國立交通大學

應用數學系

碩士論文

排除混色圈的著色完全二分圖

Forbidding Multicolored Cycles in an Edge-colored $K_{m,n}$

研 究 生：裴若宇

指導教授：傅恆霖 教授

中 華 民 國 九 十 八 年 六 月

排除混色圈的著色完全二分圖

Forbidding Multicolored Cycles in an Edge-colored $K_{m,n}$

研 究 生：裴若宇　　Student：Ryo-Yu Pei

指導教授：傅恆霖　　Advisor：Hung-Lin Fu

國 立 交 通 大 學

應 用 數 學 系

碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

June 2009

Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 八 年 六 月

排除混色圈的著色完全二分圖

研究生：裴若宇　　　指導老師：傅恆霖 教授

國 立 交 通 大 學

應 用 數 學 系

摘 要

在一個邊已著色的圖中，若有一個子圖它的每個邊的顏色皆不相同，我們稱這種子圖為混色子圖。在這篇論文中，我們先整理了一些以往有關混色子圖的定理與猜測，我們將依照子圖的種類分成四類來介紹；接下來我們討論在一個完全二部圖 $K_{m,n}$ 中，是否存在一種恰用了 n 色的邊著色可以避免混色的圈出現，我們證明出來當 $2 \le m \le n$ 及 $n \ge 4$ 時，在 $K_{m,n}$ 中一定會產生混色的 C_4。而在下列兩種情形：(1) $m \ge 3$ 且 $n \ge 9$ 或 (2) $m \ge 4$ 且 $n = 7$ 時，在 $K_{m,n}$ 中也會產生混色的 C_6。更進一步的，對於 $k \le m \le 2k$ 且 k 為奇數時，我們找到一種 $2k$ 個顏色的著色法使得 $K_{m,2k}$ 中能避免混色的 C_{2k} 出現。

Forbidding Multicolored Cycles in an Edge-colored $K_{m,n}$

Student: Ryo-Yu Pei

Department of Applied Mathematics

National Chiao Tung University

Hsinchu, Taiwan 30050

Advisor: Hung-Lin Fu

Department of Applied Mathematics

National Chiao Tung University

Hsinchu, Taiwan 30050

Abstract

In an edge-colored graph, a subgraph whose edges are of distinct colors is known as a multicolored (or rainbow) subgraph. In this thesis, we shall first introduce several known results and conjectures related to multicolored subgraph in an edge-colored K_n, according to four categories of multicolored subgraphs. Then, we extend this study to consider whether there is a proper edge-coloring in a complete bipartite graph which forbids multicolored cycles. First, we claim that it is impossible to forbid multicolored 4-cycles in any proper n-edge-coloring of $K_{m,n}$ where $2 \le m \le n$ and $n \ge 4$. Second, we prove that any n-edge-colored $K_{m,n}$ ($m \le n$) contains a multicolored C_6 if (i) $m \ge 3$ and $n \ge 9$; or (ii) $m \ge 4$ and $n = 7$. Finally, if k is odd, we obtain a proper $2k$-edge-coloring of $K_{m,2k}$ which forbids multicolored $(2k)$-cycles where $k \le m \le 2k$.

Acknowledgement

三年前，我進入了交通大學應用數學研究所就讀；也在同一年，我順利擠進教師甄試的窄門，成為新竹高商的正式老師；而因為本身熱愛打籃球，我還加入了交大校女籃；這三年來，自己同時扮演著教師、學生與球員的三重身份，無力感常油然而生；但人總是會在壓力下成長，在困境中學習，在這三年中，我感覺確實學到不少，但也覺得自己一口氣增加了不只三歲！

首先，我最感謝的就是我的指導老師：傅恆霖教授，傅老師對我相當包容，不管是在課業、工作，還是在運動方面，都提供我相當大的協助，如果不是遇到傅老師，那我想我的研究生涯將會是黑白的。再來感謝交大校女籃教練：鄭智仁老師，在忙碌之餘，還能讓我在球場上盡情揮灑汗水，甚至也給我機會讓我在球場上為交大爭光，這種體驗十分難得。接下來要感謝的是新竹高商王承先校長，願意讓我在職進修，半工半讀，甚至在我要離開時也不為難我，王校長真的是我見過最棒、最有教育熱忱的校長。

除了老師們，在我研究生涯中最要感謝的就是貓頭大大，貓頭大大的存在對我來說就像是天上掉下來的禮物、漂流在怒海中的一根浮木、或是上完廁所後的唯一一張衛生紙，另外還有同門的臭賓賓、敏筠、Robin、軒軒、舜婷、施智懷、雁婷學姊、惠蘭學姊等人，都曾在學術上提供過我協助，非常感謝。

再來是球隊學妹們，跟你們玩樂的時光，是我在交大最快樂的回憶，尤其是小瑋跟小易，我們在交大室外場奮勇殺敵到頭破血流的日子，是我非常難忘的。三年來在新竹也交了不少朋友，新商同事、偉小慈、黑 jacky、小培、(偽)應數系女籃、交大田徑隊、…等等，你們都讓我的生活更豐富精采。還有貼心的新商學生們，對於有個忙碌的導師從不抱怨，取而代之的是貼心問候，能教到你們我真的很幸福。

最後，要感謝的是我的家人以及一個愛吃罐頭、個性執著、睡覺四腳朝天、不時會到床上撒尿的孩子－Gucci，Gucci 讓我學到知足常樂、天天開心這些最基本的人生道理，疲累的時候看著 Gucci 的笑臉好像壓力都解除了，一起來吃起士雞肉條吧！

Contents

List of Figures

1 Introduction and Preliminaries

In the study of graph theory, graph decomposition and coloring are two important topics. A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. In graph coloring, we study the vertex-coloring and edge-coloring which deal with the assignments of colors onto the vertex set of G and the edge set of G respectively.

