國 立 交 通 大 學

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碩 士 論 文

在 m 乘 n 陣列裡的橫截

Transversals in m × n Arrays

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中 華 民 國 九 十 六 年 六 月

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在 m 乘 n 陣列裡的橫截

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摘　要

當$2\le m\le n$，一個m乘n的陣列是由m個列和n個行組成的mn個格子。在m乘n的陣列裡的一個部分橫截是收集m個格子的集合，這些格子是來自不同行不同列。在m乘n的陣列裡的一個橫截是一個部分橫截，這個部分橫截裡的m個符號都是不一樣的。定義L(m,n)是一個最大的整數使得如果每一個符號在m乘n的陣列裡出現最多L(m,n)次，則這個陣列一定會有一個橫截。在本篇論文，我們把找拉丁方陣的橫截的研究延伸到找m乘n陣列的橫截的研究。大體上，我們對於對某些正整數m和n的L(m,n)值感到興趣。

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Transversals in $m \times n$ Arrays

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Abstract

An m by n array consists of mn cells in m rows and n columns, where $2 \leq m \leq n$. A partial transversal in an m by n array is a set of m cells, one from each row and no two from the same column. A transversal in an m by n array is a partial transversal which m symbols are distinct. Define $L(m, n)$ as the largest integer such that if each symbol in an m by n array appears at most $L(m, n)$ times, then the array must have a transversal. In this thesis, we extend the study of finding transversals in a Latin square to find transversals in $m \times n$ arrays. Mainly, we are interested in determining the value $L(m, n)$ for certain pairs of positive integers m and n.

Contents

1 Introduction and Preliminaries

1.1 Introduction

A *Latin square* M of order n based on an n-set S is an $n \times n$ array such that each symbol of S occurs in each row and each column exactly once. For convenience, we may use $S = \{1, 2, 3, ..., n\}$ and the symbol appears in the i-th row and j-th column is called the (i, j)-entry of the Latin square, denoted by $M(i, j)$. Then, the following figures are examples of a Latin square of order 4 and a Latin square of order 5 respectively.

$M_1 =$

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

$M_2 =$

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

A *transversal* T of a Latin square is a set of n cells such that no two are in the same row and the same column and the symbols occur in T are distinct. It is not difficult to see that the above squares have transversals respectively. For examples, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\{(1, 1), (2, 3), (3, 5), (4, 2), (5, 4)\}$. These two sets are the transversals of M_1 and M_2 respectively. But, not every Latin square has a transversal. For example,

$M_3 =$

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

It is easy to check that M_3 has no transversal. Therefore, to determine whether a Latin square has a transversal or not is an interesting problem. More than 250 years ago, Euler conjectured that there do not exist two orthogonal Latin squares of order $4k+2$ for each positive integer k. It is believed that the idea is mainly originated from the fact that there exists a Latin square of order $4k+2$ which does not have a transversal. This is easy to see from M_3.

Now, we known that a pair of orthogonal Latin squares of order $4k+2$, $k \geq 2$, does not exist [9]. But, for a given Latin square, to determine whether a transversal exists is still an open problem. Toward solving this problem, in 1967, Ryser [7] conjectured that every Latin square of odd order has a transversal, and the number of transversals of a Latin square has the same parity as the order of the square. But, Parker pointed out that many Latin square of order 7 have an even number of transversals in 1989. Balasubramanian [2] proved that a Latin square of even order has an even number of transversals in 1990.

