国 立 交 通 大 學

應用數學系

碩 士 論 文

完全圖的路徑分割

Decomposing the Complete Graph into Paths

研 究 生：張澍仁

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中 華 民 國 九 十 六 年 六 月

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完全圖的路徑分割

研 究 生：張澍仁　　　　指導老師：傅恆霖　教授

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摘　要

已知當m可以整除完全圖的邊數時，在1≦m≦v-1的情況下一個v點的完全圖可以分割成全部都是長度m的路徑。可是，任取一個正整數m滿足1≦m≦v-1，m並不一定能夠整除v點的完全圖邊數。所以我們討論在這種情況下是否仍有類似的漂亮結果，即當m不整除完全圖的邊數時，分割完全圖成為一些長度為m的路徑及一個長度為餘數的路徑。在本論文中，我們證明：完全圖可以分割成為k個長度為m的路徑加上一個長度為r的路徑，若且為若完全圖的邊數等於km+r且0≦r＜m≦v-1。

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Decomposing the Complete Graph into Paths

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Abstract

It is known that if $m \mid \binom{v}{2}$, then the complete graph K_v can be decomposed into paths of length m as long as $1 \leq m \leq v-1$. But, for a given positive integer $1 \leq m \leq v-1$, m may not be a factor of $\binom{v}{2}$. Therefore, we are interested in the case $m \nmid \binom{v}{2}$. In these cases, we need a path of distinct length. Let P_t denote a path with t edges. Then, it is proved in this thesis that the complete graph K_v can be decomposed into k P_m's and one P_r if and only if $\binom{v}{2} = km + r$ where $0 \leq r < m \leq v-1$.

Contents

1 Introduction

Graph decomposition has been one of the most important topics in Graph Theory. Not only the study is close related to discover the structures of graphs, but also give rise another approach to construct combinatorial designs. It is well-known that the existence of a balanced incomplete block design (BIBD), (v, k, λ)-design, is equivalent to the decomposition of the multi-graph λK_v into edge-disjoint complete graphs K_k. As an analog, a λ-fold k-cycle system of order v is a decomposition of λK_v into cycles of length k and a λ-fold k-path system of order v is a decomposition of λK_v into paths of length k.

To construct a BIBD for each admissible triple v, k and λ is not an easy task, see [8, 12] for references. But, path systems have been obtained for all possible parameters v, k and λ, see [6, 13].

A bit of reflection, if K_v can be decomposed into paths of length k, then $v \geq k+1$ and $k \mid \binom{v}{2}$. But, $k \mid \binom{v}{2}$ does not occur very often. In case that $\binom{v}{2} = qm + r$, $0 < r < m$, then decomposing K_v into paths of length m is not possible. Instead, we try to decompose K_v into q paths each of length m and one path which is of length r. This is the main theme of this thesis.

2 Preliminaries

Note that the following definitions and notations can be referred to the book : *INTRODUCTION TO GRAPH THEORY* [14].

A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each two vertices (not necessary distinct) called its endpoints.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges.

A *complete graph* K_v is a simple graph whose vertices are pairwise adjacent. A complete graph with v vertices is denoted K_v.

A *walk* is a list $v_0, e_1, v_1, e_2, v_2, ..., e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. If the edges $e_1, e_2, ..., e_k$ are distinct, then this walk is called a *trail.* If, in addition, the vertices $v_0, v_1, ..., v_k$ are distinct, then this walk is called a *path.* If this walk is a path and $v_0 = v_k$, then this walk is called a *cycle.* A path with k vertices is denoted P_k and a cycle with k vertices is denoted C_k. But, for convenience, we use P_k to denote a path with k edges throughout this thesis.

The *length* of a walk, trail, path, or cycle is its number of edges. A walk or trail is closed if its endpoints are the same.

A *factor* of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph.

A *matching* of size k in a graph G is a set of k pairwise disjoint edges. The vertices belonging to the edges of a matching are saturated by the matching; the others are unsaturated. If a matching saturates every vertex of G, then it is a perfect matching or 1-factor. We shall denote 1-factor by I.

A *Hamiltonian graph* is a graph with a spanning cycle, also called a *Hamiltonian cycle*. A *Hamiltonian path* is a path containing all the vertices. An *Eulerian circuit* (or Eulerian trail) in a graph is a circuit (or trail) containing all the edges.

A *separating set* or *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* of G is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is *k-connected* if its connectivity is at least k.

In this thesis, let v be an even integer and let Z_v denote the vertex set of K_v. The operations, addition and multiplication, are taken modulo v. We have listed some fundamental definitions of Graph Theory above. For our specific subject we also need the following notations :

1. Let d_i (difference i) be an edge set where $1 \leq i \leq \frac{v}{2}$, then

   $$\langle x, y \rangle \in d_i \Leftrightarrow i = min\{| x - y |, v- | x - y |\}.$$

   Note : $| d_1 | = | d_2 | = ... = | d_{\frac{v}{2}-1} | = v$, $| d_{\frac{v}{2}} | = \frac{v}{2}$.

2. Let $A = \langle a_0, a_1, ..., a_s \rangle$ be an ordered trail, then $d(A)$ is the least number of edges between two appearances of the same vertex along A (the length of the minimal cycle).

3. Let $A = \langle a_0, a_1, ..., a_s \rangle$, $B = \langle b_0, b_1, ..., b_t \rangle$ be two ordered trails and $a_s = b_0$, then $A + B = \langle a_0, a_1, ..., a_s, b_1, b_2, ..., b_t \rangle$.

4. Let $A = \langle a_0, a_1, ..., a_s \rangle$ and $k \in \mathbb{Z}$,

   then $A + k = \langle a_0 + k, a_1 + k, ..., a_s + k \rangle$ (right shift k).

5. For $A = \langle a_0, a_1, ..., a_s \rangle$, $A^t = \langle a_s, a_{s-1}, ..., a_0 \rangle$.

6. For $x, y \in Z_v$, $I(x, y) = \langle x, x+1, x+2, ..., y-1, y\rangle$.

7. For v even, $x \in Z_v$, $a, b \in \{1, 2, ..., \frac{v}{2}-1\}$ such that $1 \leq a < b \leq \frac{v}{2}-1$, $S(v, x, a, b) = \langle x, x+a, x-1, x+a+1, x-2, ..., y\rangle$, where the vertex y is chosen to make the length of the path $(b-a+1)$.

   Note : The first edge is in d_a, the second edge is in d_{a+1}, and so on, and the last edge is in d_b. Thus $S(v, x, a, b)$ is a path containing exactly one edge of $d_a, d_{a+1}, ..., d_b$ respectively.

8. For v, x, a, b satisfying the conditions of the previous construction, $L(v, x, a, b) = S(v, x, a, b) + \langle y, y+\frac{v}{2}\rangle + S(v, x+\frac{v}{2}, a, b)^t$.

   Note : This is a path of length $2(b-a)+3$ which uses two edges in differences $a, a+1, ..., b$ and one edge in difference $\frac{v}{2}$.

9. For v, x, a, b satisfying the conditions of the previous construction, $M_{v,a,b} = \{L(v, x, a, b) \mid 0 \leq x \leq \frac{v}{2}-1\}$.

   Note : This is a set of $\frac{v}{2}$ paths with length $2(b-a)+3$ which uses all the edges in $d_a, d_{a+1}, ..., d_b$ and $d_{\frac{v}{2}}$. Each vertex is the end vertex of exactly one path.