We combine these two topics together in this thesis. In an edge-colored graph, a subgraph whose edges are of distinct colors is known as a multicolored (or rainbow) subgraph. First, in the study of the edge-colorings of the complete graphs. In 2006, Akbari, Alipour, Fu and Lo [2] showed that there exists an edge-coloring of K_{2n} such that all the edges can be partitioned into edge-disjoint multicolored isomorphic spanning trees. Then consider the complete graph of odd order. In 2005, Constantine [10] partitioned K_n into multicolored Hamiltonian cycles by a given proper n-edge-coloring if n is an odd prime. In addition, he proposed a new conjecture that for any proper n-edge-coloring of K_n, the edges can be partitioned into multicolored unicyclic isomorphic subgraphs. Several years later, Fu and Lo [15] improved above result from n is an odd prime to n is an odd integer and therefore verify the conjecture.

Montellano-Ballesteros and Neumann-Lara [20] presented that if the edges of K_n are colored by n or more colors actually appearing, then there is a multicolored C_3 somewhere. That means, there is no edge-coloring of K_n with n or more colors actually appearing which forbids multicolored cycles. With the same idea, we discuss whether there exists a proper edge-coloring in a complete bipartite graph which forbids multicolored cycles. It is impossible to forbid multicolored 4-cycles in any proper n-edge-coloring of $K_{m,n}$ where $2 \le m \le n$ and $n \ge 4$. How about forbidding multicolored $(2k)$-cycles? In this thesis, the first part of the main results are concerned about the discussion of forbidding multicolored C_6 in a proper n-edge-colored $K_{m,n}$ where $3 \le m \le n$ and $n \ge 6$. We discuss the lower bound of n such that in any proper n-edge-coloring of

$K_{m,n}$, there is a multicolored 6-cycle somewhere. Then, for each smaller m, n, we will give a specific proper n-edge-coloring which forbids multicolored 6-cycles. If k is an odd integer, furthermore, there exists a proper $(2k)$- edge-coloring of $K_{m,2k}$ which forbids multicolored $(2k)$-cycles, where $k \le m \le 2k$.

Now, we introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West [22].

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. In this thesis, all the graphs we consider are simple.

The size of the vertex set $V(G)$, denoted by $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, denoted by $|E(G)|$, is called the *size* of G. When u and v are the endpoints of an edge, written uv in short, they are *adjacent* and are *neighbors*. If vertex v is an endpoint of edge e, then v and e are *incident*. The *neighborhood* of v, written $N(v)$, is the set of vertices adjacent to v. The *degree* of v, written $deg(v)$, is the number of neighbors of v; that is, $deg(v) = |N(v)|$.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G, denoted by $H \subseteq G$. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. A *matching* in G is a set of edges with no shared endpoints. A *perfect matching* in a graph G is a matching that saturates all vertices. A *k-factor* is a spanning subgraph with each degree equal to k. Then a 1-factor and a perfect matching are almost the same thing.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n. A *Hamiltonian cycle* is a graph with a spanning cycle. A graph with no cycles is called *acyclic* and a graph with exactly one cycle is *unicyclic*. A *tree* is a connected acyclic graph. A *spanning tree* is a spanning subgraph that is a tree.

A *complete graph* is a simple graph whose vertices are pairwise adjacent, and the complete graph with n vertices is denoted by K_n. An *independent set* in a graph is a set of pairwise nonadjacent vertices. A graph G is *bipartite* if $V(G)$ is the union of two disjoint sets, called *partite sets* of G. A graph G is *m-partite* if $V(G)$ can be expressed as the union of m independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes s and t, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size n, the complete bipartite graph is called *balanced*, denoted by $K_{n,n}$. Similarly, the complete m-partite graph is denoted by $K_{s_1, s_2, \cdots, s_m}$ if the sets have the sizes $s_1, s_2, \cdots$ and s_m. The balanced complete m-partite graph is denoted by $K_{m(n)}$ where each partite set has n vertices.

An *isomorphism* from a graph G to a graph H is a bijection $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H.

A *k-edge coloring* of G is a labeling from $E(G)$ into a set S, where $|S| = k$. In this thesis, we use $S = \{1, 2, 3, \cdots, k\}$. The labels are *colors*, and the edges which have the same color form a *color class*. A k-edge coloring is *proper* if all incident edges have different labels (i.e., each color class is a matching). The *chromatic index* of a graph G, $\chi'(G)$, is the minimum number k for which G has a proper k-edge coloring. A subgraph in an edge-colored graph is said to be *multicolored* if no two edges have the same color.

If the edges of a graph G are colored by r colors $\{1, 2, \cdots, r\}$, then its *color distribution* $(a_1, a_2, \cdots, a_r)$ means that the number of edges with color i is equal to a_i for every $1 \leq i \leq r$. An edge-coloring of a graph G is called an edge coloring with *complete bipartite decomposition* if each color class forms a complete bipartite subgraph of G. If the edges of G are colored so that no color is appeared in more than k edges, we refer to this as a *k-bounded coloring*. For a vertex v of G, the *color degree* of v, denoted by $\deg_{col}(v)$, is the number of colors on the edges which are incident with v.

