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摘　　要

我們研究二次微分同胚$H_{\alpha,b}(x,y)=(1+y_t-\frac{1}{\alpha}x^2,bx)$的總體行為。在固定 b 之後，我們可以把 Henon 映射轉化成差分方程式H_α，此時α為參數。當$\alpha=0$，差分方程式H_α可以簡化成雙符號的全轉移，我們研究從這種狀態延續的軌跡。首先，我們證明H_0是混沌的並且在微小的擾動之後，此系統仍會滿足混沌的充分條件，亦即當α足夠靠近0，H_α會是混沌。在這篇論文中，我們估計α使得H_α為混沌，並由此去得到 Henon 映射的混沌區域。

An estimate of chaotic region for the Henon map

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Abstract

We investigate the global behavior of the quadratic diffeomorphism of the plane given by $H_{\alpha,b}(x,y)=(1+y_t-\frac{1}{\alpha}x^2,bx)$. If we fix b, the Henon map can be considered as H_α, a difference equation, where α is the parameter. For α =0, the difference equation, H_α, reduces to the full shift on two symbols, and we study orbits that continues from these states. We first show that the system H_0 is chaotic and under a small perturbation, the system satisfies the su fficient condition of chaos, that is for α close enough to zero, H_α display chaotic dynamics. In this paper, we estimate for which α, H_α display chaotic to get the chaotic region for the Henon map.

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目　　　　錄

1 Introduction

Let $H_{\alpha,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$H_{\alpha,b}(x,y) = (1 + y - \frac{1}{\alpha}x^2, bx),$$

where α,b are the parameters. This map is called the Henon map. It was written down by Henon[1] to realize the Smale horseshoe for a specific function which could be iterated on the computer. Instead of being a model of any particular physical situation, the Henon map is a map with a simple algebraic form which could easily be studied by means of computer simulation.

In 1979, Devaney and Nitecki used a geometrical argument to get the bound on the existence of the horseshoe [2]. In 1999, Kamihigashi provided easy-to-verify sufficient conditions for chaos [3]. If we fix b, the Henon map can be considered as G_α, a difference equation, where α is the parameter. We first show that for $\alpha = 0$ the system is chaotic and under a small perturbation, the system satisfies the sufficient condition of chaotic; that is, for α close enough to zero, it also displays chaotic dynamics. More precisely, we have the estimate of chaotic region below:

Theorem 1 (Main) *Fixing b and considering the difference equation G_α, If $\gamma < (2\sqrt{2} - 2)[\frac{1}{(2-\sqrt{1-\gamma\frac{\gamma+\sqrt{\gamma^2+4}}{2}})}]$, where $\gamma \equiv \sqrt{\alpha}(1 + |b|)$, then G_α is chaotic.*

We conclude this introductory section by mentioning the structure of the thesis as follows. In Section 2, we give some defintions and show that for $\alpha = 0$, the Henon map is chaotic. In Section 3, we estimate α such that G_α has some propositions. Finally, in Section 4, we prove these propositions to be the sufficient conditions of chaos.

2 Preliminary

Let's start from giving some definitions. For any function f, we denote its domain by Dom(f). We consider difference equation of the form

$$G(x_{n-1}, x_n, x_{n+1}) = 0, \tag{1}$$

where G :Dom$(G)\subset \mathbb{R}^3 \to \mathbb{R}$. A bi-infinite sequence $\{x_t\}_{t=-\infty}^{\infty}$ is called the *orbit of* G provided for all $t \in \mathbb{Z}$, $G(x_{t-1},x_t,x_{t+1}) = 0$. The *orbit* is *$n$-periodic orbit* provided there exists $n \in \mathbb{N}$ such that $y_{t+n} = y_t$ for all $t \in \mathbb{Z}$ and $y_{t+j} \neq y_t$ for $0 < j < n$. The orbit $\{y_t\}_{t=-\infty}^{\infty}$ is called *asymptotically periodic* if there is a periodic sequence $\{y_t^*\}$ such that $\|y_t - y_t^*\| \to 0$ as $t \to \infty$. If y is not asymptotically periodic, then y is called *asymptotically nonperiodic.* For any sequence $\{y_t\}$, let $\|\cdot\|$ denote the sup norm whenever its argument is a vector or a sequence.

