# 國 立 交 通 大 學

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# Numerical Ranges of Companion Matrices and Normal Operators

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## 摘　要

在本篇論文中，我們研究兩類算子的數值域。針對友矩陣部分，我們證明$3\times 3$可分解友矩陣的數值域會包含相對應的$2\times 2$友矩陣的數值域的充要調件為$3\times 3$可分解友矩陣的行列式值的絕對值會大於 1。然而，相對應的結果在一般的$n\times n$可分友矩陣並不正確。針對規範算子的部份，我們將數值域以乘積算子的表現函數來表示。

中 華 民 國 九 十 六 年 六 月

# Numerical Ranges of Companion Matrices and Normal Operators


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**Abstract**

In this thesis, we study the numerical ranges of two kinds of operators. For companion matrices, we show that the numerical range of a 3-by-3 reducible companion matrix $C(p)$ contains the numerical range of the 2-by-2 companion matrix $C((1/3)p')$ if and only if the absolute value of its determinant is greater than 1. However, the corresponding assertion for $n$-by-$n$ reducible companion matrices is false. For a normal operator, we express its numerical range in terms of the function in its multiplication operator representation.

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# Contents

# 1 Introduction

Let $A$ be a bounded linear operator on the complex Hilbert space $H$. The numerical range of $A$ is the set of complex numbers of the form $\langle Ax, x\rangle$, where $x$ is any unit vector in $H$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $H$. We denote it by $W(A)$. Namely,

$$W(A) = \{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}.$$

If $H$ is an $n$-dimensional space, then $A$ can be seen as an $n \times n$ complex matrix.

The information about the numerical range of an $n \times n$ complex matrix has been quite well known. The shape of $W(A)$ for a $2 \times 2$ matrix $A$ is known to be a (possibly degenerate) elliptic disc. More generally, the numerical range of an $n \times n$ ($n \geq 3$) matrix can be expressed in terms of an algebraic curve associated with the matrix [6]. Consider a $3 \times 3$ reducible companion matrix $A$ associated with a monic polynomial $p$ and the companion matrix $B$ associated with the monic polynomial $(1/3)p'$. In Section 2, we will discuss the relation between $W(A)$ and $W(B)$. We prove that $W(A)$ contains $W(B)$ if and only if $|\det A| \geq 1$. We will also show that the corresponding result does not hold for a $4 \times 4$ reducible companion matrix.

If $A$ is a normal operator on the Hilbert space $H$, it is known that the numerical range of $A$ can be expressed by its spectral measure [2]. If $H$ is separable, then there exists a $\sigma$-finite measure space $(X, \Omega, \mu)$ and a function $f$ in $L^\infty(\mu)$ such that $A$ is unitarily equivalent to the multiplication operator $M_f$. In Section 3, we will show that $W(A)$ can be described by the behavior of $f$.

We now introduce the notations to be used in the following sections. The boundary of a subset $\triangle$ in the plane is denoted by $\partial\triangle$. The convex hull of a set $\triangle$, denoted by $\triangle^\wedge$, is the smallest convex set including $\triangle$. The interior of $\triangle$ is denoted by Int $\triangle$. The closure of $\triangle$ is denoted by $\overline{\triangle}$. $A^*$ is the adjoint operator of $A$. Next, we list properties of the numerical range of an operator.

**Proposition 1.1.** *Let $A$ be an operator on $H$. Then the following hold:*

(1) *$W(A)$ is bounded. Moreover, if $A$ acts on a finite-dimensional space, then*

*$W(A)$ is compact.*

(2) *$W(A+aI) = W(A) + a$ for any complex number $a$.*

(3) *$W(bA) = bW(A)$ for any complex number $b$.*

(4) *$W(A)$ is a convex subset of $\mathbb{C}$.*

(1), (2) and (3) are easily obtained from the definition. (4) appeared in [4, p. 315].

It follows from the definition of the numerical range that the diagonal entries $a_{ii}$ of a matrix $A$ are all in $W(A)$.

**Theorem 1.2.** *Let $A$ and $B$ be operators (on possibly different spaces).*

(1) *If $A$ is unitarily equivalent to $B$, then $W(A) = W(B)$.*

(2) *If $B$ dilates to $A$, then $W(B)$ is contained in $W(A)$.*

(3) *If $A = A_1 \bigoplus A_2$, then $W(A)$ equals the convex hull of $W(A_1) \bigcup W(A_2)$.*

Recall that $B$ is said to dilate to $A$ if $A$ is unitarily equivalent to an operator matrix of the form $\begin{bmatrix} B & * \\ * & * \end{bmatrix}$. Both (1) and (2) can be derived directly from the definition. The proof of (3) can be found in [4, p. 116].

In what follows, we explore the relations between the numerical range and the spectrum of an operator. The spectrum of an operator $A$, denoted by $\sigma(A)$, is the set of scalars $z$ for which $A - zI$ is not invertible. The point spectrum $\sigma_p(A)$ of $A$ is the set of eigenvalues of $A$.

**Theorem 1.3.** *For an arbitrary operator $A$, we have*

(1) *$\sigma_p(A) \subseteq W(A)$, and*

(2) *$\sigma(A) \subseteq \overline{W(A)}$.*

(1) can be proved from the definition, and (2) is justified in [4, Problem 214].