Let S be an n-set. A *latin square of order n* based on S is an $n{\times}n$ array in which every element of S is arranged such that each element occurs exactly once in each row and column. For convenience, let $S = \{1, 2, \cdots, n\}$. We denote a latin square of order n based on S by $LS(n) = [\ l_{i,j}\ ]_{n\times n}$ where $l_{i,j} \in S$. An *$m{\times}n$ latin rectangle* ($m \le n$) is an $m{\times}n$ array in which n distinct elements are arranged such that each element occurs at most once in each row and column, denoted by $LR(m, n)$. A *partial latin square of order r* is an $r{\times}r$ array in which n distinct elements are arranged, $n > r$, such that each element occurs at most once in each row and column. A *circulant latin square of order n* is a special $LS(n)$ where each row is rotated one element to the right relative to the preceding row, denoted by L_n. A *transversal* of a $LS(n)$ is a set of n entries from each column and each row such that these n entries are all distinct. Replace $LS(n)$ by partial latin square of order r, its transversal is a set of r entries from each column and each row such that these r entries are all distinct.

1	2
2	1

1	2	3
3	1	2
2	3	1

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Figure 1: Circulant latin squares of order 2, 3, and 4

There is a corresponding relationship between an $m{\times}n$ latin rectangle and a proper n-edge-colored $K_{m,n}$ where $m \le n$. Let $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ be the two partite sets of $K_{m,n}$ and the edge u_iv_j be colored with $l_{i,j}$ where $LR(m, n) = [l_{i,j}]_{m\times n}$ is an $m{\times}n$ latin rectangle, then we have a proper n-edge-colored $K_{m,n}$.

1	2	3	4	5
2	4	5	1	3
3	1	4	5	2

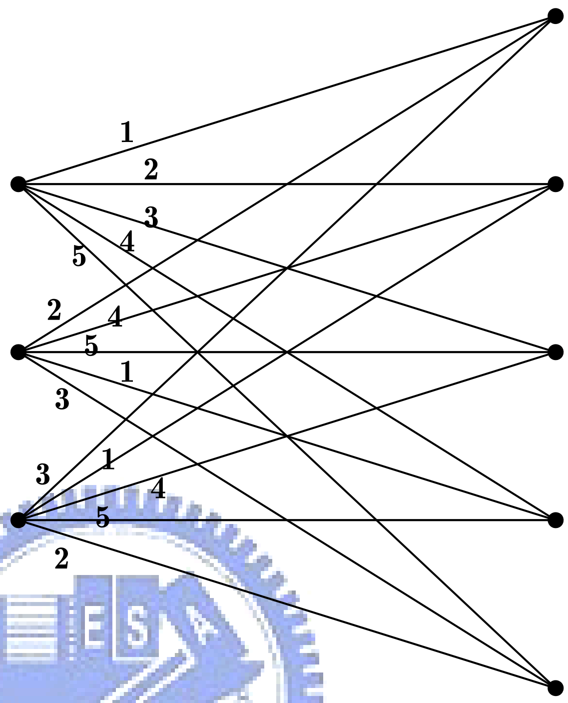

Figure 2: A 3×5 latin rectangle and its corresponding 5-edge colored $K_{3,5}$

2 Known Results

In this section, some theorems and conjectures related to multicolored subgraph in an edge-colored K_n will be reorganized. It can be introduced according to the following four categories of multicolored subgraph.

2.1 Multicolored Spanning Tree

First, consider a non-proper coloring in K_n. Assume that it uses r colors. The following two results were proposed by Akbari and Alipour [1] in 2006.

Theorem 2.1. [1] *If the complete graph K_n, $n \geq 3$, is r-edge-colored and $r \geq \binom{n-2}{2}+2$, then K_n has a multicolored spanning tree.*

Theorem 2.2. [1] *If the complete graph K_n, $n \geq 6$, is r-edge-colored and $r \geq \binom{n-2}{2}+3$, then K_n has two edge-disjoint multicolored spanning trees.*

In the same paper, they also used a different perspective, color distribution, to deal with this problem as follows.

Theorem 2.3. [1] *If the r-edge-colored K_n has a color distribution $(a_1, \cdots, a_r)$ with $1 \leq a_1 \leq \cdots \leq a_r \leq (n+3)/2$ and $r \geq n-1$, then K_n has a multicolored spanning tree.*

Theorem 2.4. [1] *If the r-edge-colored K_n has a color distribution $(a_1, \cdots, a_r)$ with $1 \leq a_1 \leq \cdots \leq a_r \leq n/2$, then K_n has two multicolored spanning trees.*

As early as in 1991, however, Alon, Brualdi and Shader [4] discussed the existence of multicolored spanning trees from the perspective of complete bipartite decomposition.

Theorem 2.5. [4] *Every K_n having an edge-coloring with complete bipartite decomposition contains a multicolored spanning tree.*

On the other hand, the existence of multicolored spanning trees in a proper edge-colored complete graph was discussed. Since $\chi'(K_{2n}) = 2n - 1$, it is natural to ask if there exists a partition of the edges of an edge-colored K_{2n} into multicolored subgraphs each has $2n - 1$ edges. Here are three conjectures related to this problem.

Conjecture 2.6. [11] *For $n > 2$, there exists a proper $(2n–1)$-edge-coloring of K_{2n} such that all edges can be partitioned into n isomorphic multicolored spanning trees.*

Conjecture 2.7. [7] *If $n > 2$, then in any proper edge-coloring of K_{2n} with $2n–1$ colors, all edges can be partitioned into n multicolored spanning trees.*

Conjecture 2.8. [11] *If $n > 2$, then in any proper edge-coloring of K_{2n} with $2n–1$ colors, all edges can be partitioned into n isomorphic multicolored spanning trees.*

For the first conjecture, it has been verified by Akbari, Alipour, Fu and Lo [2] in 2006.