Definition 2 *We say that a difference equation G is chaotic if (T1) and (T2) are held below:*

(T1) There exists $m \in \mathbb{N}$ such that for all $n \geq m$, G has an n-periodic orbit.

(T2) G has an uncountable set χ of asymptotically nonperiodic orbit such that for all x,$y \in \chi : (x \neq y)$

$$\limsup_{t\to\infty} \|x_t - y_t\| > 0, \tag{2}$$

$$\liminf_{t\to\infty} \|(x_{t-n}, ..., x_{t+n}) - (y_{t-n}, ..., y_{t+n})\| = 0, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3}$$

For $Dom(G) \subset \mathbb{R}^3$, we define $Dom(G)_0 = \{x_0 \in \mathbb{R} \mid (x_{-1},x_0,x_1) \in Dom(G)\}$. We define a static system as a difference equation that depends only on x_t. More precisely, we say that G is *static* or a *static system* if there is a function $G^s : Dom(G)_0 \to \mathbb{R}$, such that for all $(x_{-1},x_0,x_1) \in Dom(G)$, $G^s(x_0) = G(x_{-1},x_0,x_1)$. If G is static, we defin a *static point* of G as a point $\xi \in Dom(G)_0$ such that $G^s(\xi) = 0$.

Now return to the theme. Investigating the behavior of the quadratic diffeomorphism of Henon map, the Henon map is a two dimensional quadratic map $(x_t,y_t) \to (x_{t+1},y_{t+1})$ defined by

$$\begin{cases} x_{t+1} = 1 + y_t - \frac{1}{\alpha}x_t^2, \\ y_{t+1} = bx_t. \end{cases} \tag{4}$$

Substituting the second equation into the first yields $x_{t+1} - 1 - bx_{t-1} + \frac{1}{\alpha}x_t^2 = 0$. Let $z_t = \frac{x_t}{\sqrt{\alpha}}$, we have$\sqrt{\alpha}z_{t+1} - 1 - b\sqrt{\alpha}z_{t-1} + z_t^2 = 0$. We can reduce the Henon map to a difference equation $G_\alpha : \mathbb{R}^3 \to \mathbb{R}$

$$G_\alpha(x_{n-1}, x_n, x_{n+1}) = \sqrt{\alpha}x_{n+1} - 1 - b\sqrt{\alpha}x_{n-1} + x_n^2, \tag{5}$$

where α is the parameter. For $\alpha = 0$, this equation reduces to a *static system*:

$$G_0(x_{n-1}, x_n, x_{n+1}) = -1 + x_n^2. \tag{6}$$

We know that G_0 is a static system with static point $\xi_1 = \{1\}$, $\xi_2 = \{-1\}$. Next, we will show that for $\alpha = 0$, G_0 is chaotic.

Proposition 3 *The difference equation $G_0(x_{n-1},x_n,x_{n+1}) = -1 + x_n^2$ is chaotic.*

Proof. First of all, we verify (T1). $G_0(x_{n-1},x_n,x_{n+1}) = 0$ has two solutions $\xi_1 = 1$, $\xi_2 = -1$. Since G_0 is a function of x_n alone, the orbit $\{x_t\}_{t=-\infty}^{\infty}$ must be $x_t \in \{\xi_1,\xi_2\}$, for all t. Given any $n \in \mathbb{N}$, let $x_0 = x_1 =...= x_{n-2} = \xi_1$, $x_{n-1} = \xi_2$ and $x_{n+k} = x_k$, for all $k \in \mathbb{Z}$. This sequence $\{x_n\}_{n=-\infty}^{\infty}$ is the orbit of G_0 and $n \in \mathbb{N}$ is the smallest number that satisfies $x_{k+n} = x_k$, which means that for all $n \geq 1$, G_0 has an n-periodic orbit. The above illustrates (T1).