## 2 Companion Matrices

Recall that for every complex monic polynomial $p(z) = z^n + a_1 z^{n-1} + ... + a_n$ of degree $n$, there is associated an $n \times n$ matrix

$$(2.1) \qquad C(p) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \end{pmatrix},$$

called the companion matrix of $p$. Note that the characteristic polynomial and minimal polynomial of $C(p)$ are both equal to $p$. We say that a matrix is reducible if it is unitarily equivalent to the direct sum of two other matrices. The numerical ranges of 2-by-2 and 3-by-3 matrices have been known before. Here we give a brief sketch.

**Proposition 2.1.** *Let $A$ be a $2 \times 2$ matrix unitarily equivalent to* $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

(1) *If $b = 0$, then $W(A)$ is the line segment with endpoints $a$ and $c$.*

(2) *If $b \neq 0$ and $a = c$, then $W(A)$ is the circular disc centered at $a$ with radius $|b|/2$.*

(3) *If $b \neq 0$ and $a \neq c$, then $W(A)$ is the elliptic disc with foci $a$ and $c$ and with the length of minor axis $|b|$.*

The proof of this proposition is provided in [5, pp. 20–23].

To describe the numerical ranges of $3 \times 3$ matrices, we need some extra notions. A point in homogeneous coordinates is an ordered triple

$$(x, y, z)$$

of complex numbers $x$, $y$ and $z$ which are not all zero. Two such points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are equivalent if and only if $x_2 = ax_1$, $y_2 = ay_1$, $z_2 = az_1$ for some

$a \neq 0$. Then the complex projective plane is the set of all the equivalence classes $[x, y, z]$. That is,

$$\mathbb{CP}^2 = \{[x, y, z] : (x, y, z) \in \mathbb{C}^3 - \{0\}\}.$$

The point $[x, y, z]$ in $\mathbb{CP}^2$ with $z \neq 0$ can be mapped to the point $(x/z, y/z)$ in nonhomogeneous coordinates. On the other hand, the point $(x, y)$ in nonhomogeneous coordinates becomes $[x, y, 1]$ in $\mathbb{CP}^2$. The points $[x, y, 0]$ in $\mathbb{CP}^2$ are points at infinity. If $C$ is an algebraic curve in $\mathbb{CP}^2$ given by $p(x, y, z) = 0$, where $p$ is a homogeneous polynomial in $x, y$ and $z$, then its dual curve $C^*$ is given by

$$\{[u, v, w] \in \mathbb{CP}^2 : ux + vy + wz = 0 \text{ is a tangent line of } C\}.$$

For an $n \times n$ matrix $A$, Re $A = (A + A^*)/2$ and Im $A = (A - A^*)/(2i)$ denote the real and imaginary parts of $A$, respectively. Define the degree-$n$ homogeneous polynomial $p_A$ in $x, y$ and $z$ by

$$p_A(x, y, z) = \det(x\text{Re}A + y\text{Im}A + zI_n). \tag{2.2}$$

Kippenhahn [6] proved that the numerical range of an $n \times n$ matrix $A$ can be described in terms of $p_A$ as follows.

**Theorem 2.2.** *The numerical range of $A$ equals the convex hull of the real points of the dual curve of $p_A(x, y, z) = 0$.*

Next we state the classification for the numerical ranges of $3 \times 3$ matrices, which is also given by Kippenhahn [6].

**Proposition 2.3.** *If $A$ is a $3 \times 3$ matrix and $p_A$ is defined as in (2,2), then $W(A)$ can be classified into four classes:*

(1) *If $p_A$ is the product of three linear factors:*

$$p_A(x, y, z) = \prod_{j=1}^{3} (z + a_j x + b_j y),$$

*then $A$ is normal and $W(A)$ is the closed triangular region with vertices $(a_j, b_j), j = 1, 2, 3$ (it may degenerate to a line segment or a point ).*

(2) *If $p_A$ is the product of a linear and an irreducible quadratic factor:*

$$p_A(x, y, z) = (z + ax + by)q(x, y, z),$$

*then $W(A)$ is the convex hull of the point $(a, b)$ and the ellipse given by the dual curve of $q(x, y, z) = 0$. Hence $W(A)$ is an elliptic disc possibly with a cone added to it; in the latter case, $A$ is reducible.*

(3) *If $p_A$ is irreducible and the dual curve of $p_A = 0$ has degree four, then $W(A)$ has a line segment on the boundary.*

(4) *If $p_A$ is irreducible and the dual curve of $p_A = 0$ has degree six, then $W(A)$ is an oval set.*

We now start to consider our problem on the numerical ranges of companion matrices. The next two results are from [3].

**Proposition 2.4.** *If $A$ is a companion matrix, then $\lambda A$ is unitarily equivalent to a companion matrix for any $\lambda$, $|\lambda| = 1$.*

This proposition says that if A is of the form

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_n & a_{n-1} & \dots & a_1 \end{bmatrix},$$

then $\lambda A$ is unitarily equivalent to the companion matrix

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \lambda^n a_n & \lambda^{n-1} a_{n-1} & \dots & \lambda a_1 \end{bmatrix}.$$

The detailed proof can be found in [3, Lemma 2.8].