Theorem 2.9. [2] *For $n \geq 3$, K_{2n} can be properly edge-colored with $2n–1$ colors in such a way that the edges can be partitioned into edge-disjoint multicolored isomorphic spanning trees.*

As for Conjecture 2.7, proposed by Brualdi and Hollingsworth [7], they also proved the existence of two multicolored spanning trees in the same paper. Then, the existence of three multicolored spanning trees has been proven by Krussel, Marshall and Verrall [19] in 2002.

Theorem 2.10. [7] *If $n > 2$, then in any proper edge-coloring of K_{2n} with $2n-1$ colors, there exist two edge-disjoint multicolored spanning trees.*

Theorem 2.11. [19] *If $n > 2$, then in any proper edge-coloring of K_{2n} with $2n-1$ colors, there exist three edge-disjoint multicolored spanning trees.*

Later, Kaneko, Kano and Suzuki [18] extended the above theorem from K_{2n} to K_n in 2003.

Theorem 2.12. [18] *Every properly edge-colored K_n ($n \geq 6$) has three edge-disjoint multicolored spanning trees.*

Conjecture 2.8 can imply Conjecture 2.7 easily; therefore, it has not been completely solved yet. A partial result, however, was proposed by Fu and Lo [14] recently.

Theorem 2.13. [14] *In any proper edge-coloring of K_{2n} with $2n-1$ colors, if $n > 2$, then there exist two edge-disjoint isomorphic multicolored spanning trees; and if $n > 13$, then there exist three edge-disjoint isomorphic multicolored spanning trees.*

2.2 Multicolored Cycle

In an edge-colored K_n, it is clear that there is no multicolored cycle if and only if there is no multicolored C_3. Notice that there exists a cycle somewhere in a subgraph of K_n with n edges. Montellano-Ballesteros and Neumann-Lara [20] presented the following results.

Theorem 2.14. [20] *If the edges of K_n are colored by n or more colors actually appearing, then there is a rainbow K_3 somewhere.*

This theorem infers that there is no edge-coloring of K_n with n or more colors which forbids multicolored cycles. Analogous to the multicolored trees, the existence of multicolored cycles in a proper edge-colored complete graph was discussed. It is natural to think about a multicolored Hamiltonian cycle in a proper $(2n+1)$-edge colored K_{2n+1}.

Theorem 2.15. [10] *If $2n+1$ is an odd prime, then there exists a proper $(2n+1)$-edge-coloring of K_{2n+1} such that all edges can be partitioned into n multicolored Hamiltonian cycles.*

Above theorem was provided by Constantine [10] in 2005, and he also gave a relative conjecture.

Conjecture 2.16. [10] *Any proper coloring of the edges of a complete graph on an odd number of vertices allows a partition of the edges into multicolored isomorphic unicyclic subgraphs.*

Theorem 2.15 was improved by Fu and Lo [15] in 2009.

Theorem 2.17. [15] *For any odd integer $2n+1$, there exists a proper $(2n+1)$-edge-coloring* *of K_{2n+1} such that all edges can be partitioned into n multicolored Hamiltonian cycles.*

Now, we consider a k-bounded coloring. For any positive integer k, the problem is to find a positive integer n which is large enough so that every k-bounded edge-colored K_n contains a multicolored Hamiltonian cycle. Here are three relative results. We list them in historical order.

Theorem 2.18. [16] *There exists a constant number c such that if $n \geq ck^3$, then every k-bounded edge-colored K_n has a multicolored Hamiltonian cycle.*

Theorem 2.19. [13] *There exists a constant number c such that if n is sufficiently large and $k \leq n/(c \ln n)$, then every k-bounded edge-colored K_n contains a multicolored Hamiltonian cycle.*

Theorem 2.20. [3] *Let $c < 1/32$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k-bounded edge-colored K_n contains a multicolored Hamiltonian cycle.*

Theorem 2.18 was obtained by Hahn and Thomassen [16] in 1986 and implied that k could grow as fast as $n^{1/3}$ to guarantee that a k-bounded edge-colored K_n contains a multicolored Hamiltonian cycle. In 1993, Frieze and Reed [13] made further progress, see Theorem 2.19. Few years later, in 1995, Albert, Frieze and Reed [3] improved Theorem 2.19 and proved the growth rate of k could in fact be linear.

2.3 Multicolored Matching

The perfect matching only exists in K_{2n} and the general case has been mentioned in 1998 by Woolbright and Fu [23].

Theorem 2.21. [23] *For* $n \geq 3$*, every properly* $(2n-1)$*-edge-colored* K_{2n} *has a rainbow perfect matching.*

There is a conjecture concerning matching a long time ago.

Conjecture 2.22. [6, 21] *In any proper edge-coloring of* $K_{n,n}$ *with n colors,*

(1) *If n is even, then there exists a multicolored matching M with* $|M| = n-1$.

(2) *If n is odd, then there exists a multicolored matching M with* $|M| = n$.

Notice that there is a corresponding relation between a matching in $K_{n,n}$ and a partial transversal in $LS(n)$. We have the following theorem.

Theorem 2.23. [17] *Every latin square has a partial transversal of length at least* $n - 11.053 \log^2 n$.

2.4 Multicolored Path

The length of a multicolored path will increase along with the number of colors. So we can get the following.

Theorem 2.24. [12] *Every r-edge-colored graph G of order n has a multicolored path of length at least* $\lceil (2r)/n \rceil$.

In 2005, Broersma, Li, Woeginger and Zhang [5] obtained the following result.