10. For v even, $k \in \{1, 2, ..., \frac{v}{2}-2\}$, $C(v, k) = \langle 0, k+1, 1, k+2, 2, k+3, ..., k-1, v-1, k, 0\rangle$.

    Note : This is a circuit of length $2v$ which contains all the edges in d_k and d_{k+1}.

11. For v even, $a, b \in \{1, 2, ..., \frac{v}{2}-1\}$ such that $1 \leq a < b \leq \frac{v}{2}-1$ and $b-a$ is odd, $PC(v, a, b) = C(v, a) + C(v, a+2) + ... + C(v, b-1)$.

    Note : This is a circuit which contains all the edges in differences $a, a+1, ..., b$.

3 Known Results

3.1 Some results in cycle decomposition

Theorem 3.1. *[9] (1) For all odd integers n and all non-negative integer r satisfying $3r = \dfrac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r satisfying $3r = \dfrac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 3-cycles which partitions the edges set of $K_n - F$.*

We can establish the existence of cycle systems not only the 3-*cycle* system but also the m-*cycle* system for any m.

Theorem 3.2. *[11] (1) For all odd integers n and all non-negative integer r and m satisfying $mr = \dfrac{n(n-1)}{2}$ there is a decomposition of K_n into r m-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r and m satisfying $mr = \dfrac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r m-cycles which partitions the edge set of $K_n - F$.*

Theorem 3.3. *[1] (1) For all odd integers n and all non-negative integers r, and s satisfying $3r + 5s = \dfrac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles and s 5-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r, and s satisfying $3r + 5s = \dfrac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 3-cycles and 5-cycles which partitions the edge set of $K_n - F$.*

Theorem 3.4. *[7] (1) For all odd integers n and all non-negative integer r, s and t satisfying $3r + 4s + 6t = \dfrac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles, s 4-cycles, and t 6-cycles which partition the edge set of K_n. (2) For all even integers n and all non-negative integer r, s and t satisfying $3r + 4s + 6t = \dfrac{n(n-2)}{2}$ there is* *a decomposition of $K_n - F$ into r 3-cycles, s 4-cycles, and t 6-cycles which partition the edge set of $K_n - F$.*

Theorem 3.5. *[3] (1) For all odd integers n and all non-negative integer r and s satisfying $4r + 5s = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r 4-cycles, s 5-cycles which partition the edge set of K_n. (2) For all even integers n and all non-negative integer r and s satisfying $4r + 5s = \frac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 4-cycles, s 5-cycles which partition the edge set of $K_n - F$.*

The following useful contains three different lengths which are n, $n-1$, $n-2$.

Theorem 3.6. *[7] Let $S = \{n-2, n-1, n\}$. If n is odd and $a(n-2)+b(n-1)+cn = \frac{n(n-1)}{2}$, then $K_n = aC_{n-2}+bC_{n-1}+cC_n$. If n is even and $a(n-2)+b(n-1)+cn = \frac{n(n-2)}{2}$, then $K_n - F = aC_{n-2}+bC_{n-1}+cC_n$.*

Alspach Conjecture is also true if the cycles lengths m_i are bounded by some linear function of n and n is sufficiently large.

Theorem 3.7. *[2] Assume n must be larger than N_2 which is very large absolute constants. If $m_1, ..., m_t$ are integers with $3 \le m_i \le \lfloor \frac{n-112}{20} \rfloor$ and $\sum_{i=1}^{t} m_i = \binom{n}{2}$ (n odd) or $\binom{n}{2} - \frac{n}{2}$ (n even), then one can pack K_n (n odd) or $K_n - I$ (n even) with cycles of lengths $m_1, ..., m_t$.*

3.2 Some results in path decomposition

Theorem 3.8. *[5] Let n be an even positive integer. Then K_n can be decomposed into $\frac{n}{2}$ hamiltonian paths.*

Theorem 3.9. *[10] If n is odd and $\{a_i : 1 \le i \le r\}$ is a multiset of r positive integers satisfying $1 \le a_i \le n-2$ and $\sum_{i=1}^{r} a_i = \binom{n}{2}$. Then K_n can be decomposed into $\{P_{a_i} \mid 1 \le i \le r\}$.*

Theorem 3.10. *[13] Let* $m \mid \lambda\binom{n}{2}$*, and* $m \leq n-1$*. Then* λK_n *can be decomposed into isomorphic paths of length* m*.*

Theorem 3.11. *[4] If* v *is odd. Let* $m_1, m_2, ..., m_t$ *be* t *positive integers such that* $1 \leq m_i \leq n-2$, $\sum_{i=1}^{t} m_i + k(n-1) = \binom{n}{2}$*, and* $k \in \{1, 2, \frac{n-1}{2}\}$*, then* K_v *can be decomposed into* $t+k$ *paths* $P^1, P^2, ..., P^{t+k}$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$ *and the length of* P^i *is* $n-1$ *for* $i > t$*.*

Theorem 3.12. *[4] If* v *is odd. Let* $n-1 \geq m_1 \geq m_2 \geq ... \geq m_t \geq 1$ *and* $h \leq m_t \leq n-h-1$ *such that* $\sum_{i=1}^{t} m_i = \binom{n}{2}$, $m_1 = m_2 = ... = m_h = n-1$*. Then* K_v *can be decomposed into* t *paths* $P^1, P^2, ..., P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$*. Moreover, if there exists a* $h < t' \leq t$ *such that* $h \leq m_{t'} \leq n-h-1$ *or* $h \leq \sum_{i=t'}^{t} m_i \leq n-h-1$*, then* K_v *can be decomposed into* t *paths* $P^1, P^2, ..., P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$*.*

Theorem 3.13. *[4] If* v *is odd. Let* $n-1 \geq m_1 \geq m_2 \geq ... \geq m_t \geq 1$, $m_t < h$*, and* $m_{t-1} - m_t \leq n-h-1$ *such that* $\sum_{i=1}^{t} m_i = \binom{n}{2}$, $m_1 = m_2 = ... = m_h = n-1$*. Then* K_v *can be decomposed into* t *paths* $P^1, P^2, ..., P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$*.*

Theorem 3.14. *[4] If* v *is odd. Let* $n-1 \geq m_1 \geq m_2 \geq ... \geq m_t \geq 1$ *and* $n+h-2 \leq m_t + m_{t-1} \leq 2n-h-3$ *such that* $\sum_{i=1}^{t} m_i = \binom{n}{2}$, $m_1 = m_2 = ... = m_h = n-1$*. Then* K_v *can be decomposed into* t *paths* $P^1, P^2, ..., P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$*. Moreover, if there exists a* $h < t' \leq t$ *such that* $n+h-2 \leq \sum_{i=t'}^{t} m_i \leq 2n-h-3$*, then* K_v *can be decomposed into* t *paths* $P^1, P^2, ..., P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, ..., t$*.*

4 Main Result

We shall prove the main theorem in what follows.

Theorem 4.1. *K_v can be decomposed into k P_m's and one P_r if and only if $\binom{v}{2} = km + r$ where $0 \le r < m \le v-1$.*

First of all, we obtain some lemmas below by using the preliminary definitions.

Lemma 4.2. *The union of the i-th path of $M_{v,2,\frac{m}{2}}$ (the endpoints are $i-1$ and $i-1+\frac{v}{2}$) and one of $\langle i-1, i\rangle$ and $\langle i-1+\frac{v}{2}, i+\frac{v}{2}\rangle$ is a simple path of length m.*

Proof. By the definition, the i-th path : $\langle i-1, i+1, i-2, i+2, ..., y\rangle + \langle y, y+\frac{v}{2}\rangle + \langle y+\frac{v}{2}, ..., i+2+\frac{v}{2}, i-2+\frac{v}{2}, i+1+\frac{v}{2}, i-1+\frac{v}{2}\rangle$, where y is chosen to make the length of the path $(m-1)$.

