y,\, f^*(y))]^{-1}D_1F^*(y,\, f^*(y))\right\| \\
\leq\ & \left\|D_1F(y,\, f^F(y))([D_2F(y,\, f^F(y))]^{-1} - [D_2F^*(y,\, f^*(y))]^{-1})\right\| + \\
& \left\|[D_2F^*(y,\, f^*(y))]^{-1}(D_1F(y,\, f^F(y)) - D_1F^*(y,\, f^*(y)))\right\| \\
\leq\ & \left\|D_1F(y,\, f^F(y))\right\| \left\|[D_2F(y,\, f^F(y))]^{-1} - [D_2F^*(y,\, f^*(y))]^{-1})\right\| + \\
& \left\|[D_2F^*(y,\, f^*(y))]^{-1}\right\| \left\|D_1F(y,\, f^F(y)) - D_1F^*(y,\, f^*(y))\right\| \\
\leq\ & (1+N_1)[2M^2(\delta_2 + \overline{\delta})] + M(\delta_3 + \delta_2)
\end{aligned}$$

for any $y \in Y$ and $F \in N_{\delta_3}(F^*,\, H)$.

That is,

$$\sup_{F \in N_{\delta_3}(F^*,H)} \sup_{y \in Y} \left\|Df^F(y) - Df^*(y)\right\| \to 0 \text{ as } \overline{\delta} \to 0. \tag{6}$$

By Eqs. (2) and (6), we have

$$\sup_{F \in N_{\delta_3}(F^*,H)} (\max\{\sup_{y\in Y} \left\|f^F(y) - f^*(y)\right\|, \sup_{y \in Y}\left\|Df^F(y) - Df^*(y)\right\|\}) \to 0 \text{ as } \delta_3 \downarrow 0.$$

This completes the proof of (c2). ■

3 Chaos

3.1. Chaos

For any function G^*, we denote its domain by $Dom(G^*)$. We consider difference systems of the form

$$G^*(x_{t-\mu},..., x_{t-1}, x_t, x_{t+1},..., x_{t+\nu}) = 0, \tag{7}$$

where each side of Eq. (7) is an $N \times 1$ column vector and $G^* : Dom(G^*) \subset (\mathbb{R}^N)^{\mu+1+\nu} \to \mathbb{R}^N$ with N, μ, $\nu \in \mathbb{N}$. By a *difference system*, we always mean a system of the form in Eq. (7), which we denote simply by G^*.

We define an *orbit* of G^* as a bi-infinite sequence $\{x_t\}_{t=-\infty}^{\infty}$ such that for all $t \in \mathbb{Z}$, $G^*(x_{t-\mu},..., x_{t-1}, x_t, x_{t+1},..., x_{t+\nu}) = 0$.

Let $\| \, . \, \|$ be the sup norm whenever its argument is a vector or a sequence. Let $y = \{y_t\}_{t=l}^{\infty}$, $l \geq -\infty$, be any sequence. If there exists a $n \in \mathbb{N}$ for all $t \geq l$ such that $y_{t+n} = y_t$, then y is called *periodic*. If $n \in \mathbb{N}$ is the smallest such number, then y is called *n-periodic*. Suppose y is a sequence in $\mathbb{R}^m$, $m \in \mathbb{N}$. We say y is called *asymptotically periodic* if there is a periodic sequence $\{y_t^*\}$ such that $\|y_t - y_t^*\| \to 0$ as $t \to \infty$. If y is not asymptotically periodic, then y is called *asymptotically nonperiodic*.

Definition 2 *We say that a difference system G^* is chaotic if (T1) and (T2) below hold:*

(T1) There exists a $m \in \mathbb{N}$, for all $n \geq m$, G^* has an n-periodic orbit.

(T2) G^* has an uncountable set χ of asymptotically nonperiodic orbit such that

for all x, $y \in \chi$ $(x \neq y)$

$$\limsup_{t \to \infty} \|x_t - y_t\| > 0, \tag{8}$$

$$\textit{for all } n \in \mathbb{N}, \liminf_{t \to \infty} \|(x_{t-n}, \ldots, x_{t+n}) - (y_{t-n}, \ldots, y_{t+n})\| = 0. \tag{9}$$

Condition (T2) means that any two orbit in χ never converge to each other but they become arbitrarily close infinitely often.