Theorem 2.25. [5] *Let G be an edge-colored graph. If* $deg_{col}(x) \geq k$ *for every vertex x of G, then for every vertex v of G, there exists a multicolored path starting at v and of length at least* $\lceil (k+1)/2 \rceil$.

Then Chen and Li [8] improved theorem 2.25.

Theorem 2.26. [8] *Let G be an edge-colored graph and* $k \geq 1$ *be an integer. If* $deg_{col}(x) \geq k$ *for every vertex x of G, then there exists a multicolored path of length at least* $\lceil (3k)/5 \rceil + 1$. *Moreover, if* $1 \leq k \leq 7$, *there exists a multicolored path of length at least* $k - 1$.

Theorem 2.27. [9] *Let G be an edge-colored graph and* $k \geq 8$ *be an integer. If* $deg_{col}(x) \geq k$ *for every vertex x of G, then there exists a multicolored path of length at least* $\lceil (2k)/3 \rceil + 1$.

We can get the following corollary by Theorem 2.27.

Corollary 2.28. *In any proper coloring of* K_n, *if* $n \geq 9$, *then there exists a multicolored path of length at least* $\lceil (2n-2)/3 \rceil + 1$.

3 Main Results

Now, we will discuss whether there exists a proper n-edge-coloring in a complete bipartite graph $K_{m,n}$ which forbids multicolored $(2k)$-cycles. For $k \geq 2$ and $2 \leq m \leq n$, we define the *forbidding multicolored* $(2k)$-*cycles set*, $FMC\,(2k)$ in short, by $(m, n) \in FMC\,(2k)$ if there exists a proper n-edge-coloring of $K_{m,n}$ which forbids multicolored $(2k)$-cycles. Obviously, $(i, j) \in FMC\,(2k)$ if $i < k$ or $j < 2k$. In this thesis, we completely determine the two sets $FMC\,(4)$ and $FMC\,(6)$. Furthermore, for k is odd, we find several elements in the set $FMC\,(2k)$. Besides, we denote an $m \times n$ latin rectangle which forbids multicolored $(2k)$-cycles in its corresponding $K_{m,n}$ by $L_{m,n}(2k)$.

3.1 Forbidding Multicolored 4-cycles and 6-cycles

It is impossible to forbid multicolored 4-cycles in any proper n-edge-coloring of $K_{m,n}$ where $2 \leq m \leq n$ and $n \geq 4$. Thus we have the following theorem.

Theorem 3.1. $FMC\,(4) = \{(2, 2), (2, 3), (3, 3)\}$.

Proof. It suffices to show that there exists a multicolored C_4 in a proper 4-edge-colored $K_{2,4}$. Let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3, v_4\}$ be the two partite sets of $K_{2,4}$. Without loss of generality, assume the colors on u_1v_1, u_2v_1 are 1 and 2. There must be one vertex v_i where $i \in \{2, 3, 4\}$ such that the colors on u_1v_i, u_2v_i are different from $\{1, 2\}$. Thus we have a multicolored C_4. □

Then we will have a discussion on forbidding multicolored C_6 in a proper n-edge-colored $K_{m,n}$ where $3 \leq m \leq n$ and $n \geq 6$. Notice that every proper n-edge-coloring of $K_{m,n}$ has its corresponding $m \times n$ latin rectangle using n distinct entries. In an $m \times n$ latin rectangle, consider a 3×3 partial latin square. If there exist 2 disjoint transversals using 6 distinct entries in the 3×3 partial latin square, then there exists a multicolored C_6 in its corresponding $K_{3,3} \subseteq K_{m,n}$. On the other hand, we can regard the existence of 2 disjoint transversals as omitting three positions that no two of them are in the same row or column. Figure 3 is an example of a 3×3 partial latin square, and the two disjoint transversals, which can be combined to a multicolored C_6, are discovered by omitting the three "gray" positions.

7	3	5
1	4	2
2	8	6

Figure 3: A 3×3 partial latin square

Obviously, in a 3×3 partial latin square, if there appear 9 kinds of entries, then a multicolored C_6 must occur somewhere. And if there appear 8 kinds of entries, then we can omit the two positions which have the repeated entry to obtain a multicolored C_6.

Proposition 3.2. *Let L be a 3×3 partial latin square with 7 distinct entries. There is no multicolored C_6 in its corresponding $K_{3,3}$ if and only if L has an L_2.*

Proof. Assume that L has no L_2.

Case 1. If there is one entry appearing 3 times, then omitting these three positions yields a multicolored C_6, a contradiction.

Case 2. There are two entries appeared twice separately. Without loss of generality, let the two entries be 1 and 2, and let the positions of entry 1 be arranged at the diagonal, see Figure 4.

1		
	1	

Figure 4: Case 2 of Proposition 3.2

Now, consider the positions where entry 2 may appear. Since there is no L_2, there must be at least one position which labels entry 2 in the third column or the third row. Name this position be A. Then we just omit position A and one of the positions labeled 1 which is not in the same row and column with A. Thus, we have a multicolored C_6.

Conversely, suppose the two entries in L_2 be 1 and 2. Since there is none or two 1's (or 2's) in any transversal of L, any two disjoint transversals couldn't have 6 kinds of entries. Then, there is no multicolored C_6 in its corresponding $K_{3,3}$. □

Proposition 3.3. *Let L be a 3×3 partial latin square with 6 distinct entries. There is no multicolored C_6 in its corresponding $K_{3,3}$ if one of the following conditions occurs:*

(i) There exists 2 columns (or rows) in L used exactly 3 distinct entries.

(ii) Some entry appears three times in L.

(iii) There is an L_2 in L.

