國 立 交 通 大 學

應用數學系

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A Study of Berge's Strong Path Partition Conjecture

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Student: Yen-Ting Chang Advisor: Hung-Lin Fu

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050

摘 要

令$\mathcal{P}$是一個由路徑 (path) 所形成的集合。若$\mathcal{P}$裡的路徑兩兩交集爲空集合, 而且$\mathcal{P}$中所有路徑的點聯集爲圖G所有的點, 則$\mathcal{P}$是圖G的一組路徑分割 (path partition)。令k是一個正整數, 則我們對任一組路徑分割$\mathcal{P}$可以定義它的k範數 (k-norm): $|\mathcal{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\}$。若一組路徑分割擁有最小的$k$範數, 則此路徑分割被稱爲是最優化的$k$範數路徑分割 ($k$-optimal path partition)。

令C^k是圖G的一組k著色, 即圖G中k個由點形成的獨立集所成之集合, 而且兩兩獨立集交集爲空集合。若路徑分割$\mathcal{P}$裡任一條路徑中有 $\min\{|P_i|, k\}$個點分別落在C^k裡不同的獨立集, 則稱此k著色C^k正交於路徑分割$\mathcal{P}$。

Berge 猜測對於任一組最優化的k範數路徑分割, 都可找到一組k著色與之正交。這個猜測至今尚未被解決, 只有一些特別的情形被證明; 而在這篇論文裡, 我們藉由一些特殊的圖來驗證 Berge 的猜測是對的。

Abstract

A family $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ of paths is called a path partition of G if its members are vertex disjoint and $V[\mathcal{P}] = V(G)$. Let k be a positive integer, then the k-norm of a path partition $\mathcal{P}$ is defined by $|\mathcal{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\}$. A path partition $\mathcal{P}$ minimizes $|\mathcal{P}|_k$ is called k-optimal.

A k-coloring of G is a family $\mathcal{C}^k = \{C_1, C_2, \ldots, C_k\}$ of k vertex disjoint independent sets called color classes. A k-coloring $\mathcal{C}^k$ is orthoganal to a path partition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ if $\mathcal{C}^k$ meets every path in $\mathcal{P}$ in $\min\{|P_i|, k\}$ different color classes. Berge conjectured that for every k-optimal path partition $\mathcal{P}$, there exists a k-coloring orthogonal to it, and this is known as Berge's strong path partition conjecture.

This conjecture is still open today, but several results have been obtained in some special cases. In this thesis, we verify this conjecture to be true for certain special digraphs.

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Contents

Chapter 1

Introduction and Preliminaries

1.1 Basic Notation

A graph G is composed of two types of objects. It has a finite set of elements called vertices and a set of unordered pairs of vertices called edges. The vertex set is denoted by $V(G)$ or V, and the edge set is denoted by $E(G)$ or E.

A directed graph or digraph G is composed of two types of objects. It has a vertex set V and an edge set E, and the edge set is a set of ordered pairs of vertices. For each edge of G, the first vertex of the ordered pair is the tail of the edge and the second is the head; together, they are endpoints of the edge. If there is an edge (u, v), then u is a predecessor of v, and v is the successor of u. A loop in a digraph is an edge whose endpoints are the same. Multiple edges are edges having the same ordered pair of endpoints. The graphs or digraphs we consider hereinafter contain no loops and no multiple edges.

A subgraph of a graph (or digraph) G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of a graph (or digraph) G is a subgraph with vertex set $V(G)$.

Let x and y be two vertices of a graph (or digraph) G. An x-y walk of G is an alternating sequence $(x = v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k = y)$ of vertices and edges, such that $e_i = (v_{i-1}, v_i) \in E(G)$ for $i = 1, 2, ..., k$. A trivial walk contains no edges, i.e., the walk contains only one vertex. An x-y walk is closed if $x = y$, and an x-y walk is not closed (or open) if $x \neq y$. An x-y trail is an x-y walk in which no edge is repeated. A nontrivial closed trail of a graph (or digraph) G is called a circuit of G. A circuit $(v_1, v_2, ..., v_n, v_1)$ is a cycle if the n vertices are all distinct. A graph (or digraph) G is acyclic if there is no cycle (or directed cycle respectively) in G.

A path P in a digraph $G = (V, E)$ is a sequence of distinct vertices $(v_1, v_2, ..., v_l)$ such that $(v_i, v_{i+1}) \in E$, for $i = 1, 2, ..., l - 1$. Let $V(P)$ denote the set of vertices $v_1, v_2, ..., v_l$ of a path P, and $|P|$ denote the cardinality of P, i.e. $|P| = |V(P)|$. A path of cardinality one is called a trivial path. For any path P in a digraph G, we denote the first vertex in P by $in(P)$, and the last vertex by $ter(P)$. For a path P and a vertex $x \in P$, let $P_-(x)$ denote the vertex preceding x on P, and $P_+(x)$ denote the vertex following x on P.

For a set of vertex disjoint paths $\mathcal{P}$, we denote $\bigcup\{V(P); P \in \mathcal{P}\}$ by $V[\mathcal{P}]$, and $\bigcup\{E(P); P \in \mathcal{P}\}$ by $E[\mathcal{P}]$. A family $\mathcal{P}$ of paths is called a path partition of G if its members are vertex disjoint paths and $V[\mathcal{P}] = V(G)$. Note that a digraph may have many path partitions. The trivial path partition, where every path is a trivial path, is an example of a path partition. The cardinality of a path partition $\mathcal{P}$ is the number of paths in $\mathcal{P}$, and we denote it by $|\mathcal{P}|$. Let $\pi(G)$ denote the minimum number of paths in any path partition of G. In other words, $\pi(G) = \min|\mathcal{P}|$, where the minimum is taken over all path partitions $\mathcal{P}$. A path partition $\mathcal{P}$ of G is called optimal if $|\mathcal{P}| = \pi(G)$.

A vertex subset X in a graph (or digraph) G is independent if for any pair of vertices in X, there is no edge in G adjoining them. The size of X, denoted by $|X|$, is the number of vertices in X. The indenpendence number $\alpha(G)$ of a graph G is the size of a maximum independent vertex set in G.

1.2 Motivation

Before talking about Berge's conjecture, we introduce the Dilworth's theorem and the Greene-Kleitman theorem which are generalized and extended by Berge's conjecture. Now, we need some definitions and notations.

Let S be a nonempty set. A binary relation R is a subset of $A \times A$. R is reflexive on A if for all $x \in A$, $(x, x) \in R$. R is transitive on A if for all $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R$ would imply that $(x, z) \in R$. R is antisymmetric on A if for all $x, y \in A$, $(x, y) \in R$ and $(y, x) \in R$ would imply that $x = y$.