3.2. Static system and chaos

Let $G^* : Dom(G^*) \subset (\mathbb{R}^N)^{\mu+1+\nu} \to \mathbb{R}^N$ with N, μ, $\nu \in \mathbb{N}$ be a difference system. We denote

$$Dom(G^*)_0 = \{x_0 \in \mathbb{R}^N \mid (x_{-\mu}, \ldots, x_{-1}, x_0, x_1, \ldots, x_\nu) \in Dom(G^*)$$

$$\text{where } x_{-\mu}, \ldots, x_{-1}, x_1, \ldots, x_\nu \in \mathbb{R}^N\}.$$

We say that G^* is *static* or a static system if there is a function $G^s : Dom(G^*)_0 \subset \mathbb{R}^N \to \mathbb{R}^N$ such that $G^s(x_0) = G^*(x_{-\mu}, \ldots, x_{-1}, x_0, x_1, \ldots, x_\nu)$ for all $(x_{-\mu}, \ldots, x_{-1}, x_0, x_1, \ldots, x_\nu) \in Dom(G^*)$. If G^* is static, we defined a *static point* of G^* as a point $\xi \in Dom(G^*)_0$ such that $G^s(\xi) = 0$.

Let $K_1, \ldots, K_M \subset Dom(G^*)_0$. We defined a *pattern* as a vector of $\mu + 1 + v$ natural numbers; a sequence of natural number is called a *symbolic sequence*. We say that a pattern $p = (p_{-\mu}, \ldots, p_v)$ is a *feasible pattern (w.r.t. G^* and $K_1, \ldots, K_M$)* if $K_{p_{-\mu}} \times \ldots \times K_{p_v} \subset Dom(G^*)$. Let $P(G^*, K_1, \ldots, K_M)$ be the set of pattern feasible w.r.t. G^* and $K_1, \ldots, K_M$. We say that a symbolic sequence $\{s_t\}_{t=l}^{\infty}$, $-\infty \leq l \leq -\mu \leq \infty$, is *feasible* (w.r.t. G^* and $K_1, \ldots, K_M$) if for all $t = l + \mu, \ldots, \mu - v$, $(s_{t-\mu}, \ldots, s_{t+v}) \in P(G^*, K_1, \ldots, K_M)$.

Let $p,\ q \in P(G,\ K_1,\ldots,\ K_M)$. We say that q *is reachable from* p if one of the following three cases holds: (i) there exists a $n \in \mathbb{N}$, there is a symbolic sequence $\{s_t\}_{t=1}^n$ such that $\{p_{-\mu},\ldots,\ p_\nu,\ s_1,\ldots,\ s_n,\ q_{-\mu},\ldots,\ q_{-\nu}\}$ is feasible; (ii) $\{p_{-\mu},\ldots,\ p_\nu,\ q_{-\mu},\ldots,\ q_{-\nu}\}$ is feasible; (iii) there exists a $m \in \{1,\ldots,\ \mu+\nu\}$, $\{p_{-\mu},\ldots,\ p_\nu,\ q_{\nu-m+1},\ldots,\ q_\nu\}$ is feasible and for all $i = -\mu+m,\ldots,\ \nu$ $p_i = q_{i-m}$.

Theorem 3 *Let G^* be a static system with static points $\xi_1,\ldots,\ \xi_M \in Dom(G^*)_0$ and there are $p,\ q \in P(G^*,\ \xi_1,\ldots,\ \xi_M)$ with $p \neq q$ such that $p_{-\mu} =\ldots = p_{-\nu}$ and $p,\ q$ are reachable from each other. Then G^* is chaotic.*

Proof. Without loss of generality, assume $p_{-\mu} =\ldots = p_{-\nu} = 1$. If q is reachable from p with case (ii) or (iii) holding, then case (i) also holds for any $n \in \mathbb{N}$, if we let $S_t = 1$ for all $t = 1,\ldots,\ n$. Hence in any case, there is a symbolic sequence $S \equiv \{s_i\}_{i=1}^n$ such that $\{p_{-\mu},\ldots,\ p_\nu,\ s_1,\ldots,\ s_n,\ q_{-\mu},\ldots,\ q_{-\nu}\}$ is feasible. Define $T \equiv \{t_i\}_{i=1}^m$, similarly. Let

$$v^2 = \{p_{-\mu},\ldots,\ p_\nu,\ s_1,\ldots,\ s_n, q_{-\mu},\ldots,\ q_{-\nu},\ t_1,\ldots,\ t_m,\ p_{-\mu},\ldots,\ p_\nu\}.$$