Secondly we verify (T2). For $r \in \mathbb{R}$, let $[r]$ denote the largest integer less than or equal to r. For $w \in (0,1)$, define a bi-infinite sequence of natural number χ^w as follows. For $i \leq 0$, let $x_i^w = \xi_1$. For $i \geq 1$, we defined x_i^w as follows:

$$\begin{aligned}
x^w_{1,10} &= \{\overbrace{\xi_1, ..., \xi_i}^{[10w]\xi_1^{'}s},\ \overbrace{\xi_2, ..., \xi_2}^{(10-[10w])\xi_2^{'}s}\},\\
x^w_{11,110} &= \{\overbrace{\xi_1, ..., \xi_i}^{[100w]\xi_1^{'}s},\ \overbrace{\xi_2, ..., \xi_2}^{(100-[100w])\xi_2^{'}s}\},\\
x^w_{111,1110} &= \{\overbrace{\xi_1, ..., \xi_i}^{[1000w]\xi_1^{'}s},\ \overbrace{\xi_2, ..., \xi_2}^{(1000-[1000w])\xi_2^{'}s}\},
\end{aligned}$$

and so on. More precisely, let $T_n = 1 + 10 + ... + 10^n$, for $n \in \mathbb{N}$, we have for all $n \in \mathbb{N}$,

$$\begin{aligned}
x_i^w &= \xi_1 \text{ for all } i = T_n, ..., T_n + [10^n w] - 1,\\
x_i^w &= \xi_2 \text{ for all } i = T_n + [10^n w], ..., T_n - 1.
\end{aligned}$$

Let $\chi = \{x^w \mid w \in (0,1)\}$.We show that χ satisfies (T2). Clearly, χ is an uncountable set. Let $w \in (0,1)$. Since $[10^n w] \to \infty$ as $n \to \infty$, x^w is asymptotically nonperiodic. Note that for all w, $w^{'} \in (0,1)$ where $w \neq w^{'}$. We have $[10^n w] \neq [10^n w^{'}]$, for n is large enough. Thus, for all w, $w^{'} \in (0,1)$ $(w \neq w^{'})$ $x_i^w \neq x_i^{w^{'}}$ for infinitely many $i^{'}s$. So $\limsup_{t\to\infty} \left\| x_t^w - x_t^{w^{'}} \right\| > 0$. Let w, $w^{'} \in (0,1)$ with $w \neq w^{'}$. Let $\overline{w} = \min\{w,w^{'}\}$ and $m \in \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $u_n = [\frac{10^n \overline{w}}{2}]$. We have $\left\| (x^{\overline{w}}_{u_n - m, u_n + m}) - (\xi_1, ..., \xi_1) \right\| = 0$, for n is large enough. So

$$\begin{aligned}
&\liminf_{t\to\infty} \left\| (x^w_{t-m}, ..., x^w_{t+m}) - (x^{w^{'}}_{t-m}, .., x^{w^{'}}_{t+m}) \right\| \\
\leq\ &\liminf_{t\to\infty} \left\| (x^w_{t-m}, ..., x^w_{t+m}) - (\xi_1, ..., \xi_1) \right\| + \liminf_{t\to\infty} \left\| (\xi_1, ..., \xi_1) - (x^{w^{'}}_{t-m}, ..., x^{w^{'}}_{t+m}) \right\| \\
=\ &0.
\end{aligned}$$

The above proves (T2). So G_0 is chaotic. ■

3 The estimate of α

We first show that we can choose λ such that the orbit of G_α contains in $W = I \times I \times I$, where $I = [-1-\lambda, -1+\lambda] \cup [1-\lambda, 1+\lambda]$. The method was mentioned by D. Sterling and J.D Meiss[4]. We will write an orbit $z(\varepsilon)$ of the Henon map as a fixed point of an operator T whose t-th component is

$$T_t(z) \equiv \pm\sqrt{1-\sqrt{\alpha}(x_{n+1}-bx_{n-1})}.$$

Define the l^∞ norm, let $\|x\|_\infty = \sup_t |x_t|$ and define B_M to be the closed ball of radius M around the point s,

$$B_M(s) = \{z : \|z-s\|_\infty \leq M\}.$$

Proposition 4 *For any G_α,we define $\gamma \equiv |\sqrt{\alpha}|(1+|b|)$. If $\gamma < \frac{1}{\sqrt{2}}$,*

$$\text{and } \lambda > M_\infty(\gamma) = 1-\sqrt{1-\gamma\frac{\gamma+\sqrt{\gamma^2+4}}{2}}, \tag{7}$$

