國 立 交 通 大 學

應用數學系

碩 士 論 文

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The Poset constructed from a Hermitian Forms Graph

研 究 生：卜文強

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摘要

在這篇論文裡，我們專注在荷米爾遜圖上。首先由一個半徑為 D 的荷米爾遜圖建構一個偏序集 P。在 P 中的元素是半徑不大於 D 的荷米爾遜子圖。我們從反序的包含關係來定義 P 的順序。我們獲得一些 P 的計算性質。然後，我們試著在 P 中建構一個拉鍊的結構，計算在 P 中有多少拉鍊的結構。

The Poset constructed from a Hermitian Forms Graph

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Abstract

In this thesis, we focus on Hermitian forms graphs. Firstly, we construct a poset P from a Hermitian forms graph $Her_q(D)$, where D is the diameter. The elements in P are those subgraphs of $Her_q(D)$ which are isomorphic to $Her_q(t)$ for $0 \leq t \leq D$. We order P by reversed inclusion. Some counting properties of P are obtained. Then, we try to construct a zigzag-like structure in P so that we can count the number of zigzags inside P.

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Contents

1 Introduction

It is well-known that a Hermitian forms graph is a distance-regular graph that contains many subgraphs, each of them isomorphic to a Hermitian forms graph with smaller diameter. In this thesis, we fix a Hermitian forms graph $Her_q(D)$ and construct a poset P from $Her_q(D)$, where D is the diameter. The elements in P are those subgraphs of $Her_q(D)$ which are isomorphic to $Her_q(t)$ for $0 \leq t \leq D$. We order P by reversed inclusion. P is known to be a raked poset. What we aim in this thesis is to study other properties of P in counting aspects. After introducing the background, in chapter 4, we count the number of the subgraphs with diameter $D - t$ in $Her_q(D)$ by Theorem 4.1, i.e., we know the $|P_t|$ in P. It helps us to count the number of $|w^+ \cap P_j|$ because w is the Hermitian forms graph $Her_q(D - i)$, where $w \in P_i$ and $0 \leq i \leq j \leq D$. Then, we get Lemma 5.1. Besides, we are also interested in counting $|[0, z] \cap P_i|$, where $z \in P_j$ and $0 \leq i \leq j \leq D$. By 2-way counting, we solve this question, namely Lemma 5.2. In Lemma 5.3, we get the general case of Lemma 5.2, namely we get $|[w, z] \cap P_h|$, where $w \in P_i$, $z \in P_j$, and $0 \leq i \leq h \leq j \leq D$. Those lemmas are basic tools that help us to count the complex structure in P. In Lemma 5.4, we try to construct a zigzag-like structure in P so that we can count the number of zigzags inside P. Then, we solve it by multiplication principle.

In chapter 6, we count more numbers based on Lemma 5.4. We add some conditions on x and y for $x \in P_i$ and $y \in P_{i+1}$, thus, we let x, y meet in

P_{i-1} and x, u, v, y to form a zigzag. We name $t_i(x, y)$ the number of zigzags based on x, y. But it will be difficult if there is no element in $x \vee y$. There is a simple case with $x \vee y$ not existing in Lemma 6.1, we find $t_{D-1}(x, y)$=0 or 1. But in the cases of $1 \leq i \leq D-2$, it is hard to find out. There are a lot of difficulties that we need to overcome in the future.

2 Preliminaries

Let P denote a finite set. By a *partial order* on P, we mean a binary relation $\leq$ on P such that

1. $x \leq x \qquad (\forall x \in P)$,
2. $x \leq y$ and $y \leq z \rightarrow x \leq z \qquad (\forall x, y, z \in P)$,
3. $x \leq y$ and $y \leq x \rightarrow x = y \qquad (\forall x, y \in P)$.