Let $\overline{m} = 3(\mu+1+\nu)+n+m$ and $v^1 = \{1,\ 1,\ldots,\ 1\}$ with $\overline{m}$ 1's; $v^1$and v^2 have the same dimension. For each bi-infinite sequence τ of 1 and 2 (i.e.., $\tau_i \in \{1,\ 2\}$ for all $i \in \mathbb{Z}$), let $s(\tau)$ be the symbolic sequence such that for all $i \in \mathbb{Z}$

$$s(\tau)_{i\overline{m},\ldots,\ (i+1)\overline{m}-1} = v^{\tau_i}. \tag{10}$$

Note that the mapping $\tau \to s(\tau)$ is one-to-one and that $s(\tau)$ is always feasible.

We first verify (T2). For $r \in \mathbb{R}$, let $[r]$ denote the largest integer less than or equal to r . For $w \in (0,\ 1)$, define a bi-infinite symbolic sequence τ^w as follows. For $i \leq 0$, let $\tau_i^w = 1$. For $i \geq 1$, define τ_i^w as follows:

$$\tau_{1,10}^w = \{\underbrace{1,\ldots,\ 1}_{[10w]1\text{'s}},\ \underbrace{2,\ldots,\ 2}_{(10-[10w])2\text{'s}}\}$$

$$\tau^w_{11,110} = \{\underbrace{1,..., 1}_{[100w]1's}, \underbrace{2,..., 2}_{(100-[100w])2's}\}$$

$$\tau^w_{111,1110} = \{\underbrace{1,..., 1}_{[1000w]1's}, \underbrace{2,..., 2}_{(1000-[1000w])2's}\}$$

and so on. More precisely, letting $T_n = 1 + 10 + ... + 10^n$ for $n \in \mathbb{N}$, we have for all $n \in \mathbb{N}$

$$\tau^w_i = 1, \text{ for all } i = T_n, ..., T_n + [10^n w] - 1, \quad (11)$$

$$\tau^w_i = 2, \text{ for all } i = T_n + [10^n w], ..., T_{n+1} - 1. \quad (12)$$

Note that for any w, $w' \in (0, 1)$, $w \neq w'$. $[10^n w] \neq [10^n w']$ for n large enough. Thus

$$\tau^w_i \neq \tau^{w'}_i \text{ for infinitely many } i's. \quad (13)$$

Therefore, for any $w \in (0, 1)$ there is an orbit x^w such that for all $t \in \mathbb{Z}$, $x^w_t = \xi_{s(\tau^w)_t}$. Let $\chi = \{x^w \mid w \in (0, 1)\}$; we show that χ satisfies (T2). Clearly χ is an uncountable set. Let $w \in (0, 1)$. Since $[10^n w] \uparrow \infty$ as $n \uparrow \infty$, τ^w is asymptotically nonperiodic; thus x^w is asymptotically nonperiodic. It remains to show Eqs. (8) and (9). Let w, $w' \in (0, 1)$ with $w \neq w'$. Let $\overline{w} = \min\{w, w'\}$. Let $m \in \mathbb{Z}\backslash\{0\}$. For $n \in \mathbb{N}$, let $\mu_n = T_n + [\frac{10^n \overline{w}}{2}]$. By Eqs. (11) and (12), we have

$$\left\|(x^w_{\mu_n - m}, ..., x^w_{\mu_{n+m}}) - (x^{w'}_{\mu_n - m}, ..., x^{w'}_{\mu_{n+m}})\right\| = 0 \text{ as } n \to \infty.$$

That is

$$\liminf_{t\to\infty} \left\|(x^w_{\mu_n - m}, ..., x^w_{\mu_{n+m}}) - (x^{w'}_{\mu_n - m}, ..., x^{w'}_{\mu_{n+m}})\right\| = 0.$$

By Eq. (13), we also have

$$\limsup_{t\to\infty} \left\|x^w_t - x^{w'}_t\right\| > 0$$

Since w, w', and m were arbitrary, we have verified Eqs. (8) and (9) and thus (T2).