Unfortunately, the above results do not provide any assistance in determining whether there exists a transversal in a given Latin square or not. An intuitive approach is to find as many distinct elements from distinct rows and columns as possible. A *partial transversal* of a Latin square is a set of n cells from distinct rows and columns. The *size* of a partial transversal is the number of distinct symbols which appears in the partial transversal. For example, $P_1 = \{(1,1),(2,3),(3,2),(4,4)\}$ is a partial transversal of M_1 of size 2. $P_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ is a partial transversal of M_2 of size 1. It is easy to see that we can always find a partial transversal of size at least $n/2$ in a Latin square of order n. (Pick any cell in the first row, then a cell in the second row with a different symbol, and so on.) But, for larger size, it takes a while to get to the best known result today. First, in 1969, Koksma [6] showed that the length of a partial transversal in a Latin square is at least $n-(1/3)n$. Later Drake [3] showed that the lower bound is $n-(1/4)n$ in 1977. Then, by using the idea of matchings in the bipartite graph $K_{n,n}$, Woolbright [11] improved this lower bound to $n - \sqrt{n}$ in 1978. Four years later, 1982, Shor [10] gave a better bound $n - (5.53)(\ln n)^2$. Finally, by using a careful calculation in Shor's technique, Fu et al. [5] improved this the lower bound to $n - (5.518)(\ln n)^2$ in 2002.

Recently, the notion "*transversals in Latin square*" has been converted to that of arrays where we allow common symbols in both rows and columns. For positive integers m and n, where $2 \leq m \leq n$, an m by n *array* contains m rows and n columns. An m by n array A consists of mn cells and each cell contains one symbol and for $1 \leq i \leq m$ and $1 \leq j \leq n$, we use $A(i,j)$ to denote the symbol which appears in the row i and column j. A *partial transversal* in an m by n array is a set of m cells such that no two are in the same row and the same column. A partial transversal of *size* k contains exactly k distinct symbols which appears in the partial transversal. A *transversal* is a partial transversal of size m. Let $L(m,n)$ be the largest integer such that if each symbol in an m by n array appears at most $L(m,n)$ times, then the array must have a transversal. For example,

$A =$

1	1	2	3
4	2	4	1
2	5	3	2

$B =$

1	1	2	2
2	2	3	3
3	3	1	1

Then A and B are 3 by 4 arrays. Each symbol in A appears at most 4 times. Each symbol in B appears at most 4 times. $T = \{(1,1),(2,2),(3,3)\}$ is a transversal of A. $P = \{(1,1),(2,2),(3,3)\}$ is a partial transversal of B of size 2. It is easy to check that B has no transversal. By the array B, $L(3,4) < 4$. In 1991, P. Erdős and J. Spencer [4] showed that an array of order n in which each symbol appears at most $(n-1)/16$ times has a transversal. This implies $L(n,n) \geq \lfloor (n-1)/16 \rfloor$. Recently, S. Akbari. et al. [1] proved that $L(m,n) = \lfloor (mn-1)/(m-1) \rfloor$ for $m \geq 2$ and

$n \geq 2m^3 - 6m^2 + 6m - 1$. In this thesis, we study the value $L(m, n)$ for certain pairs of positive integers m and n.

1.2 Preliminaries

1.2.1 Probabilistic method: Lovász Local Lemma

Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. Let $\bar{A}_i$ denote the complement of event A_i. Then the probability of A_1 given A_2 is $Pr(A_1|A_2) = \dfrac{Pr(A_1 \cap A_2)}{Pr(A_2)}$. If $Pr(A_1|A_2) = Pr(A_1)$, we say that A_1 and A_2 are mutually independent. Let S be a set of events. In general, A_i is mutually independent of S if $Pr(A_i | \bigcap_{A_j \in T} A_j) = Pr(A_i)$ for all $T \subseteq \{A_j | A_j \in S \text{ or } \bar{A}_j \in S\}$.

Definition 1.1. Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a lopsidependency graph for the events $A_1, A_2, ..., A_n$ if $Pr(A_i | \bigcap_{j \in S} \bar{A}_j) \leq Pr(A_i)$ for each $i \in V$ and each $S \subseteq V \setminus N_G[i]$.

Definition 1.2. Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a dependency digraph for the events $A_1, A_2, ..., A_n$ if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$.

Theorem 1.3. ***[Lopsided Lovász Local Lemma]*** *Let A_1, A_2,...,A_n be events with lopsidependency graph G and suppose all the events have probability at most p and that each $i \in G$ has degree at most d. Assume $4pd \leq 1$. Then $Pr(\bigcap_{i=1}^{n} \bar{A}_i) > 0$.*

The following lemma, first proved in Erdős and Lovász in 1975, is an extremely powerful tool.