Checking the segment $\langle i-1, i+1, i-2, i+2, ..., y\rangle$ which contains $(\frac{m}{2}-1)$ edges. The subsequence of even indices which starts at the vertex $i+1$ is $\{i+1, i+2, i+3, ...\}$. The subsequence of odd indices which starts at the vertex $i-1$ is $\{i-1, i-2, i-3, ...\}$. Since $\frac{m}{2}-1 < \frac{v}{2}-1$ and the length of these two subsequences are less than $\frac{v}{4}$. Thus the segment $\langle i-1, i+1, i-2, i+2, ..., y\rangle$ does not contain the vertex i.

Now, consider the segment $\langle y+\frac{v}{2}, ..., i+2+\frac{v}{2}, i-2+\frac{v}{2}, i+1+\frac{v}{2}, i-1+\frac{v}{2}\rangle^t$. Since the length is $(\frac{m}{2}-1)$, the subsequence of even indices is an increasing sequence which starts at the vertex $i+1+\frac{v}{2}$ and the subsequence of odd indices is a decreasing sequence which starts at $i-1+\frac{v}{2}$. Then it is proved by the same way as above.

Similarly, since the i-th path does not contain the vertex $i+\frac{v}{2}$, the union of the i-th path of $M_{v,2,\frac{m}{2}}$ (endpoints are $i-1$ and $i-1+\frac{v}{2}$) and one of $\langle i-1, i\rangle$ and $\langle i-1+\frac{v}{2}, i+\frac{v}{2}\rangle$ is a path of length m. ∎

Lemma 4.3. $d(C(v,k)) = 2k+1$.

Proof. Since $C(v,k) = \langle 0, k+1, 1, k+2, 2, k+3, ..., k-1, v-1, k, 0\rangle$. The odd places of $C(v,k)$ is the subsequence $\{0,1,2,...,v-1\}$ and the even places of $C(v,k)$ is the subsequence $\{k+1, k+2, ..., v-1, 0, 1, ..., k-1, k\}$. Thus, each vertex of K_v appears twice in $C(v,k)$; one in the odd places of $C(v,k)$ and the other one in the even places of $C(v,k)$. Let x_{even} (x_{odd}) denote the vertex $x \in Z_v$ which appears in the even (odd) places of $C(v,k)$. If x_{odd} appears before x_{even}, then the distance from x_{odd} to x_{even} is $2v-2k-1 \geq 2k+1$. Else, the distance from x_{even} to x_{odd} is $2k+1$. This concludes the proof. ∎

Lemma 4.4. $d(PC(v,a,b)) = 2a+1$.

Proof. Looking at Lemma 4.3 and the structure of $PC(v,a,b)$, it suffices to check whether the length of the cycle C which begins in $C(v,k)$ and ends in $C(v,k+2)$ is larger than $2k+1$.

Let x_{e1} denote the vertex $x \in Z_v$ which appears in the even places of $C(v,k)$ and x_{o1} be the other one which appears in the odd places of $C(v,k)$. Similarly, let x_{e2} and x_{o2} denote the vertex x which appear in $C(v,k+2)$. Let $d(x,y)$ be the distance from x to y. If the cycle C begins at x_{o1} and ends at x_{o2}, then $d(x_{o1}, x_{o2}) = 2v \geq 2k+1$. If the cycle C begins at x_{e1} and ends at x_{e2}, then $d(x_{e1}, x_{e2}) \geq 2v-4 \geq 2k+1$. If the cycle C begins at x_{o1} and ends at x_{e2}, then $d(x_{o1}, x_{e2}) = 2v-2k-5 \geq 2k+1$. If the cycle C begins at x_{e1} and ends at x_{o2}, then $d(x_{e1}, x_{o2}) = 2k+1$. Thus, $d(PC(v,a,b)) = min\{d(C(v,a)), d(C(v,a+2)), ..., d(C(v,b-1))\} = min\{2a+1, 2a+5, ..., 2b-1\} = 2a+1$. ∎

Lemma 4.5. *When v is odd, $\binom{v}{2} = km + r$ and $0 \leq r < m = v-1$. K_v can be decomposed into k P_m's and one P_r.*

Proof. Let $V(K_v) = \{x_\infty, x_1, x_2, ..., x_{v-1}\}$.

Let $C^i = \langle x_\infty, x_i, x_{v+i-2}, x_{i+1}, x_{v+i-3}, x_{i+2}, ..., x_{i+\frac{v-3}{2}}, x_{i+\frac{v-1}{2}}, x_\infty \rangle$ (Indices take modulo $v-1$). Then K_v can be decomposed into $\{C^i \mid 1 \le i \le \frac{v-1}{2}\}$ and each C^i is a hamiltonian cycle. Observe that $\langle x_i, x_{v+i-2}\rangle \in C^i$ for $1 \le i \le \frac{v-1}{2}$. Thus, cutting these $\frac{v-1}{2}$ edges from each C^i, we have $\frac{v-1}{2}$ hamiltonian path of length $v-1 = m$. Now, the proof follows by combining the above $\frac{v-1}{2}$ edges into the path $\langle x_{v-1}, x_1, x_2, ..., x_{\frac{v-1}{2}}\rangle$. ∎

Proof of Theorem 4.1.

Since the necessary part is easy to see, it is left to prove the sufficiency. Note that when v is odd and $1 \le m \le v-2$, the condition is proved to be sufficient by [10]. Moreover, if v is odd and $m = v-1$, the condition is proved to be sufficient by Lemma 4.5. Thus, we put the accent on the case : when v is even. The proof is split into four cases by taking $M_{v,a,b}$ into consideration.

<u>**Case 1**</u> **:** $v - m \equiv 1 \ (mod\ 4)$

Case 1.1 : $m \le v - 5$

Because $M_{v,1,\frac{m-1}{2}}$ covers all the edges in $d_1, d_2, ..., d_{\frac{m-1}{2}}$ and $d_{\frac{v}{2}}$ exactly once, $PC(v, \frac{m+1}{2}, \frac{v}{2}-1)$ covers all the edges in $d_{\frac{m+1}{2}}, d_{\frac{m+3}{2}}, ..., d_{\frac{v}{2}-1}$ exactly once. Hence, these two parts cover all the edges of K_v exactly once, i.e., $E(K_v) = M_{v,1,\frac{m-1}{2}} \cup PC(v, \frac{m+1}{2}, \frac{v}{2}-1)$.

By definition, $M_{v,1,\frac{m-1}{2}}$ is a set of $\frac{v}{2}$ paths of length $2(\frac{m-1}{2}-1)+3 = m$. By Lemma 4.3, $d(PC(v, \frac{m+1}{2}, \frac{v}{2}-1)) = 2(\frac{m+1}{2})+1 = m+2 > m$. Then we can partition $PC(v, \frac{m+1}{2}, \frac{v}{2}-1)$, starting from it's beginning, into paths of length m, and the remainder is a path of length r.

Case 1.2 : $m = v - 1$

The proof follows by decomposing K_v into Hamiltonian paths.