then the orbit of G_α is contained in the ball $B_\lambda(s)$

Proof. Let T be defined by (10). For any $z \in B_\lambda(s)$, it is easy to see that

$$c_n \leq \|T^n(z)\|_\infty \leq d_n, \tag{8}$$

where the sequences c_n and d_n are given by the iterations

$$\begin{aligned} d_{n+1} &= f(d_n) \equiv \sqrt{1+\gamma d_n}, \\ c_{n+1} &= \sqrt{1-\gamma d_n}, \end{aligned}$$

with the initial conditions $c_0 = 1+\lambda$ and $d_0 = 1-\lambda$. The map $f(d)$ has a single attracting fixed point

$$d_\infty = \frac{\gamma+\sqrt{\gamma^2+4}}{2}.$$

Each of the c_n must be real, so we must have $1 - \gamma d_n \geq 0$. This requirement gives a right boundary to the region in the (γ, λ) plane where T^n exists. As $n \to \infty$ these boundaries approach the vertical line defined by

$$1 - \gamma d_\infty = 0 \Rightarrow \gamma = \frac{1}{\sqrt{2}},$$

which gives one of the bounds in the lemma.

Finally, (8) implies that

$$\|T^n - s\|_\infty \leq \max(|c_n - 1|, |d_n - 1|) = 1 - c_n.$$

Thus the requirement M_n for which $1 - c_n - M_n = 0$ converges monotonically to M_∞ from above; therefore for any $\lambda > M_\infty(\gamma)$, there is an N such that $T^n{:}B_\lambda(s) \to B_\lambda(s)$, for all $n > N$. The orbit is fixed point of T^n. So it is contained in the ball $B_\lambda(s)$. ■

Let $K_1 = [-1-\lambda, -1+\lambda]$, $K_2 = [1-\lambda, 1+\lambda]$, where $M_\infty(\gamma) < \lambda < 1$. Such that $K_1$and K_2 are disjoint, compact, and convex. Let $P = \{(p_{-1}, p_0, p_1) \in \mathbb{R}^3 \mid p_i \in \{1,2\}$, for all $i = 1,2,3\}$.

Definition 5 *We say that G_α is quasi-static if for all $p = (p_{-1}, p_0, p_1) \in P$, for all $x_{-1} \in K_{p_{-1}}$, and for all $x_1 \in K_{p_1}$, there is a unique $x_0 \equiv g_p(x_{-1}, x_1) \in K_{p_0}$ such that $G_\alpha(x_{-1}, x_0, x_1) = 0$.*

Remark 6 *G_0 is quasi-static since we can define $g_p(x_{-1}, x_1) = \xi_{p_0}$.*

Proposition 7 *For all $p = (p_{-1}, p_0, p_1) \in P$, define $Y_p = K_{p_{-1}} \times K_{p_0}$, $Z_p = K_{p_0}$, $W_p = K_{p_{-1}} \times K_{p_0} \times K_{p_1}$. If $\sup\limits_{(y,\, z) \in W_p} \|G_\alpha(y, z) - G_0(y, z)\| < 1$, then G_α is quasi-static.*

Proof. We have $\sup\limits_{(y,\, z) \in W_p} \|G_\alpha(y, z) - G_0(y, z)\|$

$= \sup_{(x_{-1},x_0,x_1)\in K_{\xi_{p_{-1}}}\times K_{\xi_{p_0}}\times K_{\xi_{p_1}}} \|\sqrt{\alpha}x_{n+1} - b\sqrt{\alpha}x_{n-1}\| < 1$. Then we can define $g_p : Y_p \to Z_p$ by

$$g_p(x_{-1}, x_1) = \begin{cases} \sqrt{1 + b\sqrt{\alpha}x_{n-1} - \sqrt{\alpha}x_{n+1}}, \text{ if } p_0 = 1 \\ -\sqrt{1 + b\sqrt{\alpha}x_{n-1} - \sqrt{\alpha}x_{n+1}}, \text{ if } p_0 = -1 \end{cases},$$

then $G_\alpha(x_{-1}, g_p(x_{-1}, x_1), x_1) = 0$. ■

For any $m \times n$ matrix $B = (Bij)$, let $\|B\|$denote the operator norm of B :

$$\|B\|_\infty = \max_{\xi\in\mathbb{R}^n} \frac{\|B\xi\|_\infty}{\|\xi\|_\infty} = \max_{1\le i\le m} \sum_{j=1}^{n} |B_{ij}|.$$

Proposition 8 *For* $\sup_{(y,\, z)\in W_p} \|G_\alpha(y, z) - G_0(y, z)\| < 2\sqrt{2}-2$, *we have* $\max_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1}, x_1)\| < 1$.