Theorem 1.4. ***[Lovász Local Lemma; General Case]*** *Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. Suppose that $D = (V, E)$ is a dependency* *digraph for the above events and suppose there are real numbers* $x_1, x_2, ..., x_n$ *such that* $0 \leq x_i < 1$ *and* $Pr(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_i)$ *for all* $1 \leq i \leq n$. *Then* $Pr(\bigcap_{i=1}^{n} \bar{A}_i) \geq \prod_{i=1}^{n}(1 - x_i)$. *In particular, with positive probability for no event* A_i *holds.*

Theorem 1.5. ***[Lovász Local Lemma; Symmetric Case]*** *Let* $A_1, A_2, ..., A_n$ *be events in an arbitrary probability space. Suppose that each event* A_i *is mutually independent of a set of all the other events* A_j *but at most* d*, and that* $Pr(A_i) \leq p$ *for all* $1 \leq i \leq n$. *If* $ep(d + 1) \leq 1$ *then* $Pr(\bigcap_{i=1}^{n} \bar{A}_i) > 0$.

In 1985, Shearer proved that the constant "e" is the best possible constant in the above lemma. In Lovász Local Lemma of general case, we can replace the two assumptions that each "A_i is mutually independent of $\{A_j : (i, j) \notin E\}$ " and that "$Pr(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_i)$ " by the weaker assumption that "for each i and each $S \subset \{1, 2, ..., n\} \backslash \{j : (i, j) \in E\}$, $Pr(A_i | \bigcap_{j\in S} \bar{A}_j) \leq x_i \prod_{(i,j)\in E}(1 - x_i)$ ".

1.2.2 Ideas in direct argument

Besides probabilistic method, we also use a direct argument to find the lower bound of $L(m, n)$. The idea is based on the following fact which is easy to see.

Proposition 1.6. Let A be an m by n array such that A has a transversal. Then, the new array A' obtained by the following three operations also has a transversal.

1. a permutation of rows

2. a permutation of columns

3. a permutation of symbols

So, without loss of generality, we may assume the transversal of an m by n array A lies on the following set of cells: $\{(1,1),(2,2),...,(m,m)\}$. For convenience, we also use $A(1,1),A(2,2),...,A(m,m)$ to denote the transversal of A.

Thus, we are ready to introduce several known results.

2 Known Results

For completeness, we also include their proofs.

Theorem 2.1. *[4] Given an $n \times n$ array A. Let $k \leq (n-1)/16$ and suppose that no entry of A appears more than k times. Then A has a transversal.*

Proof. We use Lopsided Lovász Local Lemma. Let S_n be a set of permutations on an n-set. Let $V = \{(s,t,u,v)|\ s < u,\ t \neq v \text{ and } A(s,t) = A(u,v)\}$. For each $(s,t,u,v) \in T$, let $A_{stuv} = \{\sigma|\ \sigma \in S_n,\ \sigma(s) = t \text{ and } \sigma(u) = v\}$. Then A has a transversal if and only if $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$. Hence we will show that $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$.

Note that $Pr(A_{stuv}) = (n-2)!/n! = 1/n(n-1)$.

Define a graph G with vertex set V and (s,t,u,v) adjacent to (x,y,z,w) if and only if $\{s,u\} \cap \{x,z\} \neq \varnothing$ or $\{t,v\} \cap \{y,w\} \neq \varnothing$. Then we can count the maximal degree of G. Given $(s,t,u,v) \in V$, there are at most $4n$ choices of (x,y) with either $x \in \{s,u\}$ or $y \in \{t,v\}$ and k choices for (z,w) with $A(x,y) = A(z,w)$. Either (x,y,z,w) adjacent to (s,t,u,v) or (z,w,x,y) adjacent to (s,t,u,v). Thus G has maximal degree at most $4nk$. Then $4 \cdot 4nk \cdot (1/n(n-1)) \leq 1$.