<u>Case 2</u> : $v - m \equiv 3 \ (mod\ 4)$

Case 2.1 : $m \leq v - 7$

First, we claim that $E(K_v) = M_{v,2,\frac{m+1}{2}} \cup I(0,m) \cup (I(m,0) + PC(v, \frac{m+3}{2}, \frac{v}{2} - 1))$. This is by the fact that $M_{v,2,\frac{m+1}{2}}$ contains all the edges in $d_2, d_3, ..., d_{\frac{m+1}{2}}$ and $d_{\frac{v}{2}}$, $\{I(0,m) \cup I(m,0)\}$ contains all the edges in d_1, and $PC(v, \frac{m+3}{2}, \frac{v}{2} - 1)$ contains all the edges in $d_{\frac{m+3}{2}}, ..., d_{\frac{v}{2}-1}$.

Next, we show that this construction provide a set of P_m's and exactly one P_r. By definition $M_{v,2,\frac{m+1}{2}}$ is a set of $\frac{v}{2}$ P_m's and $I(0,m)$ is a P_m. Now the proof follows by claiming that $d(I(m,0)+PC(v, \frac{m+1}{2}, \frac{v}{2}-1)) > m$. Since $I(m,0)+PC(v, \frac{m+1}{2}, \frac{v}{2} - 1) = \langle m, m+1, m+2, ..., v-1, 0, \frac{m+3}{2}+1, 1, \frac{m+3}{2}+2, 2, \frac{m+3}{2}+3, ...\rangle$. Therefore, $I(m,0)$ is increasing and $PC(v, \frac{m+1}{2}, \frac{v}{2} - 1)$ is alternately increasing. The first repeat vertex between $I(m,0)$ and $PC(v, \frac{m+1}{2}, \frac{v}{2}-1)$ is m. So, if the first vertex m of $PC(v, \frac{m+1}{2}, \frac{v}{2}-1)$ appears in the even (index) part, then the distance between these two vertices is $(v-m)+(m-4) > m$. Otherwise, the distance is $(v-m)+2m > m$. Thus $d(I(m,0) + PC(v, \frac{m+1}{2}, \frac{v}{2} - 1)) > m$.

Case 2.2 : $m = v - 3$

Since $M_{v,2,\frac{v}{2}-1}$ is a set of $\frac{v}{2}$ paths of length $v - 3 = m$, it covers all the edges of K_v exactly once except these edges in d_1. Now, the cycle $\langle 0, 1, 2, ..., v-1, 0\rangle$ covers all the edges in d_1 and each segment of length less than v on the cycle is a path. This concludes the proof of this case.

<u>Case 3</u> : $v - m \equiv 2$ $(mod\ 4)$

Case 3.1 : $m \le \frac{v}{2}$

Note that $E(K_v) = (M_{v,2,\frac{m}{2}} \cup I(0, \frac{v}{2})) \cup (I(\frac{v}{2}, 0) + PC(v, \frac{m}{2}+1, \frac{v}{2}-1))$. Now, we prove that the construction is a set of P_m's and exactly one P_r. By Lemma 4.2, The union of the i-th path of $M_{v,2,\frac{m}{2}}$ (the endpoints are $i-1$ and $i-1+\frac{v}{2}$) and one of $\langle i-1, i\rangle$ and $\langle i-1+\frac{v}{2}, i+\frac{v}{2}\rangle$ is a path of length m. Since $I(0, \frac{v}{2}) = \{\langle i, i-1\rangle \mid i = 1, 2, ..., \frac{v}{2}\}$, $M_{v,2,\frac{m}{2}} \cup I(0, \frac{v}{2})$ can be decomposed into paths of length m.

Similar to the proof of the claim in Case 2.1. We have to prove that the distance between repeating vertices is larger than m. Since the first repeat vertex between $I(\frac{v}{2}, 0)$ and $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$ is $\frac{v}{2}$, $m \le \frac{v}{2}$ and the length of $I(\frac{v}{2}, 0)$ is $\frac{v}{2}$. No matter the first repeat vertex $\frac{v}{2}$ belongs to the even part of $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$ or not, the distance between these two vertices is larger than m. This concludes the proof of this subcase.

Case 3.2 : $\frac{v}{2} < m \le v - 6$

Before the proof, we need some notations. Let $f = min\{[\frac{v}{2(v-m)}], [\frac{v(v-m-2)}{2(2m-v)}]\}$ (Denote the integer part of x as $[x]$). Let $S_i = I(m, 0) + (i-1)(m - \frac{v}{2})$, where $1 \le i \le f$ and $S_R = I(0, \frac{v}{2} - f(v-m)) + (f-1)\frac{v}{2}$. Denote by T_i, $1 \le i \le f$, f paths of length $2m - v$ each, cut along $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$, and denote by T_R, the final segment remaining of $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$ after taking out $T_1, T_2, ..., T_f$. Finally, let $D_i = S_i + T_i$, $1 \le i \le f$ and $D_R = S_R + T_R$. Since the end of S_i is $0 + (i-1)(m - \frac{v}{2})$ (mod v). And because $l(T_i) = 2m - v$ is even and each T_i is gotten by cut along $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$. Thus, the beginning of $T_i \equiv$ the end of T_{i-1} $\equiv$ the beginning of $T_{i-1} + \frac{2m-v}{2} \equiv ... \equiv$ the beginning of $T_i + (i-1)(m - \frac{v}{2}) \equiv 0 + (i-1)(m - \frac{v}{2}) \equiv$ the end of S_i (mod v). So D_i and D_R are well defined. And by definition, $(M_{v,2,\frac{m}{2}} \cup [d_1 \setminus (S_1 \cup S_2 \cup ... \cup S_f \cup S_R)]) \cup D_1 \cup D_2 \cup ... \cup D_f \cup D_R$ is obtained from $M_{v,2,\frac{m}{2}} \cup d_1 \cup PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$ which contains all the edges of K_v exactly once. Then we have $E(K_v) = (M_{v,2,\frac{m}{2}} \cup [d_1 \setminus (S_1, S_2, ..., S_f, S_R)]) \cup D_1 \cup D_2 \cup ... \cup D_f \cup D_R$.

Now, let x be the end vertex of S_i. Then $S_i = \langle x+m, x+m+1, ..., x\rangle$ and $T_i = \langle x, x+k, x+1, x+k+1, x+2, ..., x+m-\frac{v}{2}\rangle$. By the property of $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$, we have $\frac{m}{2} < k < \frac{v}{2}$. Thus, the vertices in the even places of T_i belong to the open interval $(x+\frac{m}{2}, x+m)$. And because the vertices in the odd places of T_i is $\{x, x+1, ..., x+m-\frac{v}{2}\}$, $S_i = \{x+m, x+m+1, ..., x\}$. The vertex x is the only common vertex of S_i and T_i. Hence, for each i, $D_i = S_i + T_i$ is a path of length $l(S_i) + l(T_i) = (v-m) + (2m-v) = m$. Since

$$S_i = \begin{cases} I(\frac{v}{2} - (i)(v-m), \frac{v}{2} - (i-1)(v-m)) & i: \text{even} \\ I(v - (i)(v-m), v - (i-1)(v-m)) & i: \text{odd} \end{cases} \text{; and}$$

$$S_R = \begin{cases} I(\frac{v}{2}, v - (f)(v-m)) & f: \text{even} \\ I(0, \frac{v}{2} - (f)(v-m)) & f: \text{odd} \end{cases},$$

$S_1, S_2, ..., S_f, S_R$ are all distinct and that exactly one of $\langle x, x+1\rangle$ and $\langle x+\frac{v}{2}, x+\frac{v}{2}+1\rangle$ belongs to $[d_1 \setminus (S_1 \cup S_2 \cup ... \cup S_f \cup S_R)]$. Then we can obtain $\frac{v}{2}$ P_m's by Lemma 4.2.