Proof. Since there are just 6 kinds of entries, we should keep every kind of entries left and omit the other repeated ones. Thus we have done. □

Consider an n-edge-colored $K_{m,n}$, $m \le n$, the larger n is, the more colors we can use. Therefore, the possibility to forbid multicolored 6-cycles in an n-edge-colored $K_{m,n}$ gets lower as n increases.

Proposition 3.4. *For any proper n-edge-coloring of $K_{m,n}$ where $n \ge 9$ and $m \le n$, there exists a multicolored C_6.*

Proof. It is sufficient to consider $m = 3$. Suppose NOT. There exists a proper n-edge-coloring of $K_{3,n}$ which forbids multicolored C_6's. Let $L_{3,n}(6)$ be the corresponding latin rectangle. Without loss of generality, let the three entries of the first column in $L_{3,n}(6)$ be 1, 2 and 3.

Except the first column, the three entries 1, 2 and 3 can occur in at most 6 columns. So, there is at least one column which has no entries 1, 2 and 3. We can assume the three entries of the second column be 4, 5 and 6. There are $n-6$ unused entries left and each of them must appear in the remaining $n-2$ columns exactly three times. Consider the inequality: $\dfrac{3(n-6)}{n-2} = 1+\dfrac{2n-16}{n-2} > 1$, if $n \geq 9$. By Pigeon-hole principle, there must be one column which has at least two entries disjoint from the set $\{1, 2, 3, 4, 5, 6\}$. Combining this column with the first two ones, there will be a multicolored C_6 in its corresponding $K_{3,3}$. It leads a contradiction. □

So far, we have narrowed the two indices n and m down to $6 \leq n \leq 8$ and $3 \leq m \leq n$.

Lemma 3.5. *For* $3 \leq m \leq 6$, $(m, 6) \in FMC(6)$.

Proof. Let $L_{6,6}(6) = L_3 \times L_2$ be composed of four copies of L_3, and suppose the entries in the top-left and bottom-right copies are from $\{1, 2, 3\}$ while the entries in the other two copies are from $\{4, 5, 6\}$. For convenience, name the four copies A, B, C and D clockwise from the top-left one, see Figure 5.

$L_{6,6}(6) =$

1	2	3	4	5	6
3	1	2	6	4	5
2	3	1	5	6	4
1	2	3	4	5	6
3	1	2	6	4	5
2	3	1	5	6	4

A	B
D	C

Figure 5: $L_{6,6}(6)$ and the four copies of L_3

Suppose that there exist 6 positions somewhere which induce a multicolored C_6. Let L be the 3×3 partial latin square which contains the 6 positions. By Proposition 3.2 (i), we can assume L cross all four copies. Without loss of generality, suppose there are four positions of L locating on A. Since A has only 3 kinds of entries, some entry must appear twice, say a.

Then consider the only one entry of L in C. By Proposition 3.2 (ii), let the entry be

b, where $b \neq a$. Moreover, there is exactly one repeated entry in the other four positions of L in B and D. Recall that we can obtain a multicolored C_6 by omitting three positions that no two of them are in the same row or column. If we omit the position in C, then there must be a repeated entry left in B and D. Otherwise, the two positions having entry a in A will be left. It's a contradiction. □

Lemma 3.6. *For* $3 \leq m \leq 8$, $(m, 8) \in FMC(6)$.

Proof. Let $L_{8,8}(6) = L_2 \times L_2 \times L_2$ be composed of 8 copies of L_2. Similar to the proof of Lemma 3.5, suppose the entries in the top-left and bottom-right copies are from {1, 2, 3, 4} while the entries in the other two copies are from {5, 6, 7, 8}, and the four copies are arranged as following Figure 6. For convenience, let $L_{8,8}(6) = [\, l_{i,j} \,]$ where $1 \leq i, j \leq 8$.

$L_{8,8}(6) =$

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

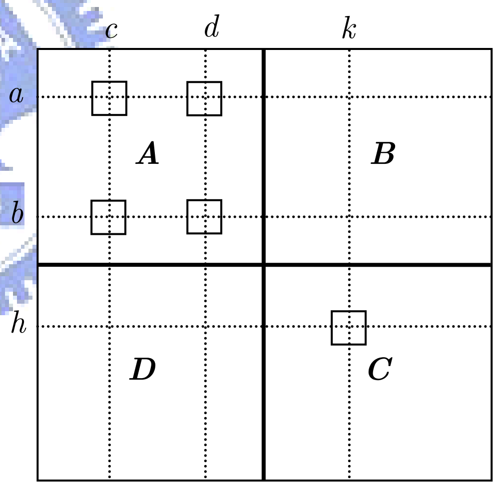

Figure 6: $L_{8,8}(6)$ and the four copies of $(L_2)^2$

Suppose that there are 6 positions somewhere which induce a multicolored C_6. Let L be the 3×3 partial latin square which contains the 6 positions. It is easy to see that any 2×3 partial latin rectangle in $L_2 \times L_2$ contains an L_2. By Proposition 3.1, we can assume L cross all four copies. Without loss of generality, suppose there are four positions of L locating on A. Let the four positions in A be (a, c), (a, d), (b, c), (b, d), and the only one position in C be (h, k), where $1 \leq a, b, c, d \leq 4$ and $5 \leq h, k \leq 8$.

By Proposition 3.2, $l_{a,c} \neq l_{b,d}$ or $l_{a,d} \neq l_{b,c}$. Actually, the four entries $l_{a,c}$, $l_{b,d}$, $l_{a,d}$, $l_{b,c}$ are distinct. Assume $l_{h,k} \neq l_{a,c}$, then $l_{a,k} \neq l_{h,c}$ because of $L_{8,8}(6) = (L_2)^3$. Thus, we have an L_2 in L, a contradiction. □

Lemma 3.7. $(3, 7) \in FMC(6)$.