By a *partially ordered set* (or *poset*, for short), we mean a pair $(P, \leq)$, where P is a finite set, and where $\leq$ is a partial order on P. Abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P denote a poset, with partial order $\leq$, and let x and y denote any elements in P. As usual we write $x < y$ whenever $x \leq y$ and $x \neq y$. We say y *covers* x whenever $x < y$, and there is no $z \in P$ such that $x < z < y$. An element $x \in P$ is said to be *minimal* whenever there is no $y \in P$ such that $y < x$. Let $\min(P)$ denote the set of all minimal elements in P. Whenever min(P) consists of a single element, we denote it by 0, and we say P has 0.

Suppose P has a 0. By an *atom* in P, we mean an element in P that covers 0. We let A_p denote the set of atoms in P.

Suppose P has 0, By a *rank function* on P, we mean a function

$$\text{rank}: P \rightarrow Z$$

such that rank(0)=0, and such that for all $x, y \in P$,

$$y \text{ covers } x \rightarrow \text{rank}(y) \text{ - rank}(x)=1.$$

Observe the rank function is unique if it exits. P is said to *ranked* whenever P has a rank function. In this case, we set

$$\text{rank}(P) := max\{\text{rank}(x) \mid x \in p\},$$

$$P_i := \{x \mid x \in P, \text{rank}(x) = i\} \ (i \in Z),$$

and observe $P_0 = 0$, $P_1 = A_p$.

Let P denote any poset, and let S denote any subset of P. Then there is a unique partial order on S such that for all $x, y \in S$,

$$x \leq y \text{ (in } S) \leftrightarrow x \leq y \text{ in } P.$$

This partial order is said to be *induced* form P. By a *subposet* of P, we mean a subset of P, together with the partial order induced from P. Pick any $x, y \in P$ such that $x \leq y$. By the *interval* $[x, y]$, we mean the subposet

$$[x, y] := \{z \mid z \in P, x \leq z \leq y\}$$

of P.

Let P denote any poset, and pick any $x, y \in P$. By a *lower bound* for x, y, we mean an element $z \in P$ such that $z \leq x$ and $z \leq y$. Suppose the subposet of lower bounds for x, y has a unique maximal element. In this case we denote this maximal element by $x \wedge y$, and say $x \wedge y$ *exists*. This element $x \wedge y$ is known as the *meet* of x and y. P is said to be (*meet*)*semi-lattice* whenever P is nonempty, and $x \wedge y$ exists for all $x, y \in P$. A semi-lattice has 0. Suppose P is a semi-lattice, and pick $x, y \in P$. By an *upper bound* for x and y, we mean an element $z \in P$ such that $x \leq z$ and $y \leq z$.Observe that subset of upper bounds for x and y is closed under $\wedge$; in particular, it has a unique minimal element iff it is nonempty. In this case we denote this minimal element by $x \vee y$, and say that $x \vee y$ exists. The element $x \vee y$ is known as the *join* of x and y.

Let P be a semi-lattice. Then P is said to be *atomic* whenever each element of P that is neither 0 nor an atom is a join of atoms. Observe if P is a ranked atomic semi-lattice, then $\mid [0, x] \cap P_1 \mid \geq \text{rank}(x)$ for all $x \in P$. A semi-lattice P is atomic iff each element of P that is not 0 and not an atom covers at least 2 elements of P.

Let $\Gamma = (X, R)$ denote a finite undirected graph without loops or multiple edges with vertex set X, edge set R and diameter D. For all $x \in X$ and for all integers $0 \le i \le D$, set

$$\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}.$$

Γ is said to be *distance-regular* whenever for all integers $0 \le h, i, j \le D$ and for all $x, y \in \Gamma$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := \left|\Gamma_i(x) \cap \Gamma_j(y)\right|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}$, $a_i := p^i_{1i}$, $b_i := p^i_{1i+1}$, and $k_i := p^0_{ii}$. Note that $c_1 = 1$, $a_0 = 0$, $b_D = 0$, $k_1 = c_i + a_i + b_i$,