Finally, if $f = [\frac{v}{2(v-m)}]$, then $l(S_R) < l(S_i) = v - m$. Therefore, by the same argument as above, we obtain $d(D_R) > d(D_i) = m$.

On the other hand, if $f = [\frac{l(PC(v, \frac{m}{2}+1, \frac{v}{2}-1))}{2m-v}] = [\frac{v(v-m-2)}{2(2m-v)}] < [\frac{v}{2(v-m))}]$. This implies $(v-m-2)(v-m) < (2m-v)$. Because $v - m \equiv 2 \pmod 4$ which implies that $l(PC(v, \frac{m}{2}+1, \frac{v}{2}-1)) = \frac{v(v-m-2)}{2} = (\frac{v-m-2}{2})(2m-v) + (v-m-2)(v-m)$, $f = \frac{(v-m-2)}{2}$ and $l(T_R) = (v-m-2)(v-m) := q$ is even. Since $(v-m-2)(v-m) < (2m-v) < 2v$, T_R is contained in the last circuit $C(v, \frac{v}{2}-2)$ of $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$ and $T_R = \langle v-\frac{q}{2}, \frac{v-q}{2}-1, ..., v-1, \frac{v}{2}-2, 0\rangle$. And $f = \frac{(v-m-2)}{2}$ implies that $S_R = I(\frac{v}{2}, v-\frac{q}{2})$. Hence $D_R = S_R + T_R$ is also a path. This concludes the proof of this subcase.

Case 3.3 : $m = v - 2$

The proof follows by the fact that $E(K_v) = (M_{v,2,\frac{v}{2}-1} \cup I(0, \frac{v}{2})) \cup I(\frac{v}{2}, 0)$.

<u>**Case 4**</u> **:** $v - m \equiv 0 \ (mod\ 4)$

Note that if $tm < v$, then the proof follows by decomposing K_v into paths of length tm and a path of length less than tm (may be zero). Therefore, it suffices to consider the cases $v > m \geq \frac{v}{2}$.

Case 4.1 : $\frac{v}{2} \leq m < \frac{3v}{4}$, $m \leq v - 8$ *and* $v \equiv 0 \ (mod\ 4)$ **or**
$\frac{v}{2} \leq m < \frac{3v}{4} - \frac{1}{2}$, $m \leq v - 8$ *and* $v \equiv 2 \ (mod\ 4)$

First, we need some notations.
Let $L'(v, x, 2, \frac{m}{2}) = [S(v, x, 2, \frac{m}{2})]^t + \langle x, x + \frac{v}{2} + 1, x + \frac{v}{2}\rangle + S(v, x + \frac{v}{2}, 2, \frac{m}{2})$,
$M = \{L'(v, x, 2, \frac{m}{2}) \mid 0 \leq x \leq \frac{v}{2} - 1\}$,
$A = I(0, \frac{v}{2})$ and $B = \langle 0, \frac{v}{2}, 1, \frac{v}{2} + 1, 2, \frac{v}{2} + 2, 3, ..., v - 1, \frac{v}{2}\rangle$.
Let $\overline{A}$ be obtained from A by replacing the last $(m - \frac{v}{2})$ edges with the last $2(m - \frac{v}{2})$ edges of B and $\overline{B}$ be obtained from B by replacing the last $2(m - \frac{v}{2})$ edges with the last $(m - \frac{v}{2})$ edges of A. Let $\overline{B} = D + E$, where $l(D) = \frac{3v}{2} - 2m$, $l(E) = m$. Then, we have $E(K_v) = M \cup \overline{A} \cup E \cup (PC(v, \frac{m}{2} + 1, \frac{v}{2} - 2) + D)$.

So, it is sufficient to prove that $M \cup \overline{A} \cup E \cup (PC(v, \frac{m}{2} + 1, \frac{v}{2} - 2) + D)$ is a set of P_m's and one P_r by the following steps :

(1) M is a set of $\frac{v}{2}$ paths of length m.

(2) $\overline{A}$ is a path of length m.

(3) E is a path of length m.

(4) $d(PC(v, \frac{m}{2}+1, \frac{v}{2}-2)+D) = m+3$.

Step (1) : By definition, M is a set of $\frac{v}{2}$ paths of length m.

Step (2) : Because $\overline{A} = \langle 0, 1, 2, ..., v-m, \frac{3v}{2}-m, v-m+1, \frac{3v}{2}-m+1, ..., v-1, \frac{v}{2}\rangle$. Thus, $\overline{A}$ is a path if and only if there is no repeat vertex of the following vertex sets : $\{0, 1, 2, ..., v-m\}$, $\{v-m+1, v-m+2, ..., \frac{v}{2}\}$ and $\{\frac{3v}{2}-m, \frac{3v}{2}-m+1, ..., v-1\}$. Since $\frac{v}{2} < \frac{3v}{2} - m \Leftrightarrow m < v$, the proof follows.

Step (3) : Notice that if $v \equiv 0 \pmod 4$, then

$$C = \langle \frac{3v}{4}-m, \frac{5v}{4}-m, \frac{3v}{4}-m+1, \frac{5v}{4}-m+1, ..., \frac{3v}{2}-m-1, v-m, v-m+1, v-m+2, ..., \frac{v}{2}\rangle.$$

Thus, E is a path if and only if there is no repeat vertex of the following vertex sets : $\{\frac{3v}{4}-m, \frac{3v}{4}-m+1, ..., v-m\}$, $\{v-m+1, v-m+2, ..., \frac{v}{2}\}$ and $\{\frac{5v}{4}-m, \frac{5v}{4}-m+1, ..., \frac{3v}{2}-m-1\}$. Since $\frac{v}{2} < \frac{5v}{4} - m \Leftrightarrow m < \frac{3v}{4}$, we have the claim.

On the other hand, if $v \equiv 2 \pmod 4$, then

$$C = \langle \frac{5v}{4}-m-\frac{1}{2}, \frac{3v}{4}-m+\frac{1}{2}, \frac{5v}{4}-m+\frac{1}{2}, \frac{3v}{4}-m+\frac{3}{2}, ..., \frac{3v}{2}-m-1, v-m, v-m+1, v-m+2, ..., \frac{v}{2}\rangle.$$

Thus, E is a path if and only if there is no repeat vertex of the following vertex sets : $\{\frac{3v}{4}-m+\frac{1}{2}, \frac{3v}{4}-m+\frac{3}{2}, ..., v-m\}$, $\{v-m+1, v-m+2, ..., \frac{v}{2}\}$ and $\{\frac{5v}{4}-m-\frac{1}{2}, \frac{5v}{4}-m+\frac{1}{2}, ..., \frac{3v}{2}-m-1\}$. Since $\frac{v}{2} < \frac{5v}{4}-m-\frac{1}{2} \Leftrightarrow m < \frac{3v}{4}-\frac{1}{2}$, the proof follows.

Step (4) : Because the length of E is larger then $(m-\frac{v}{2})$. Thus, D is contained in B. Then D is a segment of the first $(\frac{3v}{2}-2m)$ edges of $C(v, \frac{v}{2}-1)$. By Lemma 4.4, we are done.