Definition 1.2.1. A partially ordered set (or poset for short) S is a set with a binary relation R, such that R is reflexive, transitive, and antisymmetric on S. R is called a partial order of S.

If there is a poset (S, R), we can define a digraph G of this poset as following: let $V(G) = S$ and for any $x, y \in V(G)$, $(x, y) \in E(G)$ if and only if $(x, y) \in R$. For convenience, we delete the loops in G. Note that every path in G induces a clique, in other words, a set of pairwise adjacent vertices (ignoring the direction) because of the transtivity of S. The maximum independent set in a clique is exactly one vertex. Hence if G is a digraph of a poset, then every path partition can meet an independent set at most once. Therefore, the size of maximum independent set is at most the number of paths in any path partition, then we have $\pi(G) \geqslant \alpha(G)$. Together with the follwing theorem due to Gallai and Milgram, we have the Dilworth's theorem.

Theorem 1.2.2. (*Gallai and Milgram* [5]). *Every directed graph satisfies* $\pi(G) \leqslant \alpha(G)$.

Theorem 1.2.3. (*Dilworth* [*4*]). *If G is a digraph of a poset, then $\pi(G) = \alpha(G)$. Furthermore, every path in an optimal path partition in G meets every maximum independent set exactly once.*

Dilworth's theorem was generalized by Greene and Kleitman by considering a collection of k disjoint independent sets ($1 \leq k \leq n$), instead of one independent set.

For a positive integer k, a k-coloring of a graph G is a labeling $f : V(G) \rightarrow S$, where $|S| = k$ (often we use $S = \{1, 2, ..., k\}$). The labels are colors, and the vertices of the same color form a color class. A k-coloring is proper if the labeled endpoints of each edge have different labels. We say that a vertex subset X of a graph (or a digraph) is an independent set if $\forall x, y \in X$, there is no edge adjoin x and y. Observe that each color class is an independent set in a proper k-coloring, and hence we could denote a proper k-coloring by $\mathcal{C}^k = \{C_1, C_2, ..., C_k\}$, where each C_i is an independent vertex set, and for any two color classes C_i and C_j, they are vertex disjoint. On the other hand, a family $\mathcal{C}^k = \{C_1, C_2, ..., C_k\}$ of k vertex disjoint independent sets in G could be thought as a proper k-coloring. Now, we give a formal definition of a k-coloring.

Definition 1.2.4. A k-coloring of G is a family $\mathcal{C}^k = \{C_1, C_2, ..., C_k\}$ of k vertex disjoint independent sets called color classes. The cardinality of a k-coloring $\mathcal{C}^k = \{C_1, C_2, ..., C_k\}$ is $|\mathcal{C}^k| = \sum_{i=1}^{k} |C_i|$ and $\mathcal{C}^k$ is optimal if $|\mathcal{C}^k|$ is as large as possible. Denote by $\alpha_k(G)$ the cardinality of an optimal k-coloing in G.

Berge called a k-coloring of a graph (or digraph) G a partial k-coloring, because it is a partial coloring of the vertex set of G with k colors, i.e., we don't need to color all vertices of G. We prefer the shorter name k-coloring for convenience. In Theorem 1.2.2 and Theorem 1.2.3, we considered the minimum number of paths in all path partitions, and now we extend the value to k-norm.

Definition 1.2.5. (k-norm of a path partition). For each positive integer k, the k-norm $|\mathcal{P}|_k$ of a path partition $\mathcal{P} = \{P_1, P_2, ..., P_m\}$ is defined by

$$|\mathcal{P}|_k = \sum_{i=1}^{m} min\{|P_i|, k\}.$$

A path partition $\mathcal{P}$ that minimizes $|\mathcal{P}|_k$ is k-optimal. Denote by $\pi_k(G)$ the k-norm of a k-optimal path partition in G.

In a k-optimal path partition, those paths of cardinality at least k are called long paths, and denote by $\mathcal{P}^+$ the set of long paths. Those paths of cardinality less than k are called short paths, and denote by $\mathcal{P}^0$ the set of short paths. Note that a 1-optimal path partition is a partition with the minimum number of paths in all path partitions of G, i.e., $\pi_1(G) = \pi(G)$. We are able to state the Greene-Kleitman theorem now.

Theorem 1.2.6. (*Greene-Kleitman theorem* [*6*]). *Let G be a digraph of a poset, and let k be a positive integer. Then $\alpha_k(G) = \pi_k(G)$.*

Observe that for $k = 1$, Theorem 1.2.6 is identical to Theorem 1.2.3 (Dilworth's theorem). If G is a digraph of a poset, then each path P_i meets each k-coloring at most $\min\{|P_i|, k\}$ vertices. Hence for any k-coloring $\mathcal{C}^k = \{C_1, C_2, .., C_k\}$ and any path partition $\mathcal{P} = \{P_1, P_2, ..., P_m\}$, the following holds:

$$|\mathcal{C}^k| = \sum_{i=1}^{m} |V[\mathcal{C}^k] \cap V(P_i)| \leq \sum_{i=1}^{m} \{\min |P_i|, k\} = |\mathcal{P}|_k.$$

Therefore, for an optimal k-coloring and a k-optimal path partition, $\alpha_k(G) \leq \pi_k(G)$ holds in a digraph of poset. The following corollary is from above and Theorem 1.2.6.

Corollary 1.2.7. *Let G be a digraph of a poset, k be a postive integer, $\mathcal{P}$ be a k-optimal path partition, and $\mathcal{C}^k$ be an optimal k-coloring. Then $\mathcal{C}^k$ meets every path in $\mathcal{P}$ in exactly* $\min\{|P_i|, k\}$ *vertices.*

If G is not a digraph of a poset, then a color class may meet a path more than once, and hence a k-coloring may meet a path more than k times. Lineal's conjecture extends the Greene-Kleitman theorem to all digraphs.

Conjecture 1.2.8. (Linial [8]) Let G be a digraph and k a positive integer. Then $\alpha_k(G) \geq \pi_k(G)$.

Conjecture 1.2.8 can also be called the "weak path partition conjecture", and the strong path partition conjecture is also known as the Berge's strong path partition conjecture. We give a formal definition of the relation between coloring and path partition.

Definition 1.2.9. (Orthogonality of path partitions and k-colorings). A k-coloring $\mathcal{C}^k$ is orthogonal to a path partition $\mathcal{P} = \{P_1, P_2, ..., P_m\}$ if $\mathcal{C}^k$ meets every path in $\mathcal{P}$ in $\min\{|P_i|, k\}$ different color classes.