Now to verify (T1), let $\tau = \{...,1, 2, 1, 2,...\}$ with $\tau_0 = 2$. Clearly $s(\tau)$ is feasible and $(2\overline{m})$-periodic. Let $m = 2\overline{m}$. Let $s^m = s(\tau)$. Note from Eq. (10) that $s^m_t = 1$

for all t expect that if $t = im$ for some $i \in \mathbb{Z}$, $s^m_{t,\ldots,\ t+\overline{m}-1} = v^2$. For $n > m$, let s^n be the symbolic sequence such that $s^n_t = 1$ for all t expect that if $t = in$ for some $i \in \mathbb{Z}$, $s^m_{t,\ldots,\ t+\overline{m}-1} = v^2$. Clearly for all $n \geq m$, s^n is feasible and n-periodic. That is for all $n \geq 2\overline{m}$, G^* has an n-periodic orbit. This completes the proof of theorem 3. ■

4 Stability of chaos

4.1. Quasi-static system

We introduce the concept of *quasi-static system*. Quasi-static systems are difference systems that are in a certain sense relatively close to static systems.

Definition 4 *We say that G^* is quasi-static (w.r.t. K_1,..., K_M) if (K1) and (K2) below hold:*

(K1) K_1,..., K_M are disjoint, compact, and convex.

(K2) For all $p \in P(G^*, K_1,..., K_M)$, $(\xi_{p_{-\mu}},..., \xi_{p_{-1}}) \in K_{p_{-\mu}} \times ... \times K_{p_{-1}}$, and $(\xi_{p_1},..., \xi_{p_\nu}) \in K_{p_1} \times ... \times K_{p_\nu}$ there is a unique $\xi \equiv g_p(\xi_{p_{-\mu}},..., \xi_{p_{-1}}, \xi_{p_1},..., \xi_{p_\nu}) \in K_{p_0}$ such that $G^*(\xi_{p_{-\mu}},..., \xi_{p_{-1}}, \xi, \xi_{p_1},..., \xi_{p_\nu}) = 0$.

Lemma 5 *G^* is quasi-static w.r.t. K_1,..., $K_M \subset Dom(G^*)_0$ then*

(c1) for each bi-infinite feasible symbolic sequence $\{s_t\}_{t=-\infty}^{\infty}$, G^* has an orbit $\{x_t\}_{t=-\infty}^{\infty}$ such that for all $t \in \mathbb{Z}$, $x_t \in K_{s_t}$.

(c2) For each n-periodic bi-infinite feasible symbolic sequence $\{s_t\}_{t=-\infty}^{\infty}$, G^* has an n-periodic asymptotic orbit $\{x_t\}_{t=-\infty}^{\infty}$ such that for all $t \in \mathbb{Z}$, $x_t \in K_{s_t}$.

Remark 6 *For all $p \in P(G^*, K_1,..., K_M)$, $g_p^* : D_p \to K_{p_0}$ is continuous, where $D_p = (K_{p_{-\mu}} \times ... \times K_{p_{-1}}) \times (K_{p_1} \times ... \times K_{p_\nu})$.*

Remark 7 (Brouwer fixed point theorem) *Suppose that M is a nonempty, convex, compact subset of $\mathbb{R}^n$, $n \geq 1$ and that $f : M \to M$ is a continuous mapping then f has a fixed point.(see [5], p. 51)*

