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圖的圈覆蓋

Cycle Cover of Graphs

研 究 生：連敏筠

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摘要

一個圖G的圈覆蓋(cycle cover)是收集一些G裡面的圈(cycle)，使得G裡面的邊都被覆蓋住(cover)。一個整數流(integer flow)是在圖上賦予方向，且給定一個 整數函數ϕ對應到G上所有的邊，使得對所有G裡的點v都滿足由v流出的量$\sum_{e\in E^+(v)}\phi(e)y`\sum_{e\in E^-(v)}\phi(e)$。當所有的 $\phi(e)$，都滿足$-k<\phi(e)<k$，則ϕ稱為一個k-整數流(k-flow)，如果對所有G上的邊，給的值都不為零， 則ϕ稱為處處不為零的k-整數流(nowhere-zero k-flow)。在這篇論文中，我們證明：如果Tutte的3-整數流猜測(Tutte's 3-flow conjecture)是對的，則當k是奇數時，對所有的$(k-1)$-邊連通圖滿足最小度數為k，會有一個處處不為零的6-整數流(6-flow) ϕ，使得那些給定奇數值的邊數會大於等於$\frac{k-1}{k}$的總邊數，這會推得圖G有一個圈覆蓋最多只需要$\frac{13k+5}{9k}$的總邊數。當k是偶數時，對所有的$(k-1)$-邊連通圖滿足最小度數為k，會有一個處處不為零的6-整數流(6-flow) ϕ，使得那些給定奇數值的邊數會大於等於$\frac{k-2}{k-1}$的總邊數，這會推得圖G有一個圈覆蓋最多只需要$\frac{13k-8}{9(k-1)}$的總邊數。

Cycle Cover of Graphs

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Abstract

A *cycle cover* of a graph G is a collection of cycles of G which covers all edges of G. The *size* of a cycle cover is the sum of the lengths of the cycles in the cover. A *flow* in G under orientation D is an integer-valued function ϕ on $E(G)$ such that the output value $\sum_{e\in E^+(v)} \phi(e)$ is equal to the input value $\sum_{e\in E^-(v)} \phi(e)$ for each $v \in V(G)$. The *support* of ϕ is defined by $S(\phi) = \{e \in E(G) : \phi(e) \neq 0\}$. For a positive integer k, if $-k < \phi(e) < k$ for every $e \in E(G)$, then ϕ is called a *k-flow*, and furthermore, if $S(\phi) = E(G)$, then ϕ is called a *nowhere-zero k-flow*. In this thesis we prove: (1) if Tutte's 3-Flow Conjecture is true, then every $(k-1)$-edge-connected graph G with $\delta(G) = k$ has a nowhere-zero 6-flow ϕ such that when k is odd $|E_{odd}(\phi)|\geq \frac{k-1}{k}|E(G)|$ and when k is even $|E_{odd}(\phi)|\geq \frac{k-2}{k-1}|E(G)|$; (2) If a $(k-1)$-edge-connected graph G with $\delta(G) = k$ has a nowhere-zero 6-flow ϕ such that when k is odd $|E_{odd}(\phi)|\geq \frac{k-1}{k}|E(G)|$, then G has a cycle cover in which the size of the cycle cover is at most $\frac{13k+5}{9k}|E(G)|$ and when k is even $|E_{odd}(\phi)|\geq \frac{k-2}{k-1}|E(G)|$, then G has a cycle cover in which the size of the cycle cover is at most $\frac{13k-8}{9(k-1)}|E(G)|$, where $E_{odd}(\phi)$=$\{e \in E(G) : \phi(e)$ is odd $\}$.

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Contents

Chapter 1

Introduction and Preliminaries

"Cycle cover" has been one of the most important topics in Graph Theory. A *cycle cover* of a graph G is a collection of cycles of G which covers all edges of G. Since a graph is even if and only if it has a decomposition into edge-disjoint cycles, we also regard a cycle cover as a collection of even subgraphs. The *size* of a cycle cover is the sum of the lengths of the cycles in the cover. The problem of finding a cycle cover of small size (a short cycle cover) has been studied by several authors[3, 12, 21]. The best known result on this subject is the one by Bermond, Jackson and Jaeger [3], and independently by Alon and Tarsi [1], that every bridgeless graph G has a cycle cover of size at most $\frac{5}{3}|E(G)|$. Later, this upper bound was improved to $\frac{44}{27}|E(G)|$ by Fan in [8] under the assumption that Tutte's 3-flow conjecture is true. The focus of this study will be on the graphs G with larger edge-connectivity. In which, under the same assumption, we are able to obtain a smaller upper bound with respect to the ratio of $|E(G)|$ and the size of cycle cover.

1.1 Basic Notations

A *graph* G is composed of two types of objects. It has a finite set of elements called vertices and a set of unordered pairs of vertices called edges. The vertex set is denoted by $V(G)$ or V, and the edge set is denoted by $E(G)$ or E.

A *directed graph* or *digraph* G is composed of two types of objects. It has a vertex set $V(G)$ and an edge set $E(G)$, and the edge set is a set of ordered pairs of vertices. The first vertex of the ordered pair is the *tail* of the edge, and the second is the *head*. We say that an edge in a digraph is an edge from its tail to its head.

A *loop* is an edge whose endvertices are equal. *Multiple edges* are edges having the same pair of endvertices. A *simple graph* is a graph having no loops or multiple edges.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* of a graph G is a subgraph with vertex set $V(G)$. The *components* of a graph G are its maximal connected subgraphs.

The *degree* of vertex v in a graph G is the number of edges incident to v, written $deg(v)$. The *maximum degree* is $\Delta(G)$, the *minimum degree* is $\delta(G)$, and G is *regular* if $\Delta(G) = \delta(G)$. It is *k-regular* if the common degree is k. The *neighborhood* of v, written $N(v)$, is the set of vertices adjacent to v.

In a graph G, *contraction* of edge e with endvertices u, v is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v.

A graph is called a *weighted graph* if each edge e is assigned a nonnegative real number $w(e)$, called the *weight* of e. Let G be a weighted graph and H a subgraph of G. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For graphs G and H, the *symmetric difference* $G \oplus H$ is the subgraph of $G \cup H$ whose edges are the edges of $G \cup H$ appearing in exactly one of G and H. The *symmetric difference* of two even subgraphs Z_1 and Z_2, is the even subgraph $(Z_1 \cup Z_2) \backslash (Z_1 \cap Z_2)$.

