# 國 立 交 通 大 學

應用數學系

碩 士 論 文

# 利用快速傅立葉轉換增加蒙地卡羅方法評價衍生性金融商品的效率

# Efficient Valuation of Financial Derivatives by Monte Carlo Simulations with Fast Fourier Transform

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## 摘　要

在本篇文章中，我們提出了使用快速傅立葉演算法來加速蒙地卡羅評價選擇權的方法。我們以快速傅立葉演算法計算選擇權的Delta值，並且利用這些Delta值去建立馬丁格爾控制變異數項來降低估計值的變異數。我們發現結合快速傅立葉演算法與馬丁格爾控制變異數方法在增加運算效率方面是非常有用的，並且也保留了運算的正確性。同時我們也討論了利用快速傅立葉演算法的誤差分析。

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## ABSTRACT

In this paper, we proposed the use of Fast Fourier transform (FFT) method to accelerate Monte Carlo simulations in option pricing. The method of FFT is applied to compute the Deltas of the options. These Deltas are essential in construct martingale control for variance reduction. We find that the combination of the FFT method with the martingale control variate method is very useful to reduce the computational time while preserving the accuracy of simulations. The error analysis of using FFT method is also discussed.

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# 目　　　錄

# 1 Introduction

The method of Monte Carlo simulations is a very popular technique which is applied in many scholastic fields, such as physics, engineering, statistics, finance, and so on. This method is based on the analogy between probability and volume. The measure theory formalizes the intuitive notion of probability of the event to be its volume or measure relatives to that of a universe of possible outcomes. Monte Carlo uses this identity in reverse, to calculate the volume of a set by interpreting it as a probability. For example, we can randomly sample from a universe of possible outcomes and take the fraction of random draws that fall in a given set as an estimate of the set's volume. According to the law of large numbers, this estimate converges to the correct value as the number of draws increases. The advantage of Monte Carlo simulations is that it is no or less sensitive to dimensionality of the underlying problem and suitable for parallel computations. However the main disadvantage of this method is that the rate of convergence is slow because it is limited by the central limit theorem. It is relatively slow compared to deterministic schemes for low dimensional problems.

To improve the efficiency of Monte Carlo methods, there are two main possible approaches : Quasi Monte Carlo simulation (QMC) and variance reduction technique. Quasi Monte Carlo simulations are also called low-discrepancy methods. The main difference between QMC and the Monte

Carlo method is that QMC makes no attempt to mimic the underlying randomness. Indeed, it seeks to increase the accuracy specifically by generating points evenly to obtain the randomness. QMC forms a class of methods where low-discrepancy numbers are generated in a deterministic way while basic Monte Carlo uses pseudo-random numbers. Variance reduction method exploits information about the errors to reduce the errors in estimates of unknown variables. On the other hand, this method seeks probabilistic ways to reformulate the undertaken problem in order to gain significant variance reduction. For example, control variate methods take into account the correlation properties of random variables, but the efficiency of these techniques is often restricted to certain undertaken problems.

In financial applications such as pricing derivatives, taking the control as a (local) martingale is a very useful method. This method is called "martingale control variate method." The martingale control variate method can be well understood in finance terminology. The constructed control variate corresponds to a continuous Delta hedge strategy taken by a trader who sells an option. So this method is also known as "hedging martingale variance control method". Fouque and Han [7] apply this method to price European option, American option, and Barrier option(down and out) in stochastic volatility models. Also they show the variance analysis of this method. However, the weakness of this method is that it takes time to compute the parameter values for each path. If we want to estimate an option price by Monte Carlo simulations, we will construct many simulated paths. The martingale control is a stochastic integral consisting of a partial derivative, known as Delta. We need to compute Delta at each simulated time step. Section 2 will describe the martingale control variate method in detail. The purpose of this paper is to apply the fast Fourier transform (FFT) methodology to reduce the computing time in Monte Carlo simulation for option pricing.

FFT option pricing method is first developed by Carr and Madan [14]. They use Fourier transform to change the option pricing problem from the real domain to the complex domain. This Fourier transform can be represented by the characteristic function of the natural logarithm of the underlying at the expiration date. The reason for using characteristic function is that under some models or processes, the characteristic functions are easier to compute. For example, under Levy processes, we can have general forms of characteristic functions. See, Bertoin [6] in detail. We can take inverse Fourier transform to get the option price. FFT is used to approximate this inverse Fourier transform. In other words, this method requires only the characteristic function of the natural logarithm of the underlying at maturity. Borak, Detlefsen, and Härdle [21] use this method to price call option under Heston model and Bates model. Itkin [2] applies this in variance gamma (VG) process. Lee [15] offers the error bound of this method. The restriction of this method is it is only suitable for pricing European option. Based on this method, we estimate the Delta in the related Black-Scholes model. Moreover, we give the error analysis of this estimation.

The rest of this paper is arranged as the following. In section 2, we review the martingale control variate method, and discuss its computational issue of this method. Section 3 introduces the FFT option pricing method and discusses the models which can apply this method. In section 4, we apply FFT option pricing method to compute the Delta of the call option in geometric Brownian motion (GBM) environment using martingale control variate method, and illustrate numerical results in figures. Moreover, the error bound of this method is computed. Finally, we conclude in section 5.

# 2 Hedging Martingale Control with Variance Reduction

## 2.1 Review of Martingale Control Variate Method

Under the risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t<\infty}, P^*)$, we consider the risky underlying asset $S_t$ which is governed by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t^*, \tag{1}$$

where $r$ is a risk-free rate and $\sigma$ is the volatility. Both $r$ and $\sigma$ are constants. $W_t^*$ is the Brownian motion under risk-neutral probability. There are two corollaries of this model.

**Corollary 1** *The logarithm of the underlying asset $S_t$ follows the normal distribution with mean $(r-\frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. i.e.*

$$\log S_t \sim N((r-\frac{1}{2}\sigma^2)t, \sigma^2 t) \tag{2}$$

**Corollary 2** *The closed form solution of this model is*

$$S_T = S_t \exp((r-\frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}^*), \tag{3}$$

*where $S_t$ is the price of the underlying asset at time $t$.*

Given this model, the fair price of a European-style derivative with maturity $T<\infty$, denoted by $P$, is simply a conditional expectation

$$P(t,x) = E_{t,x}^*[e^{-r(T-t)}H(S_T)|\mathcal{F}_t], \tag{4}$$

where $E^*_{t,x}$ denotes the expectation with respect to $P^*$ conditioned on the current states $S_t = x$, $H(x)$ the payoff function satisfying the integrability condition. For example, $H(x) = \max\{x - K, 0\}$ for strike price $K > 0$, it is a call payoff. A'financial contract with the call or put payoff is called a European call option or a European put option respectively. From the simulation point of view, it is straightforward to construct the basic Monte Carlo estimator of the option price $P(0, S_0)$ at time 0 by

$$\frac{1}{Q}\sum_{i=1}^{Q} e^{-rT} H(S_T^{(i)}), \tag{5}$$

where $Q$ is the total number of independent sample paths and $S_T^{(i)}$ denotes the $i$-th independent replication of the underlying asset price at time $T$.

Assuming that the European option price $P(t, x)$ is smooth enough, we apply Ito's lemma to its discounted price $e^{-rt}P$, and then integrate from time 0 to the maturity $T$. The following martingale representation is obtained

$$P(t, x) = e^{-rT} H(S_T) - M_0(P; T) \tag{6}$$

where centered martingale is defined by

$$M_0(P; T) = \int_o^T e^{-rs} \frac{\partial P}{\partial x}(s, S_s)\sigma S_s dW_s^*. \tag{7}$$

**Remark 3** *$M_0(P)$ is a martingale and it has mean zero.*

This martingale plays the role of "perfect" control for Monte Carlo simulations and the integrand consists of the perfect Delta hedge if the partial derivative $\frac{\partial P}{\partial x}(t, x)$ is known so that the option price $P(t, x)$ would be known in advance. In reality, $P(t, x)$ is not known. Therefore, equation (6) is not feasible for a direct computation for the option price. Nevertheless by employing a martingale as a control we can formula the unbiased control variate estimator

$$\frac{1}{Q}\sum_{i=1}^{Q} [e^{-rT} H(S_T^{(i)}) - M_0^{(i)}(P_{BS}; T)] \tag{8}$$

for the option price $P_0 = E^*[e^{-rT}H(S_T) - M_0(P_{BS};T)|\mathcal{F}_0]$ where the martingale control $M_0(P_{BS};T)$ consists of the price approximation $P_{BS}$ of the actual option price $P$. That is

$$M_0(P_{BS};T) = \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW_s^*, \tag{9}$$

where $P_{BS}$ is the solution of Black-Scholes partial differential equation with the terminal condition $P_{BS}(T,x) = H(x)$. In financial interpretation $M_0(P_{BS};T)$ represents the Delta hedging portfolio accumulated up to time T, so the term $M_0(P_{BS};T)$ is called the hedging martingale be the price $P_{BS}$ so that the estimator defined by (9) is called the martingale control variate estimator. Apply Ito's isometry, the variance of the controlled payoff $P_0$ is simply the sum of quadratic variations of martingale :

$$\begin{aligned} & Var(e^{-rT}H(S_T) - M_0(P_{BS};T)) \qquad (10) \\ = \;& E_{0,x}^*\{\int_o^T e^{-2rs}(\frac{\partial P}{\partial x}(s,S_s) - \frac{\partial P_{BS}}{\partial x}(s,S_s))^2\sigma^2 S_s^2 ds.\}. \end{aligned}$$

Therefore, if the Delta trading $\frac{\partial P_{BS}}{\partial x}(t,x)$ is closed to the actual hedging strategy, the variance of the martingale control estimator should be small.

Now, we introduce the algorithm to estimate the martingale control.

**Step1.** Simulate the underlying asset's paths in order to obtain the terminal prices of the asset. Compute the sample paths of $H(S_T)$ and its discounted value $e^{-rT}\ H(S_T)$.

**Step2.** Discretize the martingale control of the $e^{-rT}\ H(S_T)$. Use the lower Riemann sum to approximate the integral (9) . i.e.

$$\begin{aligned} & M_0(P_{BS};T) = \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW_s^* \qquad (11) \\ \approx \;& \sum_{j=1}^{M} e^{-r\frac{T}{M}(j-1)}\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S_{\frac{T}{M}(j-1)})\sigma S_{\frac{T}{M}(j-1)}\sqrt{\frac{T}{M}}\varepsilon_j, \end{aligned}$$

where $\{0, \frac{T}{M}, \frac{T}{M}*2, ..., \frac{T}{M}*(M-1)\}$ is the partition of the interval $[0,T]$ , $K$ is the strike price of option, and $\varepsilon_j$ are identical and independent (I.I.D) standard normal random variables.

**Step3.** An estimator of the martingale control variate for the option price is the following,

$$E^*_{0,x}[e^{-rT}H(S_T) - \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW^*_s] \quad (12)$$

$$\approx \frac{1}{Q}(\sum_{l=1}^{Q} e^{-rT}H(S_T^{(l)}) - \sum_{l=1}^{Q}\sum_{j=1}^{M} e^{-r\frac{T}{M}(j-1)}\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S^{(l)}_{\frac{T}{M}(j-1)})\sigma S^{(l)}_{\frac{T}{M}(j-1)}\sqrt{\frac{T}{M}}\varepsilon_j,$$

where $Q$ is the number of sample paths.

## 2.2 Examples

We take two examples in European call option to observe the efficiency of the martingale control variance method. The payoff function $H(x) = (x-K)^+$, where $K$ is the strike price. We suppose the risk-free rate $r = 0.1$ , the volatility $\sigma = 0.25$ , and the maturity $T = 1$. The current time is assumed 0, and the initial underlying asset price $S_0$ is 100. In the first example, we take the strike price $K$ as 80 to fit the case of in-the-money. The other example, we take $K = 120$ which is in the out-of-the money environment. Let the number of sample paths $Q = 10000$ and the partitions of time interval $[0,T]$ $= 100$. We compare the standard error (SE) of Monte Carlo simulations with and without martingale control. We also show the CPU time spending in these two conditions. The variance reduction ratio is also represented.

