國立交通大學應用數學系

博 士 論 文

多維度有限型移位的ζ–函數

Zeta Functions for Multi-dimensional Shifts of Finite Type

研究生：胡文貴

指導教授：林松山教授

中華民國九十八年五月

多維度有限型移位的ζ–函數

Zeta Functions for Multi-dimensional Shifts of Finite Type

研究生：胡文貴　　Student：Wen-Guei Hu

指導教授：林松山　　Advisor：Song-Sun Lin

國 立 交 通 大 學

應 用 數 學 系

博 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Applied Mathematics

May 2009

Hsinchu, Taiwan, Republic of China

中華民國九十八年五月

多維度有限型移位的ζ-函數

研 究 生：胡文貴　　　　　　　　　　　　指導教授：林松山 教授

國 立 交 通 大 學

應 用 數 學 系

摘要

本論文主要研究二維以上有限型移位的ζ-函數。關於$\mathbb{Z}^d$作用ϕ的ζ-函數$\zeta^0(s)$是由林德推廣阿廷-馬蘇爾ζ-函數所得到。首先，研究二維的情況。定義跡算子T_n爲在 x 方向 n 週期且高度 2 之花樣的轉移矩陣，此T_n具有旋轉對稱性。根據T_n的旋轉對稱性，引進約化跡算子τ_n，進一步推得ζ-函數$\zeta=\prod_{n=1}^{\infty}\left(\det\left(I-s^n\tau_n\right)\right)^{-1}$是一個多項式的無窮乘積的倒數。此外，對於任何由$GL_2(\mathbb{Z})$中的單位模變換決定的傾斜坐標皆可得到相同結果。所以有一族ζ-函數都是解析函數$\zeta^0(s)$的半純擴張，在此我們也研究自然邊界問題。這些ζ-函數在原點的泰勒級數展開式皆相同，並且其係數皆爲整數。因此，

可以得到一族在數論上有趣的恆等式。此方法在三維以上的情況也適用，而且可應用到有限範圍交互作用之伊辛模型的熱力學ζ-函數。

Zeta Functions for Multi-dimensional Shifts of Finite Type

Student：Wen-Guei Hu　　　　　　　　Advisor：Prof. Song-Sun Lin

Department of Applied Mathematics
National Chiao Tung University

Abstract

This dissertation investigates zeta functions for d-dimensional shifts of finite type, $d \geq 2$. A d-dimensional zeta function $\zeta^0(s)$ which generalizes the Artin-Mazur zeta function was given by Lind for $\mathbb{Z}^d$ action ϕ. First, the two-dimensional case is studied. The trace operator T_n which is the transition matrix for x-periodic patterns of period n with height 2 is rotationally symmetric. The rotational symmetry of T_n induces the reduced trace operator τ_n. The zeta function $\zeta = \prod_{n=1}^{\infty}\left(\det\left(I - s^n \tau_n\right)\right)^{-1}$ is now a reciprocal of an infinite product of polynomials. The results hold for any inclined coordinates, determined by unimodular transformation in $GL_2\left(\mathbb{Z}\right)$. Therefore, there exists a family of zeta functions that are meromorphic extensions of the same

analytic function $\zeta^0(s)$. The natural boundary of zeta function is studied. The Taylor series expansions at the origin for these zeta functions are equal with integer coefficients, yielding a family of identities which are of interest in number theory. The methods used herein are also valid for d-dimensional cases, $d \geq 3$, and can be applied to thermodynamic zeta functions for the Ising model with finite range interactions.

誌　　謝

在交大總共有六年的時間，碩士班兩年，博士班四年，當初剛來到交大的時候，沒想到自己會再繼續攻讀博士班，如今能順利拿到了博士學位，內心除了高興之外，同時也非常感謝許多這一路來幫助我、陪伴我的人。

首先最感謝的就是我的指導教授林松山老師，在研究方面，老師從如何讀書和讀論文慢慢一步步地教導我，接著老師帶著我做研究，讓我明白做研究的態度和方法，當我迷網不知所措的時候，總是會適時地指引我方向，鼓勵我繼續往前邁進，在老師身上，我看到一個學者在研究上的專業、熱情與堅持，是我將來學習的一個好榜樣;除了研究之外，在做人處世方面，老師也教導我如何應對進退。老師給我的幫助不是簡短的三言兩語所能言盡的，真心感謝老師這六年來的教導。