A *matching* in a graph G is a set of non-loop edges with no shared endvertices. The vertices incident to the edges of a matching M are saturated by M. A *perfect matching* in a graph G is a matching that saturates every vertex of G.

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* is a spanning k-regular subgraph. An *odd component* of a graph is a component of odd order; the number of odd component of H is $o(H)$.

A *disconnecting set* of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. A graph is *k-edge-connected* if every disconnecting set has at least k edges. Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endvertices in S and the other in T. An *edge cut* is an edge set of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$.

In a graph G, *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w. A subdivision of H is a graph obtained from a graph H by successive edge subdivisions.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

Two graphs G and G' are *homeomorphic* if there is an isomorphism from some subdivision of G to some subdivision of G'.

1.2 Preliminaries

First, we introduce the *cycle cover* and *integer flow*. Now, we need some definitions and notations.

Definition 1.2.1. A *cover* of a graph G is a collection $\mathcal{H}$ of subgraphs of G which covers all edges of G. A *cycle cover* of a graph G is a collection $\mathcal{C}$ of cycles of G which covers all edges of G; $\mathcal{C}$ is called a *cycle m-cover* of G if each edge of G is covered exactly m times by the members in $\mathcal{C}$. A cover is called an (l, m)-cover if each edge of G is covered either exactly l or exactly m times.

Definition 1.2.2. An *orientation* D of an undirected graph G is an assignment of a direction to each edge $e \in E(G)$. Let G be a graph with orientation D. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail v, and $E^-(v)$ the set of non-loop edges with head v. A *flow* in G under orientation D is an integer-valued function ϕ on $E(G)$ such that

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e) \text{ for each } v \in V(G).$$

The *support* of ϕ is defined by

$$S(\phi) = \{e \in E(G) : \phi(e) \neq 0\}.$$

For a positive integer k, if $-k < \phi(e) < k$ for every $e \in E(G)$, then ϕ is called a *k-flow*, and furthermore, if $S(\phi) = E(G)$, then ϕ is called a *nowhere-zero k-flow*.

Definition 1.2.3. For a flow ϕ in G, we define

$$E_{odd}(\phi) = \{e \in E(G) : \phi(e) \text{ is odd}\}$$

and

$$E_{even}(\phi) = \{e \in E(G) : \phi(e) \text{ is even}\}.$$

Conjecture 1.2.4. (Tutte's 3-flow Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow.

A basic result on integer flows is the following one due to Tutte((6.3) of [22]).

Lemma 1.2.5. [22] *If G has a flow ϕ, then for any integer $k > 0$, G has a k-flow ϕ' such that $\phi'(e) \equiv \phi(e) (mod\ k)$ for every $e \in E(G)$.*

The following is an easy consequence of Lemma 1.2.5.

Lemma 1.2.6. *Let $F \subseteq E(G)$ and let G' be obtained from G by contracting the edges in F. If G' has a k-flow ϕ', then G has a k-flow ϕ such that $S(\phi') \subseteq S(\phi)$.*

A flow in G is always associated with some orientation of G. By changing signs, one can arrange for two flows in G to have the same orientation. If ϕ_1 and ϕ_2 are two flows in G under the same orientation D, then for any integers l and m the linear combination $\phi = l\phi_1 + m\phi_2$ is a flow under D. Let ϕ be a flow in G under a given orientation and $e \in E(G)$. If we reverse the direction of e and change $\phi(e)$ to $-\phi(e)$, we still have a flow in G under the new orientation. Thus, we have

Proposition 1.2.7. If G has a flow ϕ under some orientation D, then for any orientation D', G has a flow ϕ' under D' with $|\phi'(e)| = |\phi(e)|$ for every edge e.

Definition 1.2.8. Let ϕ be a flow in G. A flow f in G is called a *sub-flow* of ϕ if

1. f has the same orientation as ϕ;
2. $|f(e)| \le |\phi(e)|$ and $f(e)\phi(e) \ge 0$ for every $e \in E(G)$.

Moreover, if f is a k-flow, then we call f a *sub-k-flow* of ϕ. By the definition, if f is a sub-flow of ϕ, then $\phi - f$ is also a sub-flow of ϕ.

Lemma 1.2.9. [16] *Let ϕ be a flow in a graph G and r a real number, $r \ge 1$. Then G has a sub-flow f of ϕ in which $f(e) = \lfloor \frac{\phi(e)}{r} \rfloor$ or $\lceil \frac{\phi(e)}{r} \rceil$ for every edge $e \in E(G)$.*

By this result, if a graph G has a k-flow ϕ, where $k \ge 2$, then it has a sub-2-flow f of ϕ (using $r = k-1$). Since $\phi - f$ is a sub-(k-1)-flow of ϕ, we may apply the lemma to $\phi - f$ using $r = k-2$ (if $k \ge 3$). Repeatedly, we decompose the k-flow ϕ into (k-1) sub-2-flows. This is a result obtained by Little, Tutte, and Younger [17]. We state it as Lemma 1.2.10 below.

Lemma 1.2.10. [17] *Every k-flow ϕ is a sum of $(k-1)$ sub-2-flows of ϕ.*

Lemma 1.2.10 plays a key role in the proof of the next lemma. For a flow ϕ in a graph G, we set

$$E_i(\phi) = \{e \in E(G) : \phi(e) = i\} \text{ and } E_{\pm i}(\phi) = E_i(\phi) \cup E_{-i}(\phi).$$

Lemma 1.2.11. [8] *Let f be a k-flow in G. Then there is a k-flow ϕ in G such that $S(\phi) = S(f)$ and*

$$|E_{\pm 1}(\phi)| \geq \frac{k-1}{k}(|E_{\pm 1}(f)| + |E_{\pm(k-1)}(f)|).$$

Proof. By Lemma 1.2.10, we have that $f = \sum_{i=1}^{k-1} f_i$, where f_i is a sub-2-flow of f. Let $\phi_i = kf_i - f, 1 \leq i \leq k-1$. Since $f_i(e)$ and $f(e)$ have the same sign for each $e \in E(G)$, each ϕ_i is a k-flow in G with the same support as f. Let $e \in E(G)$; if $|f(e)| = 1$, then there is exactly one j such that $|f_j(e)| = 1$, and so $|\phi_i(e)| = 1$ for all $i \neq j$, $1 \leq i \leq k-1$; if $|f(e)| = k-1$, then for every i, $1 \leq i \leq k-1$, $|f_i(e)| = 1$ and so $|\phi_i(e)| = 1$. Therefore,

$$\sum_{i=1}^{k-1} |E_{\pm 1}(\phi_i)| \geq (k-2)|E_{\pm 1}(f)| + (k-1)|E_{\pm(k-1)}(f)|.$$

That is,

$$|E_{\pm 1}(f)| + \sum_{i=1}^{k-1} |E_{\pm 1}(\phi_i)| \geq (k-1)(|E_{\pm 1}(f)| + |E_{\pm(k-1)}(f)|).$$

Choosing $\phi \in \{f, \phi_1, ..., \phi_{k-1}\}$ with $|E_{\pm 1}(\phi)|$ maximum,

$$k|E_{\pm 1}(\phi)| \geq (k-1)(|E_{\pm 1}(f)| + |E_{\pm(k-1)}(f)|).$$

Dividing both sides by k yields the required result. ∎

Proposition 1.2.12. Let ϕ be a flow in G and $Z = E_{odd}(\phi)$. Then Z is an even subgraph of G.