(1) $K = 80$

| | Call Price | Standard Error | Time(Seconds) |
|---|---|---|---|
| Using martingale control variate | 28.581 | 0.0107 | 63.5897 |
| Without using martingale control variate | 28.683 | 0.2416 | 0.4594 |

The variance reduction ratio =590.83

(2) $K = 120$

| | Call Price | Standard Error | Time(Seconds) |
|---|---|---|---|
| Using martingale control variate | 6.638 | 0.0195 | 65.7116 |
| Without using martingale control variate | 6.589 | 0.1407 | 0.4317 |

The variance reduction ratio =52.062

From numerically results, we can observe that when we use martingale control variate to reduce variance, the standard error is diminished so much. As we know, the convergence rate of Monte Carlo simulations is governed by $1/\sqrt{Q}$, where $Q$ is the number of sample paths. In the first example, if we want to reduce the standard error from the method of without using martingale control variate to that of using martingale control variate, the sample paths should be increased from 10000 to 6250000 approximately. This is the power of the martingale control variate method. But there comes a disadvantage of this method, it spends a lot of time. The time using martingale control variate method is much greater that it without using this method. Observe the martingale control in equation (9), we can find that the most part of time spending in martingale control variate method is to compute the term $\frac{\partial P_{BS}}{\partial x}(t,x)$, which is call the Delta of the option. We would estimate Delta values $\frac{\partial P_{BS}}{\partial x}(t,x)$ in every pairs $(t, S_t)$, where $t \in \{0, \frac{T}{M}, \frac{T}{M}*2, ..., \frac{T}{M}*(M-1)\}$. For this reason, we want to search some methods to increase the efficiency in computing martingale control. We find that take advantage of FFT option pricing method to compute Delta may be a feasible way. Next, we introduce the FFT option pricing method and explain how to use this method to make the martingale control variate method more efficiently.

# 3 Apply FFT Method to Price Option

In this section, we will introduce how to apply Fast Fourier transform (FFT) method to price the call option. The approach has been addressed by Carr and Madan [14]. The big attraction of this method is the Fast Fourier transform (FFT) could be used to make computation more efficient. This efficiency is even boosted by the possibility of the pricing algorithm to calculate prices for a whole range of strikes. The other advantage for this method is that if we know the characteristic function of nature logarithm of underlying asset price at maturity, this method can be applied directly. The characteristic function often has a simple form for many models while the probability density functions of the log price is often not get in the closed form.

## 3.1 Introduction of FFT

FFT is first developed by Cooley and Tukey [8]. It is an efficient algorithm for computing the summation

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j) \qquad \text{for } k = 1, \ldots, N, \tag{13}$$

where N is typically a power of 2. The power of FFT is that the method can compute the element of the sequence $\{w(1), w(2), ..., w(N)\}$ rapidly. The algorithm reduces the number of multiplications in the require $N$ summations from an order of $N^2$ to that of $N \log_2 N$, a very considerable reduction. We go into details this algorithm in appendix B.

## 3.2 Fourier Transform of Option Price

We define some notations and these notations are used around this section. Let $C_T(k)$ be the discounted value of the call option with maturity $T$ at the current time 0. $k$ is the natural logarithm of the strike price $K$ of the option. And we use $S_t$ to represent the underlying asset price at time $t$. The initial price of asset is denoted by $S_0$. In this method, we usually suppose $S_0 = 1$ for convenience. We also define $X_T = \log S_T$.

**Definition 4** *The characteristic function of the natural logarithm of the price of the underlying asset , $X_T$, is defined by*

$$\phi_T(u) \triangleq E^*[\exp(iuX_T)] = \int_{-\infty}^{\infty} e^{ius} q_T(s) ds \tag{14}$$

*where $q_T(s)$ is the probability density function of $X_T$ under the risk-neutral world.*

**Definition 5** *Define $A_{X_T}$ as the set*

$$A_{X_T} \triangleq \{v \in R^n : E^*[e^{u \cdot X_T}] < \infty\}, \tag{15}$$

*where $\cdot$ is the inner product.*

**Definition 6** *Define $\Lambda_{X_T}$ as the set*

$$\Lambda_{X_T} \triangleq \{\zeta \in C^n : -Im(\zeta) \in A_{X_T}\}, \tag{16}$$

*the complex vectors whose negated imaginary parts are in $\Lambda_{X_T}$ form a "strip"or "tube".*

**Lemma 7** *The characteristic function $\phi_T$ is well-defined and analytic (infinitely differentiable) in $\Lambda_X$, which is a convex set. Partial derivative of $\phi_T$ may be taken through the expectation.*

Define the initial (discounted) call value $C_T(k)$ is related to the risk-neutral density $q_T(s)$ by:

$$
\begin{aligned}
C_{0,T}(k;S_0) &\triangleq E^*[e^{-rT}(\max(e^{X_T}-e^k,0))] \\
&= \int_{-\infty}^{\infty} e^{-rT}\max(e^s-e^k,0)q_T(s)ds \\
&= \int_{k}^{\infty} e^{-rT}(e^s-e^k)q_T(s)ds.
\end{aligned}
$$

Here, simply, we suppose the risk-free rate, $r$ is a constant.

**Theorem 8** *For any $p>0$,*

$$
C_{0,T}(k;S_0) \le \frac{e^{-rT}E^*[\exp(p+1)X_T]}{(p+1)\exp(pk)}(\frac{p}{p+1})^p \quad \textit{and} \quad C_{0,T}(k;S_0) \le e^{-rT}E^*[\exp X_T] \tag{17}
$$

**Proof.** For all $s \ge 0$ we have

$$
s-e^k \le \frac{s^{p+1}}{(p+1)\exp(pk)}(\frac{p}{p+1})^p,
$$

because the left-hand and right-hand sides, as function of s, have equal values and first derivatives at $s=(p+1)exp(k)/p$, and the second derivative of the right-hand side is always positive. Moreover, since the right-hand side is positive, the left side can improve to $(s-\exp(k))^+$.Now, substitute $s = exp(X_T)$, take expectations, and discount both sides to obtain the first bound. The second bound is obvious. ■

**Definition 9** *The Fourier transform of the function $f$ on $\mathbb{R}$ is defined by*

$$
\varphi(v) = \int_{\mathbb{R}} f(x)e^{ivx}dv \tag{18}
$$

*and its attached inversion is given by*

$$
f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(v)e^{-ivx}dv. \tag{19}
$$

**Lemma 10** *The Fourier transform of the function $f$ on $\mathbb{R}$ exists if $\|f\|_1$ is finite or $f \in L^1(\mathbb{R})$. i.e.*

$$\int_{\mathbb{R}} |f(x)|\, dx < \infty. \tag{20}$$

**Lemma 11** *If $f \in L^1(\mathbb{R})$, then $\lim_{x\to\pm\infty} f(x) = 0$.*

But, we know that when $k = logK$ tends to negative infinity, in other words, the strike price $K$ tends to zero, the option is deeply in the money, and the discounted option price tends to the initial underlying price. That is

$$\lim_{k\to-\infty} C_{0,T}(k; S_0) = S_0 \neq 0. \tag{21}$$

By the lemma 11 , we know that the Fourier transform of the discounted call option price does not exist. To make $C_{0,T}(k; S_0)$ to be an absolutely integral function, we add a parameter $\alpha$ into $C_{0,T}(k; S_0)$, the parameter is usually called the "damping parameter". Consider the modified call price defined by

$$c_{0,T}(k; S_0) \triangleq \exp(\alpha k) C_T(k) \tag{22}$$

for $\alpha > 0$. [The reason for $\alpha > 0$ is that we want $c_{0,T}(k; S_0)$ tends to zero as $k$ tends to negative infinity. For a range of positive value of $\alpha$, we expect that $c_{0,T}(k; S_0)$ is integrable in $k$ over the entire real line. How to choose the value of $\alpha$ will be discussed later. Consider the Fourier transform of $c_{0,T}(k; S_0)$,

$$\Psi_T^{X(T)}(v) = \int_{\mathbb{R}} e^{ivk} c_{0,T}(k; S_0) dk. \tag{23}$$

**Lemma 12** *We develop an analytical expression for $\Psi_T^{X(T)}(v)$ in terms of $\phi_T(v)$. i.e.*

$$\Psi_T^{X(T)}(v) = \frac{e^{-rT}\phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}. \tag{24}$$

**Proof.**

$$\begin{aligned}
\Psi_T^{X(T)}(v) &= \int_R c_{0,T}(k;S_0)e^{ivk}dk \\
&= \int_{-\infty}^{\infty} e^{ivk}\int_k^{\infty} e^{\alpha k}e^{-rT}(e^s-e^k)q_T(s)dsdk \\
&= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s} e^{\alpha k}(e^s-e^k)e^{ivk}dkds \\
&= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_{-\infty}^{s} \left(e^{s+\alpha k+ivk}-e^{k+\alpha k+ivk}\right)dkds \\
&= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\left[\frac{e^{s+\alpha k+ivk}}{\alpha+iv}-\frac{e^{k+\alpha k+ivk}}{\alpha+1+iv}\right]_{-\infty}^{s} ds \\
&= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\left[\frac{e^{s+\alpha s+ivs}}{\alpha+iv}-\frac{e^{s+\alpha s+ivs}}{\alpha+1+iv}\right] ds \\
&= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\frac{e^{s+\alpha s+ivs}}{(\alpha+iv)(\alpha+1+iv)}ds \\
&= \frac{e^{-rT}}{(\alpha+iv)(\alpha+1+iv)}\int_{-\infty}^{\infty} q_T(s)e^{s+\alpha s+ivs}ds \\
&= \frac{e^{-rT}}{(\alpha+iv)(\alpha+1+iv)}\int_{-\infty}^{\infty} q_T(s)e^{i(v-((\alpha+1)i))s}ds \\
&= \frac{e^{-rT}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}
\end{aligned}$$

■

Then, we take the inverse Fourier transform of $\Psi_T(v)$ and undamp it to get the call price,

$$C_{0,T}(k;S_0) = e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_T^{X(T)}(v)e^{-ivk}dv. \tag{25}$$

**Lemma 13** *The call option price can be simplified to the following form*

$$C_{0,T}(k;S_0) = e^{-\alpha k}\frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}(\Psi_T^{X(T)}(v)e^{-ivk})dv. \tag{26}$$

**Proof.** See appendix A. ■

We note that the integration (26) is a direct Fourier transform and lends itself to an application of the FFT. Also note that in the denominator of (24) vanishes when $v = 0$, this is another reason for using damping parameter or something similar is required. Positive value of $\alpha$ assist the integrability of the modified call value (22) over the negative nature logarithm of strike price axis, but aggravate the same condition for positive nature logarithm of strike price axis. For the modified call value $c_{0,T}(k; S_0)$ to be integrable in the positive nature logarithm of strike price direction, a sufficient condition is provided by $\Psi_T^{X(T)}(0)$ being finite. From (24), we observe that $\phi_T(-(\alpha+1)i)$ should be finite. It means that

$$(\alpha + 1) \in \Lambda_{X_T} \tag{27}$$

is a sufficient condition. Carr and Madan [14] proposed that one fourth of the upper bound which satisfying the condition (27) serves as good choice for $\alpha$. Schoutens [17] found that 0.75 is a good choice for $\alpha$ and led to stable algorithms.

## 3.3 Evaluation of Option Price by FFT Method

The remainder work is to estimate the integral (26) numerically. Using the Trapezoid rule for the integral on the right-hand side of (26) and setting $v_j = \eta(j-1)$, an approximation for $C_T(k)$ is:

$$C_{0,T}(k; S_0) \approx \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(e^{-iv_j k}\Psi_T(v)\eta). \tag{28}$$

The effective upper limit for the integration is now

$$U = N\eta. \tag{29}$$

Here, we called $U$ as the truncated upper bound and $\eta$ as discretization size of out FFT method. We are mainly interested in at-the-money call option value $C_T(k)$, which correspond to $k$ near 0. The FFT returns $N$ values of $k$ and we employ a regular spacing of size $\lambda$, so that our values for $k$ are

$$k_u = -b + \lambda(u-1), \quad \text{for } u = 1,\ldots,N. \tag{30}$$

This gives us log strike levels ranging from $-b$ to $b$ where

$$b = \frac{N\lambda}{2}. \tag{31}$$

On the other hand, the sequence of strike price $K$ correspond to $k_u$ is

$$K = \{\exp(-b), \exp(-b+\lambda), ..., \exp(-b+\lambda(N-1))\}. \tag{32}$$

Substituting (30) into (28) yields:

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(e^{-iv_j(-b+\lambda(u-1))}\Psi_T(v_j)\eta), \quad \text{for } u = 1,\ldots,N. \tag{33}$$

Noting that $v_j = \eta(j-1)$ and after arranging the summation, we get

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(e^{-i\lambda\eta(j-1)(u-1)}e^{ibv_j}\Psi_T(v_j)\eta). \tag{34}$$

Taking $\lambda\eta = \frac{2\pi}{N}$, then the summation (34) becomes that

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(e^{-i\frac{2\pi}{N}(j-1)(u-1)}e^{ibv_j}\Psi_T(v_j)\eta). \tag{35}$$

Then the equation (35) fits the FFT form (13). FFT method can be applied to compute the summation (35). Note that if we want to use FFT option pricing to evaluate call option price, the condition is we should know $\Psi_T(v)$. By (24), $\Psi_T(v)$ can be represented as the function of characteristic function of nature logarithm of underlying asset price at maturity, $\phi_T(u)$. That is, knowing $\phi_T(u)$ is the only condition to use the FFT option pricing method. This is very powerful. In many models, $\phi_T(u)$ can be computed easily. Next , we offer characteristic functions, $\phi_T(u)$, in some models. And these models can directly apply FFT option pricing method to pricing European derivatives.