**Theorem 2.5.** *An $n \times n$ $(n \geq 2)$ companion matrix $A$ is reducible if and only if its eigenvalues are of the form:*

$$a{\omega_n}^{j_1}, \ldots, a{\omega_n}^{j_p}, (\frac{1}{\bar{a}}){\omega_n}^{j_{p+1}}, \ldots, (\frac{1}{\bar{a}}){\omega_n}^{j_n},$$

*where $a \neq 0$, $\omega_n = \exp(2\pi i/n)$ denotes the nth primitive root of 1, $1 \leq p < n$, and $\{j_1, \ldots, j_p\}$ and $\{j_{p+1}, \ldots, j_n\}$ form a partition of $\{0, 1, \ldots, n-1\}$. In this case, $A$ is unitarily equivalent to $A_1 \bigoplus A_2$ with $\sigma(A_1) = \{a{\omega_n}^{j_1}, \ldots, a{\omega_n}^{j_p}\}$ and $\sigma(A_2) = \{(1/\bar{a}){\omega_n}^{j_{p+1}}, \ldots, (1/\bar{a}){\omega_n}^{j_n}\}$.*

This theorem is verified in [3, Theorem 1.1].

The next theorem is our main result in this section. It partially solves a question posed by J. Zemánek.

**Theorem 2.6.** *Let $A$ be a $3 \times 3$ reducible companion matrix and $p$ be its associated polynomial. For the monic polynomial $(1/3)p'$, there is associated a $2 \times 2$ companion matrix $B$. Then their numerical ranges $W(A)$ and $W(B)$ have the following containment relations:*

(1) *$W(A) \bigcap W(B) \neq \emptyset$.*

(2) *If $|\det A| < 1$, then $W(B) \nsubseteq W(A)$.*

(3) *If $|\det A| \geq 1$, then $W(B) \subseteq W(A)$. Moreover, $\partial W(A) \bigcap \partial W(B) = \emptyset$ if $|\det A| > 1$, and $\partial W(A)$ intersects $\partial W(B)$ at exactly three points if $|\det A| = 1$.*

*Proof.* Since $A$ and $B$ are companion matrices, 0 is in both $W(A)$ and $W(B)$. This proves (1).

For (2) and (3), we may assume by Theorem 2.5 that the eigenvalues of $A$ are

$$a, \frac{1}{\bar{a}}\omega, \frac{1}{\bar{a}}\omega^2,$$

where $\omega = \exp(2\pi i/3)$ is the 3rd primitive root of 1. Let $e^{-i\theta}$ be such that $ae^{-i\theta} = t > 0$. Since the characteristic polynomial of $A$ is

$$\begin{aligned} p(z) &= (z-a)(z-\frac{1}{a}\omega)(z-\frac{1}{a}\omega^2) \\ &= z^3 - (a-\frac{1}{a})z^2 - (\frac{a}{a}-\frac{1}{a^2})z - \frac{a}{a^2} \\ &= z^3 - (t-\frac{1}{t})e^{i\theta}z^2 - (1-\frac{1}{t^2})e^{2i\theta}z - \frac{1}{t}e^{3i\theta}, \end{aligned}$$

the $3\times 3$ reducible companion matrix $A$ is of the form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t}e^{3i\theta} & (1-\frac{1}{t^2})e^{2i\theta} & (t-\frac{1}{t})e^{i\theta} \end{bmatrix}.$$

A direct computation shows that

$$\frac{1}{3}p'(z) = z^2 - \frac{2}{3}(t-\frac{1}{t})e^{i\theta}z - \frac{1}{3}(1-\frac{1}{t^2})e^{2i\theta},$$

and thus $B$ is of the form

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{3}(1-\frac{1}{t^2})e^{2i\theta} & \frac{2}{3}(t-\frac{1}{t})e^{i\theta} \end{bmatrix}.$$

It follows from Proposition 2.4 that $e^{-i\theta}A$ is unitarily equivalent to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t} & 1-\frac{1}{t^2} & t-\frac{1}{t} \end{bmatrix}.$$

Similarly, $e^{-i\theta}B$ is unitarily equivalent to

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{3}(1-\frac{1}{t^2}) & \frac{2}{3}(t-\frac{1}{t}) \end{bmatrix}.$$

Hence we can assume that $a > 0$. Under this assumption, $\det A = 1/a > 0$. Obviously,

$$\begin{aligned} &\det A > 1 \Leftrightarrow a < 1, \\ &\det A = 1 \Leftrightarrow a = 1, \\ &\det A < 1 \Leftrightarrow a > 1. \end{aligned}$$

Using Theorem 2.5, we derive that the reducible companion matrix $A$ is unitarily equivalent to

$$A_1 \bigoplus A_2 = \begin{bmatrix} a \end{bmatrix} \bigoplus \begin{bmatrix} \frac{1}{a}\omega & b \\ 0 & \frac{1}{a}\omega^2 \end{bmatrix},$$

where the entry $b$ satisfies

$$a^2 + |\frac{1}{a}\omega|^2 + |b|^2 + |\frac{1}{a}\omega^2|^2 = 1 + 1 + (\frac{1}{a})^2 + (1 - \frac{1}{a^2})^2 + (a - \frac{1}{a})^2,$$

and can be taken to be nonnegative. A simple computation yields

$$b = |1 - \frac{1}{a^2}|.$$

Clearly, if $a = 1$, then $b = 0$. In this case, $A$ is normal and $W(A)$ is the regular triangular region with vertices $1, \omega$ and $\omega^2$. When $a \neq 1$, the numerical range of $A_1$ is the singleton $\{a\}$, and the numerical range of $A_2$ is the elliptic disc with foci $\omega/a$ and $\omega^2/a$ and with the length of minor axis $|1 - 1/a^2|$ by Proposition 2.1. Therefore the boundary of $W(A_2)$ is given by the equation