Lemma 1.2.13. *Let ϕ be a 4-flow in G. Then G contains two even subgraphs Z_1 and Z_2 such that*

$$Z_1 \cup Z_2 = S(\phi) \text{ and } Z_1 \cap Z_2 = E_{\pm 2}(\phi).$$

Proof. We apply Lemma 1.2.9 to ϕ using $r = 2$ to obtain a sub-3-flow f, and let $f' = \phi - f$. Then $E_{\pm 1}(f)$ and $E_{\pm 1}(f')$ are two even subgraphs of the required properties. ∎

In second part, we introduce "Cover by perfect matchings".

Definition 1.2.14. An *r-graph* G is an r-regular graph such that for each vertex subset $X \subset V(G)$ with $|X| \equiv 1 \pmod 2$ and $0 < |X| < |V(G)|$, $[X, \overline{X}] \geq r$.

Lemma 1.2.15. *Let G be an r-regular and r-edge-connected graph then G is an r-graph.*

The following theorem is a corollary of Edmonds' Matching Polyhedron Theorem [5].

Theorem 1.2.16. (Edmonds [5]) *Let G be an r-graph. Then there is an integer p and a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in exactly p members of $\mathcal{M}$.*

Theorem 1.2.17. *Let G be an r-graph where $r \geq 3$ is odd. Then G contains a perfect matching M such that $w(M) \leq \frac{1}{r} w(G)$.*

Proof. By Theorem 1.2.16, the r-graph G has a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in exactly p members of $\mathcal{M}$. Thus the number of perfect matchings in $\mathcal{M}$, $|\mathcal{M}| = rp$. Let A be an r-edge-cut of G. For any $M \in \mathcal{M}$, $|A \cap (E(G) \backslash M)|$ is even because $E(G) \backslash M$ is a disjoint union of 2-factors of G, and so $|A \cap M|$ is odd. It follows that

$$|A \cap M| \geq 1 \text{ for every } M \in \mathcal{M}.$$

Since each edge of A belongs to exactly p members of $\mathcal{M}$,

$$\sum_{M \in \mathcal{M}} |A \cap M| = rp = |\mathcal{M}|.$$

Consequently,

$$|A \cap M| = 1 \text{ for any } r\text{-edge-cut } A \text{ of } G \text{ and every } M \in \mathcal{M}.$$

Since each edge of G belongs to exactly p members of $\mathcal{M}$,

$$\sum_{M \in \mathcal{M}} w(M) = pw(G).$$

Hence there is some $M^* \in \mathcal{M}$ such that

$$w(M^*) \leq \frac{pw(G)}{|\mathcal{M}|} = \frac{1}{r} w(G).$$

∎

Chapter 2

Known Results and Conjectures

In this chapter, we introduce several conjectures and known results related to cycle cover and integer flows.

2.1 Conjectures

The following problem was first considered by Szekeres [20] for bridgeless cubic graphs, and independently, formulated as a conjecture by Seymour [18] for general bridgeless graphs. It is now known as the "Cycle Double Cover Conjecture".

Conjecture 2.1.1. (Cycle Double Cover Conjecture) Every bridgeless graph has a cycle 2-cover.

A stronger form of this conjecture was proposed later by Celmins [4].

Conjecture 2.1.2. (Celmins [4]) Every bridgeless graph has a 2-cover by 5 even subgraphs.

If the graph is cubic, then Fulkerson expects that we have better outcome.

Conjecture 2.1.3. (Fulkerson [10]) Every bridgeless cubic graph has a 2-cover by 6 perfect matchings.

In fact, Conjecture 2.1.3 is equivalent to

Conjecture 2.1.4. Every bridgeless cubic graph has a 4-cover by 6 even subgraphs.

Since every bridgeless graph is a contraction of some bridgeless cubic graph (”split” each vertex into a cycle), the following conjecture is equivalent to Conjecture 2.1.4.

Conjecture 2.1.5. Every bridgeless graph has a 4-cover by 6 even subgraphs.

There are other conjectures on the study of the existence of nowhere-zero flows, we list some of them in what follows.

Conjecture 2.1.6. (Goddyn [11]) Every bridgeless graph has a cycle 6-cover.

Conjecture 2.1.7. (Tutte) Every bridgeless graph without 3-cuts has a nowhere-zero 3-flow.

Conjecture 2.1.8. (Tutte) Every bridgeless graph containing no subgraph contractible to the Petersen graph has a nowhere-zero 4-flow.

Conjecture 2.1.9. (Tutte's 3-flow Conjecture) Every 4-edge-connected graph has a nowhere-zero 3-flow.

Conjecture 2.1.10. (Tutte's 4-flow Conjecture [1966]) Every bridgeless graph containing no subdivision of the Petersen graph has a nowhere-zero 4-flow.

Conjecture 2.1.11. (Tutte's 5-flow Conjecture [1954]) Every bridgeless graph has a nowhere-zero 5-flow.

In order to investigate the structure of a cycle cover, Fan started to consider the total size of cycles used in a cover. He obtained a substantial result as we shall see in next section and also posed the following.

Conjecture 2.1.12. (Fan [7]) Every 4-edge-connected graph G can be covered by two even subgraphs of total size at most $\frac{6}{5}|E(G)|$.

The following two conjectures are due to Pyber (Problem 19 in [2]).

Conjecture 2.1.13. Every bridgeless graph G has a cycle cover such that every vertex of G is contained in at most $\Delta(G)$ cycles of the cover.

Conjecture 2.1.14. Every bridgeless graph G has a cycle cover such that every vertex of G is contained in at most $\frac{2}{3}\Delta(G) + 1$ cycles of the cover.