## 3.4 Examples

(1) Merton model:

The price of the underlying asset follows the dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t + dZ_t, \tag{36}$$

where $Z_t$ is a compound Poisson process with a log-normal distribution of jump sizes. The jumps follow a Poisson process $N_t$ with intensity $\lambda$ which is independent of $W_t$. The log-jump sizes $Y_i \sim N(\mu, \delta^2)$ are i.i.d random variables with mean $\mu$ and variance $\delta^2$, which are independent of both $N_t$ and $W_t$. The dynamics of asset price is then given by:

$$S_t = S_0 \exp(\mu^M t + \sigma W_t + \sum_{i=1}^{N_t} Y_i), \tag{37}$$

where $\mu^M = r - \sigma^2 - \lambda(\exp(\mu + \frac{1}{2}\delta^2) - 1)$. The characteristic function of $X_T = \log S_T$ is

$$\phi_T(u) = \exp[T(-\frac{\sigma^2 u^2}{2}) + i\mu^M u + \lambda(\exp(-\frac{\delta^2 u^2}{2} + i\mu u - 1)]. \tag{38}$$

(2) Heston model:

The price of the underlying asset follows the dynamics

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t} dW_t^{(1)} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}, \end{aligned} \tag{39}$$

where $v_t$ is another unobservable stochastic process and follows the square root process. So, this model is a type of "stochastic volatility model". And the two Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$ are correlated with rate $\rho$. i.e.

$$Cov(dW_t^{(1)}, dW_t^{(2)}) = \rho dt. \tag{40}$$

Parameter $\delta$ measures the speed of mean reversion, $\theta$ is the average level of volatility and $\sigma$ is the volatility of volatility. In (40) the correlation $\rho$ is typically negative, which is known as the "leverage effect".For the natural logarithm price of the underlying asset $X_t = \log S_t$, one obtains the equation:

$$dX_t = (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t^{(1)}. \tag{41}$$

The characteristic function of $X_T = \log S_T$ is

$$\phi_T(u) = \frac{\exp(\frac{\kappa\theta T(\kappa - i\rho\sigma u)}{\sigma^2} + iuTr + iux_0)}{(\cosh\frac{\gamma T}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma T}{2})^{\frac{2\kappa\theta}{\sigma^2}}} * \exp(-\frac{(u^2 + iu)v_0}{\gamma\coth\frac{\gamma T}{2} + \kappa - i\rho\sigma u}), \tag{42}$$

where $\gamma = \sqrt{\sigma^2(u^2 + iu) + (\kappa - i\rho\sigma u)^2}$, and $x_0$ and $v_0$ are the initial values for the log-price process and volatility process, respectively.

(4) Bates model:

The price of the underlying asset follows the dynamics

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t}dW_t^{(1)} + dZ_t \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^{(2)} \end{aligned} \tag{43}$$

$$Cov(dW_t^{(1)}, dW_t^{(2)}) = \rho dt. \tag{44}$$

As in (43) $Z_t$ is a compound Poisson process with intensity $\lambda$ and log-normal distribution of jump sizes independent of $W_t^{(1)}$ and $W_t^{(2)}$. If $J$ denotes the jump size then $\log(1 + J) \sim N(\log(1 + \varsigma) - \frac{1}{2}\delta^2, \delta^2)$ for some $\varsigma$. Under the risk neutral probability one obtains the equation for the logarithm of the asset price:

$$dX_t = (r - \lambda\varsigma - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t^{(1)} + \tilde{Z}_t, \tag{45}$$

where $\tilde{Z}_t$ is a compound Poisson process with normal distribution of jump magnitudes. Since the jumps are independent of the diffusion part in (36), the characteristic function of $X_T = \log S_T$ is

$$\phi_T(u) = \phi_T^D(u)\phi_T^J(u), \tag{46}$$

where

$$\phi_T^D(u) = \frac{\exp(\frac{\kappa\theta T(\kappa - i\rho\sigma u)}{\sigma^2} + iuT(r - \lambda\varsigma) + iux_0)}{(\cosh\frac{\gamma T}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma T}{2})^{\frac{2\kappa\theta}{\sigma^2}}} * \exp(-\frac{(u^2 + iu)v_0}{\gamma\coth\frac{\gamma T}{2} + \kappa - i\rho\sigma u}) \tag{47}$$

is the diffusion part characteristic function and

$$\phi_T^J(u) = \exp(T\lambda(\exp(-\frac{\delta^2 u^2}{2} + i(\ln(1 + \varsigma) - \frac{1}{2}\delta^2)u) - 1), \tag{48}$$

is the jump part characteristic function.

(5) Variance Gamma (VG) process:

The VG process is obtained by evaluating arithmetic Brownian motion with drift $\theta$ and volatility $\sigma$ at a random time given by a gamma process having a mean rate per unit time of 1 and the variance rate of $\nu$. The resulting process $X_t(\sigma, \theta, v)$ is a pure jump process with two additional parameters $\theta$ and $v$ relative to the Black Scholes model, providing control over skewness an kurtosis respectively.See [3] in detail. The underlying asset follows the process

$$S_t = S_0 \exp(rt + X_t(\sigma, \theta, v) + \omega t) \quad t > 0, \tag{49}$$

where by setting $\omega = (1/v)\log(1 - \theta v - \sigma^2 v/2)$, the mean rate of return on the asset equals the interest rate $r$. The characteristic function of $X_T = \log S_T$ is

$$\phi_T(u) = \exp(\log(S_0 + (r + \omega)T)(1 - i\theta vu + \sigma^2 u^2 v/2))^{-T/v}. \tag{50}$$

The VG process is hard to using Monte Carlo Simulation to pricing option when underlying asset follows this process. But in FFT option pricing method, since we know the characteristic function of $\log S_T$, we can apply this method to pricing call option. This is the advantage of this method. Next, we will apply this method to estimate the Delta of the call option.

# 4 Delta Estimation Using FFT Option Pricing Method

## 4.1 Introduction

In this subsection, we discuss how to use the FFT pricing option method to compute the Delta of the call option. We also suppose that the price of underlying asset follows the geometric Brownian motion which is described in section 2. We use the notation $\Delta_t(k)$ to represent the Delta of the call option at the time $t$. Simply, we let the current time is 0. $k$ is the natural logarithm of the strike price $K$. The definition of delta $\Delta_t(k)$ is

$$\Delta_t(k) = \frac{\partial C_t(t, S_t)}{\partial S_t}, \tag{51}$$

where T is the maturity of the call option. We will deduce it to our wanted form which can apply FFT option pricing method. Note that

$$C_t(t, S_t) = e^{-r(T-t)} E^*[(S_T - K)^+ | S_0] = e^{-r(T-t)} E^*[(S_T - K)^+], \tag{52}$$

where the $E^*$ is the expectation under the risk-neutral probability $P^*$. Recall that the closed form solution of $S_T$ under geometric Brownian motion is

$$S_T = S_t \exp((r - \frac{1}{2}\sigma^2)(T - t) + \sigma W^*_{T-t}), \tag{53}$$

$W^*_{T-t}$ is the Brownian motion under risk-neutral probability with mean 0 and variance $T-t$. Then substitute (3) and (2) into (1), we obtain that

$$\begin{aligned}\Delta_t(k) &= \frac{\partial e^{-r(T-t)}E^*[(S_t\exp((r-\frac{1}{2}\sigma^2)T+\sigma W^*_{T-t})-K)^+]}{\partial S_t} &(54)\\ &= e^{-r(T-t)}E^*[\frac{\partial(S_t\exp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma W^*_{T-t})-K)^+}{\partial S_t}] &(1)\\ &= e^{-r(T-t)}E^*[I_{\{S_T\geq K\}}\exp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma W^*_{T-t})], &(2)\end{aligned}$$

where $I_{\{S_T\geq K\}}$ is the indicator function. i.e.

$$I_{\{S_T\geq K\}}=\begin{cases}1, & \text{if } S_T\geq K\\ 0, & \text{otherwise.}\end{cases}.$$

In order to simplify $\Delta_t(k)$, we use the following lemma, called Girsanov's theorem.

**Lemma 14** *Let $W(t)$, $0\leq t\leq T$, be a Brownian motion on a probability space $(\Omega,F,P)$, and let $F(t)$, $0\leq t\leq T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0\leq t\leq T$ , be an adapted process. Define*

$$Z(t)=exp\{-\int_0^t\Theta(u)dW(u)-\frac{1}{2}\int_0^t\Theta^2(u)du\},$$

$$\tilde{W}(t)=W(t)+\int_0^t\Theta(u)du,$$

*and assume that*

$$E[\int_0^T\Theta^2(u)Z^2(u)du]<\infty.$$

*Set $Z=Z(T)$. Then $E[Z]=1$ and under the probability measure*

$$\tilde{P}(A)=\int_A Z(w)dP(w)\ for\ all\ A\in F,$$

*the process $\tilde{W}(t)$, $0\leq t\leq T$, is a Brownian motion.*

In our case, we define

$$\tilde{W}_t=W^*_t-\sigma t=W^*_t+\int_0^t-\sigma du \tag{55}$$

and

$$Z(t) = exp\{\int_0^t \sigma dW_u^* - \frac{1}{2}\int_0^t \sigma^2 du\} = \exp((-\frac{1}{2}\sigma^2)t + \sigma W_t^*), \quad (56)$$

then under the new probability measure

$$\tilde{P}(A) = \int_A Z(w)dP(w) \quad (57)$$

$\tilde{W}_t$ is a Brownian motion. Then

$$\begin{aligned} \Delta_t(k) &= e^{-r(T-t)}E^*[I_{\S_T} \geq K\} \exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}^*)] \quad (58) \\ &= \tilde{E}[I_{\{\tilde{S_T} \geq K\}}], \end{aligned}$$

where $\log(\tilde{S_T}) \sim N((r + \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$. Let $\tilde{X}_T = \log(\tilde{S_T})$, then $\Delta_t(k)$ could be rewrote as

$$\Delta_t(k) = \tilde{E}[I_{\{\tilde{X}_T \geq k\}}] \quad (59)$$

Now, we apply the FFT option pricing method to estimate this expectation. Note $\Delta_t(k)$ tends to 1 when $k$ tends to $-\infty$, damping parameter $\alpha$ should be used to let $\Delta_t(k) \in L^1(\mathbb{R})$. Define

$$\tilde{\Delta}_t(k) = \exp(\alpha k)\Delta_t(k). \quad (60)$$

The Fourier transform of $\tilde{\Delta}_t(k)$ is

$$\Psi_T(v) = \int_R e^{ivk}\tilde{\Delta}_t(k)dk. \quad (61)$$

**Lemma 15** *We develop an analytical expression for $\Psi_T(v)$ in terms of the characteristic function of $\tilde{X}_T$, $\phi_T(v)$. i.e.*

$$\Psi_T(v) = \frac{1}{\alpha + iv}\phi_T(v - \alpha i). \quad (62)$$

**Proof.**

$$
\begin{aligned}
\Psi_T(v) &= \int_R \tilde{\Delta}_t(k) e^{ivk} dk \\
&= \int_{-\infty}^{\infty} e^{ivk} \int_k^{\infty} e^{\alpha k} q_T(s) ds dk \\
&= \int_{-\infty}^{\infty} q_T(s) \int_{-\infty}^{s} e^{\alpha k} e^{ivk} dk ds \\
&= \int_{-\infty}^{\infty} \frac{e^{i(v-\alpha i)s}}{\alpha + iv} q_T(s) ds \\
&= \frac{1}{\alpha + iv} \phi_T(v - \alpha i),
\end{aligned}
$$

where $q_T(s)$ is the density function of $\tilde{X}_T$ under the risk-neutral probability.