Case 4.2 : $\frac{3v}{4} - \frac{1}{2} \leq m < \frac{3v}{4}$**,** $m \leq v-8$ *and* $v \equiv 2 \pmod 4$

These conditions implies that $m = \frac{3v}{4} - \frac{1}{2}$. We shall use the same notations for $L', M, A, B, \overline{A}, \overline{B}$ as in Case 4.1. Let $\overline{B} = F + G$, where $l(F) = m$, $l(G) = 1$. Then, we have $E(K_v) = M \cup \overline{A} \cup F \cup (PC(v, \frac{m}{2}+1, \frac{v}{2}-2) \cup G)$.

By applying the idea of the proof in Case 4.1, we only have to check F is a P_m and $PC(v, \frac{m}{2}+1, \frac{v}{2}-2) \cup G$ can be decomposed into P_m's and exactly one P_r.

For $m = \frac{3v}{4} - \frac{1}{2}$, we know that $\overline{B} = \langle 0, \frac{v}{2}, 1, \frac{v}{2}+1, 2, ..., \frac{3v}{2}-m-1, v-m, v-m+1, v-m+2, ..., \frac{v}{2}-1, \frac{v}{2}\rangle$ of length $m+1$. Because the length of the path $\langle 0, \frac{v}{2}, 1, \frac{v}{2}+1, 2, ..., \frac{3v}{2}-m-1, v-m\rangle$ is even. Which implies that the number of vertices in F is $\frac{3v}{2} - m = m+1 = l(F)+1$. Thus, F is a path of length m.

Let $C'(v,k) = \langle \frac{v}{2}-1, \frac{v}{2}-1+k+1, \frac{v}{2}, \frac{v}{2}-1+k+2, \frac{v}{2}+1, ..., \frac{v}{2}-1+k, \frac{v}{2}-1\rangle$, where $1 \leq k \leq \frac{v}{2}-2$. Then $PC(v,a,b) = C(v,a) + C(v,a+2) + ... + C(v,b-1) = C'(v,a) + C'(v,a+2) + ... + C'(v,b-1)$. By Lemma 4.4, we have $d(C'(v,a) + C'(v,a+2) + ... + C'(v,b-1)) = 2a+1$. Because the end vertex of $C'(v, \frac{v}{2}-3)$ is the beginning vertex of G. Thus $PC(v, \frac{m}{2}+1, \frac{v}{2}-2) \cup G = C'(v, \frac{m}{2}+1) + C'(v, \frac{m}{2}+3) + ... + C'(v, \frac{v}{2}-3) + \langle \frac{v}{2}-1, \frac{v}{2}\rangle$. Since the distance between the last vertex (the second vertex) $\frac{v}{2}$ of $C'(v, \frac{v}{2}-3)$ and the end vertex $\frac{v}{2}-1$ of $C'(v, \frac{v}{2}-3)$ is $2(v-4-\frac{v}{2}) = v-8 \geq m$. Thus, we can cut $C'(v, \frac{m}{2}+1) + C'(v, \frac{m}{2}+3) + ... + C'(v, \frac{v}{2}-3) + \langle \frac{v}{2}-1, \frac{v}{2}\rangle$, starting from its end vertex, into paths of length m, and the remainder is a path of length r. This concludes the proof of this subcase.

Case 4.3 : $\frac{v}{2} \leq m < \frac{3v}{4}$ *and* $m = v-4$

Since $8 \leq v < 16$, there are only four cases left to prove.

i $v = 8$, $m = 4$

K_v can be decomposed into $\{\langle 0, x, x+6, x+1, x+5\rangle \mid x \in \{1,2,3,...,6,7\}\}$.

ii $v = 10$, $m = 6$

Let $V(K_v) = \{0, 1, 2, 3, ..., 8, 9\}$. Then, the decomposition is

$\langle 1, 0, 2, 9, 4, 7, 5\rangle \cup \langle 2, 1, 3, 0, 5, 8, 6\rangle \cup \langle 3, 2, 4, 1, 6, 9, 7\rangle \cup \langle 4, 3, 5, 2, 7, 0, 8\rangle \cup$

$\langle 5, 4, 6, 3, 8, 1, 9\rangle \cup \langle 0, 4, 8, 9, 3, 7, 1\rangle \cup \langle 1, 5, 9, 0, 6, 7, 8\rangle \cup \langle 8, 2, 6, 5\rangle$.

iii $v = 12$, $m = 8$

Let $V(K_v) = \{0, 1, 2, 3, ..., 10, 11\}$. Then, the decomposition is

$\langle 0, 2, 11, 3, 9, 5, 8, 6, 7\rangle \cup \langle 2, 1, 3, 0, 4, 10, 6, 9, 7\rangle \cup \langle 3, 2, 4, 1, 5, 11, 7, 10, 8\rangle \cup$

$\langle 4, 3, 5, 2, 6, 0, 8, 11, 9\rangle \cup \langle 5, 4, 6, 3, 7, 1, 9, 0, 10\rangle \cup \langle 6, 5, 7, 4, 8, 2, 10, 1, 11\rangle \cup$

$\langle 7, 0, 1, 6, 11, 10, 3, 8, 9\rangle \cup \langle 11, 0, 5, 10, 9, 2, 7, 8, 1\rangle \cup \langle 11, 4, 9\rangle$.

iv $v = 14$, $m = 10$

Let $V(K_v) = \{0, 1, 2, 3, ..., 12, 13\}$. Then, the decomposition is

$\langle 1, 0, 2, 13, 3, 12, 5, 10, 6, 9, 7\rangle \cup \langle 2, 1, 3, 0, 4, 13, 6, 11, 7, 10, 8\rangle \cup$

$\langle 3, 2, 4, 1, 5, 0, 7, 12, 8, 11, 9\rangle \cup \langle 4, 3, 5, 2, 6, 1, 8, 13, 9, 12, 10\rangle \cup$

$\langle 5, 4, 6, 3, 7, 2, 9, 0, 10, 13, 11\rangle \cup \langle 5, 7, 4, 8, 3, 10, 1, 11, 0, 12, 13\rangle \cup$

$\langle 7, 6, 8, 5, 9, 4, 11, 2, 12, 1, 13\rangle \cup \langle 0, 6, 5, 11, 12, 4, 10, 9, 8, 7, 13\rangle \cup$

$\langle 5, 13, 0, 8, 2, 10, 11, 3, 9, 1, 7\rangle \cup \langle 6, 12\rangle$.

Case 4.4 : $m \geq \frac{3v}{4}$**,** $m = v - 4$ *and* $v \equiv 0$ $(mod\ 4)$

Let $A = \langle \frac{v}{2}+4, 5, \frac{v}{2}+6, 7, \frac{v}{2}+8, ..., \frac{v}{2}-1, 0\rangle + I(0, 4) + \langle 4, \frac{v}{2}+5, 6, \frac{v}{2}+7, ..., \frac{v}{2}\rangle$,

$B = \langle \frac{v}{2}, 1, \frac{v}{2}+2, 3, \frac{v}{2}+4\rangle + I(\frac{v}{2}+4, 0) + \langle 0, \frac{v}{2}+1, 2, \frac{v}{2}+3, 4\rangle$ and

$M = M_{v,2,\frac{m}{2}} \cup I(4, \frac{v}{2}+4)$. Then, we have $E(K_v) = M \cup A \cup B$.

It suffices to check A is a path of length m and B is a path of length $\frac{v}{2}+4$.