$$\frac{(x + \frac{1}{2a})^2}{\frac{1}{4}(1 - \frac{1}{a^2})^2} + \frac{y^2}{\frac{1}{4}(\frac{a^4 + a^2 + 1}{a^4})} = 1,$$

which we call $\Gamma_A$. Note that the center of $\Gamma_A$, labeled $c_A$, is at $-1/(2a)$. It is easy to see that when $a = 1$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $W(B)$ is the circular disc centered at 0 with radius 1/2. In this case, $W(A)$ contains $W(B)$ and their boundaries intersect at exactly three points. If $a \neq 1$, by a brief computation, $B$ is unitarily equivalent to

$$\begin{bmatrix} \frac{a^2 - 1 + \sqrt{a^4 + a^2 - 2}}{3a} & c \\ 0 & \frac{a^2 - 1 - \sqrt{a^4 + a^2 - 2}}{3a} \end{bmatrix},$$

where the nonnegative $c$ satisfies

$$|\frac{a^2 - 1 + \sqrt{a^4 + a^2 - 2}}{3a}|^2 + |\frac{a^2 - 1 - \sqrt{a^4 + a^2 - 2}}{3a}|^2 + c^2 = 1 + (\frac{1}{3}(1 - \frac{1}{a^2}))^2 + (\frac{2}{3}(a - \frac{1}{a})^2).$$

Hence the preceding equation yields

$$c = \frac{2a^2+1}{3a^2} \quad \text{if } a > 1,$$

and

$$c = \sqrt{\frac{4}{9}a^2 + \frac{8}{9} - \frac{4}{9a^2} + \frac{1}{9a^4}} \quad \text{if } a < 1.$$

Again, Proposition 2.1 says that the numerical range of $B$ is an elliptic disc and the boundary of $W(B)$ is given by the equation

$$\frac{(x - \frac{a^2-1}{3a})^2}{(\frac{2}{9} - \frac{1}{9a^2} + \frac{1}{36a^4} + \frac{a^2}{9})} + \frac{y^2}{(\frac{2a^2+1}{6a^2})^2} = 1,$$

which we call $\Gamma_B$. Note that the center of $\Gamma_B$, labeled $c_B$, is at $(a^2-1)/(3a)$. Obviously, the points $a, c_A$ and $c_B$ are on the $x$-axis and satisfy $a > c_B > c_A$. As the point $a$ may be in or out of $(\Gamma_A)^\wedge$, we have two different cases to consider. These are illustrated in the following figures.

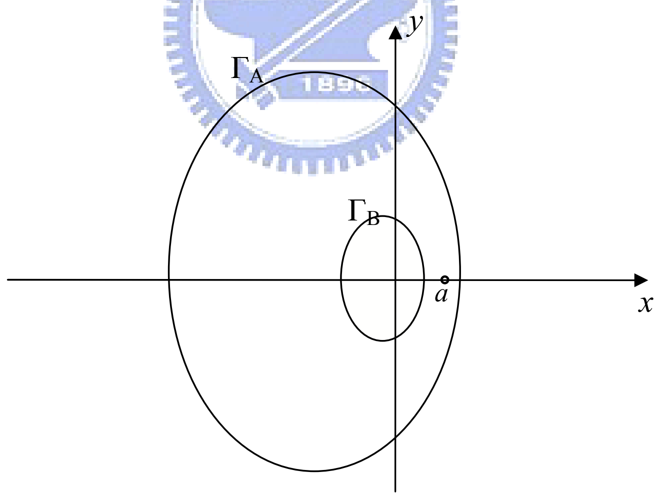


Figure 1: $a$ is in $(\Gamma_A)^\wedge$.

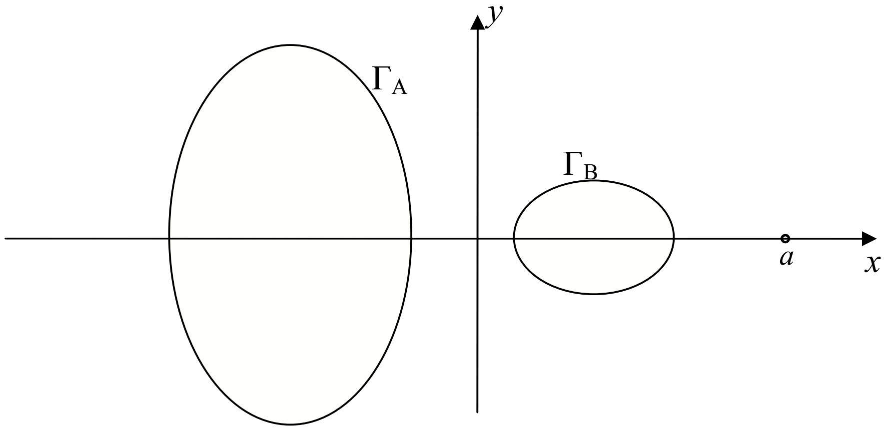


Figure 2: $a$ is not in $(\Gamma_A)^\wedge$.