Before the proof of Proposition8, we first prove the lemma.

Lemma 9 *If we can choose* α *such that* $\sup_{(y,\, z)\in W_p} \|G_\alpha(y, z) - G_0(y, z)\| < \delta$, *then for the same* α $\sup_{(y,\, z)\in W_p} \|DG_\alpha(y, z) - DG_0(y, z)\| < \delta$.

Proof. The inequality $\sup_{(y,\, z)\in W_p} \|G_\alpha(y, z) - G_0(y, z)\|$

$= \sup_{(x_{-1},x_0,x_1)\in K_{\xi_{p_{-1}}}\times K_{\xi_{p_0}}\times K_{\xi_{p_1}}} \|\sqrt{\alpha}x_{n+1} - b\sqrt{\alpha}x_{n-1}\|$

$= \sqrt{\alpha}(1 + |b|)(1 + \lambda) < \delta$ implies

$$\sqrt{\alpha} < \frac{1}{(1 + \lambda)(1 + |b|)}\delta. \tag{9}$$

On the other hand, $\sup_{(y,\, z)\in W_p} \|DG_\alpha(y, z) - DG_0(y, z)\| = \max\{\sqrt{\alpha}, |b|\sqrt{\alpha}\} < \delta$ implies $\sqrt{\alpha} < \min\{\delta, \frac{1}{|b|}\delta\}$. It is clear that $\frac{1}{(1+\lambda)(1+|b|)}\delta < \min\{\delta, \frac{1}{|b|}\delta\}$. ■

Proof of Propositon 8. Suppose $\sup_{(y,\, z)\in W_p} \|G_\alpha(y,z) - G_0(y,z)\| < \delta$, where $0 < \delta < 1$

$$\begin{aligned}
&\max_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1},x_1)\| \\
= &\left\| \begin{pmatrix} \frac{b\sqrt{\alpha}}{2\sqrt{1+b\sqrt{\alpha}x_{n-1}-\sqrt{\alpha}x_{n+1}}} \\ \frac{-\sqrt{\alpha}}{2\sqrt{1+b\sqrt{\alpha}x_{n-1}-\sqrt{\alpha}x_{n+1}}} \end{pmatrix} \right\| \\
= &\max_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \{ \left| \frac{b\sqrt{\alpha}}{2\sqrt{1+b\sqrt{\alpha}x_{n-1}-\sqrt{\alpha}x_{n+1}}} \right|, \left| \frac{-\sqrt{\alpha}}{2\sqrt{1+b\sqrt{\alpha}x_{n-1}-\sqrt{\alpha}x_{n+1}}} \right| \} \\
< &\frac{\sup_{(y,\, z)\in W_p} \|DG_\alpha(y,z) - DG_0(y,z)\|}{2\sqrt{1 - \sup_{(x_{-1},x_0,x_1)\in K_{\xi_{p_{-1}}}\times K_{\xi_{p_0}}\times K_{\xi_{p_1}}} \|\sqrt{\alpha}x_{n+1} - b\sqrt{\alpha}x_{n-1}\|}} \\
< &\frac{\delta}{2\sqrt{1-\delta}} < 1,
\end{aligned}$$

this implies $0 < \delta < 2\sqrt{2} - 2$. ■

From Proposition7, 8, we can conclude that if $\sup_{(y,\, z)\in W_p} \|G_\alpha(y,z) - G_0(y,z)\| < 2\sqrt{2} - 2$, then G_α is quasi-static and $\max_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1},x_1)\| < 1$.