Berge defined a k-coloring $\mathcal{C}^k$ to be strong for a path partition $\mathcal{P}$ if $\mathcal{C}^k$ meet every path P in $\mathcal{P}$ in exactly $\min\{|P_i|, k\}$ different color classes. We prefer using "orthogonal" to "strong" in this thesis. The following conjecture is proposed by Berge and is what we focus in this thesis.

Conjecture 1.2.10. (Berge's strong path partition conjecture [2]). Let G be a digraph and let k be a positive integer. Then for every k-optimal path partition $\mathcal{P}$ there exists a k-coloring orthogonal to it.

Conjecture 1.2.10 is still open today, but several results have been obtained in some special cases. In the next chapter, we will introduce some known results and present their proofs for some of them.

Chapter 2

Known Results

Berge observed in [2] that Conjecture 1.2.10 holds in the following special cases:

1. For $k = 1$.

2. In the case that the k-optimal path partition contains no path of cardinality more than k, i.e. the paths in the path partition are all short paths.

3. For digraphs containing a Hamilton path.

4. For bipartite graphs.

If G is acyclic then Conjecture 1.2.10 was shown to be true in [9, 3, 1, 7].

2.1 A Proof of Berge's Conjecture for Acyclic Digraphs

In this section, we will introduce the proof given by Hartman and Berger in [7] for acyclic digraphs. For any path partition $\mathcal{P}$, Hartman and Berger gave an algorithm for acyclic digraphs to find out either a k-coloring orthogonal to $\mathcal{P}$ or a path partition $\mathcal{P}'$ such that $|\mathcal{P}'|_k < |\mathcal{P}|_k$, i.e. $\mathcal{P}$ is not k-optimal.

2.1.1 Notations and Definitions

Before we talk about the algorithm, we need some notations and definitions. Review that for a given path partition $\mathcal{P}$, $\mathcal{P}^+$ denote the set of all long paths (i.e. of cardinality at least k) in $\mathcal{P}$, and $\mathcal{P}^0$ denote the set of all short paths (i.e. of cardinality less than k) in $\mathcal{P}$. We assume that all paths in $\mathcal{P}^0$ are of cardinality one by breaking each short path of cardinality greater than one into single vertex. Additionally, we denote by $\overline{\mathcal{P}^0}$ the set of trivial paths in $\mathcal{P}$ and $\overline{\mathcal{P}^+}$ the set of nontrivial paths in $\mathcal{P}$. If x is a vertex on a path $P \in \mathcal{P}$, then $\mathcal{P}_-(x)$ denote the unique vertex that precedes x on P. When x is the initial vertex of P, $\mathcal{P}_-(x)$ is undefined. Similarly, $\mathcal{P}_+(x)$ denote the unique vertex that follows x on P. When x is the terminal vertex of P, $\mathcal{P}_+(x)$ is undefined.

Definition 2.1.1. An undirected trail Q in G is a sequence $Q = (v_0, e_1, v_1, ..., e_l, v_l)$ such that, for each $1 \le i \le l$, either $e_i = (v_{i-1}, v_i) \in E(G)$ or $e_i = (v_i, v_{i-1}) \in E(G)$, and all edges are distinct. We assign a direction to Q from v_0 to v_l. After assigning the direction, for each edge e_i, if $e_i = (v_{i-1}, v_i) \in E(G)$, then e_i is a forward edge, and if $e_i = (v_i, v_{i-1}) \in E(G)$, then e_i is a backward edge.

Definition 2.1.2. (k-alternating trail). Given a path partition $\mathcal{P}$, an undirected trail $Q = (v_0, e_1, v_1, ..., e_l, v_l)$ is k-alternating relative to $\mathcal{P}$, if the conditions below hold:

1. All forward edges of Q are in $E(G) - E[\mathcal{P}^+]$, all backward edges are in $E[\mathcal{P}^+]$, and every forward edge (u, v), where $v \in V[\mathcal{P}^+]$ is followed by a backward edge, unless $v \in in[\mathcal{P}^+]$ and $v = v_l$.

2. There are at most $k-1$ consecutive backward edges in Q, unless the first one follows a forward edge, in which case, there are at most k consecutive backward edges.

3. For every vertex $v \in V(Q)$, there exists at most one forward edge $(u, v) \in E(Q)$ and at most one forward edge $(v, w) \in E(Q)$.

Definition 2.1.3. (Prim and proper k-alternating trail). A k-alternating trail $Q = (v_0, e_1, v_1, ..., e_l, v_l)$ is proper if either (a) $v_0 \in ter[\mathcal{P}^+]$ or (b) $v_0 \in V[\mathcal{P}^0]$. A k-alternating trail is prim if either (1) $v_l \in in[\mathcal{P}^+]$ or (2) $v_l \in V[\mathcal{P}^0]$.

According to the definition 2.1.3, we have four types of prim and porper k-alternating trails, (a-1), (a-2), (b-1) and (b-2). A prim and porper k-alternating trail is of type (a-1) if it is of type (a) and type (1). The others are defined similarly. We denote by $\mathcal{P} \oplus Q$ the spanning subgraph of G containing edges in the symmetric difference $E[\mathcal{P}] \oplus E(Q) = E[\mathcal{P}] \cup E(Q) \setminus (E[\mathcal{P}] \cap E(Q))$

Lemma 2.1.4. *Let $\mathcal{P}$ be a path partition, Q be a prim and proper k-alternating trail relative to $\mathcal{P}$, and $\mathcal{P}' = \mathcal{P} \oplus Q$. Then the spanning subgraph $\mathcal{P}'$ of G contains disjoint paths and cycles.*

Proof. Let $Q = (v_0, e_1, v_1, ..., e_l, v_l)$ be a prim and proper k-alternating trail relative to $\mathcal{P}$. Note that $v_0 \in ter[\mathcal{P}^+] \cup V[\mathcal{P}^0]$ and $v_l \in in[\mathcal{P}^+] \cup V[\mathcal{P}^0]$ by the definition 2.1.3. According to conditions (1) and (3) of the definition 2.1.2, for every $v \in V[\mathcal{P}^+] \setminus \{v_0, v_l\}$, $\deg^-_{\mathcal{P}'}(v) \leq \deg^-_{\mathcal{P}}(v) = 1$ and $\deg^+_{\mathcal{P}'}(v) \leq \deg^+_{\mathcal{P}}(v) = 1$. For $v_0 \in ter[\mathcal{P}^+]$, $\deg^+_{\mathcal{P}'}(v_0) \leq 1$ and $\deg^-_{\mathcal{P}'}(v_0) \leq 1$. For $v_l \in in[\mathcal{P}^+]$, $\deg^+_{\mathcal{P}'}(v_l) \leq 1$ and $\deg^-_{\mathcal{P}'}(v_l) \leq 1$. By the condition (3) of the definition 2.1.2, all vertices in $V[P^0]$ receive indegree and outdegree at most one respectively. Because each vertex in $\mathcal{P}'$ has indegree and outdegree at most one respectively, $\mathcal{P}'$ contains disjoint paths and cycles. ∎