Proof of Lemma. For any sequence $\{y_t\}$, let

$$y_t^- = (y_{t-\mu},..., y_{t-1}),\ y_t^+ = (y_{t+1},..., y_{t+\nu})$$

We first prove (c2). The proof of (c1) is similar to that of (c2), and is thus omitted. Let $\{s_t\}_{t=l}^{\infty}$, $-\infty \leq l \leq -\mu$, be a feasible sequence For $t \geq l$, let $S_t = K_{s_t}$.Suppose $l = -\infty$ and $\{s_t\}$ is n-periodic. Given $x_{1,n} \equiv (x_1,\ ...,\ x_n) \in K_{s_1} \times K_{s_2} \times ... \times K_{s_n}$. Let x be the n-periodic sequence such that x_1,..., x_n are as given. Define $T_{1,n}$: $K_{s_1} \times K_{s_2} \times ... \times K_{s_n} \rightarrow K_{s_1} \times K_{s_2} \times ... \times K_{s_n}$ by $T_{1,n}(x_{1,n}) = g_{s_{t-\mu,t+\nu}}(x_t^-,\ x_t^+)$. Since $T_{1,n}$ is continuous and $S_{1,n}$ is compact and convex, $T_{1,n}$ has a fixed point $x_{1,n}^* = (x_1^*$,..., $x_n^*)$ (by the Brouwer fixed point theorem) Clearly, the associated n-periodic orbit x^* is an orbit of G such that for all $t \in \mathbb{Z}$, $x_t^* \in K_{s_t}$. ■

Note that if G^* is a static system with static points ξ_1,..., ξ_M, then G^* is a quasi-static w.r.t.$\{\xi_1\}$,..., $\{\xi_M\}$, and the conclusions (c1)-(c2) trivially hold with $x_t \in \{\xi_1$,..., $\xi_M\}$ for all t. The lemma says that they continue to hold for a quasi-static system with appropriate compact convex sets replacing static points.

4.2. Stability of chaos for regular static system under small C^1 perturbations

Let G^* is static; we say that G^* is *regular* if G^* is C^1, if G^* has only a finite number of static points ξ_1,..., $\xi_M \in Dom(G)_0$, and $DG^s(\xi_i)$ is nonsingular for all $i = 1$,..., M.

Let G^* be a C^1 static system with static points ξ_1,..., $\xi_M \in Dom(G^*)_0$. Denote

$$J(G^*,\xi_1,..., \xi_M) = \{(\xi_{i_{-\mu}},..., \xi_{i_0},..., \xi_{i_\nu}) \in Dom(G^*) \mid 1 \leq i_{-\mu},..., i_0,..., i_\nu \leq M\},$$

and

$$J(G^*,\overline{N_\varepsilon}(\xi_1),..., \overline{N}_\epsilon(\xi_M)) = \{\overline{N_\varepsilon}(\xi_{i_{-\mu}}) \times ... \times \overline{N}_\epsilon(\xi_{i_\nu}) \subset Dom(G^*) \mid$$

$$\text{for all } (i_{-\mu},..., i_\nu) \in P(G^*, \overline{N_\varepsilon}(\xi_1), ..., \overline{N_\varepsilon}(\xi_M))\}.$$

Theorem 8 *Let G^* be a regular static system with static points $\xi_1,..., \xi_M \in K_0$ and $Dom(G^*)$ is open. Let $K \subset Dom(G^*)$ be a compact set such that $J(G^*,\xi_1,...,\xi_M) \subset \overset{\circ}{K}$ (the interior of K). Then there exist ε and $\delta_\varepsilon > 0$ such that for all $G \in N_{\delta_\varepsilon}(G^*, K)$*

(i) we have G is quasi-static (w.r.t. $\overline{N}_\epsilon(\xi_1)$, $\overline{N}_\epsilon(\xi_2)$,..., $\overline{N}_\epsilon(\xi_M)$) and

(ii) for G^* , if there are p, $q \in P(G^*, \xi_1,..., \xi_M)$ with $p \neq q$ such that $p_{-\mu} =... = p_{-\nu}$ and p, q are reachable from each other. Then G is chaotic.