Therefore, the Cycle Double Cover Conjecture has the following equivalent formulation.

Conjecture 2.1.15. Every bridgeless graph G has a cycle cover such that every vertex v of G is contained in at most $d(v)$ cycles of the cover.

All the above conjectures remain open to date.

2.2 Known results

Bermond, Jackson and Jaeger [3] established the following theorem.

Theorem 2.2.1. [3] *Every bridgeless graph has a 4-cover by 7 even subgraphs.*

Godden also made some effort and prove.

Theorem 2.2.2. (Goddyn [11]) *For each bridgeless graph G, there exists an integer k (depending on $|E(G)|$) such that G has a cycle $(4k + 2)$-cover.*

Theorem 2.2.3. (Goddyn [11]) *Every bridgeless graph G has a 6-cover by 10 even subgraphs.*

Clearly, Theorem 2.2.1 implies that every bridgeless graph has a cycle m-cover for any $m \equiv 0 (\text{mod } 4)$ and $m \geq 4$. For odd m, the following result tells everything.

Theorem 2.2.4. *If m is odd, a graph G has a cycle m-cover if and only if G is an even graph.*

Proof. If G is even, then m copies of G give the required cover. Conversely, suppose that $\{H_1, H_2, \ldots, H_l\}$ is a cycle m-cover of G and m is odd. Then $G = H_1 \oplus H_2 \oplus \ldots \oplus H_l$ and so G is even. ∎

Theorem 2.2.3 together with Theorem 2.2.1 yields

Theorem 2.2.5. *Every bridgeless graph has an m-cover for any even number $m \geq 4$.*

Toward a proof of Conjecture 2.1.11, Jaeger [14] established

Theorem 2.2.6. [14] *Every bridgeless graph has a nowhere-zero 8-flow.*

Theorem 2.2.6 was improved by Seymour [19] in the following famous theorem.

Theorem 2.2.7. [19] *Every bridgeless graph has a nowhere-zero 6-flow.*

By a result of Jaeger [13] derived from Tutte's work, every graph with a nowhere-zero 4-flow can be covered by two even subgraphs, and these two subgraphs together with their symmetric difference, form a 2-cover by three even subgraphs. Thus we have the following result.

Theorem 2.2.8. *Every graph with a nowhere-zero 4-flow has a 2-cover by 3 even subgraphs.*

On the size of cover, Fan obtained the following results.

Theorem 2.2.9. [7] *If G has a nowhere-zero 4-flow, then the minimum total size of two even subgraphs which together cover G is at most $|E(G)| + |V(G)| - 1$.*

Theorem 2.2.10. (Fan [6]) *Every bridgeless weighted graph G has a $(2, 4)$-cover by four even subgraphs of total weight at most $\frac{20}{9}w(G)$.*

Theorem 2.2.11. (Fan [9]) *Every bridgeless graph G has a cycle cover such that each vertex v of G is contained in at most $d(v)$ cycles of the cover if $d(v) \geq 3$ and in at most three cycles of the cover if $d(v) = 2$.*

Our research is in fact motivated by the following result. We intend to find a cycle cover with minimum total size.

Theorem 2.2.12. (Alon and Tarsi [1]) *Every bridgeless G has a cycle cover of size at most $\frac{5}{3}|E(G)|$.*

An improved version is as follows.

Theorem 2.2.13. (Bermond, Jackson and Jeager [3]) *Every bridgeless G has a cycle cover of size at most $|E(G)| + \min\{\frac{2}{3}|E(G)|, \frac{7}{3}(|V(G)| - 1)\}$.*

The following result is the best known result so far.

Theorem 2.2.14. (Fan [8])

1. *If Tutte's 3-flow conjecture is true, then every bridgeless graph G has a nowhere-zero 6-flow ϕ such that $|E_{odd}(G)| \geq \frac{2}{3}|E(G)|$.*

2. *If G has a nowhere-zero 6-flow ϕ such that $|E_{odd}(G)| \geq \frac{2}{3}|E(G)|$, then G has a cycle cover in which the sum of lengths of the cycles in the cycle cover is at most $\frac{44}{27}|E(G)|$.*

Chapter 3

Main Results and Conclusion

3.1 The Main Results

In [8], Fan proved that if Tutte's 3-flow Conjecture is true, then every bridgeless graph G has a nowhere-zero 6-flow ϕ such that $|E_{odd}(G)| \geq \frac{2}{3}|E(G)|$ and he utilized this conclusion to show such a graph G has a cycle cover in which the sum of lengths of cycles in the cycle cover is at most $\frac{44}{27}|E(G)|$. Clearly, bridgeless graphs are what he was concerned. In what follows, motivated by this result, we consider the graphs which are (in some sense) more dense. That is, we shall focus on these graphs G with $\delta(G) = k$ and G is $(k-1)$-edge-connected. We expect that for larger k, the total size of a cycle cover comparing to $\frac{44}{27}|E(G)|$ will be smaller.

Suppose that F is an even subgraph of a graph G. If G has a 3-flow ϕ with $E(G)\backslash E(F) \subseteq S(\phi)$, then we let f be a 2-flow in G with $S(f) = F$; it is easy seen that $g = 2\phi + f$ is a nowhere-zero 6-flow with $E_{odd}(g) = F$. Conversely, if G has a nowhere-zero 6-flow g with $E_{odd}(g) = F$, then by a result of Tutte (Lemma 1.2.5), G has a 3-flow ϕ in which $\phi(e) = 0$ only if $|g(e)| = 3$, and thus, only if $e \in E(F)$. These results show the following two conjectures are equivalent.

Conjecture 3.1.1. Every $(k-1)$-edge-connected graph G with $\delta(G) = k$ has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $|E(F)| \geq \frac{k-1}{k}|E(G)|$.

Conjecture 3.1.2. Every $(k-1)$-edge-connected graph G with $\delta(G) = k$ has a nowhere-zero 6-flow g such that $|E_{odd}(g)| \geq \frac{k-1}{k}|E(G)|$.

In what follows, we shall first prove the case when k is odd.

Theorem 3.1.3. *If Tutte's 3-Flow Conjecture is true, then every graph which is homeomorphic to a $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is odd) has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $|E(F)| \geq \frac{k-1}{k}|E(G)|$.*

Note that unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. On the other hand, for a weighted graph with a weight function w; since the set of rational members are dense, we may assume that w is rational-valued. By multiplying out denominators, we have a weighted graph in which each edge has an integer weight $w'(e)$. Replacing each edge e by a path of length $w'(e)$ gives an unweighted graph. By these observation we see that Theorem 3.1.3 is equivalent to

Theorem 3.1.4. *If Tutte's 3-Flow Conjecture is true, then every weighted $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is odd) has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $w(F) \geq \frac{k-1}{k}w(G)$.*

The following lemmas are essential to the proof of the main theorem.