To show G is a lopsidependency graph. By symmetric, it suffices to show $Pr(A_{1122}|\bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}) \leq 1/n(n-1)$ where $s,t,u,v \neq 1,2$.

Let $N_{ij} = \{\sigma|\ \sigma(1) = i,\ \sigma(2) = j \text{ and } \sigma \in \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}\}$

Claim: $|N_{12}| \leq |N_{ij}|$ for all $i \neq j$.

subpf: If $i, j > 2$. Let $\sigma \in N_{12}$. There exist a, b with $\sigma(a) = i$, $\sigma(b) = j$. Define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = j$, $\sigma^*(a) = 1$, $\sigma^*(b) = 2$, and $\sigma^*(x) = \sigma(x)$ for all $x \neq 1,2,a,b$. Since $(1,i)$, $(2,j)$, $(a,1)$, $(b,2)$ are not part of any element in S, σ^* is in N_{ij}. Then $f: N_{12} \to N_{ij}$ is injective. Thus $|N_{12}| \leq |N_{ij}|$. The case $\{1,2\} \cap \{i,j\} \neq \varnothing$ is similar.∎

Hence,

$$Pr(A_{1122}|\bigcap_{(s,t,u,v)\in S}\bar{A}_{stuv}) = |N_{12}|/\sum_{i\neq j}|N_{ij}| \leq |N_{12}|/\sum_{i\neq j}|N_{12}| = 1/n(n-1).$$

By Lopsided Lovász Local Lemma, $Pr(\bigcap_{(s,t,u,v)\in V}\bar{A}_{stuv}) \neq 0$. So A has a transversal. ∎

The followings are direct proofs

Lemma 2.2. *[8] (1) $L(m+1,n) \leq L(m,n)$ and (2) $L(m,n) \leq L(m,n+1)$.*

Proof. (1) Suppose that $L(m+1,n) = k$. Consider an m by n array A in which each symbols appears at most k times. Without loss of generality, the symbols in A are positive integers. Then we add a row to get an $(m+1)\times n$ array B and the symbols in that row are negative integers and each symbol in that row appears at most k times. Hence B has a transversal. This implies that A must have a transversal.

(2) Suppose that $L(m,n) = k$. Consider an m by $(n+1)$ array A in which each symbols appears at most k times. Deleting the first column, then we get an $m\times n$ array B. Hence B has a transversal. This implies that A must have a transversal. ∎

Theorem 2.3. *[8] If $n \leq 2m-2$, then $L(m,n) \leq n-1$.*

Proof. We illustrated for the cases when $(m,n) = (3,3)$, and $(m,n) = (3,4)$:

1	1	3
2	2	1
3	3	2

1	1	3	3
2	2	1	1
3	3	2	2

It is easy to check that the above arrays have no tranversals.

Theorem 2.4. *[8] $L(m,n) < mn/(m-1)$.*

Proof. If only $m-1$ distinct symbols appear in an $m\times n$ array, the array has no transversal. Hence, if each of $(m-1)$ symbols appears at most $mn/(m-1)$ times, the symbols can fill all the cells. ∎

Theorem 2.5. *[8] $L(2, n) = 2n - 1$ for $n \geq 3$.*

Proof. Consider a 2 by n array A in which each symbol appears at most $2n - 1$ times. Suppose A has no transversal. Then A is equivalent to the following array:

1	b	b	b	b	b	...
a	1	a	a	a	a	...

It is easy to check that a, b stand for 1.

Then 1 appears $2n$ times, a contradiction. ∎

Lemma 2.6. *[8] Assume that in a 3 by n array, $n \geq 4$, some symbol occurs at most three times. Then, if there is no transversal some symbol occurs at least $2n - 2$ times, hence at least $3n/2$ times.*

Proof. There are 10 inequivalent cases when one symbol appears at most three times. We list the 10 cases.

1		

1	1	

1		
1		

1	1	

1	1	1

1	1	
		1

1	1	
1		

1		
1		
1		

1		
1		
	1	

1		
	1	
		1

We illustrate the case when 1 appears one time. Then we have the following array:

1							
	2	b	b	b	b	b	...
	a	2	2	2	2	2	...