Proof. (i) Let G^* be a C^1 static system with static points $\xi_1,..., \xi_M \in Dom(G^*)_0$, and $DG^s(\xi_i)$ is nonsingular for all $i = 1,..., M$, and $Dom(G^*)$ is open. Let $K \subset Dom(G^*)$ be a compact set such that $J(G^*,\xi_1,...,\xi_M) \subset \overset{\circ}{K}$ (the interior of K). Then there is $\varepsilon > 0$ such that $J(G^*,\xi_1,...,\xi_M) \subset J(G^*, \overline{N_\varepsilon}(\xi_1),..., \overline{N_\varepsilon}(\xi_M)) \subset K \subset Dom(G^*)$, and $P(G^*, \overline{N_\varepsilon}(\xi_1),..., \overline{N_\varepsilon}(\xi_M)) = P(G^*, \xi_{1,M}) \equiv P$. For this $\varepsilon > 0$, G^* is a quasi-static system w.r.t. $\overline{N_\varepsilon}(\xi_1)$,..., $\overline{N_\varepsilon}(\xi_M)$. Let $K_i = \overline{N_\varepsilon}(\xi_i)$ for $i = 1,..., M$. For $p \in P$, let $g_p^* : D_p \to K_{p_0}$ be defined as in (K2). Note that for all $p \in P$ and for all $\zeta \in D_p$, we have $g_p^*(\zeta) = \xi_{p_0} \in \overset{\circ}{K}_{p_0}$ and $\left\|Dg_p^*(\zeta)\right\| = 0$. Hence by theorem 1, there is $\delta_\varepsilon > 0$ such that for all $G \in N_{\delta_\varepsilon}(G^*, K)$ for all $p \in P$, there is a unique function $g_p^G : D_p \to K_{p_0}$ such that for all $\zeta \in D_p$, $G(\xi^-, g_p^G(\zeta), \xi^+) = 0$, g_p^G is C^1, and $\max_{\zeta \in D_p}$ $\left\|Dg_p^G(\zeta)\right\|$ $=\max_{\zeta \in D_p}$ $\left\|Dg_p^G(\zeta) - Dg_p^*(\zeta)\right\| < 1$. Therefore for all $G \in N_\delta(G^*, K)$, G satisfies (K1) and (K2). So, we have G is quasi-static (w.r.t. $\overline{N}_\epsilon(\xi_1)$, $\overline{N}_\epsilon(\xi_2)$,..., $\overline{N}_\epsilon(\xi_M)$). From Lemma 5, note that if $\{s_t\}_{t=-\infty}^{\infty}$ is the orbit of G^*, then there exists a correspond orbit of G called $\{x_t\}_{t=-\infty}^{\infty}$, where $x_t \in \overline{N}_\varepsilon(s_t)$ for all $t \in \mathbb{Z}$. Moreover the correspond orbit preserve the period of $\{s_t\}_{t=-\infty}^{\infty}$. That is if $\{s_t\}_{t=-\infty}^{\infty}$ is $n-$periodic then $\{x_t^{\alpha}\}_{t=-\infty}^{\infty}$ is $n-$periodic.

(ii) For G^*, if there are p, $q \in P(G^*, \xi_1,..., \xi_M)$ with $p \neq q$ such that $p_{-\mu} =... = p_{-\nu}$ and p, q are reachable from each other, then G^* satisfies (T1) and (T2) and is chaotic. (From Theorem 3). In fact, for all $G \in N_{\delta_\varepsilon}(G^*, K)$ G also satisfies (T1) and there exists an uncountable set χ^G of asymptotically nonperiodic orbit such that for all x^G, $y^G \in \chi^G$ $(x^G \neq y^G)$ we have

$$\limsup_{t\to\infty} \left\| x_t^G - y_t^G \right\| > 0.$$

We remain Eq.(3) to be check. Now we give two remarks as following, and postpone the proof of Remark 9 to the appendix.

Remark 9 *Let G be a C^0 system. Let $H \subset Dom(G^*)_0$ is compact. Suppose (a) there is a unique orbit x^* such that $x_t^* \in H$ for all $t \in \mathbb{Z}$. Then x^* is a constant sequence and for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$ and for any orbit x, if $x_i \in H$ for all $i = t-n,..., t+n$, we have $\|x_t - x_t^*\| < \varepsilon$.*

Remark 10 *If G is quasi-static (w.r.t. $\overline{N}_\epsilon(\xi_1), \overline{N}_\epsilon(\xi_2),..., \overline{N}_\epsilon(\xi_M)$) and there are p, $q \in P(G^*, \overline{N}_\epsilon(\xi_1),..., \overline{N}_\epsilon(\xi_M))$ with $p \neq q$ such that $p_{-\mu} =... = p_{-\nu}$ and p, q are reachable from each other and g_p^G as given by (K2) is C^1 and $max_{\zeta\in D_p} \left\| Dg_p^G(\zeta) \right\| < 1$. Then there is a constant sequence $\{..., \xi^*, \xi^*, \xi^*,...\}$ is the unique orbit $\{x_t\}$ such that $x_t \in \overline{N}_\epsilon(\xi_{p_0})$ for all $t \in \mathbb{Z}$.*