Let $e_i = (v_i, v_{i-1})$ be a backward edge of Q. If e_i does not follow a forward edge in Q, then v_{i-1} is a trivial path in $\mathcal{P}' = \mathcal{P} \oplus Q$. Similarly, if e_i is not followed by a forward edge in Q, then v_i is also a trivial path in $\mathcal{P}'$. Denote by $w^+(Q)$ the set of such trivial paths, then

$$w^+(Q) = \{v_{i-1} \in V(Q) \cap V[\mathcal{P}^+] \mid e_i = (v_i, v_{i-1}) \text{ does not follow a forward edge}\} \cup \{v_i \in V(Q) \cap V[\mathcal{P}^+] \mid e_i = (v_i, v_{i-1}) \text{ is not followed by a forward edge}\}.$$

On the other hand, if a vertex $v \in V[\mathcal{P}^0]$ is an endpoint of a forward edge in Q, then v is on a nontrivial path or cycle in $\mathcal{P}'$. We denote by $w^-(Q)$ the set of such vertices, then $w^-(Q) = \{v \in V(Q) \cap V[P^0] \mid v \text{ is an endpoint of a forward edge}\}$. Now, we define the weight of a k-alternating trail.

Definition 2.1.5. (Weight of k-alternating trail). The weight of Q is defined as $w(Q) = |w^+(Q)| - |w^-(Q)|$. By the description above, we have $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}|$.

Definition 2.1.6. (k-improving trail). A prim and proper k-alternating trail Q is k-improving, or for short, improving, if one of the following conditions holds:

1. Q is of type (a-1) with $w(Q) \leq k - 1$.

2. Q is of type (a-2), (b-1), or (b-2) with $in(Q) = ter(Q)$ and $w(Q) \leq -1$.

3. Q is of type (b-2), Q is not a closed trail and $w(Q) \leq -(k + 1)$

Lemma 2.1.7. *Let Q be a k-improving trail relative to a path partition $\mathcal{P}$ and $\mathcal{P}' = \mathcal{P} \oplus Q$. Then $\mathcal{P}'$ is a path partition with $|\mathcal{P}'|_k < |\mathcal{P}|_k$ provided that $\mathcal{P}'$ contains no cycles.*

Proof. For each type of improving trails, we first show that $k|\overline{\mathcal{P}'^+}| + |\overline{\mathcal{P}'^0}| < k|\overline{\mathcal{P}^+}| + |\overline{\mathcal{P}^0}|$.

Type(a-1): In this case, $|\overline{\mathcal{P}'^+}| = |\overline{\mathcal{P}^+}| - 1$, and $w(Q) \leq k - 1$ by definition 2.1.6, thus we have $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}| \leq k - 1$, and then $|\overline{\mathcal{P}'^0}| \leq |\overline{\mathcal{P}^0}| + k - 1$.

Type(a-2) or (b-1): Since $|\overline{\mathcal{P}'^{+}}| = |\overline{\mathcal{P}^{+}}|$ and $w(Q) \le -1$, we have $|\overline{\mathcal{P}'^{0}}| \le |\mathcal{P}^{0}| - 1$.

Type(b-2): If Q is a closed trail, then $|\overline{\mathcal{P}'^{+}}| = |\overline{\mathcal{P}^{+}}|$. Since $w(Q) \le -1$, $|\overline{\mathcal{P}'^{0}}| \le |\overline{\mathcal{P}^{0}}| - 1$. Therefore, if Q is not a closed trail, then $|\overline{\mathcal{P}'^{+}}| \le |\overline{\mathcal{P}^{+}}| + 1$, and $|\overline{\mathcal{P}'^{0}}| \le |\overline{\mathcal{P}^{0}}| - (k+1)$.

Since $\mathcal{P}'$ is acyclic, $\mathcal{P}'$ is a path partition by lemma 2.1.4. According to the discussion above, we have $|\mathcal{P}'|_k \le k|\overline{\mathcal{P}'^{+}}| + |\overline{\mathcal{P}'^{0}}| < k|\overline{\mathcal{P}^{+}}| + |\overline{\mathcal{P}^{0}}| = |\mathcal{P}|_k$. ∎

Definition 2.1.8. (k-transversal). Let $\mathcal{P} = \{P_1, P_2, ..., P_m\}$ be a path partition of G. A vertex subset $X = X_1 \cup X_2 \cup \ldots \cup X_k$, where $X_i \cap X_j = \emptyset$ for $i \neq j$ is a k-transversal of $\mathcal{P}$ if $V[P^0] \subseteq X$ and $|X_i \cap P_j| = 1$, for $P_j \in \mathcal{P}^{+}$, $1 \le i \le k$, $1 \le j \le m$, .

Conjecture 1.2.10 is equivalent to showing that for every k-optimal path partition $\mathcal{P}$, there exists a k-transversal $X = \{X_1, X_2, ..., X_k\}$ of $\mathcal{P}$, where X_i is an independent set for all $1 \le i \le k$. The output of the algorithm is either a k-transversal $X = \{X_1, X_2, ..., X_k\}$ of $\mathcal{P}$ where X_i is an independent set for all $1 \le i \le k$, or a path partition $\mathcal{P}'$ with less k-norm than $\mathcal{P}$ where $\mathcal{P}' = \mathcal{P} \oplus Q$ and Q is a k-improving trail. Remind that for a set $\mathcal{P}$ of disjoint paths, we denote by $ter[\mathcal{P}]$ the set of all terminal vertices of paths in $\mathcal{P}$, by $ter_{-}[\mathcal{P}]$ the set of vertices preceding the terminal vertices in $\mathcal{P}$ and by $ter_{-2}[\mathcal{P}]$ the set of vertices preceding the vertices in $ter_{-}[\mathcal{P}]$. Similarly, let $ter_{-i}[\mathcal{P}]$ denote the set of vertices preceding the vertices in $ter_{-(i-1)}[\mathcal{P}]$.