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \tag{2.1}$$

for $1 \le i \le D$, and

$$|X| = 1 + k_1 + \cdots + k_D. \tag{2.2}$$

We give an example of distance-regular graph. Let q denote a prime power, and let U denote a finite vector space of dimension D over the field $GF(q^2)$. Let H denote the D^2-dimensional vector space over $GF(q)$ of the Hermitian forms on U. Thus $f \in H$ if and only if $f(u, v)$ is linear in v, and $f(v, u) = \overline{f(u, v)}$ for all $u, v \in U$. Pick $f \in H$. We define

$$\mathrm{rk}(f) = \dim(U \setminus \mathrm{Rad}(f)),$$

where

$$\text{Rad}(f) = \{u \in U | f(u, v) = 0 \text{ for all } v \in U\}.$$

The *Hermitian forms graph* $Her_q(D) = (X, R)$ is the graph with vertex set $X = H$ and vertices $x, y \in R$ iff $\text{rk}(x - y) = 1$ for $x, y \in X$. It is well known that $Her_q(D)$ is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^{i-1}(q^i - (-1)^i)}{q+1}, \tag{2.3}$$

$$b_i = \frac{q^{2D} - q^{2i}}{q+1} \tag{2.4}$$

for $0 \le i \le D$ [1, Theorem 9.5.7]. Note that

$$|X| = |H| = q^{D^2}. \tag{2.5}$$

3 Subgraphs in a Hermitian forms graph

The following theorem about Hermitian forms graphs will be used in the thesis.

Theorem 3.1. *([2], [3], [4]) Let $\Gamma = (X, R)$ be the Hermitian forms graph $Her_q(D)$. Then the following hold.*

(i) For two vertices $x, y \in X$ with distance t, there exists a subgraph $\Delta(x, y)$ such that $\Delta(x, y)$ is isomorphic to $Her_q(t)$.

(ii) Let $\Delta(x, y)$ be as in (i). Then for any $u, v \in \Delta(x, y)$ and $w \in X$ we have

$$\partial(u, w) + \partial(w, v) \le \partial(u, v) + 1 \implies w \in \Delta(x, y). \tag{3.1}$$

In particular $\Delta(x,y)$ *has intersection numbers*

$$c_i(\Delta(x,y)) = \frac{q^{i-1}(q^i-(-1)^i)}{q+1}, \tag{3.2}$$

$$b_i(\Delta(x,y)) = \frac{q^{2t}-q^{2i}}{q+1} \tag{3.3}$$

for $0 \leq i \leq D$.

(iii) Set $P := \{\Delta(x,y) \mid x,y \in G\}$, *and order* P *by reversed inclusion. Then* P *is a ranked meet semi-lattice with each interval isomorphic to a projective space over a finite field of* q^2 *elements.*

(iv) The set P_i *of rank* i *elements in* P *is*

$$P_i = \{\Delta \in P \mid \text{diameter}(\Delta) = D - i\}$$

for $0 \leq i \leq D$, *the meet is defined by*

$$\Delta \wedge \Delta' := \bigcap_{\substack{\Omega \in P \\ \Delta,\Delta' \subseteq \Omega}} \Omega,$$

and the join (if it exists) is

$$\Delta \vee \Delta' = \Delta \cap \Delta' \qquad (\textit{assuming } \Delta \cap \Delta' \neq \emptyset)$$

for $\Delta, \Delta' \in P$. □

4 The shape of P

Throughout the remaining of the thesis, fix a Hermitian forms graph $\Gamma = (X,R) = Her_q(D)$, and let P denote the corresponding poset as described in Theorem3.1(iii). Let P_i be as defined in Theorem3.1(iv) for $0 \leq i \leq D$. The following theorem counts the number of elements in P_i.