Proof. Let $L_{3,7}(6)$ be the corresponding latin rectangle of the specific proper 7-edge-coloring which forbids multicolored C_6's, see Figure 7.

It is easy to see that any two columns of the first 4 columns have an L_2, and any two columns of the last 3 columns used exactly 3 distinct entries. By proposition 3.3 (i) and (iii), we have done. □

$L_{3,7}(6) =$

1	2	3	4	5	6	7
2	1	4	3	6	7	5
3	4	1	2	7	5	6

Figure 7: $L_{3,7}(6)$

Lemma 3.8. *There exists a 3-edge-colored $K_{3,3}$ in a proper 7-edge-colored $K_{3,7}$ which forbids multicolored C_6's.*

Proof. Let $L_{3,7}(6)$ be the corresponding latin rectangle of a proper 7-edge-colored $K_{3,7}$. It suffices to show there must be a latin subsquare of order 3.

Claim 1. There exist two columns having disjoint entries.

Suppose NOT. Let the entries of the first column be 1, 2 and 3. Notice that each entry in {1, 2, 3} must appear twice in the other columns. By our assumption, each remaining column has exactly one position with entry in {1, 2, 3}. Without loss of generality, let the second column contain entries 1, 4, and 5. Except the first two columns, there are at most 4 columns having entries 4 or 5. Therefore, there exists one column having exactly one entry from {1, 2, 3} but no entries from {4, 5}. By proposition 3.2, this column and the first two columns will create a multicolored C_6, a contradiction.

Claim 2. There exists a latin subsquare of order 3.

By Claim 1, we can assume the entries of the first two columns be 1, 2, 3 and 5, 6, 7 respectively. Consider the first two columns and the three columns which have entry 4. By proposition 3.1, the other two entries in the column which has entry 4 must be both from {1, 2, 3} or {5, 6, 7}.

Case 1. The entries in the three columns with entry 4 are all from {1, 2, 3} or {5, 6, 7}.

Assume the six entries are all in {5, 6, 7} by symmetry. Then combining the first column and the last two ones, we have a latin square of order 3, see Figure 8.

1	5	4				
2	6		4			
3	7			4		

Figure 8: Case 1

Case 2. The entries in the three columns with entry 4 are NOT all from {1, 2, 3} or {5, 6, 7}.

We will use Figure 9 and Figure 10 to illustrate our arguments. First, look at Figure 9. Without loss of generality, suppose the entries in position A are from {1, 2, 3} while the entries in position B are from {5, 6, 7}.

1	5	4	***A***	***B***		
2	6	***A***	4	***B***		
3	7	***A***	***A***	4		

Figure 9: Case 2

By proposition 3.2, since combining the first two columns and one of the columns with entry 4 will form a partial latin square with 7 kinds of entries, the entries in position A and position B are uniquely determined as Figure 10. Meanwhile, the entries in some positions of the last two columns are determined except positions denoted as C. Note that the entries in position C must be from the set $\{5, 6\}$.

1	5	4	3	6	7	2
2	6	3	4	5	1	7
3	7	2	1	4	***C***	***C***

Figure 10: Case 2

Consider column 1, column 5, and column 6, they use 7 distinct entries but without L_2. By Proposition 3.2, there exists a multicolored C_6, a contradiction. □

Corollary 3.9. *For any proper 7-edge-coloring of* $K_{m,7}$ *,* $4 \leq m \leq 7$*, there exists a multicolored* C_6.

Proof. It is sufficient to consider the case $m = 4$. Suppose NOT. There exists some proper 7-edge-coloring of $K_{4,7}$ which forbids multicolored C_6's. Consider its corresponding latin rectangle $L_{4,7}$. By Lemma 3.7, there exists a latin subsquare of order 3 in the first three rows of $L_{4,7}(6)$. Without loss of generality, we put the latin subsquare of order 3 in the last three columns and let the entries be 5, 6 and 7, see Figure 11. Then, consider the last three rows. It's impossible to find a latin subsquare of order 3. It contradicts Lemma 3.7. □

				5	6	7
				7	5	6
				6	7	5

Figure 11: $L_{4,7}(6)$

To sum up, we have the following conclusion.

Theorem 3.10. *For each m, n ($m \le n$) satisfying one of the follow conditions, any n-edge-colored $K_{m,n}$ contains a multicolored C_6:*

(i) $m \ge 3$ *and* $n \ge 9$;

(ii) $m \ge 4$ *and* $n = 7$.

Proof. It can be easily proved by Proposition 3.4, Lemma 3.5, Lemma 3.6, Lemma 3.7, Lemma 3.8 and Corollary 3.9. □

3.2 Forbidding Multicolored $(2k)$-cycles

In this subsection, we consider the general version: forbidding multicolored $(2k)$-cycles. In the followings, we extend the method of Lemma 3.4, which shows a proper 6-edge-coloring of $K_{6,6}$ that forbids multicolored 6-cycles, to the case that forbids multicolored $(2k)$-cycles.

Theorem 3.11. *If k is odd, then $(m, 2k) \in FMC\,(2k)$ for $k \leq m \leq 2k$.*

Proof. It suffices to show $(2k, 2k) \in FMC\,(2k)$. Let $L_{2k,2k}(2k) = L_k \times L_2$, where L_k is the circulant latin square of order k. Similar to above proofs, suppose the top-left and bottom-right copies of L_k are based on $\{1, 2, \cdots, k\}$ while the other two copies are based on $\{k+1, k+2, \cdots, 2k\}$. Now, we claim that there are no two disjoint transversals using $2k$ kinds of entries. For convenience, name the four copies A, B, C and D clockwise from the top-left one, see Figure 12.