■

**Lemma 16** *If $Y$ is a random variable whose probability density function follows the normal distribution with mean $\mu$ and variance $\sigma^2$, then the characteristic function of $Y$, $\varphi(t)$ is*

$$
\exp(i\mu t - \frac{1}{2}\sigma^2 t^2). \tag{63}
$$

**Proof.**

$$
\begin{aligned}
\varphi(t) &= \int_R e^{iyt} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-\mu)^2}{2\sigma^2}) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-y^2 + 2y\mu - \mu^2 + 2\sigma^2 ity}{2\sigma^2}) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-(y-(\mu+\sigma^2 it))^2}{2\sigma^2}) \exp(\frac{2\mu\sigma^2 it - \sigma^4 t^2}{2\sigma^2}) dy \\
&= \exp(\frac{2\mu\sigma^2 it - \sigma^4 t^2}{2\sigma^2}) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-(y-(\mu+\sigma^2 it))^2}{2\sigma^2}) dy \\
&= \exp(\mu it - \frac{1}{2}\sigma^2 t^2)
\end{aligned}
$$

■

Since $\tilde{X}_T \sim N((r+\frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$, by the above lemma, we obtain that

$$
\phi_T(v) = \exp((r + \frac{1}{2}\sigma^2)(T-t)iv - \frac{1}{2}\sigma^2(T-t)v^2). \tag{64}
$$

Taking the inverse Fourier form of (61) and undamped, we obtain

$$\Delta_t(k) = \exp(-\alpha k)\frac{1}{2\pi}\int_{\mathbb{R}} e^{-ivk}\Psi_T(v)dv = \exp(-\alpha k)\frac{1}{\pi}\int_0^{\infty} \mathrm{Re}(e^{-ivk}\Psi_T(v))dv. \tag{65}$$

Follow the FFT option pricing method, we transform $\Delta_t(k)$ into the FFT form (13),

$$\Delta_t(k_u) \approx \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(e^{-iv_j(-b+\lambda(u-1))}\Psi_T(v_j)\eta), \quad \text{for } u = 1, \ldots, N, \tag{66}$$

the choice of b, $\lambda$, $\eta$ and $k$ is the same as in FFT option pricing method in (29) (30) (31).

## 4.2 Examples

Now, we use this method to compute the Delta and compare the results of this method to the closed form solution.We choose $N = 256$, the truncated upper is 500, the damping parameter $\alpha$ is 0.7, the risk-free rate $r$ is 0.03, and volatility $\sigma = 0.25$. We compare two environments of maturity $T = 0.5$ and 1. In our setting, the sequence of natural logarithm of strike price

$$k_u = \{-\frac{64}{125}\pi, -\frac{64}{125}\pi + \frac{\pi}{250}, -\frac{64}{125}\pi + 2 * \frac{\pi}{250}, ..., -\frac{64}{125}\pi + 255 * \frac{\pi}{250}\}.$$

We can find that in these two cases, the value of Delta is very near to the closed form solution when the strike price $K$ is larger than 0.5. But when the strike price is smaller than 0.5, the method seems to be not suitable for estimating Delta. The reason may be the convergence rate of the FFT option pricing method is slow. If we focus on the at-the-money, out-the-money, even deeply out-the-money option, this method is good for computing Delta. In the next section, we will discuss the error of using FFT option pricing method to estimate Delta. We give the the upper bound of this error theoretically.

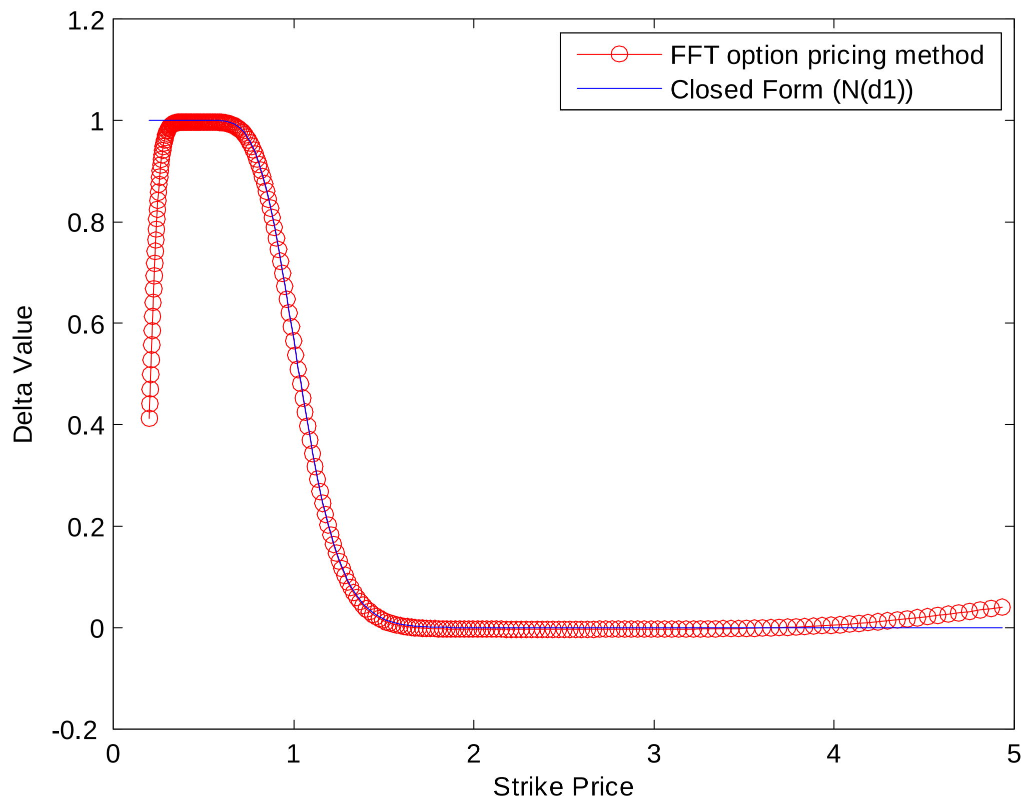


Figure 1: Delta estimating using FFT option pricing method and closed form solution (T=0.5)

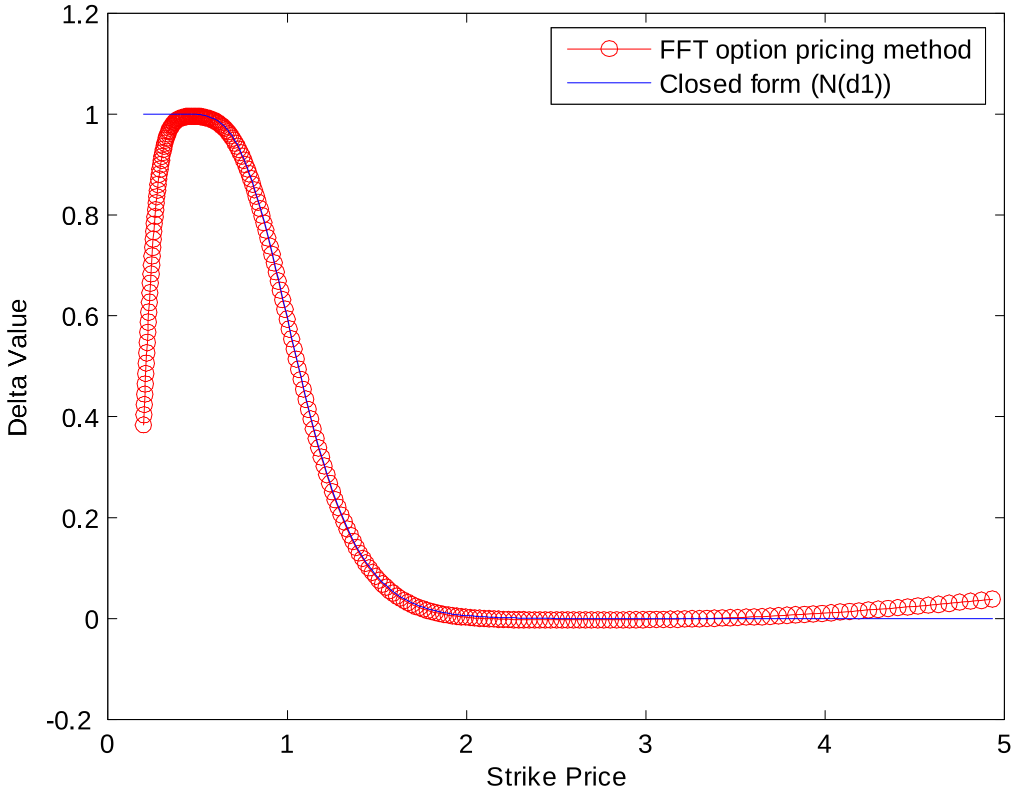


Figure 2: Delta estimating using FFT option pricing method and closed form solution (T=1)

## 4.3 Error of Using FFT Option Pricing Method to Estimate Delta

The total error is defined as the absolute difference between the true value

$$\Delta_t(k) = \exp(-\alpha k)\int_0^\infty \mathrm{Re}(e^{-ivk}\Psi_T(v))dk,$$

and the discrete approximation given by the $N$-point sum

$$\sum{}^N(k) = \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}(\sum_{j=1}^{N}(\Psi_T(j\eta)\exp(-ij\eta k)\eta).$$

The total error is bounded by the sum of the sampling error and the truncation error

$$\left|\Delta_t(k) - \sum{}^N(k)\right| \leq \left|\Delta_t(k) - \sum{}^\infty(k)\right| + \left|\sum{}^\infty(k) - \sum{}^N(k)\right|,$$

where $\sum{}^\infty(k)$ is defined as $\sum{}^N(k)$ is expect with an infinite upper limit of summations. Truncation error because the upper limit of the numeric integration is finite, and the sampling error because the integrand is evaluated numerically only at the grid points.

### 4.3.1 Truncation Error

**Theorem 17** *If $\phi_T$ is such that $\Psi_T(v)$ decays exponentially, $|\Psi_T(v)| \leq \Phi(v)\exp(-\beta v)$ for all $v \geq U_0$, where $\beta > 0$ and $\Phi(v)$ is decreasing in $v$, then the truncation error*

$$\left|\sum{}^\infty(k) - \sum{}^N(k)\right| \leq \exp(-\alpha k)\Phi(N\eta)\frac{\eta}{\pi}\frac{\exp(-\beta\eta N)}{1-\exp(-\beta\eta)} \qquad (67)$$

*provide that $N\eta > U_0$.*

**Proof.**

$$\begin{aligned}\left|\sum{}^{\infty}(k)-\sum{}^{N}(k)\right| &\le \exp(-\alpha k)\frac{\eta}{\pi}\sum_{j=N}^{\infty}|\Phi(v_j)\exp(-\beta v_j)|\\ &\le \exp(-\alpha k)\frac{\eta}{\pi}\Phi(N\eta)\sum_{j=N}^{\infty}\exp(-\beta v_j)\\ &\le \exp(-\alpha k)\frac{1}{\pi}\Phi(N\eta)\frac{\exp(-\beta\eta N)}{1-\exp(-\beta\eta)}\end{aligned}$$

Note that by (40) and (42),

$$\phi_T(v)=\exp((r+\frac{1}{2}\sigma^2)(T-t)iv-\frac{1}{2}\sigma^2(T-t)v^2),$$

and

$$\Psi_T(v)=\frac{1}{\alpha+iv}\phi_T(v-\alpha i).$$

Then

$$\begin{aligned}|\Psi_T(v)| &\le \left|\frac{\phi_T(v-\alpha i)}{v}\right|\\ &= \left|\frac{\exp((r+\frac{1}{2}\sigma^2)(T-t)iv)\exp(\sigma^2(T-t)\alpha i)\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha))}{v}\right|\\ &\quad *\left|\frac{\exp(-\frac{1}{2}\sigma^2(T-t)(v^2-\alpha^2))}{v}\right|\\ &= \frac{\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)\exp(-\frac{1}{2}\sigma^2(T-t)(v^2-\alpha^2))}{v}\\ &= \frac{\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)\exp(-\frac{1}{2}\sigma^2(T-t)(v-\alpha)^2)}{v}\exp(-(\sigma^2\alpha(T-t))v).\end{aligned}$$

Let $\beta=\sigma^2\alpha(T-t)>0$ and $\Phi(v)=\frac{\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)\exp(-\frac{1}{2}\sigma^2(T-t)(v-\alpha)^2)}{v}$ is a decreasing function in $v$. By the theorem (17), we can prove that

$$\begin{aligned}&\left|\sum{}^{\infty}(k)-\sum{}^{N}(k)\right|\\ \le\ &\exp(-\alpha k)\frac{\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)\exp(-\frac{1}{2}\sigma^2(T-t)(N\eta-\alpha)^2)}{N\eta}\\ &*\frac{\eta}{\pi}\frac{\exp(-\sigma^2\alpha(T-t)\eta N)}{1-\exp(-\sigma^2\alpha(T-t)\eta)}\\ =\ &\frac{\exp(-\alpha k)\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)}{N\eta(1-\exp(-\sigma^2\alpha(T-t)\eta))}\\ &*\exp(-\frac{1}{2}\sigma^2(T-t)(N\eta-\alpha)^2)\exp(-\sigma^2\alpha(T-t)\eta N).\end{aligned}$$

■

### 4.3.2 Sampling Error

We first describe a lemma and will be used in this subsubsection.