Theorem 4.1.

$$|P_t| = \begin{bmatrix} D \\ t \end{bmatrix}_{q^2} q^{t(2D-t)}, \tag{4.1}$$

where

$$\begin{bmatrix} D \\ t \end{bmatrix}_{q^2} := \prod_{i=0}^{t-1} \frac{q^{2D} - q^{2i}}{q^{2t} - q^{2i}}$$

Proof. For $x \in P_D$, set

$$p_t(x) := |P_t \cap [0, x]|.$$

Note that

$$p_t(x) = \begin{bmatrix} D \\ t \end{bmatrix}_{q^2} \tag{4.2}$$

is independent of the choice of x by Theorem 3.1(iii). By the 2-way counting in the pairs (x, Δ) such that $x \in P_D$, $\Delta \in P_t$ with $x \in \Delta$ we find

$$|P_D| \times |p_t(x)| = |P_t| \times |\Delta|. \tag{4.3}$$

Note that Δ is $Her_q(D-t)$ by Theorem 3.1(iv). Solving (4.2), (4.3) for $|P_t|$, and simplifying the result using (2.5), we find (4.1).

□

Note that by Theorem 3.1, the P_t collects the subgraphs of $Her_q(D)$ which are isomorphic to $Her_q(t)$. Hence Theorem 4.1 determines the number of such subgraphs.

5 The subposets w^+ and $[0, w]$

For $w \in P$, set

$$w^+ := \{u \in P \mid u \geq w\},$$

and

$$[0, w] := \{u \in P \mid u \leq w\}.$$

We study the shape of w^+ and $[0, w]$ in this section.

Lemma 5.1. *For $w \in P_i$,*

$$|w^+ \cap P_j| = \begin{bmatrix} D-i \\ j-i \end{bmatrix}_{q^2} q^{(j-i)(2D-i-j)}, \tag{5.1}$$

where $i \leq j \leq D$.

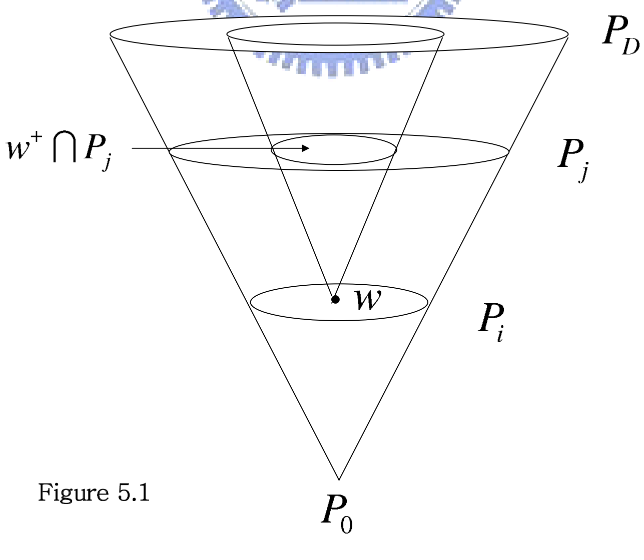

Figure 5.1

Proof. Fix $w \in P_i$. Since w is the Hermitian forms graph $Her_q(D-i)$, we know w has diameter $D-i$ and $w^+ \cap P_j$ is the rank $j-i$ elements in w^+. Hence we have (5.1) by Theorem 4.1.

□

The following is the downward version of Lemma 5.1.

Lemma 5.2. *For* $z \in P_j$

$$|[0,z] \cap P_i| = \begin{bmatrix} j \\ i \end{bmatrix}_{q^2}, \tag{5.2}$$

where $0 \le i \le j \le D$.