Because $v \equiv 0 \pmod 4$, the cycle

$$\langle 0, \frac{v}{2}+1, 2, \frac{v}{2}+3, 4, \frac{v}{2}+5, 6, ..., v-1, \frac{v}{2}, \\ 1, \frac{v}{2}+2, 3, \frac{v}{2}+4, 5, \frac{v}{2}+6, 7, ..., v-2, \frac{v}{2}-1, 0\rangle$$

contains all the edges in $d_{\frac{v}{2}-1}$ and each vertex of K_v appears exactly once. Since $A = \langle \frac{v}{2}+4, 5, \frac{v}{2}+6, 7, \frac{v}{2}+8, ..., \frac{v}{2}-1, 0\rangle + I(0,4) + \langle 4, \frac{v}{2}+5, 6, \frac{v}{2}+7, ..., \frac{v}{2}\rangle$ is obtained from the union of two segments of this cycle and a path $I(0,4)$, moreover, the vertices in $\{1,2,3\}$ appear in B (thus the vertices in $\{1,2,3\}$ will not appear in A). So, A is a path of length $(v-8)+4 = v-4 = m$. By a similar idea as above, we prove that B is a $P_{\frac{v}{2}+4}$. This concludes the proof of this subcase.

Case 4.5 : $m \geq \frac{3v}{4}$**,** $m = v-4$ *and* $v \equiv 2$ $(mod\ 4)$

Let $A = \langle 4, \frac{v}{2}+5, 6, \frac{v}{2}+7, 8, ..., 0\rangle + I(0,3) + \langle 3, \frac{v}{2}+4, 5, \frac{v}{2}+6, ..., \frac{v}{2}\rangle$,
$B = \langle \frac{v}{2}, 1, \frac{v}{2}+2, 3\rangle + I(3,4) + \langle 4, \frac{v}{2}+3, 2, \frac{v}{2}+1, 0\rangle + [I(\frac{v}{2}+4, 0)]^t$ and
$M = M_{v,2,\frac{m}{2}} \cup I(4, \frac{v}{2}+4)$. Then, $E(K_v) = M \cup A \cup B$.

Now, it suffices to check that A is a path of length m and B is a path of length $\frac{v}{2}+4$. Because $v \equiv 2 \pmod 4$, the following cycles

$$\langle 0, \frac{v}{2}+1, 2, \frac{v}{2}+3, 4, \frac{v}{2}+5, 6, ..., v-4, \frac{v}{2}-3, v-2, \frac{v}{2}-1, 0\rangle \text{ and} \\ \langle 1, \frac{v}{2}+2, 3, \frac{v}{2}+4, 5, \frac{v}{2}+6, 7, ..., v-3, \frac{v}{2}-2, v-1, \frac{v}{2}, 1\rangle$$

contain all the edges in $d_{\frac{v}{2}-1}$ and each vertex of K_v appears exactly once. Since $A = \langle 4, \frac{v}{2}+5, 6, \frac{v}{2}+7, 8, ..., 0\rangle + I(0,3) + \langle 3, \frac{v}{2}+4, 5, \frac{v}{2}+6, ..., \frac{v}{2}\rangle$ is obtained from the union of two segments of these cycles and a path $I(0,3)$, and the vertices in $\{1,2\}$ appear in B (thus the vertices in $\{1,2\}$ will not appear in A), A is a path of length $(v-7)+3 = v-4 = m$. By the same idea as above, we also prove that B is a $P_{\frac{v}{2}+4}$. This concludes the proof of this subcase.

Case 4.6 : $\frac{3v}{4} \leq m \leq v-8$ *and* $v \equiv 0$ $(mod\ 4)$

First, we need some notations.

Let $A = \langle \frac{3v}{2} - m, v-m+1, \frac{3v}{2} - m + 2, v-m+3, ..., \frac{v}{2} - 1, 0\rangle + I(0, v-m) +$
$\langle v-m, \frac{3v}{2} - m + 1, v-m+2, \frac{3v}{2} - m + 3, ..., v-1, \frac{v}{2}\rangle,$
$B = \langle \frac{v}{2}, 1, \frac{v}{2} + 2, 3, \frac{v}{2} + 4, 5, ..., v-m-1, \frac{3v}{2} - m\rangle + I(\frac{3v}{2} - m, 0) +$
$\langle 0, \frac{v}{2} + 1, 2, \frac{v}{2} + 3, 4, ..., \frac{3v}{2} - m - 1, v-m\rangle,$
$T = PC(v, \frac{m}{2} + 1, \frac{v}{2} - 2)$ and K be the last $2(2m - \frac{3v}{2})$ edges of T.

Let $\overline{B}$ be obtained from B by replacing the last $(2m - \frac{3v}{2})$ edges of the segment $I(\frac{3v}{2} - m, 0)$ with K. Also, let $f = min\{[\frac{v}{2(v-m)} - 2], [\frac{l(T \setminus K)}{2m-v}]\}$ (Denote the integer part of x as $[x]$), $S_i = I(m, 0) + (i-1)(m - \frac{v}{2})$, where $1 \leq i \leq f$ and $S_R = I(2(v-m), \frac{v}{2} - f(v-m)) + (f-1)\frac{v}{2}$. Finally, denote by T_i, $1 \leq i \leq f$, the collection of f paths of length $2m - v$ each, cut along $T \setminus K$, and denote by T_R, the final segment remaining of $T \setminus K$ after taking away $T_1, T_2, ..., T_f$. Now, let $D_i = S_i + T_i$, $1 \leq i \leq f$ and $D_R = S_R + T_R$ and $M = M_{v,2,\frac{m}{2}} \cup \{d_1 \setminus [(I(0, v-m) \cup I(\frac{3v}{2} - m, \frac{5v}{2} - 2m) \cup S_1 \cup S_2 \cup ... \cup s_f \cup S_R]\}$. Similar to the proof of Case 3.2, we know that D_i and D_R are well defined. And since $\{I(0, v-m) \cap I(\frac{3v}{2} - m, \frac{5v}{2} - 2m) \cap S_1 \cap S_2 \cap ... \cap S_f \cap S_R\} = \phi$, M is a set of $\frac{v}{2}$ paths of length m. Then, by routine checking, $E(K_v) = A \cup \overline{B} \cup M \cup D_1 \cup D_2 \cup ... \cup D_f \cup D_R$.

We prove that K_v can be decomposed into a set of P_m's and one P_r by the following steps :

(1) A is a path of length m.

(2) $\overline{B}$ is a path of length m.

(3) D_i is a path of length m.

(4) $d(D_R) > m$.

Note that the proofs of (1) and (2) are similar to that of Case 4.4, and the proofs of (3) and (4) are similar to that of Case 3.2.

Step (1) : Since $v \equiv 0 \pmod 4$, the cycle

$$\langle 0, \frac{v}{2}+1, 2, \frac{v}{2}+3, 4, \frac{v}{2}+5, 6, ..., v-1, \frac{v}{2}, \\ 1, \frac{v}{2}+2, 3, \frac{v}{2}+4, 5, \frac{v}{2}+6, 7, ..., v-2, \frac{v}{2}-1, 0\rangle$$

contains all the edges in $d_{\frac{v}{2}-1}$ and each vertex of K_v appears exactly once. Since $A = \langle \frac{3v}{2}-m, v-m+1, \frac{3v}{2}-m+2, v-m+3, ..., \frac{v}{2}-1, 0\rangle + I(0, v-m) + \langle v-m, \frac{3v}{2}-m+1, v-m+2, \frac{3v}{2}-m+3, ..., v-1, \frac{v}{2}\rangle$ which is obtained from the union of two segments of this cycle and a path $I(0, v-m)$ and the vertices in $\{1, 2, 3, ..., v-m-1\}$ appear in B (thus the vertices in $\{1, 2, 3, ..., v-m-1\}$ will not appear in A), A is a path of length $(m-\frac{v}{2}) + (v-m) + (m-\frac{v}{2}) = m$.