**Lemma 18** *For any $p > 0$,*

$$\Delta_t(k) \leq \frac{E^*[\exp(p\tilde{X}_T)]}{\exp(pk)} \quad \text{and} \quad \Delta_t(k) \leq 1. \tag{68}$$

**Theorem 19** *The sampling error*

$$\begin{aligned}
&\left|\Delta_t(k) - \sum{}^{\infty}(k)\right| \\
\leq\ & \sum_{j=1}^{\infty}(\exp(-\frac{\alpha\pi j}{\eta}) + \exp(\frac{(\alpha-p)\pi j}{\eta}) \exp(-pk)\exp((r-\frac{1}{2}\sigma^2)(T-t)p) + \frac{\sigma^2(T-t)}{2}p^2).
\end{aligned}$$

**Proof.** Recall

$$\tilde{\Delta}_t(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ivk}\Psi_T(v)dv.$$

And,

$$\begin{aligned}
&\tilde{\Delta}_t(k-\frac{\pi j}{\eta}) + \tilde{\Delta}_t(k+\frac{\pi j}{\eta}) \\
=\ & \frac{1}{2\pi}\int_R [\exp(-iv(k-\frac{\pi j}{\eta}))\Psi_T(v) + \exp(-iv(k+\frac{\pi j}{\eta}))\Psi_T(v)]dv \\
=\ & \frac{1}{2\pi}\int_R [\exp(-ivk)\exp(-\frac{i\pi vj}{\eta})\Psi_T(v) + \exp(-ivk)\exp(\frac{i\pi vj}{\eta})\Psi_T(v)]dv \\
=\ & \frac{1}{2\pi}\int_R \exp(-ivk)\Psi_T(v)(\cos(\frac{\pi vj}{\eta}) - i\sin(\frac{\pi vj}{\eta}) + \cos(\frac{\pi vj}{\eta}) + i\sin(\frac{\pi vj}{\eta}))dv \\
=\ & \frac{1}{\pi}\int_R \exp(-ivk)\Psi_T(v)\cos(\frac{\pi vj}{\eta})dv \\
=\ & 2\int_0^{\eta} F(v)\cos(\frac{\pi vj}{\eta})dv = \eta(\frac{2}{\eta}\int_0^{\eta} F(v)\cos(\frac{\pi vj}{\eta})dv),
\end{aligned}$$

where

$$F(v) \triangleq \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Psi_T(v+n\eta)\exp(-i(v+n\eta)k).$$

Note that $F$ is piesewise continuous. Let

$$A_k = \tilde{\Delta}_t(k-\frac{\pi j}{\eta}) + \tilde{\Delta}_t(k+\frac{\pi j}{\eta}),$$

then

$$F(v)\eta = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi v}{\eta})$$

is called the Fourier cosine of $F(v)\eta$. So,

$$F(v)\eta = \tilde{\Delta}_t(k) + \sum_{j=1}^{\infty} [\tilde{\Delta}_t(k - \frac{\pi j}{\eta}) + \tilde{\Delta}_t(k + \frac{\pi j}{\eta})] \cos(\frac{j\pi v}{\eta}).$$

In particular, taking $v = 0$, we have

$$\left|\tilde{\Delta}_t(k) - F(0)\eta\right| = \left|\sum_{j=1}^{\infty} \tilde{\Delta}_t(k - \frac{\pi j}{\eta}) + \tilde{\Delta}_t(k + \frac{\pi j}{\eta})\right|.$$

Multiplying by $exp(-\alpha k)$ to undamp $\tilde{\Delta}_t(k)$,

$$|\Delta_t(k) - exp(-\alpha k)F(0)\eta| = \left|\sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta})\Delta_t(k - \frac{\pi j}{\eta}) + \exp(\frac{\alpha\pi j}{\eta})\Delta_{tt}(k + \frac{\pi j}{\eta})\right|.$$

Note that, in our setting

$$exp(-\alpha k)F(0)\eta = \sum{}^{\infty}(k)$$

.Then, we apply lemma,

$$\begin{aligned}
\left|\Delta_t(k) - \sum{}^{\infty}(k)\right| &= \left|\sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta})\Delta_t(k - \frac{\pi j}{\eta}) + \exp(\frac{\alpha\pi j}{\eta})\Delta_{tt}(k + \frac{\pi j}{\eta})\right| \\
&\leq \left|\sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta}) + \exp(\frac{\alpha\pi j}{\eta})\frac{E^*[\exp(pX_T)]}{\exp(p(k + \frac{\pi j}{\eta}))}\right| \\
&= \left|\sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta}) + \exp(\frac{(\alpha - p)\pi j}{\eta})\frac{E^*[\exp(pX_T)]}{\exp(pk)}\right|,
\end{aligned}$$

in our case of geometric Brownian motion,

$$\begin{aligned}
E^*[\exp(p\tilde{X}_T)] &= \int_{-\infty}^{\infty} \exp(ps)\frac{1}{\sqrt{2\pi}\sigma\sqrt{(T-t)}} \exp(-\frac{(s - (r + \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)})ds \\
&= \exp((r + \frac{1}{2}\sigma^2)(T-t)p + \frac{\sigma^2(T-t)}{2}p^2).
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\Delta_t(k) - \sum^{\infty}(k)\right| &\leq \left|\sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta}) + \exp(\frac{(\alpha - p)\pi j}{\eta})\frac{\exp((r + \frac{1}{2}\sigma^2)(T-t)p + \frac{\sigma^2(T-t)}{2}p^2)}{\exp(pk)}\right| \\
&= \sum_{j=1}^{\infty} \exp(-\frac{\alpha\pi j}{\eta}) + \exp(\frac{(\alpha - p)\pi j}{\eta})\exp(-pk)\exp((r + \frac{1}{2}\sigma^2)(T-t)p + \frac{\sigma^2(T-t)}{2}p^2)
\end{aligned}$$

■

So, we can conclude that the upper bound of the total error in our estimating Delta method is

$$\frac{\exp(-\alpha k)\exp((r+\frac{1}{2}\sigma^2)(T-t)\alpha)\exp(-\frac{1}{2}\sigma^2(T-t)(N\eta-\alpha)^2)\exp(-\sigma^2\alpha(T-t)\eta N)}{N\eta(1-\exp(-\sigma^2\alpha(T-t)\eta))}$$
$$+\sum_{j=1}^{\infty}\exp(-\frac{\alpha\pi j}{\eta})+\exp(\frac{(\alpha-p)\pi j}{\eta})\exp(-pk)\exp((r+\frac{1}{2}\sigma^2)(T-t)p+\frac{\sigma^2(T-t)}{2}p^2).$$

**Theorem 20** *The variance of martingale control between using FFT option pricing method and true value*

$$\begin{aligned} & E^*[\int_0^T e^{-2rs}(\frac{\partial P}{\partial x}-\frac{\partial P_{FFT}}{\partial x})^2(s,\tilde{S}_s)\tilde{S}_s{}^2\sigma^2 ds] \\ \leq\ & -F_1\frac{8^4}{U^4}\frac{1}{4D_1}(1-\exp(4D_1T)-F_2\eta^4 8^4\frac{1}{4D_2}(1-\exp(4D_2T), \end{aligned}$$

*for some constants $D_1$, $D_2$, $F_1$ and $F_2$.*

The proof of this theorem is presented in appendix C.

## 4.4 Apply FFT Option Pricing Method in Martingale Control Variate Method

In section 3, we introduce how to use the FFT option pricing method to estimate Delta. In the structure of this method, it fixes the initial underlying asset price to be 1. And this method could give us the Delta values correspond to the sequence of different strike prices. But in the section 2, the problem is to compute the Delta with respect to every unbiased estimators of underlying asset in the same strike price.. Because of it, we should do some work to make us can apply FFT option pricing method. Recall that in section 2, the martingale control is

$$M_0(P_{BS};T) = \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW_s^*$$
$$\approx \sum_{j=1}^{M} e^{-r\frac{T}{M}(j-1)}\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S_{\frac{T}{M}(j-1)})\sigma S_{\frac{T}{M}(j-1)}\sqrt{\frac{T}{M}}\varepsilon_j,$$

We want to find $\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S_{\frac{T}{M}(j-1)})$ in any pairs $(\frac{T}{M}(j-1), S_{\frac{T}{M}(j-1)})$, $j = 1..N..$ If we construct $Q$ sample paths, we will compute

$$\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S^{(l)}_{\frac{T}{M}(j-1)}) \quad \text{for} \quad j = 1...M \quad \text{and} \quad l = 1..Q.$$

In the other words, we will estimate the all elements in the matrix $H$, we call this matrix is Delta matrix. i.e.

$$H = \begin{pmatrix} \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(0), S^{(1)}_{\frac{T}{M}(0)}) & \dots & \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(M-1), S^{(1)}_{\frac{T}{M}(M-1)}) \\ \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(0), S^{(2)}_{\frac{T}{M}(0)}) & \dots & \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(M-1), S^{(2)}_{\frac{T}{M}(M-1)}) \\ \vdots & \ddots & \vdots \\ \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(0), S^{(Q)}_{\frac{T}{M}(0)}) & \dots & \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(M-1), S^{(Q)}_{\frac{T}{M}(M-1)}) \end{pmatrix},$$

note that the strike price $K$ is fixed. In order to apply FFT option pricing method, we claim that

$$\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S^{(l)}_{\frac{T}{M}(j-1)}, K) = \frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), 1, \frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}). \tag{69}$$

The above equal sign is because

$$\Delta_t(\tilde{S_T}, K) = \tilde{E}[I_{\{\tilde{S_T} \geq K\}}] = \tilde{E}[I_{\{\frac{\tilde{S_T}}{\tilde{S_t}} \geq \frac{K}{\tilde{S_t}}\}}]. \tag{70}$$

So, we fix the initial underlying asset price and let the strike price is floating. Then, we can apply FFT option pricing method as following steps:

**Step 1.** Using FFT option pricing method to compute the Delta at each time, $\{0, \frac{T}{M}, 2\frac{T}{M}, ..., (M-1)\frac{T}{M}\}$.

**Step 2.** To predict whether $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is located within the range of (32) or not. If $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is located within the range of (32), the method of interpolation is applied here to approximate the Delta value with respect to the strike price $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$. In the case where $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is prior to the range of (32), Delta is set to 1, because the option is deeply out-the money.. If $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ exceeds the range of (32), then we set Delta to 0, because the option is deeply in-the money.

The numerical results show in the next section.