It is easy to check that a and b stand for 2. Hence the symbol 2 appears at least $2n-2$ times. The other cases are similar. ∎

Theorem 2.7. *[8] (a) $L(3,3)=2$ and $L(3,4)=3$. (b) For $n \geq 5$, $L(3,n) = \lfloor (3n-1)/2 \rfloor$.*

Proof. Exhaustive computer calculations shows that

$$L(3,3)=2, \quad L(3,4)=3, \quad L(3,5)=7.$$

By induction on n. Assume that the induction holds for a particular odd n. i.e. $L(3,n)=(3n-1)/2$. We will show that it holds for $n+1$, that is, $L(3,n+1) = (3n+1)/2$.

Consider a 3 by $n+1$ array A in which each symbol appears at most $(3n+1)/2$ times. If each symbol appears at most $(3n-1)/2$ times, then deleting one column to obtain a 3 by n array. By induction hypothesis, the 3 by n array has a transversal. Hence A has a transversal.

Suppose there is at least one symbol appears at least $(3n+1)/2$ times. If there are two such symbols, they appear at least $3n+1$ times. Hence some symbol appears at most three times. By Lemma 2.6, if there is no transversal, then some symbol occurs at least $3(n+1)/2$ times. So A has a transversal.

Hence there is only one symbol that appears at least $(3n+1)/2$ times. There must be a column in which it appears at least twice. Deleting that column, we get a 3 by n array in which each symbol appears at most $(3n-1)/2$ times. By induction hypothesis, the 3 by n array has a transversal. Hence A has a transversal. Thus $L(3,n+1) \geq (3n+1)/2$. By Theorem 2.4, $L(3,n+1) < (3n+3)/2$. So, $L(3,n+1)=(3n+1)/2$. When n is even, the argument is similar. ∎

Theorem 2.8. *[8] $L(m,n) \geq n-m+1$.*

Proof. We use induction on m to prove the assertion.

The theorem is true for $m = 2$ or $m = 3$. Assume that it is true for $m - 1$. We will show that it holds for m.

Assume that $L(m-1, n) \geq n - m + 2$. Consider an m by n array A in which each symbol appears at most $n - m + 1$ times. Deleting the last row of A, we get an $m - 1$ by n array. The $m - 1$ by n array has a transversal. Suppose that A has no transversal. Then A is equivalent to the following array:

1						a	a	a	a	$\ldots$
	2									$\ldots$
		3								$\ldots$
			$\ddots$							$\ldots$
				$m-1$						$\ldots$
					1	a	a	a	a	$\ldots$

An a stands for $1, 2, ..., m-1$. Then there are at least $2(n-m)+2$ cells containing a or 1. Since 1 appears at most $n - m + 1$ times in A and $2(n - m) + 2 > n - m + 1$, there must be an element in $\{2, 3, ..., m-1\}$ occuring in some cells marked a. Without loss of generality, we take the symbol to be 2. Then we have the following array:

1						2	a	a	a	$\ldots$
a	2						a	a	a	$\ldots$
		3								$\ldots$
			$\ddots$							$\ldots$
				$m-1$						$\ldots$
					1	a	a	a	a	$\ldots$

Then there are at least $3(n - m) + 3$ cells containing a, 1 or 2. Since 1 and 2 appear at most $2n - 2m + 2$ times in$A$ and $3(n - m) + 3 > 2n - 2m + 2$, there be an element in $\{3, ..., m - 1\}$ occuring in some cells marked a. Without loss of generality, we take the symbol to be 3.

1						2	3	a	a	...
a	2						a	a	a	...
a		3				a		a	a	...
			$\ddots$							
				$m-1$						...
					1	a	a	a	a	...

Continuing the analysis, the symbols $1, 2, ..., m-1$ appear at least $(m-1)(n-m)+m$ times. But, $(m-1)(n-m)+m > (m-1)(n-m+1)$, a contradiction. This concludes the proof. ∎

3 Main Result

Theorem 2.1 implies $L(n,n) \geq \lfloor (n-1)/16 \rfloor$. We improve this lower bound.