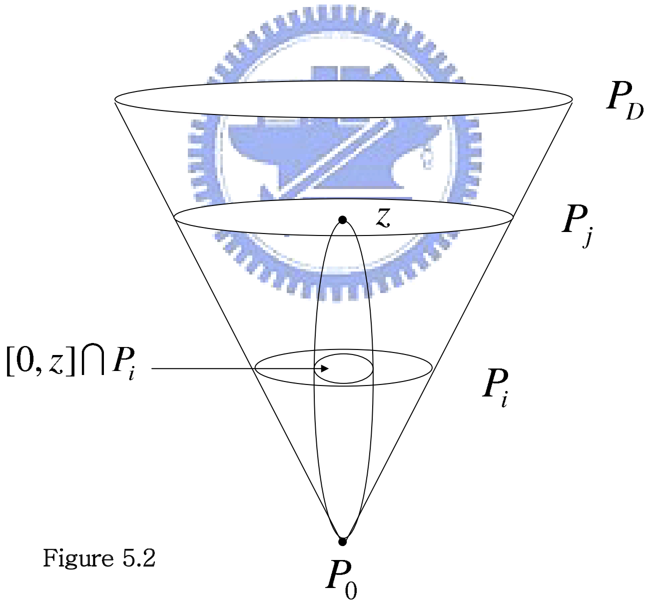

Figure 5.2

Proof. By the 2-way counting of the pairs (w, z) such that $w \in P_i$, $z \in P_j$ with $w \le z$ we find

$$|P_i| \times |w^+ \cap P_j| = |P_j| \times |[0,z] \cap P_i|. \tag{5.3}$$

Now (5.2) follows by solving (5.3) using (4.1), (5.1) for $|[0,z] \cap P_i|$.

□

The following is a generalization of Lemma 5.2.

Lemma 5.3. *For $z \in P_j$, $w \in P_i$, and $w \leq z$*

$$|[w,z] \cap P_h| = \begin{bmatrix} j-i \\ h-i \end{bmatrix}_{q^2}, \tag{5.4}$$

where $0 \leq i \leq h \leq j \leq D$.

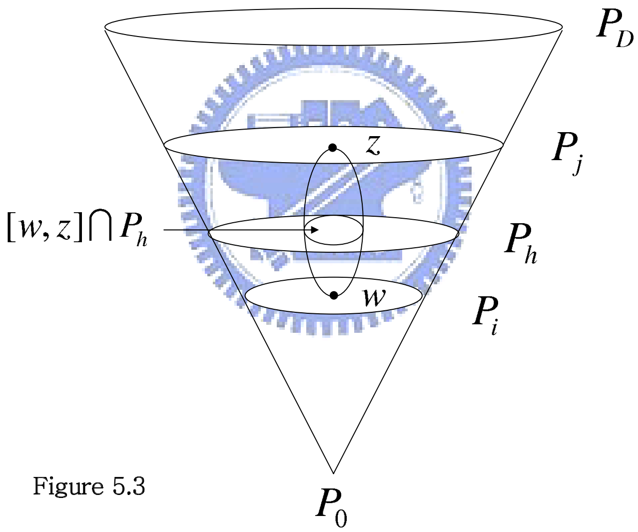

Figure 5.3

Proof. Fix $w \in P_i$. Since w is the Hermitian forms graph $Her_q(D-i)$, we know w has diameter $D-i$. Also we know that z is the rank $j-i$ elements and $[w,z] \cap P_h$ is the rank $h-i$ elements in w^+. Hence we have (5.3) by Lemma 5.2.

□

Now we want to count a structure by using the above lemmas. Let

$$Zig(i,j) := |\{(x,u,v,y) \mid x,v \in P_i, u,y \in P_j, x,v \leq u, v \leq y, x \neq v, u \neq y\}|,$$

where $0 \leq i \leq j \leq D$. The structure looks like a zigzag so we name it zigzag.

Lemma 5.4.

$$\begin{aligned} & Zig(i,j) \\ = & \begin{bmatrix} D \\ i \end{bmatrix}_{q^2} q^{i(2D-i)} \times \begin{bmatrix} D-i \\ j-i \end{bmatrix}_{q^2} q^{(j-i)(2D-i-j)} \\ \times & \left(\begin{bmatrix} j \\ i \end{bmatrix}_{q^2} - 1\right) \times \left(\begin{bmatrix} D-i \\ j-i \end{bmatrix}_{q^2} q^{(j-i)(2D-i-j)} - 1\right), \end{aligned} \qquad (5.5)$$

where $0 \leq i \leq j \leq D$.