## 4.5 Numerical Result

In this section, we compare the time we use to compute Delta value in closed form solution and in FFT option pricing method. We also show the standard error in the two conditions, without control variate and using martingale variate. In FFT option pricing method, we also take $N = 256$, the truncated upper is 500, the damping parameter $\alpha$ is 0.75, the risk-free rate $r$ is 0.03, and volatility $\sigma = 0.25$.In Monte Carlo control variance method. we take 100000 sample paths and the number of partitions of time interval $[0, T]$ is 100. The initial underlying asset price $S_0$ is 100. We test the four cases, the strike price $K = 20, 80, 120, 180$.i.e. the environment of deeply in-the-money, in-the-money, out-the-money, and deeply out-the-money. The computations are done under MATLAB-7.0 in a PC with 2.4 GHz P4 CPU. For convenience, SE means standard error and MCV means martingale control variate in brief. We summary the algorithms as followng.

**Step1.** Simulate the underlying asset's paths in order to obtain the terminal prices of the asset. Compute the sample paths of $H(S_T)$ and its discounted value $e^{-rT}\ H(S_T)$.

**Step2.** Discretize the martingale control of the $e^{-rT}\ H(S_T)$. Use the lower Riemann sum to approximate this integral. i.e.

$$M_0(P_{BS};T) = \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW_s^* \tag{71}$$
$$\approx \sum_{j=1}^{M} e^{-r\frac{T}{M}(j-1)}\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S_{\frac{T}{M}(j-1)})\sigma S_{\frac{T}{M}(j-1)}\sqrt{\frac{T}{M}}\varepsilon_j,$$

where $\{0, \frac{T}{M}, \frac{T}{M}*2, ..., \frac{T}{M}*(M-1)\}$ is the partition of the interval $[0,T]$ , $K$ is the strike price of option, and $\varepsilon_j$ are identical and independent (I.I.D) standard normal random variables.

**Step3.** An estimator of the martingale control for the option price is the following,

$$E_{0,x}^*[e^{-rT}H(S_T) - \int_o^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s,S_s)\sigma S_s dW_s^*] \tag{72}$$
$$\approx \frac{1}{Q}(\sum_{l=1}^{Q} e^{-rT}H(S_T^{(l)}) - \sum_{l=1}^{Q}\sum_{j=1}^{M} e^{-r\frac{T}{M}(j-1)}\frac{\partial P_{BS}}{\partial x}(\frac{T}{M}(j-1), S^{(l)}_{\frac{T}{M}(j-1)})\sigma S^{(l)}_{\frac{T}{M}(j-1)}\sqrt{\frac{T}{M}}\varepsilon_j,$$

where $Q$ is the number of sample paths.

**Step 4.** Using FFT option pricing method to compute the Delta at each time, $\{0, \frac{T}{M}, 2\frac{T}{M}, ..., (M-1)\frac{T}{M}\}$.

**Step 5.** To predict whether $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is located within the range of (32) or not. If $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is located within the range of (32), the method of interpolation is applied here to approximate the Delta value with respect to the strike price $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$. In the case where $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ is prior to the range of (32), Delta is set to 1, because the option is deeply out-of-the money. If $\frac{K}{S^{(l)}_{\frac{T}{M}(j-1)}}$ exceeds the range of (32), then we set Delta to 0, because the option is deeply in-the money.

(1) $K = 20$

| | Time(Seconds) | Call Price | SE(MCV) | SE(No MCV) |
|---|---|---|---|---|
| Closed Form Solution | 654.047 | 81.901 | 0.0012 | 0.0783 |
| FFT option pricing method | 91.798 | 81.961 | 0.0944 | 0.0783 |

(2) $K = 80$

| | Time(Seconds) | Call Price | SE(MCV) | SE(No MCV) |
|---|---|---|---|---|
| Closed Form Solution | 701.197 | 28.592 | 0.0033 | 0.0760 |
| FFT option pricing method | 120.547 | 28.592 | 0.0035 | 0.0760 |

(3) $K = 120$

| | Time(Seconds) | Call Price | SE(MCV) | SE(No MCV) |
|---|---|---|---|---|
| Closed Form Solution | 661.272 | 6.6379 | 0.0194 | 0.0450 |
| FFT option pricing method | 130.644 | 6.6377 | 0.0200 | 0.0450 |

(4) $K = 180$

| | Time(Seconds) | Call Price | SE(MCV) | SE(No MCV) |
|---|---|---|---|---|
| Closed Form Solution | 697.729 | 0.3085 | 0.0020 | 0.0101 |
| FFT option pricing method | 144.452 | 0.3082 | 0.0025 | 0.0101 |

The result shows that the time using FFT option pricing method to find Delta at every estimators of underlying asset is one fifth of time using closed form solution even better. And this method also preserves the effect of reducing variance expect to deeply in-the-money case. This is not surprising, we have showed that this method is not good for estimating Delta in the condition of deeply in-the-money option in section 3.1. But in other cases, this method is very well.

# 5 More Examples

In this section, we provide two extensions of the martigale control variate method we proposed above and also combine FFT method with it. In the first extension, we get rid of the assumption of constant volatility of the underlying asset. We let the volatility of the underlying asset can be stochastic and follwes some stochastic process. In our example, we suppose this volatility followes the OU process. This model is well-known as Heston model. In the other extension, we consider the American style options which allow the investors exercise the options at any time before the deadline.

## 5.1 Stochastic Volatility Model: Heston Model

Under the risk-neutral probability measure $P^*$, the Heston model is describe the the underlying asset follow the dynamics

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{y_t} S_t dW_t^{(1)*} \\ dy_t &= m(v - y_t)dt + \beta y_t dW_t^{(2)*}, \end{aligned} \tag{73}$$

where $S_t$ is the underlying asset price with a constant risk-free interesst rate $r$. Its stochastic volatility is driven by the stochastic process $y_t$. The process $y_t$ has the mean reversion property. $m$ is the mean reversion rates, $v$ is the long-run mean and $\beta$ is the volatility of the volatility. $m$,$v$ $\beta$ are constants. And $W_t^{(1)*}$ and $W_t^{(2)*}$ are independent standard Brownian motions. We consider the European style option payoff $P$, is simply a conditional expectation

$$P(t, x, y) = E^*_{t,x,y}[e^{-r(T-t)}H(S_T)],$$

where $E^*_{t,x,y}$ denotes the expectation with respect to $P^*$ conditioned on the current states $S_t = x$ and $y_t = t$. And $T$ is the maturity of the option. A basic Monte Carlo simulation estimates the option price $P(0, x_0, y_0)$ at time

0 by

$$\frac{1}{Q}\sum_{i=1}^{Q} e^{-rT} H(S_T^{(i)}),$$

where $Q$ is the total number of indenpendent sample paths and $S_T^{(i)}$ denotes the $i$-th simulated stock price at time $T$. As described in section 2, we can use Ito's lemma and follow the martingale representation theorem, then

$$P(0, x_0, y_0) = e^{-rT} H(S_T) - M_1(P) - \beta M_2(P), \tag{74}$$

where centered martingales are defined by

$$M_1(P) = \int_0^T e^{-rs} \frac{\partial P}{\partial x}(t, x, y) \sqrt{y_t} S_s dW_s^{(1)*}, \tag{75}$$

$$M_2(P) = \int_0^T e^{-rs} \frac{\partial P}{\partial y}(t, x, y) \sqrt{y_t} dW_s^{(2)*}. \tag{76}$$

These martingales play the role of "perfect" control variates for Monte Carlo simulations and their integrands would be the perfect Delta hedges if $P$ were known and volatility traded. Unfortunately neither the option price $P(t, x, y)$ nor its gradient at any time $0 \leq s \leq T$ are in any analytic form even though all the parameter of the model have been calibrated as we suppose here.

One can choose an approximation option price to substitute $P$ used in the martingale and still retain martingale properties. An approximation of the Black-Sholes type is

$$P(t, x, y) \approx P_{BS}(t, x, \upsilon). \tag{77}$$

In our setting, the martingale control variate estimator is formulated as

$$\frac{1}{Q}\sum_{i=1}^{Q} \left[ e^{-rT} H(S_T^{(i)}) - M_1(P_{BS}) \right]. \tag{78}$$

Note that there is no $M_2$ martingale term since the approximation $P_{BS}$ dose not depend on $y$ and the $y$-derivative. And the Delta term in martingale control varitate can aslo be compute using FFT option pricing method which is described above.

### 5.1.1 Numerical Result

Here, we take a example to observe the effect of martingale control variance method and also show the efficiency when combining FFT option pricing method to compute Delta rather than using closing form solution. We let the European style option is European call. Setting the initial underlying asset price $S_0$ is 100, the initial volatility of underlying asset, $\sqrt{y_0}$ is 0.1, the mean reversion rates, $m$ is 2, the long-run mean of volatility, $\upsilon$ is 0.1, the volatility of volatility, $\beta$ is 0.01. The number of sample paths we simulated $Q$ is 5000. The number of time steps in discreting the martingale control variate is 100. The parameter of FFT option pricing method is setting the same as before. We test three situations in which the strike price $K = 80, 100$ and 120. Finally, the maturity is set to be half year. The numerical results show in the following.

(1) Without Using Martingale Control Variate Method

| | Call Price | Standard Error |
|---|---|---|
| K=80 | 20.9655 | 0.7522 |
| K=100 | 3.3854 | 0.3087 |
| K=120 | 0.0307 | 0.0023 |

(2) Using Martingale Control Variate Method (computing Delta using closed form solution)

| | Call Price | Standard Error | Computational Time(s) |
|---|---|---|---|
| K=80 | 20.7976 | 5.1753e-006 | 78.4 |
| K=100 | 3.2908 | 0.0036 | 78.1 |
| K=120 | 0.0270 | 1.6844r-004 | 78.3 |

(3) Using Martingale Control Variate Method (computing Delta using

FFT option pricing method)

| | Call Price | Standard Error | Computational Time(s) |
|---|---|---|---|
| K=80 | 20.7981 | 1.1447e-005 | 8.4 |
| K=100 | 3.2914 | 0.0036 | 8.8 |
| K=120 | 0.0271 | 1.8723r-004 | 8.9 |

## 5.2 American option

The most important feature of an American option is that the option holder has the right to exercise the contract early. Under the geometric Brownian motion considered, the price of an American option with the payoff function $H$ is given by:

$$P(t,x) = (ess) \sup_{t \le \tau \le T} E^*_{t,x}[e^{-r(T-\tau)} H(S_\tau)], \tag{79}$$

where $\tau$ denotes any stopping time greater than $t$, bounded by $T$. We consider a typical American put option pricing problem, name $H(x) = (K-x)^+$, and maturity T. By the connection of optimal stopping problem and variational unequalitues [11], $P(t,x)$ can be characterized as the solution of the following variational inequalities

$$\begin{array}{c} \mathcal{L}_S P(t,x) \le 0 \\ P(t,x) \ge (K-x)^+ \\ \mathcal{L}_S P(t,x) \cdot (P(t,x) - (K-x)^+) = 0 \end{array} , \tag{80}$$

where $\mathcal{L}_S$ denotes the infinitesimal generator of the Markov process $(S_t)$. The optimal stopping time is characteristized by

$$\tau^*(t) = \{t \le s \le T, (K - S_s)^+ = P(s, S_s)\}. \tag{81}$$

The approximation by a formal expansion is

$$P(t,x) \approx P^A_{BS}(t,x,\sigma) \tag{82}$$

while $P^A_{BS}(t,x,\sigma)$ solves the homogenized variational inequality

$$\begin{aligned} \mathcal{L}_{BS}(\sigma)P^A_{BS}(t,x;\sigma) &\leq 0 \\ P^A_{BS}(t,x;\sigma) &\geq (K-x)^+ \\ \mathcal{L}_{BS}P^A_{BS}(t,x;\sigma)\cdot(P^A_{BS}(t,x;\sigma)-(K-x)^+) &= 0 \end{aligned} \quad , \tag{83}$$

where $\mathcal{L}_{BS}(\sigma)$ denotes the Black-Scholes operator with constant volatility $\sigma$. In contrast to typical European options, there is no colsed-form solution for the American put option under a constant volatility. The derivation of the accuracy of the approximation (82) is still an open problem.

As in the previous sections, we assume that the discounted American option price $e^{-rt}P(t,x)$ before exercise is smooth enough to apply Ito's lemma, then we integrate from 0 to the optimal stopping time $\tau^*$ such that we obtain

$$P(t,x) = e^{-rT}(K-S_{\tau^*})^+ - M(P(t,x)),$$

the local martingale are defined as in (7) except that the upper bounds are replaced by the optimal tine $\tau^*$.