By the proof of theorem 3 and Lemma 5 (c1), for any $w \in (0, 1)$ there is an orbit x^w such that for all $t \in \mathbb{Z}$, $x_t^w \in \overline{N}_\varepsilon(\xi_{s(\tau^w)_t})$. Let $\chi = \{x^w \mid w \in (0, 1)\}$; we show that χ satisfies (T2). Clearly χ is an uncountable set. Let $w \in (0, 1)$. Since $[10^n w] \uparrow \infty$ as

$n \uparrow \infty$, τ^w is asymptotically nonperiodic; thus x^w is asymptotically nonperiodic. It remains to show Eqs. (8) and (9). Let w, $w^{'} \in (0, 1)$ with $w \neq w^{'}$. Let $\overline{w} = \min\{w, w^{'}\}$. Let $m \in \mathbb{Z} \backslash \{0\}$. For $n \in \mathbb{N}$, let $\mu_n = T_n + [\frac{10^n \overline{w}}{2}]$. By Eqs. (11) and (12) and Remark 9 and 10, we have

$$\begin{aligned}\left\|(x^w_{\mu_n - m},\ldots, x^w_{\mu_{n+m}}) - (x^{w^{'}}_{\mu_n - m},\ldots, x^{w^{'}}_{\mu_{n+m}})\right\| &\leq \left\|(x^w_{\mu_n - m},\ldots, x^w_{\mu_{n+m}}) - (\xi^*,\ldots, \xi^*)\right\| + \\ \left\|(x^{w^{'}}_{\mu_n - m},\ldots, x^{w^{'}}_{\mu_{n+m}}) - (\xi^*,\ldots, \xi^*)\right\| &\rightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

That is

$$\liminf_{t\to\infty} \left\|(x^w_{\mu_n - m},\ldots, x^w_{\mu_{n+m}}) - (x^{w^{'}}_{\mu_n - m},\ldots, x^{w^{'}}_{\mu_{n+m}})\right\| = 0.$$

By Eq. (13), we also have

$$\limsup_{t\to\infty} \left\|x^w_t - x^{w^{'}}_t\right\| > 0$$

Since w, $w^{'}$, and m were arbitrary, we have verified Eqs. (8) and (9) and thus (T2). Therefore G satisfies (T1) and (T2) and thus chaotic. ■

5 Appendix

Here we give the proof of Remark 9.

Proof of Remark 9. Let G be a C^0 system. Let $H \subset Dom(G^*)_0$ is compact. Assume (a) above. Since $x^* = \{x_t^*\}$ is the unique orbit in H and since $\{x_{t+1}^*\}$ is clearly an orbit, we have $x_t^* = x_{t+1}^*$ for all $t \in \mathbb{Z}$, i.e., x^* is a constant sequence. Let $\xi^* = x_t^*$. Let $\varepsilon > 0$. Suppose there is no $n \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$ and for any orbit x, if $x_i \in H$ for all $i = t-n,..., t+n$, we have $\|x_t - \xi^*\| < \varepsilon$. This means that for all $n \in \mathbb{N}$ there is an orbit y^n such that for some $T_n \in \mathbb{Z}$, $\left\|y_{T_n}^n - \xi^*\right\| \geq \varepsilon$ and $y_i^n \in H$ for all $i = t-n,..., t+n$. For $n \in \mathbb{N}$, define $x^n = \{x_t^n\}$ by $x_t^n = y_{T_n}^n$. Note that for all $n \in \mathbb{N}$, x^n is an orbit and $\|x_0^n - \xi^*\| \geq \varepsilon$. Taking a subsequence if necessary, we may assume $x_t^n \to \overline{x_t} \in H$ as $n \uparrow \infty$ for all $t \in \mathbb{Z}$. Then we have $\|\overline{x_0} - \xi^*\| \geq \varepsilon$ and thus $\{\overline{x_t}\} \neq x^*$. But since G is C^0, it follows that $\{\overline{x_t}\}$ is an orbit, which contradicts (a). ■

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