Theorem 3.1. $L(n,n) \geq \lfloor (n+4e)/4e \rfloor$.

Proof. Let $k = \lfloor (n+4e)/4e \rfloor$.

Consider an n by n array A in which each symbol appears at most k times. We use Lovász Local Lemma. Let S_n be a set of permutations on an n-set. Let $V = \{(s,t,u,v) |\ s < u,\ t \neq v \ and \ A(s,t) = A(u,v)\}$. For each $(s,t,u,v) \in T$, let $A_{stuv} = \{\sigma |\ \sigma \in S_n,\ \sigma(s) = t \ and \ \sigma(u) = v\}$. Then A has a transversal if and only if $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$. Hence we will show that $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$.

Note that $Pr(A_{stuv}) = (n-2)!/n! = 1/n(n-1)$.

Define a graph G with vertex set V and (s,t,u,v) adjacent to (x,y,z,w) if and only if $\{s,u\} \cap \{x,z\} \neq \varnothing$ or $\{t,v\} \cap \{y,w\} \neq \varnothing$. Then we can count the maximal degree of G. Given $(s,t,u,v) \in V$, there are at most $4n-4$ choices of (x,y) with either $x \in \{s,u\}$ or $y \in \{t,v\}$ and $k-1$ choices for (z,w) with $A(x,y) = A(z,w)$. Either (x,y,z,w) adjacent to (s,t,u,v) or (z,w,x,y) adjacent to (s,t,u,v). Thus G has maximal degree at most $(4n-4)(k-1)-1$. Then $e \cdot ((4n-4)(k-1)-1+1) \cdot (1/n(n-1)) \leq e \cdot 4(n-1)(n/4e) \cdot (1/n(n-1)) = 1$.

To show G is a lopsidependency graph. By symmetric, it suffices to show $Pr(A_{1122} | \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}) \leq 1/n(n-1)$ where $s,t,u,v \neq 1,2$.

Let $N_{ij} = \{\sigma |\ \sigma(1) = i,\ \sigma(2) = j \ and \ \sigma \in \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}\}$

Claim: $|N_{12}| \leq |N_{ij}|$ for all $i \neq j$.

subpf: If $i,j > 2$. Let $\sigma \in N_{12}$. There exist a,b with $\sigma(a) = i$, $\sigma(b) = j$. Define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = j$, $\sigma^*(a) = 1$, $\sigma^*(b) = 2$, and $\sigma^*(x) = \sigma(x)$ for all $x \neq 1,2,a,b$. Since $(1,i)$, $(2,j)$, $(a,1)$, $(b,2)$ are not part of any element in S, σ^* is in N_{ij}. Then

$f : N_{12} \to N_{ij}$ is injective. Thus $|N_{12}| \le |N_{ij}|$. The case $\{1,2\} \cap \{i,j\} \neq \varnothing$ is similar. ∎

Hence,

$$Pr(A_{1122} | \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}) = |N_{12}| / \sum_{i\neq j} |N_{ij}| \le |N_{12}| / \sum_{i \neq j} |N_{12}| = 1/n(n-1).$$

By Lovász Local Lemma, $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$. So A has a transversal. ∎

The following results obtain from direct argument.

Lemma 3.2. $L(m,n) \le \lfloor (mn-1)/(m-1) \rfloor$

Proof. Suppose $mn = k(m-1) + r$ where $k, r \in \mathbb{Z}$, $0 \le r < m-1$. If $r = 0$. By Theorem 2.4, $L(m,n) < mn/(m-1) = k$. Then $L(m,n) \le k-1 = \lfloor (mn-1)/(m-1) \rfloor$. If $1 \le r < m-1$, $L(m,n) < mn/(m-1) = k + r/(m-1) < k+1$. Then $L(m,n) \le k = \lfloor (mn-1)/(m-1) \rfloor$. ∎

By above Lemma, if we can show that $L(m,n) \ge \lfloor (mn-1)/(m-1) \rfloor$, then $L(m,n) = \lfloor (mn-1)/(m-1) \rfloor$. The following results use the idea.