4 Main Result

In this section, we solve $\sup_{(y,\, z)\in W_p} \|G_\alpha(y,z) - G_0(y,z)\| < 2\sqrt{2} - 2$ to get the restrict of α. By (7) and (9), the conclusion is as follows.

Corollary 10 *Define $\gamma \equiv \sqrt{\alpha}(1+|b|)$. If $\gamma < (2\sqrt{2}-2)[\frac{1}{(2-\sqrt{1-\gamma\frac{\gamma+\sqrt{\gamma^2+4}}{2}})}]$, we can solve this inequality by numerical method to get that. If $0 < \gamma < A \approx 0.56$, i.e.,*

$$\sqrt{\alpha} < A\frac{1}{1+|b|} \approx 0.56\frac{1}{1+|b|}, \tag{10}$$

then we have that G_α is quasi-static w.r.t K_1, K_2 and $\max_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1},x_1)\| < 1$.

Now we want to show that if α satisfies (10), then G_α is chaotic, i.e., (1)G_α is quasi-static and (2)$\max\limits_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})}\|Dg_p(x_{-1},x_1)\|<1$ which are the sufficient conditons for chaos. Recall that we define $K_1=[-1-\lambda,\,-1+\lambda]$, $K_2=[1-\lambda,\,1+\lambda]$, where $M_\infty(\gamma)<\lambda<1$. $P=\{(p_{-1},p_0,p_1)\in\mathbb{R}^3\mid p_i\in\{1,2\}$ for all $i=1,2,3\}$. We first prove the theorem below.

Theorem 11 *Suppose $\{s_t\}_{t=-\infty}^{\infty}$ is the orbit of G_0. If α satisfies (10), then there exists a corresponding orbit of G_α called $\{x_t^*\}_{t=-\infty}^{\infty}$, where $x_t^\alpha\in K_{s_t}$ for all $t\in\mathbb{Z}$. Moreover, the corresponding orbit preserves period of $\{s_t\}_{t=-\infty}^{\infty}$; that is, if $\{s_t\}_{t=-\infty}^{\infty}$ is $n-$periodic, then $\{x_t^*\}_{t=-\infty}^{\infty}$ is $n-$periodic.*

Lemma 12 *For all $p\in P$, $g_p^*:K_{p_{-1}}\times K_{p_1}\to K_{p_0}$ is continuous.*

Lemma 13 (Brouwer fixed point theorem) *Suppose that M is a nonempty , convex, and compact subset of $\mathbb{R}^n$, and that $f:M\to M$ is a continuous mapping then f has a fixed point.*

Proof. Let $\{s_t\}_{t=-\infty}^{\infty}$ be the orbit of G_0. Suppose $\{s_t\}$ is n-periodic. Given $x_{1,n}\equiv(x_1,\ldots,x_n)\in K_{s_1}\times K_{s_2}\times\ldots\times K_{s_n}$. Let x be the n-periodic sequence such that $x_1,\ldots,x_n$ are as given. Define $T_{1,n}:K_{s_1}\times K_{s_2}\times\ldots K_{s_n}\to K_{s_1}\times K_{s_2}\times\ldots K_{s_n}$ by

$$T_{1,n}(x_{1,n})=(g_{(s_0,s_1,s_2)}(x_0,x_2),g_{(s_1,s_2,s_3)}(x_1,x_3),\ldots,g_{(s_{n-2},s_{n-1},s_n)}(x_{n-2},x_n)).$$

Since $T_{1,n}$ is continuous and $K_{s_1}\times K_{s_2}\times\ldots K_{s_n}$ is compact and convex, $T_{1,n}$ has a fixed point $x_{1,n}^\alpha=(x_1^\alpha,\ldots.,x_n^\alpha)$ (by the Brouwer fixed point theorem). Clearly, the associated n-periodic orbit x^α is an orbit of G_α such that $x_t^\alpha\in K_{s_t}$, for all $t\in\mathbb{Z}$. ∎

Since G_0 satisfies (T1), by theorem11, G_α also satisfies (T1). It remains to show that G_α satisfies (T2). From the proof of Proposition3, G_0 has an uncountable set

$\chi = \{x^w \mid w \in (0,1)\}$ of asymptotically nonperiodic orbit that satisfies (T2). If α satisfies (3), by theorem11, there exists the corresponding orbit of G_α called χ'. Then χ' is also uncountable and asymptotically nonperiodic. For all x,$y \in \chi$, we have $\limsup_{t\to\infty} \|x_t - y_t\| > 0$. For the corresponding x', y', we also have $\limsup_{t\to\infty} \left\|x_t' - y_t'\right\| > 0$. Next, we will show that χ' satisfies (3).