2.1.2 Description of Algorithm

I **Initialize Transversals**

We initialize the set X_i in X for $1 \le i \le k$ as follows:

$X_1 = ter[\mathcal{P}^{+}]$, $X_2 = ter_{-}[\mathcal{P}^{+}]$, $\ldots$, $X_i = ter_{-(i-1)}[\mathcal{P}^{+}]$, $\ldots$, $X_k = ter_{-(k-1)}[\mathcal{P}^{+}] \cup V[\mathcal{P}^{0}]$. In other words, we color the bottom k vertices with colors $1, 2, ..., k$ for each long paths and color all the vertice in short paths with color k.

II **Updating Transversals**

An admissible edge is an edge $e = (u, v)$, where $u, v \in X_i$ for some $1 \leq i \leq k$. Since admissible edges are not allowed in the objective k-transversal, we use admissible edges to update the set X_i. There are two types of admissible edges, either the tail $v \in V[\mathcal{P}^+]$ or $v \in V[\mathcal{P}^0]$.

Case 1: $v \in V[\mathcal{P}^+]$, and we assume that $v \in P_j$. We replace X_i by $X_i - v + \mathcal{P}_-(v)$, and we say that v gets 'bumped up' the path. If the preceding vertex $x = \mathcal{P}_-(v)$ is also in X, then x also gets bumped up the path and replace x by the preceding vertex of x. The process continues until either a vertex x_1 is replaced by its preceding vertex not in X (**Case 1.1**) or $x_1 \in in[\mathcal{P}^+] \cap X$, i.e. x_1 has no preceding vertex in $\mathcal{P}$ (**Case 1.2**). In Case 1.2, an improving trail Q is traced back from x_1.

Case 2: $v \in V[\mathcal{P}^0]$. If $i > 1$ (**Case 2.1**), then v is relabeled as color $i-1$ and hence v is moved from X_i to X_{i-1}. Otherwise, $i = 1$ (**Case 2.2**), then v cannot be relabeled and an improving trail Q is traced back from v.

III **Initialize Predecessors**

In order to trace back the improving trail, every vertex x involved in the algorithm has a predecessor $p[x]$ defined as follows:

For each vertex $v \in X_1 \cup V[\mathcal{P}^0]$, $p[v] \leftarrow v$. For each vertex $v \in V[\mathcal{P}^+]$, $v \in X_i$, $i > 1$, $p[v] \leftarrow \mathcal{P}_+(v)$. In other words, the predecessor of a vertex v on a long path in X_i is the vertex following v on the long path which is in X_{i-1}. Note that we trace back the improving trail Q according to the predecessors of the vertices we pass through.

IV **Updating Predecessors**

For an admissible edge $(u, v) \in X_i$, let $p(v) \leftarrow u$. If v gets bumped up the path as in Case 1.1, then let $p[\mathcal{P}_-(v)] \leftarrow v$.

2.1.3 The Algorithm

input: Graph $G = (V, E)$, path partition $\mathcal{P}$, integer $k \geq 1$.
initialize
        Transversals: As in Part I of Section 2.1.2
        Predecessors: As in Part III of Section 2.1.2
while (there exists $e = (u, v), u, v \in X_i$) **do**
      $p[v] \leftarrow u$
      Case 1: $v \in V[\mathcal{P}^+]$ (Assume $v \in P_j$)
        Case 1.1: v can be bumped up the path
              Bump v (and possibly preceding vertices) up the path as
                in Part II of Section 2.1.2
              Update predecessors as in Part IV of Section 2.1.2
        Case 1.2: $in(P_j) \in X_k$ and v cannot be bumped up the path
              Backtrack from $in(P_j)$ to find Q
              $\mathcal{P}' \leftarrow \mathcal{P} \oplus Q$
              Stop
      Case 2: $v \in V[\mathcal{P}^0]$
        Case 2.1: $i > 1$
              $X_i \leftarrow X_i - v$
              $X_{i-1} \leftarrow X_{i-1} + v$
        Case2.2: $i = 1$
              Backtrack from v to find Q
              $\mathcal{P}' \leftarrow \mathcal{P} \oplus Q$
              Stop
X_i, $1 \leq i \leq k$ are independent sets

Theorem 2.1.9. (*Hartman and Berger* [7]) *Assume G is an acyclic directed graph. Let $\mathcal{P}$ be a path partition of G, $k \geq 1$, and assume that every path in $\mathcal{P}^0$ is a trivial* *path (i.e. of cardinality one). Then the algorithm finds either a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_k < |\mathcal{P}|_k$ or a k-coloring orthogonal to $\mathcal{P}$.*

Proof. If the algorithm stops at line 24, then we have a set $X = \{X_1, X_2, ..., X_k\}$ which is a k-coloring orthogonal to $\mathcal{P}$. Otherwise, a trail Q is found in Case 1.2 and Case 2.2, and Q is a k-alternating trail. Note that Q is proper because of the initialization of predecessors, and that Q is prim of type(1) (Case 1.2) and type(2) (Case 2.2). For each case, we show that Q has weight as defined in Definition 2.1.6.

Remark: Since $v_i \in V[\mathcal{P}^0]$ decreases the color of v_i when $e_i = (v_{i-1}, v_i)$ is a forward edge in Q (Case 2.1 of the algorithm). For convenience, if $v_i \in X_1$, we shall define the color class of v_i to be X_0 after e_i is chosen.

Claim: Let $Q = (v_0, e_1, v_1, \ldots, e_l, v_l)$ be a proper (not necessarily prim) k-alternating trail found during the algorithm.

1. If Q is of type (a) (i.e. $v_0 \in X_1$), and $v_l \in X_t$ (after updating the colors), then Q is of weight $t - 1$.

2. If Q is of type (b) (i.e. $v_0 \in X_k$), and $v_l \in X_t$ (after updating the colors), then Q is of weight $t - k - 1$.

We prove the claim by induction on l, the length of Q.

For $l = 1$, it is trivial to check that the claim holds. If Q is of type (a) and $e_1 = (v_0, v_1)$ is a forward edge such that $v_1 \in V[\mathcal{P}^+]$ then $v_1 \in X_1$ and $w(Q) = 1 - 1 = 0$; if $v_1 \in V[\mathcal{P}^0]$ then $v_1 \in X_0$ after updating and $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}| = -1 = 0 - 1$. If $e_1 = (v_1, v_0)$ is a backward edge, then $v_1 \in X_2 \cap V[\mathcal{P}^+]$ and $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}| = 1 = 2 - 1$. If Q is of type (b), i.e. $v_0 \in V[\mathcal{P}^0]$, then $e_1 = (v_0, v_1)$ is a forward edge. If $v_1 \in V[\mathcal{P}^0]$, then $v_1 \in X_0$ after updating and $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}| = -2 = (k - 1) - k - 1$. If $v_1 \in V[\mathcal{P}^+]$, then $v_1 \in X_1$ and $w(Q) = |\overline{\mathcal{P}'^0}| - |\overline{\mathcal{P}^0}| = -1 = k - k - 1$. Assume $l > 1$.