If $a$ is in $(\Gamma_A)^\wedge$, then we need only check that $\Gamma_B$ is in the interior of $(\Gamma_A)^\wedge$. This can be observed from Figure 1 visually. If $a$ is not in $(\Gamma_A)^\wedge$, then let $m_A$ (resp., $m_B$) denote the slope of the tangent line from point $a$ to $\Gamma_A$ (resp., $\Gamma_B$). Since $W(B) \subseteq W(A)$ if and only if ${m_A}^2 \geq {m_B}^2$, to complete the proof, we need compare the magnitudes of ${m_A}^2$ and ${m_B}^2$. This can be observed from Figure 2. For convenience, let $b_1$ and $b_2$ denote one half of the lengths of the minor axis of $\Gamma_A$ and $\Gamma_B$, respectively.

Our discussion is now divided into two cases:

(a) If $a$ is in $(\Gamma_A)^\wedge$, then $d(a, c_A) \leq b_1$, that is,

$$|a - \frac{-1}{2a}| \leq \frac{1}{2}|1 - \frac{1}{a^2}|.$$

By computation, this holds if and only if $a \leq 1/2$. We claim that in this case

$$b_1 > d(c_A, c_B) + b_2, \tag{2.3}$$

that means

$$\frac{1}{2}(\frac{1}{a^2} - 1) > \frac{a}{3} + \frac{1}{6a} + \sqrt{\frac{2}{9} - \frac{1}{9a^2} + \frac{1}{36a^4} + \frac{a^2}{9}}, \tag{2.4}$$

and the point at the major axis of $\Gamma_B$

$$(\frac{a^2-1}{3a}, \frac{2a^2+1}{6a^2}) \qquad \text{is in } (\Gamma_A)^\wedge, \tag{2.5}$$

that means

$$(2.6)\qquad \frac{(\frac{a^2-1}{3a}+\frac{1}{2a})^2}{\frac{1}{4}(1-\frac{1}{a^2})^2}+\frac{(\frac{2a^2+1}{6a^2})^2}{\frac{1}{4}(\frac{a^4+a^2+1}{a^4})}\leqslant 1.$$

For $a \leq 1/2$, (2,4) and (2.6) are easily seen to be true. From (2.3) and (2.5), it implies that $\Gamma_B$ is contained in the interior of $(\Gamma_A)^\wedge$, and hence $W(B) \subseteq W(A)$.

(b) Assume that $a$ is not in $(\Gamma_A)^\wedge$. The tangent lines from $a$ to $\partial\Gamma_A$ are given by

$$y-0=m_A(x-(-\frac{1}{2a}))\pm\sqrt{m_A{}^2(\frac{1}{4}(1-\frac{1}{a^2})^2)+(\frac{a^4+a^2+1}{4a^4})}.$$

Since they pass through the point $(a,0)$, a calculation shows that

$$m_A{}^2=\frac{a^4+a^2+1}{4a^6+3a^4+3a^2-1}.$$

Applying this formula to $B$, one can get

$$m_B{}^2=\frac{4a^4+4a^2+1}{12a^6+8a^4+8a^2-1}.$$

Thereby it leads to

$$\frac{m_A{}^2}{m_B{}^2}=\frac{(a^4+a^2+1)(12a^6+8a^4+8a^2-1)}{(4a^4+4a^2+1)(4a^6+3a^4+3a^2-1)}.$$

Noting that

$$\begin{aligned}&(a^4+a^2+1)(12a^6+8a^4+8a^2-1)-(4a^4+4a^2+1)(4a^6+3a^4+3a^2-1)\\&=4a^2(-a^6+1)(a^2+2)\end{aligned}$$

is positive when $0 < a < 1$ and negative when $a > 1$, we have $m_A{}^2 < m_B{}^2$ when $a > 1$ and $m_A{}^2 > m_B{}^2$ when $a < 1$. Notice that $m_A{}^2 > m_B{}^2$ means that the boundary of $W(A)$ intersects $W(B)$ at no point.

As a conclusion, we have $W(B) \subseteq W(A)$ when $a < 1$ and $W(B) \nsubseteq W(A)$ when $a > 1$. Our proof is completed.

$\square$

For $n \times n$ $(n \geq 4)$ reducible companion matrices, assertion (2) in Theorem 2.6 is in general false.

**Example 2.7.** Let $p(z) = (z-20)(z+1/20)(z-i/20)(z+i/20)$ be a monic polynomial. Then

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/400 & 399/8000 & 399/400 & 399/20 \end{bmatrix}$$

is the associated $4 \times 4$ reducible companion matrix. The $3 \times 3$ companion matrix $B$ associated with the monic polynomial $(1/4)p'$ is

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 399/32000 & 399/800 & 1197/80 \end{bmatrix}.$$

Although $\det A = 1/400 < 1$, the numerical range of $B$ is contained in the numerical range of $A$. We prove this via Theorem 2.2. First, we calculate the homogeneous polynomial $p_A$ as follows.