Theorem 14 *If α satisfies (10), let χ^* be the corresponding orbit of $\chi = \{x^w \mid w \in (0,1)\}$. For all x^*,$y^* \in \chi^* : (x^* \neq y^*)$*

$$\liminf_{t\to\infty} \left\|(x_{t-n}^*, ..., x_{t+n}^*) - (y_{t-n}^*, ..., y_{t+n}^*)\right\| = 0, \text{ for all } n \in \mathbb{N} \cup \{0\}. \qquad (11)$$

Lemma 15 *Given $p = (p_{-1}, p_0, p_1)$. Suppose there is a unique orbit x^* such that $x_t^* \in K_{p_0}$, for all $t \in \mathbb{Z}$. Then for all $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that for any $t \in \mathbb{Z}$ and for any asymptotic obit x with $x_i \in K_{p_0}$, for all $i = t-n$,...,$t+n$. We have $\|x_t - x_t^*\| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and suppose there is no $n \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$, for any orbit x with $x_i \in K_{p_0}$, for all $i = t-n$,...,$t+n$, we have $\|x_t - \xi^*\| < \varepsilon$. This means that for $n \in \mathbb{N}$, there is an orbit y^n such that for some $T_n \in \mathbb{Z}$, $\left\|y_{T_n}^n - \xi^*\right\| \geq \varepsilon$ and $y_i^n \in K_{p_0}$, for $i = T_n - n$,...,$T_n + n$. For $n \in \mathbb{N}$, define x^n by $x_t^n = y_{t+T_n}^n$, for $t \in \mathbb{Z}$. Note that for $n \in \mathbb{N}$, x^n is an orbit and $\|x_0^n - \xi^*\| \geq \varepsilon$. Taking a subsequence if necessary, we may assume that for $t \in \mathbb{Z}$, $x_t^n \to \overline{x_t} \in K_{p_0}$ as $n \to \infty$. Then we have $\|\overline{x_0} - \xi^*\| \geq \varepsilon$ and thus $\{\overline{x_t}\} \neq \{\xi^*\}$. But this is a contraction since x^* is the unique orbit such that $x_t^* \in K_{p_0}$, for all $t \in \mathbb{Z}$. ∎

Lemma 16 *The constant sequence$\{...,\xi_{p_0},\xi_{p_0},...\}$ is the orbit of G_0, by theorem 11 there exists a corresponding constant sequence $\{...,\xi^*,\xi^*,...\}$ is the orbit of G_α, where $\xi_{p_0} \in K_{p_0}$. Then the constant sequence $\{...,\xi^*,\xi^*,...\}$ which is the unique orbit $\{x_t\}$ of G_α such that $x_t \in K_{p_0}$, for all $t \in \mathbb{Z}$.*

Proof. Suppose there is another orbit x with $\|x-\xi^*\| = \sup\limits_i |x_i - \xi^*| > 0$. Then by mean value theorem, for all $t \in \mathbb{Z}$,

$$\begin{aligned} x_t - \xi^* &= g_p(x_{t-1}, x_{t+1}) - g_p(\xi^*, \xi^*) \\ &= \int_0^1 Dg(\gamma(x_{t-1}, x_{t+1}))(x_{t-1} - \xi^*, x_{t-1} - \xi^*) d\gamma. \end{aligned} \tag{12}$$

Claim 17 *Let* $t \in \mathbb{Z}$:$(x_{t-1}-\xi^*$,$x_{t-1}-\xi^*) \neq 0$, $\|x_t - \xi^*\| \leq \lambda \|(x_{t-1} - \xi^*, x_{t-1} - \xi^*)\|$, *where* $\lambda = \sup\limits_{\zeta \in (K_{p_0}) \times (K_{p_0}) \backslash \{\xi^*\}} \left\| Dg_p(\zeta) \frac{\zeta - \xi^*}{\|\zeta - \xi^*\|} \right\| < 1$.