Theorem 3.3. *For* $n \ge 43$, $L(4,n) = \lfloor (4n-1)/3 \rfloor$.

Proof. Consider a 4 by n array A in which each symbol appears at most $\lfloor (4n-1)/3 \rfloor$ times. Since $L(3,n) = \lfloor (3n-1)/2 \rfloor \ge \lfloor (4n-1)/3 \rfloor$, the 3 by n array consisting of the first three rows of A has a transversal. Suppose that A has no transversal. Then A is equivalent to the following array:

1				x_1	x_2	x_3	x_4	x_5	......	x_{n-4}
	2								......	
		3							......	
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i \in \{1,2,3\}$, for all $1 \le i \le 2n-8$.

Then there are at least $2(n-4)+2$ cells containing x_i or 1. Since 1 appears at most $\lfloor(4n-1)/3\rfloor$ times in A and $2(n-4)+2 > \lfloor(4n-1)/3\rfloor$, there must be a 2 or 3 in some cells marked x_i. Without loss of generality, we take x_1 to be 2. Then we have the following array:

1				2	x_2	x_3	x_4	x_5	......	x_{n-4}
y_1	2				y_2	y_3	y_4	y_5	......	y_{n-4}
		3							......	
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i, y_j \in \{1,2,3\}$, for all $2 \le i \le 2n-8$ and $1 \le j \le n-4$.

Then there are at least $3(n-4)+3$ cells containing x_i, y_j, 1, or 2. Since 1 and 2 appear at most $2\lfloor(4n-1)/3\rfloor$ times in A and $3(n-4)+3 > 2\lfloor(4n-1)/3\rfloor$, there must be a 3 in some cell marked x_i or y_j. There are 5 inequivalent cases, $x_2 = 3$, $x_{n-3} = 3$, $x_{n-2} = 3$, $y_1 = 3$ or $y_2 = 3$.

If $x_2 = 3$, then we have the following array:

1				2	3	x_3	x_4	x_5	......	x_{n-4}
y_1	2				y_2	y_3	y_4	y_5	......	y_{n-4}
z_1		3		z_2		z_3	z_4	z_5	......	z_{n-4}
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i, y_j, z_k \in \{1,2,3\}$, for all $3 \le i \le 2n-8$ and $1 \le j \le n-4$ and $1 \le k \le n-4$. Deleting the first six columns and deleting the last row we get a 3 by $n-6$ array B in which each symbol appears at most $\lfloor(4n-1)/3\rfloor - 2$ times. Since $\lfloor(3(n-6)-1/2\rfloor \ge \lfloor(4n-1)/3\rfloor - 2$, B has a transversal T. Note that the symbols occur in T are $1, 2, 3$. Hence $A(4,1), A(4,2), A(4,3) \in \{1,2,3\}$. Otherwise, A has a transversal. Similarly, all cells contain $1, 2, 3$. Then the symbols $1, 2, 3$ appear $4n$ times. But $4n > 3\lfloor(4n-1)/3\rfloor$, a contradiction. Then A has a transversal. Since the argument of the other cases are similar, we omit the details. In fact, no matter which case, we can get an 4 by $n-6$ array consisting of the last $n-6$ columns of A in which symbols in the array are $1, 2, 3$.

Thus, $L(4,n) \geq \lfloor (4n-1)/3 \rfloor$. By Lemma 3.2, $L(4,n) \leq \lfloor (4n-1)/3 \rfloor$. So, $L(4,n) = \lfloor (4n-1)/3 \rfloor$. ∎

We can use the same technique for general case.

Theorem 3.4. *For $m \geq 2$ and $n \geq 2m^3 - 8m^2 + 12m - 5$,*
$L(m,n) = \lfloor (mn-1)/(m-1) \rfloor$.

Proof. We use induction on m to prove the assertion.