在交大應用數學系這一段時間中，在學業的學習上，系上的老師給了我很多的幫助，在動態系統與微分方程的學習上，林松山、莊重、石志文、李明佳、林琦焜、李榮耀老師們教導我很多專業學科上的知識，他們的認真熱心教導，讓我能夠正確有效地學習，以及王夏聲、許元春老師的實變課程，楊一帆老師的數論相關課程也讓我受益良多。除了課業之外，系上所有的老師，也曾給我很多協助，我才能順利地完成學業，謝謝您們。

亦師亦友般的班榮超學長和林吟衡學姐在我研究所這段期間，在研究上給我非常多的幫助，教導我許多專業知識，樂意與我討論，讓我從中得到很多寶貴的經驗，在平時，對於學弟妹們都非常照顧，那些情景如今依然歷歷在目；已經畢業的許正雄、楊智烜、楊定揮學長，雖然相處的時間不多，你們的關懷與照顧，讓我感覺到溫暖；此外，張志鴻、胡忠澤、楊其儒、陳耀漢學長在學業與生活上也給我很多的支持與協助；同研究室的昱豪學長、恭儉與明耀和隔壁研究室的明杰，與你們一起相處的時間非常輕鬆愉快，讓我度過愉快的四年；碩士班的學弟妹，慧萍、園芳、怡菁、倖綺、玉雯、佳玲、鴻勳、德展、瑛峰和晉育，有了你們我的生活更加多采多姿。

從小到大，在各方面我的父母總是盡全力支持我，家是我最好的避風港，不管發生什麼事，都會永遠在那邊守護我，讓我能夠順利完成博士學位；還有我的哥哥，在我每次回家的時候，都會和我聊聊生活以及學業上的事情，也會鼓勵我繼續努力向前；再來要感謝我的女朋友園芳(小小)，不管是在我壓力很大或者是快樂的時候，都會在我身邊陪伴我，妳的陪伴就是我最好的動力來源；胡啵啵是我家一隻很可愛的巴戈狗，牠總是看起來呆呆可愛的樣子，每次看到牠都會暫時

忘卻煩惱。

碩士班的同學和以前的朋友勝凱、維哲、雅靜、莉君、伯群、佳慧、琬琪，雖然只能偶而一起聚個餐，不過和你們相處的時光很快樂。其他我沒能一一列舉到的學長姐、同學、朋友、學弟妹，能與你們認識真的是太好了。

最後，有大家一路上的陪伴幫助，我才能順利地博士班畢業，今後我會繼續努力勇敢向前邁進。由衷地感謝大家，謝謝你們。

胡文貴 2009\6\30　於交大

Contents

1 Introduction

Various zeta functions have been investigated in the fields of number theory, geometry, dynamical systems and statistical physics. This work studies the zeta functions in a manner that follows the work of Artin and Mazur [1], Bowen and Lanford [11], Ruelle [45] and Lind [36]. First, recall the zeta function that was defined by Artin and Mazur.

Let $\phi : \mathbb{X} \longrightarrow \mathbb{X}$ be a homeomorphism of a compact space and $\Gamma_n(\phi)$ denote the number of fixed points of ϕ^n. The zeta function $\zeta_\phi(s)$ for ϕ defined in [1] is

$$\zeta_\phi(s) = \exp\left(\sum_{n=1}^{\infty} \frac{\Gamma_n(\phi)}{n} s^n\right). \tag{1.1}$$

Later, Bowen and Lanford [11] demonstrated that if ϕ is a shift of finite type, then $\zeta_\phi(s)$ is a rational function. In the simplest case, when a shift is generated by a transition matrix A in $\mathbb{Z}$, (1.1) is computed explicitly as

$$\zeta_A(s) = \exp\left(\sum_{n=1}^{\infty} \frac{trA^n}{n} s^n\right) \tag{1.2}$$

$$= (\det(I - sA))^{-1}, \tag{1.3}$$

and then

$$\zeta_A(s) = \prod_{\lambda \in \Sigma(A)} (1 - \lambda s)^{-\chi(\lambda)}, \tag{1.4}$$

where $\chi(\lambda)$ is a non-negative integer that is the algebraic multiplicity of eigenvalue λ and $\Sigma(A)$ is the spectrum of A. $\zeta_A(s)$ is a rational function which involves only eigenvalues of A.