Lemma 3.1.5. *For $S \subseteq V(G)$, let $d(S) = |[S, \overline{S}]|$. Let X and Y be nonempty proper vertex subsets of G. Then $d(X \cup Y) + d(X \cap Y) \leq d(X) + d(Y)$.*

Proof. Since

$$\begin{aligned} d(X) &= |[X, \overline{X}]| \\ &= |[X\backslash Y, Y\backslash X]| + |[X\backslash Y, V\backslash(X \cup Y)]| + |[X \cap Y, V\backslash(X \cup Y)]| + |[X \cap Y, Y\backslash X]|, \end{aligned}$$

$$
\begin{aligned}
d(Y) &= |[Y, \overline{Y}]| \\
&= |[Y \backslash X, X \backslash Y]| + |[Y \backslash X, V \backslash (X \cup Y)]| + |[X \cap Y, V \backslash (X \cup Y)]| + |[X \cap Y, X \backslash Y]|,
\end{aligned}
$$

$$d(X \cap Y) = |[X \cap Y, X \backslash Y]| + |[X \cap Y, Y \backslash X]| + |[X \cap Y, V \backslash (X \cup Y)]|$$

and

$$d(X \cup Y) = |[X \backslash Y, V \backslash (X \cup Y)]| + |[Y \backslash X, V \backslash (X \cup Y)]| + |[X \cap Y, V \backslash (X \cup Y)]|$$

.

Thus, $d(X \cup Y) + d(X \cap Y) \leq d(X) + d(Y)$. ∎

Lemma 3.1.6. *Let G be an $(r-1)$-edge-connected graph with $\delta(G) = r$. Then for each $v \in V(G)$ with $deg(v) > r$, there exist two edges uv and vw such that $G - uv - vw + uw$ is $(r-1)$-edge-connected.*

Proof. Our goal is to find u and w such that $G - uv - vw + uw$ is an $(r-1)$-edge-connected.

Fix $v \in V(G)$ with $deg(v) > r$. Among all the edge cuts that contain some edges incident to v, choose S a minimum edge cut so that the resulting disconnected graph $G - S$ has a component S_1 with the smallest order, where $v \in S_1$. Since S is a minimum edge cut, then $G - S$ has exactly two components. Let S_2 be another component of $G - S$. Clearly, $|S| \geq r - 1$.

Case 1. $|S| > r$. Select arbitrary two neighbors of v, called u and w. It is easy to see that $G - uv - vw + uw$ is also an $(r-1)$-edge-connected graph.

Case 2. $|S| = r$. Let

$$\begin{aligned} T &= \{u \in S_1 \backslash \{v\} : u \text{ is incident to } e \in S\} \text{ and} \\ U &= \{e \in S : v \text{ is incident to } e \in S\}. \end{aligned}$$

Since $deg(v) > r$ and $|T| + |U| \leq |S| = r$, there is a vertex $x \in N(v) \cap S_1 \backslash T$. Select $y \in N(v) \cap S_2$.

Claim 1. xv and vy do not belong to any edge cut S' with $|S'| \leq r$.

Suppose not. Let S' be an edge cut containing xv and vy, and $|S'| \leq r$. We only consider $|S'| = r$, since the minimum edge cut of size around v is r. Let K_1 be a component of $G - S'$ and $v \in V(K_1)$. Moreover, let $V(K_1) = T_1$. By Lemma 3.1.5, $d(S_1 \cap T_1) + d(S_1 \cup T_1) \leq d(S_1) + d(T_1)$. Since $d(S_1) = |S| = r$, $d(T_1) = |S'| = r$ and $d(S_1 \cup T_1) \geq r$, $d(S_1 \cap T_1) \leq r$. Therefore, $S_1 \cap T_1 \subsetneq S_1$ since $u \in S_1 \backslash T_1$. This contradicts to the fact that S_1 has the smallest order. ($[T_1, \overline{T_1}]$ is a smaller edge cut.) Hence S' does not exist. Now, by letting $u = x$ and $w = y$, we conclude the proof.

Case 3. $|S| = r - 1$. Again, let

$$\begin{aligned} T &= \{u \in S_1 \backslash \{v\} : u \text{ is incident to } e \in S\} \text{ and} \\ U &= \{e \in S : v \text{ is incident to } e \in S\}. \end{aligned}$$

Since $deg(v) > r$ and $|T| + |U| \leq |S| = r - 1$, there is a vertex $x \in (N(v) \cap S_1) \backslash T$. Select $y \in N(v) \cap S_2$.

Claim 2. Let S' be an edge cut containing xv and vy, then $|S'| \geq r$.

Suppose not. Let $|S'| \leq r-1$, then $|S'| = r-1$. Let K_1 be a component of $G-S'$ and $v \in V(K_1) = T_1$. By Lemma 3.1.5, $d(S_1 \cap T_1) + d(S_1 \cup T_1) \leq d(S_1) + d(T_1)$,and $d(S_1) = |S| = r-1$, $d(T_1) = |S'| = r-1$, $d(S_1 \cup T_1) \geq r-1$, then $d(S_1 \cap T_1) \leq r-1$. Therefore, $S_1 \cap T_1 \subsetneq S_1$ since $x \in S_1 \backslash T_1$. This contradicts that S_1 has the smallest order again. Hence S' does not exist.

Following **Claim 2.**, it suffices to consider the case that $|S'| \geq r$.

Case 3.1. $|S'| > r$. Let $x = u$ and $y = w$, then the lemma hold.

Case 3.2. $|S'| = r$. Among all the r-edge-cuts containing xv, vy, let S'' be a minimum r-edge-cut so that the resulting disconnected graph $G - S''$ has a component K_1 with the smallest order, where $v \in V(K_1) = T_1$. By Lemma 3.1.5, since $d(S_1 \cap T_1) + d(S_1 \cup T_1) \leq d(S_1) + d(T_1)$, $d(S_1) = |S| = r-1$, $d(T_1) = |S'| = r$, and $d(S_1 \cup T_1) \geq r-1$, $d(S_1 \cap T_1) \leq r$. Since $x, y \notin S_1 \cap T_1$. Therefore, $S_1 \cap T_1 = T_1$ which implies $T_1 \subseteq S_1$. Let

$$\begin{aligned} V &= \{u \in T_1 \backslash \{v\} : u \text{ is incident to } e \in S''\} \text{ and} \\ W &= \{e \in S'' : v \text{ is incident to } e \in S''\}. \end{aligned}$$

Since $deg(v) > r$ and $|V| + |W| \leq |S''| = r$, there is a vertex $a \in (N(v) \cap T_1) \backslash V$.