As revealed in (81), the characterization of the optimal stopping time $\tau^*(t)$ does depend on the American option price, which itself is unknow in advance. This causes an immediate difficulty to implement Monte Carlo Simulations because one does not know the time to stop in order to collect the payoff along each realized sample path. Longstaff and Schwartz [10] took a dynamic programming approach and proposed a least-squre regression to estimate the continuation value at each in-the-money stock price state. Their method exploit a decision rule for early exercise along each sample path generated. Thus an adapted stopping time, denoted by $\underline{\tau}$, is induced. It is sub-optimal because specifying any stopping time to price American option is always less than or equal to the true price by its definition:

$$E^*[e^{-r(\underline{\tau}-t)}(K-S_{\underline{\tau}})^+] \leq \sup_{0\leq\tau\leq T} E^*[e^{-r(\tau-t)}(K-S_\tau)^+]. \tag{84}$$

Like in previous sections, a local martingale control variate can be in principle contructed as

$$M(P^A_{BS};\tau^*) = \int_0^{\tau^*} e^{-rs}\frac{\partial P^A_{BS}}{\partial x}(s,S_s;\sigma)\sigma S_s dW_s^{(0)^*}.$$

The optimal stopping time $\tau^*$ is of course not known, thus we use the sub-optimal stopping time $\underline{\tau}$ obtained by Longstaff and Schwartz's method. To summarize, we contruct the following stopped martingale as a control variate

$$M((P^A_{BS};\underline{\tau}) = \int_0^{\underline{\tau}} e^{-rs}\frac{\partial P^A_{BS}}{\partial x}(s;S_s;\sigma)\sigma dW_s^{(0)*}.$$

The Monte Carlo estimator with the martingale control variate is

$$\frac{1}{Q}\sum_{i=1}^{Q}[e^{-r\underline{\tau}}(K-S^{(i)}_{\underline{\tau}})^+ - M^{(i)}((P^A_{BS};\underline{\tau})]. \tag{85}$$

As seen in (84), the estimator in (85) is low-biased. On the opposite, Rogers [17] proposed a dual formulation to contruct a high-biased estimator as follows:

$$\frac{1}{Q}\sum_{i=1}^{Q}\sup_{0\le t\le T}[e^{-rt}(K-S^{(i)}_t)^+ - M^{(i)}((P^A_{BS})].$$

In next section we perform numerical experiments to show high and low biased estimators of American option price. In particular, we see the computating time are dramatically speed up by FFT algorithm.

### 5.2.1 Numerical Result

We show the effect of the martingale control variance method in pricing American option using Monte Carlo simulations. Simultaneously, we use FFT option pricing method to acclerate this control variance method. The parameter is setting as following : $\sigma = 0.4$, $r = 0.06$, the initial underlying asset price $S_0 = 100$. The number of sample paths is 5000 and time steps is 100. The deadline of the American option is half of the year. We also test three conditions : $K = 80, 100$ and $120$. We use UB to stand for upper bound of the American put option price, LB is lower bound and CT means computational time. The standard error is presented using the bracket following the price. The numerical results is represented as following:

Case1. Computing Delta Terms Using Closed Form Solutions

| | LSM Price | UP | LB | CT |
|---|---|---|---|---|
| K=80 | 2.6306(0.0790) | 2.6410(0.0085) | 2.5466(0.0074) | 76.51 |
| K=100 | 9.9816(0.1568) | 10.0745(0.0120) | 9.8907(0.0123) | 76.61 |
| K=120 | 22.9384(0.1916) | 23.3454(0.0130) | 22.9127(0.0134) | 76.84 |

Case2. Computing Delta Terms Using FFT Option Pricing Method

| | LSM Price | UP | LB | CT |
|---|---|---|---|---|
| K=80 | 2.6306(0.0790) | 2.6369(0.0085) | 2.5539(0.0074) | 6.57 |
| K=100 | 9.9816(0.1568) | 10.0745(0.0115) | 9.8905(0.0120) | 6.76 |
| K=120 | 22.9384(0.1916) | 23.3454(0.0124) | 22.9129(0.0129) | 7.33 |

# 6 Conclusion

In this paper, we apply the FFT option pricing method to find "Delta" along every simulated price trajectory. We compare our FFT method with cases such as Black-Scholes model where the "Delta" has a closed-form solution. Simultaneously, we provide a variance analysis to show that the variance of FFT-approximation error depends on the truncated upper bound and discretization size of our FFT method. Numerical results show that: (1) our FFT algorithm outperforms the martingale control variate method in terms of computing time by 5~10 better times, (2) our method is good for out-the-money, at-the-money, out-of-the-money, and even deeply out-of-the-money call option. But it is not suitable for deeply in-the-money call option.

To find this reason or modifies this method in order to suitable for deeply in-the-money call option is a future work.

# 7 Appendix

## A.Proof claim 9 in section 3.1

**Proof.**

$$\begin{aligned}
C_T(k) &= e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_T(v)e^{-ivk}dv \\
&= e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_T(v)(\cos(-vk)+i\sin(-vk))dv \quad \text{(by Euler formula)} \\
&= e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}(\Psi_T(v))+i\,\mathrm{Im}(\Psi_T(v)))(\cos(vk)-i\sin(vk))dv \\
&= e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}(\Psi_T(v))\cos(vk)+\mathrm{Im}(\Psi_T(v))\sin(vk)) \\
&\quad +i(\mathrm{Im}(\Psi_T(v))\cos(vk)-\mathrm{Re}(\Psi_T(v))\sin(vk))dv \\
&= e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}(\Psi_T(v)e^{-ivk})dv
\end{aligned}$$

The reason of above equal sign is the call option value is real, so, the imagine part of $\Psi_T(v)e^{-ivk}$ must equals to zero. i.e.

$$\int_{-\infty}^{\infty}(\mathrm{Im}(\Psi_T(v))\cos(vk)-\mathrm{Re}(\Psi_T(v))\sin(vk))dv = 0.$$

As we know, $cos(x)$ is an even function, i.e. $cos(-v) = cos(v)$. And $sin(v)$ is an odd function, i.e. $sin(-v) = -sin(v)$. In order to make the above integration equals to zero, $\mathrm{Im}(\Psi_T(v))\cos(vk)$ and $\mathrm{Re}(\Psi_T(v))\sin(vk))$ could be odd functions with respect to $v$. Or, $\mathrm{Re}(\Psi_T(v))$ is an even function and $\mathrm{Im}(\Psi_T(v)$ is an odd function. It makes $\mathrm{Re}(\Psi_T(v))\cos(vk)$ and $\mathrm{Im}(\Psi_T(v))\sin(vk)$ are even functions. So, the real part of $\Psi_T(v)e^{-ivk}$ is a even function. Then, we can conclude that

$$C_T(k) = e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}(\Psi_T(v)e^{-ivk})dv = e^{-\alpha k}\frac{1}{\pi}\int_{0}^{\infty}(\mathrm{Re}(\Psi_T(v)e^{-ivk})dv$$

■

## B.Algorithm of FFT (Fast Fourier Transform)

**Definition 21** *Given N complex numbers*

$$\{h_j\}_{j=0}^{N-1}$$

*their N-point Discrete Fourier Transform(DFT) is denote by* $\{H_k\}$ *where* $H_k$ *is defined by*

$$H_k = \sum_{j=0}^{N-1} h_j W^{kj} \tag{B.1}$$

$$W = e^{-2\pi i/N}$$

*for all integer* $k = 0,\ 1,\ 2,...,\ N-1$

*Moreover ,* $\{h_k\}$ *is called the N-point Inverse Discrete Fourier Transform(IDFT) of* $\{H_k\}$*. And*

$$h_j = \frac{1}{N}\sum_{k=0}^{N-1} H_k W^{-kj} \tag{B.2}$$

*for all integer* $j = 0,\ 1,\ 2,...,\ N-1$

*Note : we usually choose* $N$ *is a power of* $2$*, in order to discuss the algorithm of computation easily.*

Basically , our problem is that.given the sequence $\{H_k\}$ (or $\{h_k\}$) of N complex valued numbers, how to compute its DFT(or IDFT) efficiently, according to the above-mentioned formula. Since DFT and IDFT involve basically the same type of computations, our discussion of efficiently computational algorithm for the DFT applies as well to the efficient computation of the IDFT.

We observe that for each value of $k$, direct computation of $H_k$ involves N complex multiplications and $N-1$ complex additions. Consequently, to compute all $N$ values of the DFT requires $N^2$ complex multiplications and

$N(N-1)$ complex additions. So, direct computation of the DFT or IDFT is basically inefficient primarily because it does not exploit the symmetry and periodicity properties of the phase factor $W$. In particular, these two properties are:

$$\text{Symmetry property: } W^{k+N/2} = -W^k$$

$$\text{Periodicit property: } W^{k+N} = W^k$$

then, we will use these two properties to introduce the FFT algorithm.

We begin FFT algorithm by dividing the N-point DFT in (B.1) into two sums, each of which is a (1/2) N-point DFT

$$H_k = \sum_{j=0}^{\frac{1}{2}N-1} h_{2j}(W^2)^{jk} + \sum_{j=0}^{\frac{1}{2}N-1} h_{2j+1}(W^2)^{jk}W^k. \qquad \text{(B.3)}$$

Based on (B.3) we write $H_k$ as

$$H_k = H_k^0 + W^k H_k^1 \qquad \text{(B.4)}$$

$$H_k^0 = \sum_{j=0}^{\frac{1}{2}N-1} h_{2j}(W^2)^{jk}$$

$$H_k^1 = \sum_{j=0}^{\frac{1}{2}N-1} h_{2j+1}(W^2)^{jk} \qquad (k = 0,1,...,N-1)$$

Because of the periodicit property, the periods of $\{H_k^0\}$ and $\{H_k^1\}$ are $(1/2)N$ ; they are $(1/2)N$-point DFTs of $\{h_0,h_2,...,h_{N-2}\}$ and $\{h_1,h_3,...,h_{N-1}\}$, respectively. Since $N = 2^R$ we can divide $N$ evenly by 2, and we have, by Euler formula,

$$W^{\frac{1}{2}N} = e^{-i\pi jk} = \cos(-\pi jk) + i\sin(-\pi jk) = -1. \qquad \text{(B.5)}$$

Using (B.5) we will write the first equation in (B.4) as

$$H_k = H_k^0 + W^k H_k^1 \qquad k = 0,1,...,\frac{N}{2}-1$$

$$H_{k+(1/2)N} = H_k^0 - W^k H_k^1 \qquad k = 0,1,...,\frac{N}{2}-1. \qquad \text{(B.6)}$$

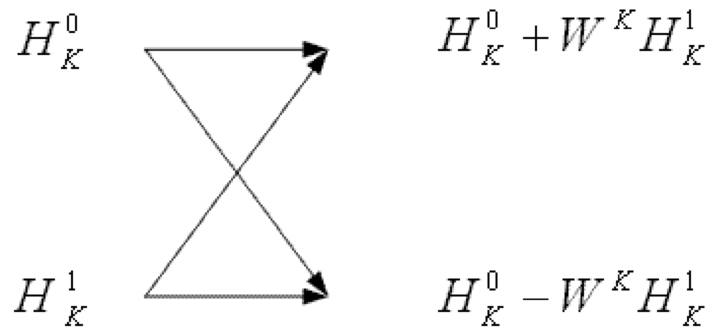


Figure 3: Butterfly in FFT algorithm

The calculation in (B.6) can be diagrammed as Figure 3.

Figure3. is called a butterfly. There are $(1/2)N$ butterflies for this stage of the FFT. We observed that the directly computation of $H_k^0$ requires $(N/2)^2$ complex multiplications. The same applies to the computation of $H_k^1$. Further, there are N/2 additional complex multiplications requires to compute $W^k H_k^1$. Hence the computataion of $H_k$ requires $2(N/2)^2 + N/2 = N^2/2 + N/2$ complex multiplications. This first step results in a reduction of the number of multiplications from $N^2$ to $N^2/2 + N/2$, which is about a factor of 2 for N large.