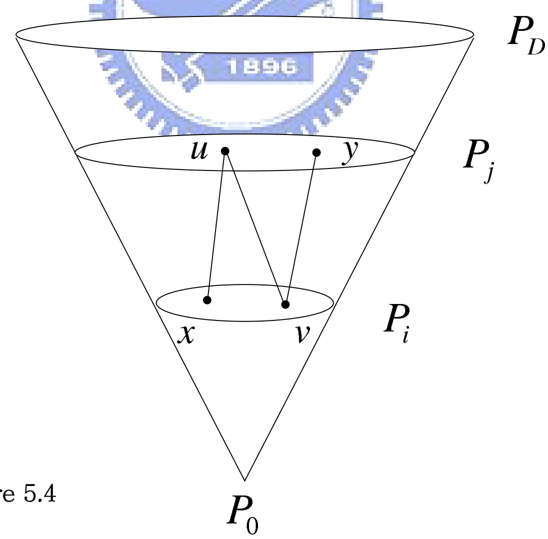

Figure 5.4

Proof. First to choose x, we have $|P_i|$ choices. Second to choose u, we have $|x^+ \cap P_j|$ choices. Third to choose v, we have $|[0,u] \cap P_i| - 1$ choices. Finally, to choose y, we have $|v^+ \cap P_j| - 1$ choices. We count those choices by Theorem 4.1, Lemma 5.1, and Lemma 5.2. Hence,

$$
\begin{aligned}
& Zig(i,j) \\
= & \begin{bmatrix} D \\ i \end{bmatrix}_{q^2} q^{i(2D-i)} \times \begin{bmatrix} D-i \\ j-i \end{bmatrix}_{q^2} q^{(j-i)(2D-i-j)} \\
\times & \left(\begin{bmatrix} j \\ i \end{bmatrix}_{q^2} - 1 \right) \times \left(\begin{bmatrix} D-i \\ j-i \end{bmatrix}_{q^2} q^{(j-i)(2D-i-j)} - 1 \right). \qquad (5.6)
\end{aligned}
$$

□

6 Zigzags in P

For $x \in P_i$, $y \in P_{i+1}$, s.t. $x \wedge y \in P_{i-1}$, set

$$t_i(x,y) := |\{(u,v) \mid u \wedge y = v, x \leq u\}|,$$

where $1 \leq i \leq D-1$. We name $t_i(x,y)$ the number of zigzags based on x, y.

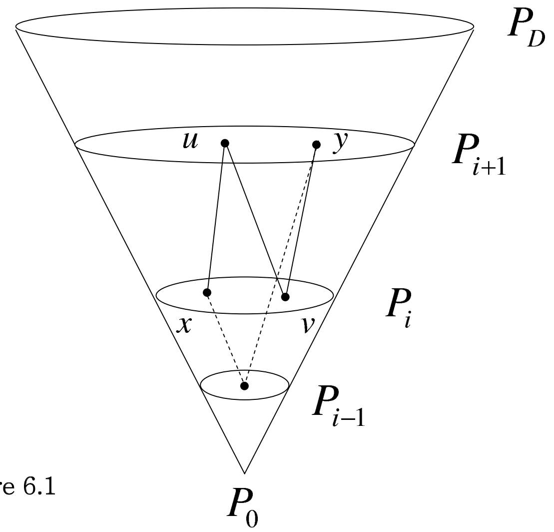

Figure 6.1

Lemma 6.1. *For $x \in P_{D-1}$ and $y \in P_D$, $t_{D-1}(x, y)$= 0 or 1 is independent of the choices of x, y.*

Proof. Given a vertex y and a maximal clique x such that $x \wedge y \in P_{D-2}$, i.e. a weak-geodetically closed subgraph of diameter 2. Now we prove this lemma by cases. Case 1: $\partial(u, y)$=2 for all u in x. Hence, $t_{D-1}(x, y)$=0. Case 2: $\partial(u, y) = 1$ for some u is in x. Then $v = \Delta(u, y)$ is the unique element in P_{D-1} containing u and y by Theorem 3.1(i), so $t_{D-1}(x, y)$=1.

□

The following problem is still open.

Problem 6.2. *For $x \in P_i$ and $y \in P_{i+1}$, determine $t_i(x, y)$, where $1 \leq i \leq D - 2$.*

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