Claim 3. va and vx do not belong to any edge cut of size not greater than r.

Suppose, to the contrary, the claim is not true. Let S''' be an edge cut containing va and vx such that $|S'''| \leq r$. Let A_1 be a component of $G - S'''$ and $v \in A_1$.

Case 1. $|S'''| = r-1$. If $|S'''| = r-1$, then $y \in A_1$. For otherwise, vx and vy are in $|S'''|$ which implies $|S'''| \geq r$, a contradiction. This contradicts to $|S'''| = r-1$. Since $y \in A_1$, $d(A_1 \cup T_1) \geq r-1$. By Lemma 3.1.5, since $d(A_1 \cap T_1) + d(A_1 \cup T_1) \leq d(A_1) + d(T_1)$, $d(T_1) = |S''| = r$, $d(A_1) = |S'''| = r-1$, and $d(A_1 \cup T_1) \geq r-1$, $d(A_1 \cap T_1) \leq r$. Furthermore $x, y \notin A_1 \cap T_1$ therefore $A_1 \cap T_1 \subsetneq T_1$ since $a \in T_1 \backslash A_1$. This contradicts to that T_1 has the smallest order. Hence S''' does not exist.

Case 2. $|S'''| = r$. By Lemma 3.1.5, since $d(A_1 \cap T_1) + d(A_1 \cup T_1) \leq d(A_1) + d(T_1)$, $d(T_1) = |S''| = r$, $d(A_1) = |S'''| = r$ and $d(A_1 \cup T_1) \geq r$, $d(A_1 \cap T_1) \leq r$. By the fact that $x, y \notin A_1 \cap T_1$, this implies that $A_1 \cap T_1 \subsetneq T_1$ since $a \in T_1 \backslash A_1$. This contradicts to that T_1 has smallest order. Hence S''' does not exist either.

Claim 3.1. $d(A_1 \cup T_1) \geq r$.

Since $x \notin A_1 \cup T_1$, we only show that vx is not belong to any edge cut of size $r-1$. Suppose, to the contrary, that the claim is not true. Let E be an edge cut contain vx, and $|E| = r-1$. Let K be a component of $G - E$ and $v \in V(K) = B$. Since vx, vy are not belong to any edge cut of size $r-1$ at the same time, then $y \in B$. By Lemma 3.1.5, since $d(S_1 \cap B) + d(S_1 \cup B) \leq d(S_1) + d(B)$, $d(S_1) = |S| = r-1$, $d(B) = |E| = r-1$, and $d(S_1 \cup B) \geq r-1$, $d(S_1 \cap T_1) \leq r-1$. Therefore, $S_1 \cap B \subsetneq S_1$ since $x \in S_1 \backslash B$. This contradicts to the fact that S_1 has the smallest order. Hence E does not exist.

In **Case 3.2.**, let $a = u$ and $x = w$, we yield an $(r-1)$-edge-connected graph by $G - uv - vw + uw$.

Combining the above three cases, we complete the proof. ∎

Now, instead of proving Theorem 3.1.3, we prove Theorem 3.1.4. Before going to the proof we need the following fact and a couple of lemmas.

Fact 1. Let G be a $(k-1)$-edge-connected graph with $\delta(G) = k$. If G has a vertex y of degree greater than $k+1$, then (by Lemma 3.1.6) there are two edges xy and yz such that deleting xy, yz and joining x and z by a new edge yield an $(k-1)$-edge-connected graph. Let G' be the new graph and assign to the new edge the weight $w(xy) + w(yz)$ so that $w(G') = w(G)$. Then, that an even subgraph of G' can be extended to an even subgraph of G with the same weight and a 3-flow in G' can be extended to a 3-flow in G with the same support.

Lemma 3.1.7. *If Tutte's 3-Flow Conjecture is true, then every weighted $(k-1)$-edge-connected graph G (k is odd) has an even subgraph F and a 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $w(F) \geq \frac{k-1}{k} w(G)$.*

Proof. We use induction on $|E(G)|$. Basic step: Consider G is $(k-1)$-regular graph and $(k-1)$-edge-connected. Since k is odd, G is an even graph then G satisfies the properties. Induction step: We assume that the claim holds for $|E(G)| \leq r$. Consider $|E(G)| = r+1$. Let G have a $(k-1)$-edge-cut $\{e_1, e_2, \ldots, e_{k-1}\}$, G' be the graph obtained by contracting e_1 and $\hat{e_1}$ be the contracted vertex. Reassign to e_i a new weigth $w(e_1)+w(e_i)$ so that $w(G')=w(G)$, where e_i is the edge close to e_1 in even subgraph of G'. Consider an even subgraph F' of G' and a 3-flow ϕ' in G'. If $\hat{e}_1 \notin F'$, then F' gives an even subgraph of G with the same weight set of edges, therefore $w_G(F') = w_{G'}(F') \geq \frac{k-1}{k}w(G') = \frac{k-1}{k}w(G)$; if $\hat{e}_1 \in F'$, then $F' \cup \{e_1\}$ is an even subgraph of G with the same weight as F', therefore, $w_G(F' \cup \{e_1\}) = w_{G'}(F') \geq \frac{k-1}{k}w(G') = \frac{k-1}{k}w(G)$. Moreover, in either case the 3-flow ϕ' can be extended to a 3-flow ϕ in G in which $\phi(e) = \phi'(e)$ for all $e \in E(G)\backslash\{e_1\}$ and $\phi(e_1) = \phi'(e_i)$ or $-\phi'(e_i)$, according to the orientation of e_i. Then G satisfies the properties. ∎

If G has a vertex y of degree greater than $k+1$, then by **Fact 1.** G satisfies the properties of Theorem 3.1.4. If G have a $(k-1)$-edge-cut, then by Lemma 3.1.7 G satisfies the properties of Theorem 3.1.4. Therefore, if G is a counterexample to the statement of Theorem 3.1.4 with a minimum number of edges, then G satisfies the following lemma.