For the induction step, we consider the different types of edges in Q:

1. Let $e_{i+1} = (v_{i+1}, v_i)$ be a backward edge in Q which is not followed by a forward edge. If $v_i \in X_{i_1}$, then $v_{i+1} \in X_{i_1+1}$, and then the value of t increases one. On the other hand, v_{i+1} is an additional trivial path in $\mathcal{P}' = \mathcal{P} \oplus Q$. Hence we have $w(v_0, \ldots, v_i, v_{i+1}) = w(v_0, \ldots, v_i) + 1$.

2. Let $e_i = (v_{i-1}, v_i)$ be a forward edge where $v_i \in V[\mathcal{P}^0]$. If $v_{i-1} \in X_{i_1}$, then $v_i \in X_{i_1-1}$ after updating in Case 2.1 of algorithm, and then the value of t decreases by one. On the other hand, $V[\overline{\mathcal{P}'^0}] = V[\overline{\mathcal{P}^0}] - \{v_i\}$. Hence $w(v_0, \ldots, v_i, v_{i+1}) = w(v_0, \ldots, v_i) - 1$.

3. For all other edges $e_i = (v_{i-1}, v_i) \in Q$, the color class of v_{i-1} is the same as v_i, and e_i contributes zero to $w(Q)$.

A prim and proper trail found in the algorithm satisfies $v_l \in in[\mathcal{P}^+] \cap X_k$ if it is of type (1)(Case 1.2), and $v_l \in V[\mathcal{P}^0] \cap X_1$ if it is of type (2)(Case 2.2).

A trail Q of type (a-1) has weight $k-1$. If Q is either of types (a-2), (b-1) or a closed trail of type (b-2), then $w(Q) = -1$. If Q is of type (b-2) then $w(Q) = -(k+1)$. In all cases, Q has weight as difined in Definition 2.1.6, then Q is a k-improving trail. Hence by Lemma 2.1.7, $\mathcal{P}' = \mathcal{P} \oplus Q$ is a path partition with $|\mathcal{P}'|_k < |\mathcal{P}|_k$. ∎

Chapter 3

New Results and Conclusion

In this chapter, we claim that Berge's strong path partition conjecture holds for certain special digraphs. First, we review a couple of definitions.

An undirected graph G is connected if for every pair of vertices x and y, there is a path from x to y. For a disconnected graph G, a component of G is a maximal connected subgraph of G. For a directed graph D, the underlying graph G of D is the graph obtained by letting the edges of D be unordered pairs. The vertex set of G is the same as the vertex set of D, but for any edge of D, it becomes undirected in G. An induced subgraph is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G)\backslash T$; this is the subgraph of G induced by T. Now, we are ready for the results.

3.1 The Main Results

Proposition 3.1.1. Given a directed graph D and a k-optimal path partition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where k is a positive integer. Then Berge's strong path partition conjecture holds for D if the underlying graph G of D satisfies the following conditions:

1. There is a vertex x of degree $n-1$, where $n = |V(G)|$.

2. $\forall v \in V(G-x)$, $deg_G(v) = 3$.

3. $G - x$ is connected.

Proof. Our goal is to show that there is a k-coloring f orthogonal to $\mathcal{P}$. Let P_1 be the path containing x in $\mathcal{P}$.

Claim 1: For any path $P_i = (a_1, a_2, \ldots, a_l) \in \mathcal{P} \setminus P_1$, $P_i + x$ forms a path of D.

Since $deg_G(x) = n - 1$, there is a directed edge adjoining x to each vertex of P_i in D. If $(x, a_1) \in E(D)$ or $(a_l, x) \in E(D)$, it is easy to check that the claim is true. Suppose the edges adjoining x to a_1 and a_l are (a_1, x) and (x, a_l) respectively. Since for each vertex $a \in V(P_i) - \{a_1, a_l\}$, either $(a, x) \in E(D)$ or $(x, a) \in E(D)$, and (a_1, x) and (x, a_l) are different directions for x, there exist two consecutive vertices a_j and a_{j+1} on P_i such that (a_j, x) and (x, a_{j+1}) appear simultaneously in D. Hence $P_i + x = (a_1, a_2, \ldots, a_j, x, a_{j+1}, \ldots, a_l)$ is also a path of D.

Claim 2: If $|P_1| \leq k$, then $|P_i| < k$ for all $P_i \in \mathcal{P} \setminus P_1$.

Suppose there is a path $P_i \in \mathcal{P} \setminus P_1$ with $|P_i| \geq k$, and let $\mathcal{P}' = \{P_1 - x, P_2, \ldots, P_i + x, \ldots, P_m\}$. Note that $P_1 - x$ is either exactly a path or two separate paths according to the position of x, and $P_i + x$ is also a path by **Claim 1**. Hence $\mathcal{P}'$ is a path partition, and $|\mathcal{P}'|_k = |\mathcal{P}|_k - 1$ because x contributes one to $|\mathcal{P}|_k$ in P_1 (since $|P_1| \leq k$), and contributes zero to $|\mathcal{P}'|_k$ in $P_i + x$ (since $P_i + x > k$). Therefore we have a path partition $\mathcal{P}'$ such that $|\mathcal{P}'|_k \leq |\mathcal{P}|_k$, a contradiction to that $\mathcal{P}$ is a k-optimal path partition.

The condition 1 together with 2 are equivalent to the statement below: For any vertex $v \in V(G')$, $deg_{G'}(v) = 2$, where G' is the graph obtained by deleting x from G. Therefore, G' is a disjoint union of cycles. And the condition 3 implies that G is a cycle.

Case 1. $|P_1| = l + 1 \leq k$. This implies that x must be colored, W.L.O.G, let the color of x be k. Since x is of degree $n-1$, x is the unique vertex with color k. Hence there are $k-1$ remaining colors left available for the rest of vertices. For convenience, we denote the underlying graph of $D - P_i$ by $G - P_i$ although G is an undirected graph and P_i is a directed path in $\mathcal{P}$.