If $m = 2$. Then $n \geq 3$. By Theorem 2.5, $L(2,n) = 2n-1$. Assume that it is true for $m-1$. That is $L(m-1,n) = \lfloor ((m-1)n-1)/(m-2) \rfloor$ for $n \geq 2(m-1)^3 - 8(m-1)^2 + 12(m-1) - 5 = 2m^3 - 14m^2 + 34m - 27$. To show it holds that for $n \geq 2m^3 - 8m^2 + 12m - 5$, $L(m,n) = \lfloor (mn-1)/(m-1) \rfloor$.

For $n \geq 2m^3 - 8m^2 + 12m - 5$, consider an m by n array A in which each symbol appears at most $\lfloor (mn-1)/(m-1) \rfloor$ times. Since $2m^3 - 8m^2 + 12m - 5 \geq 2m^3 - 14m^2 + 34m - 27$ and $\lfloor ((m-1)n-1)/(m-2) \rfloor \geq \lfloor (mn-1)/(m-1) \rfloor$, then the $m-1$ by n array consisting of the first $m-1$ rows of A has a transversal. Suppose that A has no transversal. Then A is equivalent to the following array:

1					a	a	a	a	.....	a
	2									
		$\ddots$								
			$m-1$							
				1	a	a	a	a	.....	a

where an a stands for $1, 2, ..., m-1$.

There are at least $2(n-m)+2$ cells containing a or 1. Since 1 appears at most $\lfloor (mn-1)/(m-1) \rfloor$ times and $2(n-m)+2 > \lfloor (mn-1)/(m-1) \rfloor$, there must be an element in $\{2,3,...,m-1\}$ occuring in some cells marked a. Without loss of generality, we take the symbol to be 2. Then A is equivalent to the following array:

1					2	a	a	a	.....	a
a	2					a	a	a	.....	a
		$\ddots$								
			$m-1$							
				1	a	a	a	a	.....	a

where an a stands for $1,2,...,m-1$.

If $k(n-m)+k > (k-1)\lfloor (mn-1)/(m-1)\rfloor$ for $2 \le k \le m-1$, then we can continue the argument. It is enough to show $k(m-1)(n-m)+k(m-1) > (k-1)(mn-1)$. $k(m-1)(n-m)+k(m-1)-(k-1)(mn-1) = (m-k)n - km^2 + 2km - 1$. Since $m-k \ge 1$, it is enough to show that $n > km^2 - 2km + 1 = m(m-2)k+1$. When k is getting larger, $m(m-2)k+1$ is getting larger. It is enough to show that $n > (m-1)m^2 - 2(m-1)m+1$. Since $2m^3 - 8m^2 + 12m - 5 > (m-1)m^2 - 2(m-1)m+1$, then $n > (m-1)m^2 - 2(m-1)m+1$. Hence, we can continue the argument. Then we can get an m by $n-(2m-2)$ array B consisting of the last $n-(2m-2)$ columns of A. The symbols in B are $1,2,...,m-1$. And each symbol in B appears at most $\lfloor (mn-1)/(m-1)\rfloor - 2$ times. Since $\lfloor ((m-1)(n-2m+2)-1)/(m-2)\rfloor \ge \lfloor (mn-1)/(m-1)\rfloor - 2$, the array obtained from deleting any row in B has a transversal T. Note that the symbols occur in T are $1,2,...,m-1$. Then all cells contain $1,2,...,m-1$. Otherwise, A has a transversal. Therefore, in total $1,2,...,m-1$ appear mn times. But, $mn > (m-1)\lfloor (mn-1)/(m-1)\rfloor$, a contradiction. Thus, A has a transversal. Hence $L(m,n) \ge \lfloor (mn-1)/(m-1)\rfloor$.

By Lemma 3.2, $L(m,n) \le \lfloor (mn-1)/(m-1)\rfloor$, we conclude the proof.

4 Conclusion

From the study of the "transversal problem" of an m by n array, we notice that the most difficult part remains in the situation when m is not that far from n. That is why the transversal problem of a Latin square is still one of the most difficult problem in combinatorial designs. So, for future study, we should focus on determining $L(n, n)$ or $L(m, n)$ where m is a linear function of n instead the bound we obtain in this thesis which is in cubic order.

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