Lemma 3.1.8. *If G is a counterexample to the statement of Theorem 3.1.4 with a minimum number of edges, then G is simple, k-regular and k-edge-connected.*

Proof of Theorem 3.1.4. Suppose, to the contrary, that the theorem is not true. Let G be a minimum counterexample. Then, by Lemma 3.1.8, G is a simple, k-regular and k-edge-connected. By Theorem 1.2.17, G contains an even subgraph F with $w(F) \geq \frac{k-1}{k}w(G)$ such that the graph G' obtained by contracting each component of F is 4-edge-connected. If Tutte's 3-flow conjecture is true, then G' has a nowhere-zero 3-flow ϕ with $E(G') \subseteq S(\phi)$. But $E(G') = E(G)\backslash E(F)$, and so F and ϕ have the required properties. This contraction proves Theorem 3.1.4. ∎

Now we return to unweighted graphs. Before proving our main result, we need the following lemma.

Lemma 3.1.9. *Let g be a 6-flow in G. Then there is a 6-flow ϕ in G such that $S(\phi) = S(g)$ and*

$$|E_{\pm 1}(\phi)| \geq \frac{5}{9}|E_{odd}(g)|.$$

Proof. Since $E_{odd}(g)$ is an even subgraph of G, we may define a 2-flow f_0 in G with $S(f_0) = E_{odd}(g)$. Applying Lemma 1.2.5 to the flows $g + 2f_0$ and $g - 2f_0$ yields two 6-flows in G, say f_1 and f_2, respectively, in which for every $e \in E(G)$

$$f_1(e) \equiv (g(e) + 2f_0(e)) \pmod 6 \text{ and } f_2(e) \equiv (g(e) - 2f_0(e)) \pmod 6.$$

It is not difficult to see that

$$E_{\pm 3}(f_1) \cup E_{\pm 3}(f_2) = E_{\pm 1}(g) \cup E_{\pm 5}(g).$$

Adding $E_{\pm 3}(g)$ to both sides yields

$$E_{\pm 3}(g) \cup E_{\pm 3}(f_1) \cup E_{\pm 3}(f_2) = E_{odd}(g).$$

Let $f \in \{g, f_1, f_2\}$ with $|E_{\pm 3}(f)|$ smallest. Then

$$|E_{\pm 3}(f)| \leq \frac{1}{3}|E_{odd}(g)|.$$

Since f is a 6-flow in G, by Lemma 1.2.11 with $k = 6$, there is a 6-flow ϕ in G such that $S(\phi) = S(f)(= S(g))$ and

$$\begin{aligned} |E_{\pm 1}(\phi)| &\geq \frac{5}{6}(|E_{\pm 1}(f)| + |E_{\pm 5}(f)|) = \frac{5}{6}(|E_{odd}(f)| - |E_{\pm 3}(f)|) \\ &\geq \frac{5}{6}(|E_{odd}(f)| - \frac{1}{3}|E_{odd}(g)|). \end{aligned}$$

Since $E_{odd}(f) = E_{odd}(g)$,

$$|E_{\pm 1}(\phi)| \geq \frac{5}{6}(\frac{2}{3}|E_{odd}(g)|) = \frac{5}{9}|E_{odd}(g)|.$$

This completes the proof. ∎

Theorem 3.1.10. *If a $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is odd) has a nowhere-zero 6-flow ϕ such that $|E_{odd}(\phi)| \geq \frac{k-1}{k}|E(G)|$, then G has a cycle cover in which the size of the cycle cover is at most $\frac{13k+5}{9k}|E(G)|$.*

Proof. Let g be a nowhere-zero 6-flow in G with $|E_{odd}(g)| \geq \frac{k-1}{k}|E(G)|$. It follows from Lemma 3.1.9 that G has a nowhere-zero 6-flow ϕ such that

$$|E_{\pm 1}(\phi)| \geq \frac{5}{9}|E_{odd}(g)| \geq \frac{5(k-1)}{9k}|E(G)|.$$

Applying Lemma 1.2.9 to ϕ with $r = 2$, we have a sub-4-flow of ϕ, say ϕ_1, such that $\phi_1(e) = \lfloor \frac{\phi(e)}{2} \rfloor$ or $\lceil \frac{\phi(e)}{2} \rceil$, for every $e \in E(G)$. Set $\phi_2 = \phi - \phi_1$. Then ϕ_2 is also a sub-4-flow of ϕ. Let $A = \{e \in E(G) : \phi_1(e) = 0\}$ and $B = \{e \in E(G) : \phi_2(e) = 0\}$. Then $A \cap B = \emptyset$ and $A \cup B = E_{\pm 1}(\phi)$. Moreover, $|\phi_1(e)| = 1$ if $e \in B$ and $|\phi_2(e)| = 1$ if $e \in A$. Applying Lemma 1.2.13 to ϕ_1, we have two even subgraphs X_1 and X_2, such that $X_1 \cup X_2 = S(\phi_1)$ and $X_1 \cap X_2 = E_{\pm 2}(\phi_1)$. Set $X_3 = X_1 \oplus X_2$. Then $B \subseteq X_3$ and $\{X_1, X_2, X_3\}$ covers each edge of $S(\phi_1) = E(G) - A$ exactly twice. Similarly, ϕ_2 yields three even subgraphs, say Y_1, Y_2, and Y_3, such that $A \subseteq Y_3$ and $\{Y_1, Y_2, Y_3\}$ covers each edge of $S(\phi_2) = E(G) - B$ exactly twice. Let $C_1 = \{X_1, X_2, Y_3\}$ and $C_2 = \{Y_1, Y_2, X_3\}$. Then both C_1 and C_2 are cycle covers of G. Since

$$\begin{aligned}\sum_{i=1}^{3}|X_i| + \sum_{i=1}^{3}|Y_i| &= 2|S(\phi_1)| + 2|S(\phi_2)| = 4|E(G)| - 2|A \cup B| \\ &= 4|E(G)| - 2|E_{\pm 1}(\phi)|,\end{aligned}$$

either C_1 or C_2 is of size at most

$$2|E(G)| - |E_{\pm 1}(\phi)| \leq 2|E(G)| - \frac{5(k-1)}{9k}|E(G)| = \frac{13k+5}{9k}|E(G)|.$$

∎

Next, we consider the case when k is even.

Theorem 3.1.11. *If Tutte's 3-Flow Conjecture is true, then every graph which is homeomorphic to a $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is even) has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $|E(F)| \geq \frac{k-2}{k-1}|E(G)|$.*

Similarly, we see that Theorem 3.1.11 is equivalent to

Theorem 3.1.12. *If Tutte's 3-Flow Conjecture is true, then every weighted $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is even) has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $w(F) \geq \frac{k-2}{k-1}w(G)$.*

Before the proof of Theorem 3.1.12 we need the following Lemma.