Case 1.1. $G - P_1$ is connected, and hence $G - P_1$ is a (undirected) path, and $V(P_1) \cap V(G')$ is a set of l consecutive vertices in G'. Let these l consecutive vertices be $a_1, a_2, \ldots, a_l$, where a_1 and a_l are the neighbors of the endpoints of $G - P_1$. Let $G - P_1$ be the path $(a_{l+1}, a_{l+2}, \ldots, a_{n-1})$ where a_{l+1} and a_{n-1} are the neighbors of a_l and a_1 respectively. Define a coloring $f : V(G') \to \{1, 2, \ldots, k-1\}$ by

$f(a_j) = j$, for $1 \leq j \leq l$;

$f(a_{l+p}) \equiv p \pmod{k-1}$, $1 \leq p \leq k - l - 1$, if $n - 1 - l \not\equiv 1 \pmod{k-1}$; and

$f(a_{l+p}) \equiv p + 1 \pmod{k-1}$, $1 \leq p \leq k - l - 1$, if $n - 1 - l \equiv 1 \pmod{k-1}$

Then for any segment of length less than k in $G - P_1$, the number of colors is exactly the same as the length of this segment. Note that every directed path in D could be represented as a segment in G', and the segment for P_1 is obtained by ignoring x. Therefore, f is a k-coloring orthogonal to $\mathcal{P}$.

Case 1.2. $G - P_1$ is disconnected, and hence $G - P_1$ has two paths. Note that x is not an endpoint in P_1 since $G - P_1$ is disconnected. Let $G' = (v_1, v_2, \ldots, v_{n-1})$ be a cycle where v_1 is the first vertex in P_1, v_2 is the second one, and so on. In other words, the first i vertices of G' are the first i vertices of P_1, where $v_i = \mathcal{P}_-(x)$, the unique vertex preceding to x. Suppose $j \geq q$, then either $P_1 = (v_1, v_2, \ldots, v_i, x, v_j, v_{j-1}, \ldots, v_q)$ or $P_1 = (v_1, v_2, \ldots, v_i, x, v_q, v_{q+1}, \ldots, v_j)$. We give a coloring f for P_1 first as following: $f(v_h) = h$ for $1 \leq h \leq i$ and

$f(v_{q+h}) = i+1+h$ for $0 \leq h \leq j-q$. Then the two paths separated by P_1 are $(v_{i+1}, v_{i+2}, \ldots, v_{q-1})$ and $(v_{j+1}, v_{j+2}, \ldots, v_{n-1})$. Now, we can extend f to a coloring for G. Note that $f(v_q) = i+1$, and $f(v_j) \neq 1$. For $(v_{i+1}, v_{i+2}, \ldots, v_{q-1})$, if $|(v_{i+1}, v_{i+2}, \ldots, v_{q-1})| = q-1-i \not\equiv 1 \pmod{k-1}$, then $f(v_h) \equiv h \pmod{k-1}$ for $i+1 \leq h \leq q-1$, else $f(v_h) \equiv h+1 \pmod{k-1}$ for $i+1 \leq h \leq q-1$. Note that $i \equiv i+1$ if $k=3$, i.e. v_i and v_{i+1} might be in the same color class. but it is easy to check that this case does not exist by considering the directions of edges which is restricted by the length of P_1. For $(v_{j+1}, v_{j+2}, \ldots, v_{n-1})$, if $|(v_{j+1}, v_{j+2}, \ldots, v_{n-1})| = n-j-1 \not\equiv 1 \pmod{k-1}$, then $f(v_{j+h}) \equiv h \pmod{k-1}$ for $1 \leq h \leq n-j-1$, else $f(v_{j+h}) \equiv h+1 \pmod{k-1}$ for $1 \leq h \leq n-j-1$. Similarly, for every segment of length less than k in $(v_{i+1}, v_{i+2}, \ldots, v_{q-1})$ and $(v_{j+1}, v_{j+2}, \ldots, v_{n-1})$, the number of colors is exactly the same as the length of this segment. Since every path in D could be represented as a segment, f is a k-coloring orthogonal to $\mathcal{P}$.

Case 2. If $|P_1| = l+1 > k$, then x need not to be colored. The technique that we use in this case is similar to **Case 1**. We color $V(P_1) \cap V(G')$ first, but we only need to pick k vertices in P_1 to color. After finishing the coloring of P_1, we consider the coloring of the remaining vertices in $G - P_1$ as we have done in **Case 1**. Note that the difference is that in **Case 1** the colors are taken modulo $k-1$, but in **Case 2** the color are taken modulo k. Then for any segment with length $\leq k$ in $G - P_1$, it is easy to see that the vertices in this segment have different colors from each other, and then for any segment with length greater than k in $G - P_1$, there are k colors in this segment (some colors might appear more than once). So, for $|P_1| = l+1 > k$, there also exists a k-coloring orthogonal to $\mathcal{P}$. This concludes the proof of this case.

■

Most of the notations hereunder are the same as the notations in Proposition 3.1.1. They are P_1, the path containing x; G', the graph obtained by deleting x from G; and $G - P_i$, the underlying graph of $D - P_i$.

Proposition 3.1.2. Given a directed graph D and a k-optimal path parition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where k is a positive integer. Then Berge's strong path partition conjecture holds for D if the underlying graph G of D satisfies the following conditions:

1. There is a vertex x of degree $n-1$, where $n = |V(G)|$.

2. $\forall v \in V(G-x)$, $deg_G(v) \leq 3$.

3. $G - x$ is connected.

Proof. In condition 2, we have shown the case that $deg_G(v) = 3$, $\forall v \in V(G-x)$. When $deg_G(v) < 3$, and together with condition 1 and 3, we know that $G' = G - x$ is a path (a trivial path is possible). If G' is a path of length 1 or 2, then D has a hamiltonian path. Hence this proposition holds. Let $G' = (v_1, v_2, \ldots, v_{n-1})$, $n-1 \geq 3$. Let H be the graph obtained by adding the edge (v_{n-1}, v_1) into G. Then for any digraph D' whose underlying graph is H, Berge's strong path paritition conjecture holds for D' by Proposition 3.1.1. But in Proposition 3.1.1, the coloring is found for any segment in $H - P_1$. Therefore, for the k-optimal path parition $\mathcal{P}$ in D, the coloring found by Proposition 3.1.1 is orthogonal to it. ■

Proposition 3.1.3. Given a directed graph D and a k-optimal path parition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where k is a positive integer. Then Berge's strong path partition conjecture holds for D if the underlying graph G of D satisfies the following conditions:

1. There is a vertex x of degree $n-1$, where $n = |V(G)|$.

2. $\forall v \in V(G-x)$, $deg_G(v) = 3$.