L_k based on $\{1, 2, \cdots, k\}$	L_k based on $\{k+1, \cdots, 2k\}$
L_k based on $\{k+1, \cdots, 2k\}$	L_k based on $\{1, 2, \cdots, k\}$

A	B
D	C

Figure 12: $L_{2k,2k}(2k)$ and four copies of L_k

Suppose that there exist two disjoint transversals using $2k$ kinds of entries. Let L be the $k \times k$ partial latin square containing these two transversals. Note here that each column and row contains exactly two entries from the two transversals. If L crosses only two copies of L_k, the two disjoint transversals must contain an even number of entries from $[k]$. Therefore, we can assume that L crosses all four copies. Let a, b, c and d be the numbers of entries of the two transversals from A, B, C and D respectively. Clearly, $a+c$ is even because $a+b$ and $b+c$ are both even. By the hypothesis, $a+c = k$ is odd, a contradiction. Then we complete the proof. □

4 Conclusion

In this thesis, we have obtained the following three main results:

1. $FMC(4) = \{(2, 2), (2, 3), (3, 3)\}$.
2. $FMC(6) = \{(a, b), (c, 8), (3, 7) \mid 2 \leq a \leq b \leq 6, 2 \leq c \leq 8\}$.
3. If k is odd, then $(m, 2k) \in FMC(2k)$ for $2 \leq m \leq 2k$.

For the future study, we shall try to find the smallest n such that there always exists a multicolored C_{2k} in an arbitrary proper n-edge-colored $K_{k,n}$ for $k \geq 4$. In order to solve this problem, we may find the smallest t such that there always exists a multicolored C_{2k} in an arbitrary proper t-edge-colored $K_{k,k}$ for $k \geq 4$. Hopefully, this task can be done in the near future.

References

[1] S. Akbari and A. Alipour, Multicolored trees in complete graphs, J. Graph Theory 54, 221–232. (2006)

[2] S. Akbari, A. Alipour, H. L. Fu and Y. H. Lo, Multicolored parallelism of isomorphic spanning trees, SIAM Discrete Math. (June) (2006) 564–567.

[3] M. Albert, A. Frieze and B. Reed, Multicolored Hamilton cycles, Electronic J. Combin. 2, R10 (1995)

[4] N. Alon, R. A. Brualdi, B. L. Shader, Multicolored forests in bipartite decompositions of graphs, J. Combin. Theory, Ser. B 53, 143–148 (1991)

[5] H. J. Broersma, X. Li, G. Woeginger and S. Zhang, Paths and cycles in colored graphs, Australasian J. Combin. 31, 297–309. (2005)

[6] R. A. Brualdi and H. J. Ryser, Combinatorial Matrix Theory, Cambridge Univ. Press, (1992).

[7] R. A. Brualdi and S. Hollingsworth, Multicolored trees in complete graphs, J. Combin. Theory, Ser. B 68, 310–313. (1996)

[8] H. Chen and X. Li, Long heterochromatic paths in edge-colored graphs, Electron. J. Combin. 12, R33 (2005)

[9] H. Chen and X. Li, Color degree and color neighborhood union conditions for long heterochromatic paths in edge-colored graphs, arXiv:math.CO/0512144 v1 7 Dec (2005)

[10] G. M. Constantine, Edge-disjoint isomorphic multicolored trees and cycles in complete graphs, SIAM Discrete Math. 18(2005), No. 3, 577-580.

[11] G. M. Constantine, Multicolored parallelisms of isomorphic spanning trees, Discrete Math. Theor. Comput. Sci. 5 (2002), No. 1, 121-125.

[12] P. Erdös and T. Gallai, On maximal paths and circuits of graphs, Acta. Math. Acad. Sc. Hungar. 10, 337–356. (1959)

[13] A. Frieze and B. Reed, Polychromatic Hamilton cycles. Discrete Math. 118, 69–74. (1993)

[14] H. L. Fu and Y. H. Lo, Multicolored isomorphic spanning trees in complete graphs, in preprints.

[15] H. L. Fu and Y. H. Lo, Multicolored parallelisms of Hamiltonian cycles, Discrete Math. 309 (2009), pp. 4871-4876.

[16] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and an edge coloring conjecture, Discrete Math. 62, 29–33 (1986)

[17] P. Hatami and P. W. Shor, A lower bound for the length of a partial transversal in a Latin square, J. Combin. Theory, Series A, 115 (2008) 1103–1113.

[18] A. Kaneko, M. Kano and K. Suzuki, Three edge disjoint multicolored spanning trees in complete graphs, Preprint. (2003)

[19] J. Krussel, S. Marshal and H. Verral, Spanning trees orthogonal to one-factorizations of K_{2n}, Ars Combin. 57 (2002), 77-82.

[20] J.J. Montellano-Ballesteros and V. Neumann-Lara, An Anti-Ramsey Theorem on Cycles, Graphs and Combinatorics, 21 (2005), 343-354.

[21] H.J. Ryser, Neuere Probleme der Kombinatorik, in: Vorträge über Kombinatorik, Oberwolfach, Matematisches Forschungsinstitute Oberwolfach, Germany, 24–29

[22] D. B. West (2001), Introduction to graph theory, Upper Saddle River, NJ :Prentice Hall.

[23] D. E. Woolbright and H. L. Fu, On the existence of rainbows in 1-factorizations of K_{2n}, J. Combin. Des. 6, 1–20 (1998)