Step (2) : Similar to the proof of Step (1), B is a path of length $(v-m) + (m-\frac{v}{2}) + (v-m) = \frac{3v}{2} - m$. Since $K = \langle \frac{5v}{2}-2m, \frac{5v}{2}-2m+1, ..., 0\rangle$, the vertex set of K is $\{\frac{5v}{2}-2m, \frac{5v}{2}-2m+1, ..., 0\} \cup \{2(v-m)-2, 2(v-m)-1, ..., \frac{v}{2}-3\}$ and each vertex of $\{2(v-m)-2, 2(v-m)-1, ..., \frac{v}{2}-3\}$ will not appear in B. Hence $\overline{B}$ is a path of length $(\frac{3v}{2}-m) - (2m-\frac{3v}{2}) + 2(2m-\frac{3v}{2}) = m$.

Step (3) : Let x be the end vertex of S_i. Let $S_i = \langle x+m, x+m+1, ..., x\rangle$ and $T_i = \langle x, x+k, x+1, x+k+1, x+2, ..., x+m-\frac{v}{2}\rangle$. Then by the property of $PC(v, \frac{m}{2}+1, \frac{v}{2}-1)$, we have $\frac{m}{2} < k < \frac{v}{2}$. Thus, the vertices in the even places of T_i belong to the open interval $(x+\frac{m}{2}, x+m)$, since the vertices in the odd places of T_i is $\{x, x+1, ..., x+m-\frac{v}{2}\}$ and $S_i = \{x+m, x+m+1, ..., x\}$. Moreover, the vertex x is the only common vertex of S_i and T_i. Hence, for each i, $D_i = S_i + T_i$ is a path of length $l(S_i) + l(T_i) = (v-m) + (2m-v) = m$.

Step (4) : First, if $f = [\frac{v}{2(v-m)}] - 2$, then $l(S_R) < l(S_i) = v-m$. Therefore, by the same argument as above, we obtain $d(D_R) > d(D_i) = m$.

On the other hand, $f = [\frac{l(T \setminus K)}{2m-v}] < [\frac{v}{2(v-m))}] - 2$. This implies that $(v-m-2)(v-m) < (2m-v)$. Thus, $l(T \backslash K) = v(\frac{v}{2} - \frac{m}{2} - 2) - 4m + 3v = \frac{(v-m-6)}{2}(2m-v) + (v-m-2)(v-m)$ and $f = \frac{(v-m-6)}{2}$, $l(T_R) = (v-m-2)(v-m) := q < (2m-v) < 2v$ which is even. Since $l(K) = 4m - 3v$ is even and $l(K) + l(T_R) < (4m-3v) + (2m-v) = 6m - 4v < 2v$. T_R is contained in the last circuit $C(v, \frac{v}{2} - 3)$ of $PC(v, \frac{m}{2} + 1, \frac{v}{2} - 2) \backslash K$ and $T_R = \langle (\frac{5v}{2} - 2m) - \frac{q}{2}, 2(v-m) - 2 - \frac{q}{2}, ..., \frac{5v}{2} - 2m \rangle$. Since $f = \frac{(v-m-6)}{2}$ implies that $S_R = I(2(v-m), (\frac{5v}{2} - 2m) - \frac{q}{2})$, D_R is a path.

Case 4.7 : $\frac{3v}{4} \leq m \leq v - 8$ *and* $v \equiv 2$ $(mod\ 4)$

We start with some new notations.

Let $A = \langle v-m, \frac{3v}{2} - m + 1, v - m + 2, \frac{3v}{2} - m + 3, ..., \frac{v}{2} - 1, 0 \rangle + I(0, v-m-1) +$
$\langle v-m-1, \frac{3v}{2} - m, v-m+2, \frac{3v}{2} - m + 2, ..., v-1, \frac{v}{2} \rangle$ and
$B = \langle \frac{v}{2}, 1, \frac{v}{2} + 2, 3, ..., \frac{3v}{2} - m - 2, v - m - 1 \rangle + I(v-m-1, v-m) +$
$\langle v-m, \frac{3v}{2} - m - 1, v - m - 2, \frac{3v}{2} - m - 3, ..., \frac{v}{2} + 1, 0 \rangle + [I(\frac{3v}{2} - m, 0)]^t$.
Let $\overline{B}$ be obtained from B by replacing the first $(2m - \frac{3v}{2})$ edges of the segment $[I(\frac{3v}{2} - m, 0)]^t$ with the last $2(2m - \frac{3v}{2})$ segment of $PC(v, \frac{m}{2} + 1, \frac{v}{2} - 2)$. Finally, let M, D_i, D_f and D_R be defined as in Case 4.6. Then, we have $E(K_v) = A \cup \overline{B} \cup M \cup D_1 \cup D_2 \cup ... \cup D_f \cup D_R$.

Since $v \equiv 2$ (mod 4), the following two cycles

$$\langle 0, \frac{v}{2} + 1, 2, \frac{v}{2} + 3, 4, \frac{v}{2} + 5, 6, ..., v-4, \frac{v}{2} - 3, v-2, \frac{v}{2} - 1, 0 \rangle \text{ and}$$
$$\langle 1, \frac{v}{2} + 2, 3, \frac{v}{2} + 4, 5, \frac{v}{2} + 6, 7, ..., v-3, \frac{v}{2} - 2, v-1, \frac{v}{2}, 1 \rangle$$

contain all the edges in $d_{\frac{v}{2}-1}$ and each vertex of K_v appears in these two cycles exactly once. Moreover, since $A = \langle v-m, \frac{3v}{2} - m + 1, v-m+2, \frac{3v}{2} - m + 3, ..., \frac{v}{2} - 1, 0 \rangle + I(0, v-m-1) + \langle v-m-1, \frac{3v}{2} - m, v-m+2, \frac{3v}{2} - m + 2, ..., v-1, \frac{v}{2} \rangle$ is obtained from the union of two segments of these cycles and a path $I(0, v-m-1)$ and the vertices in $\{1, 2, ..., v-m-2\}$ appear in B (thus the vertices in $\{1, 2, ..., v-m-2\}$ will not appear in A), A is a path of length $(m-\frac{v}{2}-1)+(v-1-m)+(m-\frac{v}{2}+1)=m$. By the same way as above, we can prove that B is a $P_{\frac{v}{2}+4}$. Thus, by a similar argument as in Case 4.6, we have the proof of this subcase and the theorem. ∎

5 Conclusion

In this thesis, we have generalized the idea of decomposing K_v into paths of length m to a maximum packing of K_v with paths of length m and the leave is also a path. But, our long-term project is to settle the following problem.

Problem 5.1. *Let v and t be positive integers such that $t \geq \frac{v}{2}$. Let $m_1, m_2, ..., m_t$ be t positive integers less than v such that $\sum_{i=1}^{t} m_i = \binom{v}{2}$. Prove that K_v can be decomposed into t paths $P^1, P^2, ..., P^t$ such that the length of P^i is m_i for $i = 1, 2, ..., t$.*

So far, partial results have been obtained especially when v is odd. But, for the case when v is even, not much is know.

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