3. $G - x$ is disconnected.

Proof. Let $G' = G - x$. Since G' is disconnected, G' is a disjoint union of cycles by condition 2. Let $G' = \{C_1, C_2, \ldots, C_r\}$ and $x \in V(P_1)$. Note that the vertices of $P_1 - x$ are from at most two cycles in G'. For conveninece, we denote the set of paths in $\mathcal{P} \cap D[V(C)]$ by $\mathcal{P} \cap C$.

We first color the vertices in $C \in G'$, where $V(C) \cap V(P_1) = \emptyset$. For each $C = (b_1, b_2, \ldots, b_s, b_1) \in G'$ such that $V(C) \cap V(P_1) = \emptyset$, we can define a coloring for C according to the length of the longest path in $\mathcal{P} \cap C$. W.L.O.G, let the longest path in $\mathcal{P} \cap C$ be $(b_1, b_2, \ldots, b_t)$, $t \leq s$.

Case 1. For $t < k$, we define f as following:

If $|C| = s \not\equiv 1 \pmod{k-1}$, then $f(b_i) \equiv i \pmod{k-1}$ for $1 \leq i \leq s$.

If $|C| = s \equiv 1 \pmod{k-1}$, then $f(b_i) \equiv i \pmod{k-1}$ for $1 \leq i \leq s-1$ and $f(b_s) = 2$.

Case 2. For $t = k$, we first color the longest path $(b_1, b_2, \ldots, b_t)$ by $f(b_i) = i$ for $1 \leq i \leq t$ and extend f to a coloring for C:

If $|C| = s \not\equiv 1 \pmod{k}$, then $f(b_i) \equiv i \pmod{k}$ for $1 \leq i \leq s$.

If $|C| = s \equiv 1 \pmod{k}$, then $f(b_i) \equiv i \pmod{k}$ for $1 \leq i \leq t$ and $(b_i) \equiv i+1 \pmod{k}$ for $t+1 \leq i \leq s$.

Case 3. For $t > k$, there is at least one vertex which needs not be colored, and let it be b_1. We define f as $f(b_i) \equiv i - 1 \pmod{k}$, $2 \leq i \leq s$.

In all three cases above, it is clear that $(b_1, b_2, \ldots, b_t)$ contains exactly $\min\{t, k\}$ colors. For any segment with length less than k in $(b_{t+1}, b_{t+2}, \ldots, b_s)$, it is easy to see that the number of colors in the segment equals the length of this segment. For any segment with length equals to k, the number of colors is k. Only in Case 3., there might be some paths with length greater than k. For any segment with length greater than k (in Case 3. only), it is not difficult to see that the number of colors is k. Note that any path in $(b_{t+1}, b_{t+2}, \ldots, b_s)$ could be represented as a segment. Therefore, f is a coloring orthogonal to $\mathcal{P}$ partially in C.

The remaining part is the cycle(s) which is involved by P_1 and the coloring can be obtained accordingly. Note that **Claim 2.** in Proposition 3.1.1 is also true here.

Subcase 1. x is a path itself, i.e., a trivial path. The paths in $\mathcal{P}$ are all of length less than k by **Claim 2** in Proposition 3.1.1. According to Case 1. above, we only use $k-1$ colors for the vertices of $G-x$, and we can color x in k.

Subcase 2. The vertices of $P_1 - x$ are exactly from one cycle in G'. The coloring for $C_1 \cup P_1$ is the same as the discussion in Proposition 3.1.1, i.e., $C_1 \cup P_1$ can be viewed as a connected graph.

Subcase 3. The vertices of $P_1 - x$ are from two cycles, called C_1 and C_2. Let $P_1 = (a_1, a_2, \ldots, a_i, x, a_{i+1}, \ldots, a_l)$, where $(a_1, a_2, \ldots, a_i)$ and $(a_{i+1}, a_{i+2}, \ldots, a_l)$ are paths in $D[V(C_1)]$ and $D[V(C_2)]$ respectively. First, we give a coloring f for P_1. If $|P_1| \leq k$, then all the other paths in $\mathcal{P}$ are of length less than k. We color P_1 as following: $f(a_i) = i$ for $1 \leq i \leq l$, and $f(x) = k$. The remaining vertices in C_1 and C_2 form undirected paths respectively, then we can obtain the coloring respectively as in Proposition 3.1.1 Case 1.1. Note that we might need to permute the colors appearing in C_2 to ensure that P_1 has $l+1$ different colors on it. Clearly, if $|P_1| > k$, we don't have to color x. Since the argument is similar to above, except we use k to replace $k-1$, we omit the details. This concludes the proof.

∎

Proposition 3.1.4. Given a directed graph D and a k-optimal path parition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where k is a positive integer. Then Berge's strong path partition conjecture holds for D if the underlying graph G of D satisfies the following conditions:

1. There is a vertex x of degree $n-1$, where $n = |V(G)|$.
2. $\forall v \in V(G-x)$, $deg_G(v) \leq 3$.
3. $G - x$ is disconnected.

Proof. According to the conditions, we obtain that $G - P_1$ is a disjoint union of some cycles and paths. We color P_1 first. The two cases, whether x is colored or not, are discussed the same as above. For each component of $G - P_1$, if it is a cycle, then we color the vertices of it by Proposition 3.1.3; if it is a path, then we color the vertices of it by Proposition 3.1.2. Therefore we have a k-coloring orthogonal to $\mathcal{P}$. ∎

Now we have the theorem by combining the above four propositions together.

Theorem 3.1.5. *Given a directed graph D and a k-optimal path parition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where k is a positive integer. Then Berge's strong path partition conjecture holds for D if the underlying graph G of D satisfies the following conditions:*

1. *There is a vertex x of degree $n-1$, where $n = |V(G)|$.*
2. *$\forall v \in V(G-x)$, $deg_G(v) \leq 3$.*

3.2 Conclusion

As can be seen from this study, due to the diversity of general directed graphs, to prove the truth (we believe) of Berge's strong path partition conjecture seems to be very difficult. This can also be seen from the known results obtained so far only a short list in Chapter 2. Nevertheless, we step forward to make some contributions by showing the conjecture holds for a class of graphs in this thesis. Mainly, we prove that the conjecture holds for the graphs obtained by joining a vertex to a set of vertex disjoint paths and/or cycles. Hopefully, the technique used in this thesis (considering only underlying graph) can be applied to do a better job in any future study.

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