■

Proof. Let $t \in \mathbb{Z}$ and suppose $(x_{t-1}-\xi^*$,$x_{t-1}-\xi^*) \neq 0$. Since $\max\limits_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1}, x_1)\| < 1$(and Zeidler[3])

$$\begin{aligned} \|x - \xi^*\| &\leq \int_0^1 Dg(\gamma(x_{t-1}, x_{t+1}))(x_{t-1} - \xi^*, x_{t-1} - \xi^*) d\gamma \\ &= \int_0^1 \|Dg(\gamma(x_{t-1}, x_{t+1}))(x_{t-1} - \xi^*, x_{t-1} - \xi^*)\| \, d\gamma \, \|(x_{t-1} - \xi^*, x_{t-1} - \xi^*)\| \\ &\leq \lambda \|(x_{t-1} - \xi^*, x_{t-1} - \xi^*)\|, \end{aligned}$$

where the last inequality holds since $\max\limits_{(x_{-1},x_1)\in(K_{p_0}\times K_{p_0})} \|Dg_p(x_{-1}, x_1)\| < 1$.

Note that if $(x_{t-1} - \xi^*$,$x_{t-1} - \xi^*) = 0$, then $x_t = \xi^*$(since $g_p(\xi^*$,$\xi^*) = \xi^*$). Hence,

$$\begin{aligned} \|x - \xi^*\| &= \sup_{t:\|(x_{t-1}-\xi^*,x_{t-1}-\xi^*)\|>0} \|x_t - \xi^*\| \\ &\leq \sup_{t:\|(x_{t-1}-\xi^*,x_{t-1}-\xi^*)\|>0} \lambda \|(x_{t-1} - \xi^*, x_{t-1} - \xi^*)\| \leq \lambda \|x - \xi^*\|, \end{aligned}$$

where the first inequality holds by the claim above. But this is a contraction since $\lambda < 1$. ■

Proof of Theorem. As we illustrate in the proof of Proposition 3, let w, $w' \in$ (0,1) with $w \neq w'$. Let $\overline{w} = \min\{w$,$w'\}$ and $m \in \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $u_n = [\frac{10^n \overline{w}}{2}]$. We have$\left\|(x^{\overline{w}}_{u_n-m},...,x^{\overline{w}}_{u_n+m})-(\xi_1,...,\xi_1)\right\|=0$ for n is large enough thus

$$\liminf_{t\to\infty}\left\|(x^{w}_{t-m},...,x^{w}_{t+m})-(x^{w'}_{t-m},..,x^{w'}_{t+m})\right\|=0.$$

For this x^w and $x^{w'}$, by theorem 11, theres exists the corresponding orbit of G_α called x^* and $x^{'*}$ respectively. By the above lemma, we have $\left\|(x^*_{u_n-m},...,x^*_{u_n-m})-(\xi^*,...,\xi^*)\right\|\to 0$ as $n\to\infty$ and $\left\|(x^{'*}_{u_n-m},...,x^{'*}_{u_n-m})-(\xi^*,...,\xi^*)\right\|\to 0$ as $n\to\infty$. So we have

$$\liminf_{t\to\infty}\left\|(x^{*}_{t-m},...,x^{*}_{t+m})-(x^{'*}_{t-m},..,x^{'*}_{t+m})\right\|=0.$$

■

If G_0 is chaotic , i.e., G_0 satisfies (T1) (T2) below.

(T1) For all $n\in\mathbb{N}$, G_0 has a n-periodic orbit.

(T2) G_0 has an uncountable set χ of asymptotically nonperiodic orbit such that for all x,$y\in\chi:(x\neq y)$

$$\limsup_{t\to\infty}\|x_t-y_t\|>0, \tag{13}$$

$$\liminf_{t\to\infty}\|(x_{t-n},...,x_{t+n})-(y_{t-n},....,y_{t+n})\|=0,\ \text{for all } n\in\mathbb{N}\cup\{0\}. \tag{14}$$

If α satisfies (10), by theorem 11and theorem14 there exists the corresponding orbit and it also satisfies (T1) and (T2), that is, G_α is chaotic.

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