$$\begin{aligned} & p_A(x,y,z) \\ &= \det\left( x \begin{bmatrix} 0 & 1/2 & 0 & 1/800 \\ 1/2 & 0 & 1/2 & 399/16000 \\ 0 & 1/2 & 0 & 799/800 \\ 1/800 & 399/16000 & 799/800 & 399/20 \end{bmatrix} \right. \\ &\qquad \left. + y \begin{bmatrix} 0 & -i/2 & 0 & i/800 \\ i/2 & 0 & -i/2 & 399i/16000 \\ 0 & i/2 & 0 & -i/800 \\ -i/800 & -399i/16000 & i/800 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z & (x-iy)/2 & 0 & (x+iy)/800 \\ (x+iy)/2 & z & (x-iy)/2 & 399(x+iy)/16000 \\ 0 & (x+iy)/2 & z & (799x-iy)/800 \\ (x-iy)/800 & 399(x-iy)/16000 & (799x+iy)/800 & z+399x/20 \end{bmatrix} \end{aligned}$$

$$= z^4 + \frac{399}{20}xz^3 - (\frac{383520001}{256 \times 10^6}x^2 + \frac{128160001}{256 \times 10^6}y^2)z^2$$

$$-(\frac{159201}{16 \times 10^3}x^3 + \frac{159999}{16 \times 10^3}xy^2)z + \frac{159201}{64 \times 10^4}x^4 + \frac{160801}{64 \times 10^4}x^2y^2$$

$$= (z+20x)(z^3 - \frac{1}{20}z^2x - \frac{128160001}{256 \times 10^6}z^2y - \frac{127520001}{256 \times 10^6}zx^2 + \frac{160801}{128 \times 10^5}y^2x + \frac{159201}{128 \times 10^5}x^3).$$

Similarly, $p_B(x, y, z) =$

$$z^3 + \frac{1197}{80}xz^2 - (\frac{1281440801}{4096 \times 10^6}y^2 + \frac{3324320801}{4096 \times 10^6}x^2)z - \frac{383838399}{1024 \times 10^5}y^2x - \frac{382561599}{1024 \times 10^5}x^3.$$

Hence $W(A)$ is the convex hull of the point $(20, 0)$ plus the dual curve of the polynomial

$$z^3 - \frac{1}{20}z^2x - \frac{128160001}{256 \times 10^6}z^2y - \frac{127520001}{256 \times 10^6}zx^2 + \frac{160801}{128 \times 10^5}y^2x + \frac{159201}{128 \times 10^5}x^3.$$

$W(B)$ is the convex hull of the dual curve of the polynomial

$$z^3 + \frac{1197}{80}xz^2 - (\frac{1281440801}{4096 \times 10^6}y^2 + \frac{3324320801}{4096 \times 10^6}x^2)z - \frac{383838399}{1024 \times 10^5}y^2x - \frac{382561599}{1024 \times 10^5}x^3.$$

Their figures are sketched as follows.

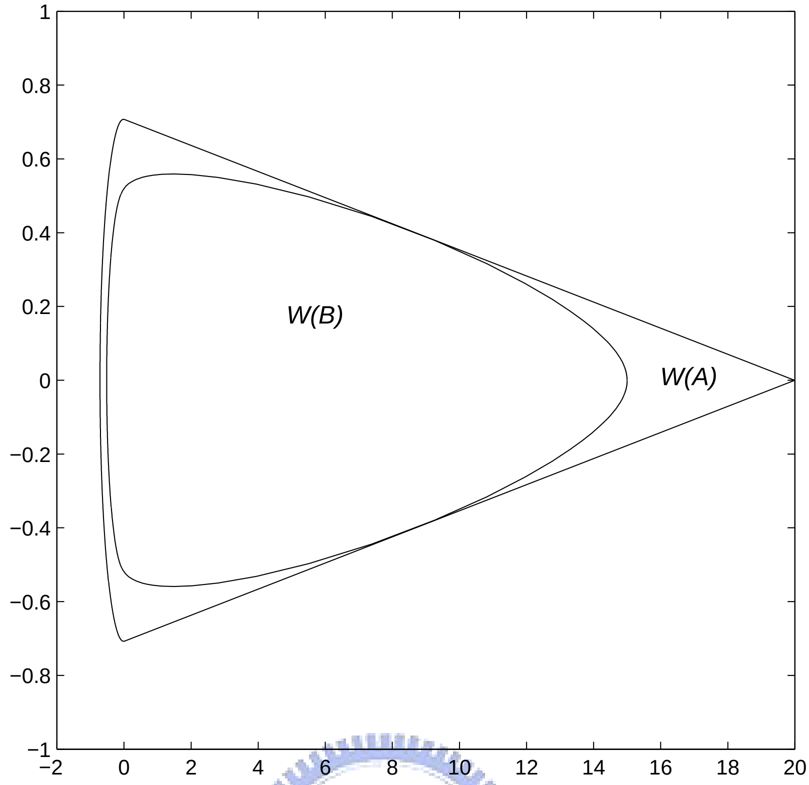


Figure 3: $W(B) \subseteq W(A)$.

We believe that Theorem 2.6(3) should be true for $n \times n$ reducible companion matrices $A$, but its proof is too complicated to be written down explicitly here.

# 3 Normal Operators

Let $A$ be a bounded linear operator on the complex Hilbert space $H$. The numerical range of an operator $A$ is closely related to its spectrum. In fact, we have $\overline{W(A)} \supseteq \sigma(A)^\wedge$. Recall that $A$ is normal if $AA^* = A^*A$. For such an $A$, $\overline{W(A)}$ and $\sigma(A)^\wedge$ are even equal.

**Theorem 3.1.** *If $A$ is a normal operator, then $\overline{W(A)} = \sigma(A)^\wedge$.*

Its proof can be found in [4, pp. 115–116].