By computing $N/4$-point DFTs, we would obtain the N/2-point DFTs $H_k^0$ and $H_k^1$ from the relations

$$H_k^0 = H_k^{00} + (W^2)^k H_k^{01}, \qquad H_{k+\frac{1}{4}N}^0 = H_k^{00} - (W^2)^k H_k^{01}$$

$$H_k^1 = H_k^{10} + (W^2)^k H_k^{11}, \qquad H_{k+\frac{1}{4}N}^1 = H_k^{10} - (W^2)^k H_k^{11} \tag{B.7}$$

for $k = 0,1,\ldots,(1/4)N - 1$.In (2.7), $\{H_K^{00}\}$ is the (1/4)N-point DFTs of $\{h_0, h_4, h_8, \ldots, h_{N-4}\}$, $\{H_k^{01}\}$ is the (1/4)N-point DFTs of $\{h_2, h_6, h_{10}, \ldots, h_{N-2}\}$, $\{H_k^{10}\}$ is the (1/4)N-point DFTs of $\{h_1, h_5, \ldots, h_{N-3}\}$, $\{H_k^{11}\}$ is the (1/4) N-point DFTs of $\{h_3, h_7, \ldots, h_{N-1}\}$.The decimation of the sequence can be repeated again and again until the resulting sequences are reduced to one-

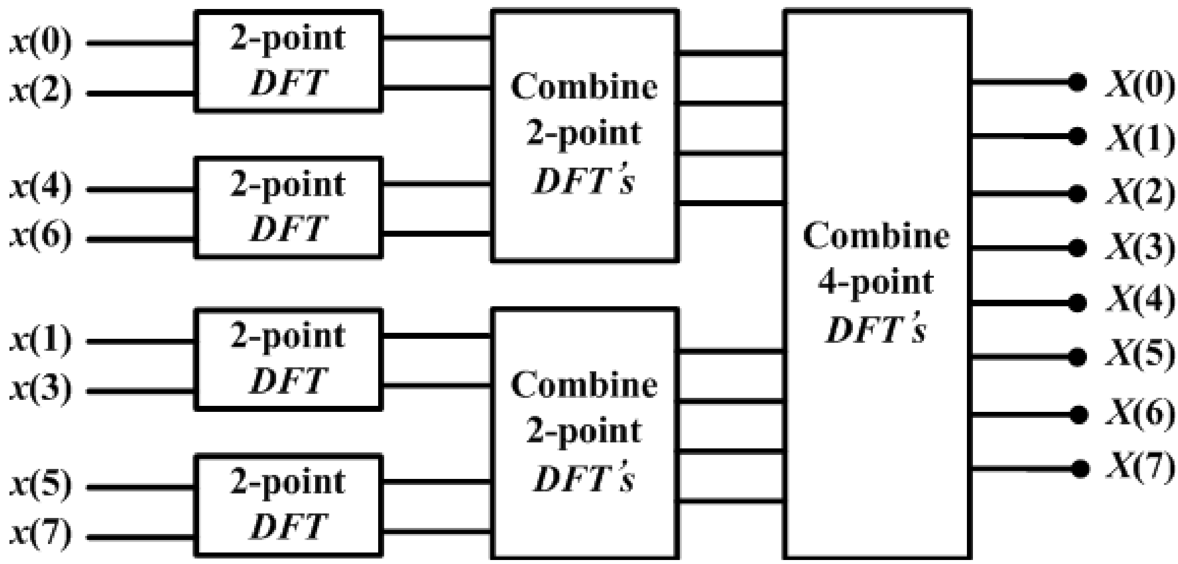


Figure 4: FFT algorithm with 8 points

point sequences. For $N = 2^R$, this can be performed $R = log_2 N$ times. Thus the total number of complex multiplications is reduced to $(N/2)log_2 N$. The number of complex additions is $Nlog_2 N$. So, use the FFT algorithm, we reduce the computation of complex multiplications from $N^2$ to $(N/2)log_2 N$. When N is large, it is a considerable method. Figure 4. is a example of FFT method with N=8.

## C.Derivation of the accuracy of the variance analysis

**Proof.** Note that by Cauchy-Schwarz inequality,

$$E^*[\int_0^T e^{-2rs}(\frac{\partial P}{\partial x} - \frac{\partial P_{FFT}}{\partial x})^2(s, \tilde{S}_s)\tilde{S}_s^{\,2}\sigma^2 ds]$$
$$\leq \left\{E^*[\int_0^T (e^{-2rs}\tilde{S}_s^{\,2}\sigma^2)^2 ds]\right\}^{1/2} \left\{E^*[\int_0^T (\frac{\partial P}{\partial x} - \frac{\partial P_{FFT}}{\partial x})^4(s, \tilde{S}_s)ds]\right\}^{1/2},$$

where $\frac{\partial P_{FFT}}{\partial x}$ is obtained by our FFT option pricing method.

(1)

$$
\begin{aligned}
& \left\{E^*[\int_0^T (e^{-2rs}\tilde{S}_s^{\,2}\sigma^2)^2 ds]\right\}^{1/2} \\
&= \left\{\sigma^4 \int_0^T E^*[(e^{-rs}\tilde{S}_s)^4]ds\right\}^{1/2} \quad \text{(by Fubini's theorem)} \\
(e^{-rs}\tilde{S}_s &= e^{-rs}S_0 e^{(r+\frac{1}{2}\sigma^2)s+\sigma\tilde{W}_s} = S_0 e^{\frac{1}{2}\sigma^2 s+\sigma\tilde{W}_s}) \\
&= \left\{\sigma^4 \int_0^T E^*[(S_0 e^{\frac{1}{2}\sigma^2 s+\sigma\tilde{W}_s})^4]ds\right\}^{1/2} \\
&= \left\{S_0^4\sigma^4 \int_0^T E^*[e^{2\sigma^2 s+4\sigma\tilde{W}_s}]ds\right\}^{1/2} \\
&= \left\{S_0^4\sigma^4 \int_0^T e^{10\sigma^2 s}ds\right\}^{1/2} \\
&= \left\{\frac{1}{10}S_0^4\sigma^2(e^{10\sigma^2 s}-1)\right\}^{1/2}
\end{aligned}
$$

(2)

$$
\begin{aligned}
& E^*[\int_0^T (\frac{\partial P}{\partial x}-\frac{\partial P_{FFT}}{\partial x})^4(s,\tilde{S}_s)ds] \\
\leq\ & E^*[\int_0^T (\left|\frac{\partial P}{\partial x}-\frac{\partial P_{FFT}}{\partial x}\right|)^4(s,\tilde{S}_s)ds],
\end{aligned}
$$

note

$$
\begin{aligned}
\left|\frac{\partial P}{\partial x}-\frac{\partial P_{FFT}}{\partial x}\right| \leq\ & \tilde{S}_s^{\,\alpha}\frac{\exp(-\alpha k)\exp((r+\frac{1}{2}\sigma^2)(T-s)\alpha)}{N\eta(1-\exp(-\sigma^2\alpha(T-t)\eta))} \\
& *\exp(-\frac{1}{2}\sigma^2(T-s)(N\eta-\alpha)^2)\exp(-\sigma^2\alpha(T-s)\eta N) \\
& +\sum_{j=1}^{\infty}[\exp(-\frac{\alpha\pi j}{\eta})+\exp(-\frac{\alpha\pi j}{\eta})\exp(-pk) \\
& *\tilde{S}_s\exp((r+\frac{1}{2}\sigma^2)(T-s)p+\frac{\sigma^2(T-s)}{2}p^2)].
\end{aligned}
$$

And

$$
\begin{aligned}
& \tilde{S}_s^{\,\alpha}\frac{\exp(-\alpha k)\exp((r+\frac{1}{2}\sigma^2)(T-s)\alpha)}{N\eta(1-\exp(-\sigma^2\alpha(T-s)\eta))} \\
& *\exp(-\frac{1}{2}\sigma^2(T-s)(N\eta-\alpha)^2)\exp(-\sigma^2\alpha(T-s)\eta N) \\
\leq\ & \frac{1}{U}\tilde{S}_s^{\,\alpha}\exp((r+\frac{1}{2}\sigma^2)(T-s)\alpha) \\
\leq\ & \frac{1}{U}\tilde{S}_s^{\,\alpha}\exp(D_1(T-s)),
\end{aligned}
$$

where $U = N\eta$ is the truncated upper bound and for some constant $D_1$.

The other series term

$$
\begin{aligned}
&\sum_{j=1}^{\infty}[\exp(-\frac{\alpha\pi j}{\eta}) + \exp(-\frac{\alpha\pi j}{\eta})\exp(-pk) * \tilde{S}_s \exp((r+\frac{1}{2}\sigma^2)(T-s)p + \frac{\sigma^2(T-s)}{2}p^2)] \\
= \;& \sum_{j=1}^{\infty}\exp(-\frac{\alpha\pi j}{\eta}) + \exp(-pk)\tilde{S}_s \exp((r+\frac{1}{2}\sigma^2)(T-s)p + \frac{\sigma^2(T-s)}{2}p^2)\sum_{j=1}^{\infty}\exp(-\frac{\alpha\pi j}{\eta}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{j=1}^{\infty}\exp(-\frac{\alpha\pi j}{\eta}) &= \frac{\exp(-\frac{\alpha\pi}{\eta})}{1-\exp(-\frac{\alpha\pi}{\eta})} \\
&\leq \frac{1}{1-\exp(-\frac{\alpha\pi}{\eta})} \\
&\approx \exp(-\frac{\alpha\pi}{\eta}) \quad \text{(if } \eta \text{ is small enough)} \\
&\leq \eta.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{j=1}^{\infty}[\exp(-\frac{\alpha\pi j}{\eta}) + \exp(-\frac{\alpha\pi j}{\eta})\exp(-pk) * \tilde{S}_s \exp((r+\frac{1}{2}\sigma^2)(T-s)p + \frac{\sigma^2(T-s)}{2}p^2)] \\
\leq \;& \eta(1+\tilde{S}_s \exp((r+\frac{1}{2}\sigma^2)(T-s)p + \frac{\sigma^2(T-s)}{2}p^2)) \\
\leq \;& \eta(1+\tilde{S}_s \exp(D_2(T-s))) \qquad \text{for some constant } D_2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&E^*[\int_0^T (\left|\frac{\partial P}{\partial x} - \frac{\partial P_{FFT}}{\partial x}\right|)^4(s,\tilde{S}_s)ds] \\
= \;& \int_0^T [E^*(\left|\frac{\partial P}{\partial x} - \frac{\partial P_{FFT}}{\partial x}\right|)^4(s,\tilde{S}_s)]ds \\
\leq \;& \int_0^T [E^*(\frac{1}{U}\tilde{S}_s{}^{\alpha}\exp(D_1(T-s)) + \eta(1+\tilde{S}_s\exp(D_2(T-s))))^4]ds \\
\leq \;& \int_0^T [E^* 8(\frac{1}{U}\tilde{S}_s{}^{\alpha}\exp(D_1(T-s))^4 + 8(\eta(1+\tilde{S}_s\exp(D_2(T-s)))))^4]ds \\
\approx \;& \int_0^T E^*[8(\frac{1}{U}\tilde{S}_s{}^{\alpha}\exp(D_1(T-s))^4]ds + \int_0^T E^*[8(\eta\tilde{S}_s\exp(D_2(T-s)))))^4]ds \\
= \;& \frac{8^4}{U^4}\int_0^T \exp(4D_1(T-s))E^*[\tilde{S}_s{}^{4\alpha}]ds \\
& + \eta^4 8^4 \int_0^T \exp(4D_2(T-s))E^*[\tilde{S}_s{}^4]ds,
\end{aligned}
$$

since $E^*[\tilde{S_s}^{4\alpha}]$ and $E^*[\tilde{S_s}^4]$ are bounded, there exists $F_1$ and $F_2$ such as $E^*[\tilde{S_s}^{4\alpha}] \leq F_1$ and $E^*[\tilde{S_s}^4] \leq F_2$, then

$$\begin{aligned}
& \frac{8^4}{U^4}\int_0^T \exp(4D_1(T-s))E^*[\tilde{S_s}^{4\alpha}]ds \\
& +\eta^4 8^4 \int_0^T \exp(4D_2(T-s))E^*[\tilde{S_s}^4]ds \\
\leq\ & \frac{8^4}{U^4}\int_0^T F_1 \exp(4D_1(T-s))ds + \eta^4 8^4 \int_0^T F_2 \exp(4D_2(T-s))ds \\
=\ & -F_1\frac{8^4}{U^4}\frac{1}{4D_1}(1-\exp(4D_1T) - F_2\eta^4 8^4\frac{1}{4D_2}(1-\exp(4D_2T)
\end{aligned}$$

■

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