Lemma 3.1.13. *If Tutte's 3-Flow Conjecture is true, then every weighted $(k-1)$-edge-connected graph G with $(k-1)$-edge-cut (k is even) has an even subgraph F and 3-flow ϕ such that $E(G)\backslash E(F) \subseteq S(\phi)$ and $w(F) \geq \frac{k-2}{k-1}w(G)$.*

Proof. By induction on $|E(G)|$. Basic step: Consider G is $(k-1)$-regular graph and $(k-1)$-edge-connected. By Theorem 1.2.17, G contains an even subgraph F with $w(F) \geq \frac{k-2}{k-1}w(G)$ such that the graph G' obtained by contracting each component of F is 4-edge-connected. If Tutte's 3-flow conjecture is true, then G' has a nowhere-zero 3-flow ϕ with $E(G') \subseteq S(\phi)$. $E(G') = E(G)\backslash E(F)$, and so F and ϕ have the required properties. Induction step: We assume that the claim holds for $|E(G)| \leq r$. Consider $|E(G)| = r+1$. Let G have a $(k-1)$-edge-cut $\{e_1, e_2, \ldots, e_{k-1}\}$, G' be the graph obtained by contracting e_1 and $\hat{e_1}$ be the contracted vertex. Reassign to e_i a new weigth $w(e_1) + w(e_i)$ so that $w(G') = w(G)$, where e_i is the edge close to e_1 in even subgraph of G'. Consider an even subgraph F' of G' and a 3-flow ϕ' in G'. If $\hat{e_1} \notin F'$, then F' gives an even subgraph of G with the same weight set of edges, therefore $w_G(F') = w_{G'}(F') \geq \frac{k-2}{k-1}w(G') = \frac{k-2}{k-1}w(G)$; if $\hat{e_1} \in F'$, then $F' \cup \{e_1\}$ is an even subgraph of G with the same weight as F', therefore, $w_G(F' \cup \{e_1\}) = w_{G'}(F') \geq$

$\frac{k-2}{k-1}w(G') = \frac{k-2}{k-1}w(G)$. Moreover, in either case the 3-flow ϕ' can be extended to a 3-flow ϕ in G in which $\phi(e) = \phi'(e)$ for all $e \in E(G)\backslash\{e_1\}$ and $\phi(e_1) = \phi'(e_i)$ or $-\phi'(e_i)$, according to the orientation of e_i. Therefore, G satisfies the properties. ∎

Proof of Theorem 3.1.12. Suppose, to the contrary, that the theorem is not true. Let G be a minimum counterexample. Then, by Lemma 3.1.8, G is a simple, k-regular and k-edge-connected. Since k is even, G is an even graph then G satisfies the properties. This contraction proves Theorem 3.1.12. ∎

Theorem 3.1.14. *If a $(k-1)$-edge-connected graph G with $\delta(G) = k$ (k is even) has a nowhere-zero 6-flow ϕ such that $|E_{odd}(\phi)| \geq \frac{k-2}{k-1}|E(G)|$, then G has a cycle cover in which the size of the cycle cover is at most $\frac{13k-8}{9(k-1)}|E(G)|$.*

Proof. Let g be a nowhere-zero 6-flow in G with $|E_{odd}(g)| \geq \frac{k-1}{k}|E(G)|$. It follows from Lemma 3.1.9 that G has a nowhere-zero 6-flow ϕ such that

$$|E_{\pm 1}(\phi)| \geq \frac{5}{9}|E_{odd}(g)| \geq \frac{5(k-2)}{9(k-1)}|E(G)|.$$

Applying Lemma 1.2.9 to ϕ with $r = 2$, we have a sub-4-flow of ϕ, say ϕ_1, such that $\phi_1(e) = \lfloor \frac{\phi(e)}{2} \rfloor$ or $\lceil \frac{\phi(e)}{2} \rceil$, for every $e \in E(G)$. Set $\phi_2 = \phi - \phi_1$. Then ϕ_2 is also a sub-4-flow of ϕ. Let $A = \{e \in E(G) : \phi_1(e) = 0\}$ and $B = \{e \in E(G) : \phi_2(e) = 0\}$. Then $A \cap B = \emptyset$ and $A \cup B = E_{\pm 1}(\phi)$. Moreover, $|\phi_1(e)| = 1$ if $e \in B$ and $|\phi_2(e)| = 1$ if $e \in A$. Applying Lemma 1.2.13 to ϕ_1, we have two even subgraphs X_1 and X_2, such that $X_1 \cup X_2 = S(\phi_1)$ and $X_1 \cap X_2 = E_{\pm 2}(\phi_1)$. Set $X_3 = X_1 \oplus X_2$. Then $B \subseteq X_3$ and $\{X_1, X_2, X_3\}$ covers each edge of $S(\phi_1) = E(G) - A$ exactly twice. Similarly, ϕ_2 yields three even subgraphs, say Y_1, Y_2, and Y_3, such that $A \subseteq Y_3$ and $\{Y_1, Y_2, Y_3\}$ covers each edge of $S(\phi_2) = E(G) - B$ exactly twice. Let $C_1 = \{X_1, X_2, Y_3\}$ and

$C_2 = \{Y_1, Y_2, X_3\}$. Then both C_1 and C_2 are cycle covers of G. Since

$$\begin{aligned} \sum_{i=1}^{3} |X_i| + \sum_{i=1}^{3} |Y_i| &= 2|S(\phi_1)| + 2|S(\phi_2)| = 4|E(G)| - 2|A \cup B| \\ &= 4|E(G)| - 2|E_{\pm 1}(\phi)|, \end{aligned}$$

either C_1 or C_2 is of size at most

$$2|E(G)| - |E_{\pm 1}(\phi)| \leq 2|E(G)| - \frac{5(k-2)}{9(k-1)}|E(G)| = \frac{13k-8}{9(k-1)}|E(G)|.$$

∎

3.2 Conclusion

As can be seen in the main results in Chapter 3, if $k = 3, 4$, then our result is exactly the same as obtained by Fan 2.2.14 [8]. Therefore, with this technique, we are not able to get a better upper bound for the total lengths of cycles used in covering the graph G (bridgeless). We do believe that this bound $\frac{44}{27}|E(G)|$ can be smaller, but not able to prove it at this moment. Since our result is obtained also under the assumption that Tutte's 3-Flow Conjecture is true, to prove this long-stand conjecture deserves to be considered as the first major goal of our future study.

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