Instead of its closure, the numerical range of a normal $A$ can be expressed more precisely in terms of its spectral measure. We introduce the spectral measure and its properties first. Let $X$ be a set, $\Omega$ be a $\sigma$-algebra of subsets of $X$, and $H$ be a Hilbert space. $B(H)$ denotes the algebra of bounded operators on $H$.

**Definition 3.2.** *A spectral measure for $(X, \Omega, H)$ is a function*

$$E : \Omega \to B(H)$$

*such that*

(1) *$E(\triangle)$ is an (orthogonal) projection for $\triangle \in \Omega$;*

(2) *$E(\emptyset) = 0$ and $E(X) = I$;*

(3) *$E(\triangle_1 \cap \triangle_2) = E(\triangle_1)E(\triangle_2)$ for all $\triangle_1$ and $\triangle_2$ in $\Omega$;*

(4) *if $\{\triangle_n\}$ is a sequence of disjoint sets in $\Omega$, then*

$$E\left(\bigcup_n \triangle_n\right) = \sum_n E(\triangle_n),$$

*where $\sum_n E(\triangle_n)$ denotes the convergence in strong operator topology.*

**Theorem 3.3.** *If $A$ is normal on $H$, then there exists a unique spectral measure*

$$E_A : \{\textit{Borel subsets of } \mathbb{C}\} \to B(H)$$

*such that*

(1) $E_A(\sigma(A)) = I$;

(2) $A = \int_{\sigma(A)} z dE_A(z)$.

This theorem is verified in [1, pp. 263–264]. This unique measure $E_A$ is called the spectral measure for $A$.

Next we describe $W(A)$ in terms of its spectral measure.

**Theorem 3.4.** *If $A$ is normal, then the numerical range of $A$ equals the interior of $\sigma(A)^\wedge$ plus the points $z$ on the boundary $\partial\sigma(A)^\wedge$ for which the longest interval $[z_1, z_2]$ in $\partial\sigma(A)^\wedge$ containing $z$ is such that both $E_A([z_1, z])$ and $E_A([z, z_2])$ are nonzero. Namely, $W(A) = (Int\ \sigma(A)^\wedge) \bigcup \{z \in \partial(\sigma(A)^\wedge) : [z_1, z_2]$ is the longest interval in $\partial(\sigma(A)^\wedge)$ containing $z$ such that $E_A([z_1, z]) \neq 0$ and $E_A([z, z_2]) \neq 0\}$.*

**Corollary 3.5.** *If $A$ is normal, then the numerical range of $A$ equals the intersection of all convex Borel subsets $\triangle$ of $\mathbb{C}$ with $E_A(\triangle) = I$. Namely,*

$$W(A) = \bigcap \{\triangle \subseteq \mathbb{C} : \triangle \text{ Borel convex},\ E_A(\triangle) = I\}.$$

Both this theorem and its corollary appeared in [2].

There is another representation for normal operators on a separable Hilbert space. In the following, we will express the numerical range of such a normal operator in terms of the ingredients in its representation. Let $\Omega$ be a $\sigma$-algebra of subsets of $X$ and $\mu$ be a positive measure on $\Omega$. For any $f$ in $L^\infty(\mu)$, $M_f$ denotes the multiplication operator $M_f g = fg$ for $g$ in $L^2(\mu)$.

**Theorem 3.6.** *If $A$ is normal on a separable Hilbert space, then there exists a $\sigma$-finite measure space $(X, \Omega, \mu)$ and a function $f$ in $L^\infty(\mu)$ such that $A$ is unitarily equivalent to $M_f$ on $L^2(\mu)$. In this situation, $\sigma(A) = \sigma(M_f) =$ essential range of $f$.*

Recall that the essential range of $f$ is defined as

$$\text{ess. ran. } (f) = \bigcap \{\overline{f(\triangle)} : \triangle \in \Omega \text{ and } \mu(X \setminus \triangle) = 0\}.$$

The proof of this theorem is provided in [1, p. 265, and pp. 272–273].

The next two results are the expressions of the numerical range of a normal $A$ in terms of the function $f$ in the above representation.

**Theorem 3.7.** *If $A$ is normal on a separable Hilbert space and $f$ is the function as in Theorem 3.6, then the numerical range of $A$ equals the interior of $\sigma(A)^\wedge$ plus the points $z$ on the boundary $\partial(\sigma(A)^\wedge)$ for which the longest interval $[z_1, z_2]$ in $\partial\sigma(A)^\wedge$ containing $z$ is such that both $\mu(f^{-1}([z_1, z]))$ and $\mu(f^{-1}([z, z_2]))$ are positive. Namely, $W(A) = (Int\ \sigma(A)^\wedge) \bigcup \{z \in \partial(\sigma(A)^\wedge) : [z_1, z_2]$ is the longest interval in $\partial\sigma(A)^\wedge$ containing $z$ such that $\mu(f^{-1}([z_1, z])) > 0$ and $\mu(f^{-1}([z, z_2])) > 0\}$.*

The following lemma is useful in proving the theorem.

**Lemma 3.8.** *Let $\triangle$ be a convex subset of $\mathbb{C}$ and $\nu$ be a probability measure on $\triangle$. Then we have*

$$\int_{\triangle} z d\nu(z) \in \triangle.$$

The lemma is trivially the consequence of the theorem in [7].

*Proof of Theorem 3.7.* According to Theorem 3.6, we may assume that $A = M_f$ on $L^2(\mu)$. If $z$ is a point in the interior of $W(A)$, then, by Theorem 3.1, $z$ is in the interior of $\sigma(A)^\wedge$.

Let $z$ be a point on $\partial\sigma(A)^\wedge$ for which the longest interval $[z_1, z_2]$ containing $z$ is such that $\mu(f^{-1}([z_1, z])) = 0$. Assume that $z$ is in $W(A)$, that is, $z = \langle Ag, g\rangle$ for some unit vector $g$ in $L^2(\mu)$. We have

$$
\begin{aligned}
z &= \langle M_f g, g \rangle \\
&= \int_X f|g|^2 d\mu \\
&= \int_{X \setminus f^{-1}([z_1,z])} f|g|^2 d\mu \\
&= \int_{f^{-1}(\sigma(A)^\wedge \setminus [z_1,z])} f|g|^2 d\mu.
\end{aligned}
$$

Define the measure $\nu$ by

$$
\nu(\triangle) = \int_\triangle |g|^2 d\mu \quad \text{for } \triangle \text{ in } \ \Omega.
$$

Then $\nu$ is a probability measure and

$$
z = \int_{\sigma(A)^\wedge \setminus [z_1,z]} w d\nu
$$

by the Randon-Nikodym theorem. Since $\sigma(A)^\wedge \setminus [z_1, z]$ is convex, Lemma 3.8 implies that $z$ is in $\sigma(A)^\wedge \setminus [z_1, z]$, a contradiction. Therefore, $\mu(f^{-1}([z_1, z])) > 0$. Similarly, $\mu(f^{-1}([z, z_2])) > 0$. This proves one direction of the containment.

For the converse, if $z$ is in the interior of $\sigma(A)^\wedge$, then trivially $z$ is in $W(A)$. Let $z$ be in $\partial\sigma(A)^\wedge$ satisfying the asserted condition. Let $g$ be a unit vector in $L^2(\mu)$ with $g = 0$ on $X \setminus f^{-1}([z_1, z])$. Then

$$
\begin{aligned}
&\langle Ag, g \rangle \\
&= \int_X f|g|^2 d\mu \\
&= \int_{f^{-1}([z_1,z])} f|g|^2 d\mu,
\end{aligned}
$$

which belongs to $[z_1, z]$ by Lemma 3.8. This shows that $[z_1, z] \bigcap W(A)$ is nonempty. In the same way, we also have $[z, z_2] \bigcap W(A)$ is nonempty. Thus $z$ is in $W(A)$ by the convexity of $W(A)$. $\square$

**Corollary 3.9.** *If $A$ is normal on a separable Hilbert space and $f$ is the function as in Theorem 3.6, then the numerical range of $A$ equals the intersection of all convex Borel subsets $\triangle$ of $\mathbb{C}$ with $\mu(X \setminus f^{-1}(\triangle)) = 0$. Namely,*

$$W(A) = \bigcap\{\triangle \subseteq \mathbb{C} : \triangle \text{ Borel convex},\ \mu(X \setminus f^{-1}(\triangle)) = 0\}.$$

*Proof.* Let $\triangle$ be a convex Borel subset of $\mathbb{C}$ with $\mu(X \setminus f^{-1}(\triangle)) = 0$. For any unit vector $g$ in $L^2(\mu)$, we have

$$\begin{aligned} z &= \langle Ag, g\rangle \\ &= \int_X f|g|^2 d\mu \\ &= \int_{f^{-1}(\triangle)} f|g|^2 d\mu, \end{aligned}$$

which is in $\triangle$ by Lemma 3.8. Therefore we conclude that $W(A) \subseteq \triangle$. This implies that

$$W(A) \subseteq \bigcap\{\triangle \subseteq \mathbb{C} : \triangle \text{ Borel convex},\ \mu(X \setminus f^{-1}(\triangle)) = 0\}.$$

Conversely, let $B \equiv \bigcap\{\triangle \subseteq \mathbb{C} : \triangle \text{ Borel convex},\ \mu(X \setminus f^{-1}(\triangle)) = 0\}$. We claim that $B$ is contained in $(\text{Int } \sigma(A)^\wedge) \bigcup \{z \in \partial(\sigma(A)^\wedge) : [z_1, z_2]$ is the longest interval in $\partial\sigma(A)^\wedge$ containing $z$ such that $\mu(f^{-1}([z_1, z])) > 0$ and $\mu(f^{-1}([z, z_2])) > 0\}$. If $z$ is in the interior of $B$, then $z$ is in the interior of $\sigma(A)^\wedge$. For the other case, $z$ is in $B$ and also on the boundary $\partial B$ of $B$. Let $[z_1, z_2]$ be the longest interval in $\partial(\sigma(A)^\wedge)$ containing $z$. If $\mu(f^{-1}([z_1, z])) = 0$, then $\triangle \equiv \sigma(A)^\wedge \setminus [z_1, z]$ is Borel convex and

$$\mu(X \setminus f^{-1}(\triangle)) = \mu(X \setminus (f^{-1}(\sigma(A)^\wedge) \setminus f^{-1}([z_1, z]))) = 0.$$

Then $B \subseteq \triangle$ and hence $z$ is in $\triangle$, a contradiction. Therefore, we must have $\mu(f^{-1}([z_1, z])) > 0$. Similarly, $\mu(f^{-1}([z, z_2])) > 0$. We conclude from Theorem 3.7 that $z$ is in $W(A)$, completing the proof. □

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