Lind [36] extended (1.1) to $\mathbb{Z}^d$-action as follows. For $\mathbb{Z}^d$-action, $d \geq 1$, let ϕ be an action of $\mathbb{Z}^d$ on $\mathbb{X}$. Denote the set of finite-index subgroups of $\mathbb{Z}^d$ by $\mathcal{L}_d$. The zeta function ζ_ϕ defined by Lind is

$$\zeta_\phi(s) = \exp\left(\sum_{L \in \mathcal{L}_d} \frac{\Gamma_L(\phi)}{[L]} s^{[L]}\right), \tag{1.5}$$

where $[L] = index[\mathbb{Z}^d/L]$ and $\Gamma_L(\phi)$ is the number of fixed points by $\phi^{\mathbf{n}}$ for all $\mathbf{n} \in L$. Lind [36] obtained some important results for ζ_ϕ, such as conjugacy invariant and product formulae, and computed ζ_ϕ explicitly for some interesting examples. Furthermore, he raised some fundamental problems for zeta functions, including the following two.

Problem 7.2. [36] For "finitely determined" $\mathbb{Z}^d$-actions ϕ such as shifts of finite type, is there a reasonable finite description of $\zeta_\phi(s)$?

Problem 7.5. [36] Compute explicitly the thermodynamic zeta function for the 2-dimensional Ising model, where α is the $\mathbb{Z}^2$ shift action on the space of configurations.

The present authors previously studied pattern generation problems in $\mathbb{Z}^d$, $d \geq 2$, and developed several approaches such as the use of higher order transition matrices and trace operators to compute spatial entropy [4; 6]. The work of Ruelle [45] and Lind [36] indicated that our methods could also be adopted to study zeta functions.

In this investigation, Problems 7.2 and 7.5 are answered when ϕ is a shift of finite type. More related results and questions are also addressed. The following paragraphs briefly introduce relevant results.

First, the two dimensional case is studied. Let $\mathbb{Z}_{m\times m}$ be the $m \times m$ square lattice in $\mathbb{Z}^2$ and $\mathcal{S}$ be the finite set of symbols (alphabets or colors). $\mathcal{S}^{\mathbb{Z}_{m\times m}}$ is the set of all local patterns (or configurations) on $\mathbb{Z}_{m\times m}$. A given subset $\mathcal{B} \subset \mathcal{S}^{\mathbb{Z}_{m\times m}}$ is called a basic set of admissible local patterns. $\Sigma(\mathcal{B})$ is the set of all global patterns defined on $\mathbb{Z}^2$ which can be generated by $\mathcal{B}$. For simplicity, only the results of $\mathbb{Z}_{2\times 2}$ with two symbols $\mathcal{S} = \{0, 1\}$ are presented here. Subsection 2.3 considers the general case.

As presented elsewhere [36], $\mathcal{L}_2$ can be parameterized in Hermite normal form [39]:

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 : \ n \geq 1,\ k \geq 1 \textit{ and } 0 \leq l \leq n-1 \right\}.$$

Given a basic set $\mathcal{B}$, denote by $P_\mathcal{B}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ the set of all $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and $\mathcal{B}$-admissible patterns and $\Gamma_\mathcal{B}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ is the number of $P_\mathcal{B}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

The zeta function, defined by (1.5), is denoted by

$$\zeta^0_\mathcal{B} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk} \Gamma_\mathcal{B}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) s^{nk} \right). \tag{1.6}$$

In [36], $\zeta^0_{\mathcal{B}}$ is shown analytically in $|s| < \exp(-g(\mathcal{B}))$, where

$$g(\mathcal{B}) \equiv \limsup_{[L]\to\infty} \frac{1}{[L]} \log \Gamma_{\mathcal{B}}(L). \tag{1.7}$$

In this work, the sum of n and k in (1.6) is treated separately as an iterated sum. Indeed, for any $n \geq 1$, define the n-th order zeta function $\zeta_n(s) \equiv \zeta_{\mathcal{B},n}(s)$ (in x-direction) as

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) s^{nk}\right); \tag{1.8}$$

the zeta function $\zeta(s) \equiv \zeta_{\mathcal{B}}(s)$ is given by

$$\zeta(s) = \prod_{n=1}^{\infty}\zeta_n(s). \tag{1.9}$$

The first observation of (1.8) is that, for $n \geq 1$ and $l \geq 1$, any $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic, where (n,l) is the greatest common divisor (GCD) of n and l. Therefore, $\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}$-periodicity of patterns must be investigated in details.

The trace operators $\mathbf{T}_n \equiv \mathbf{T}_n(\mathcal{B})$ that were introduced in [6] are useful in studying $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and the $\mathcal{B}$-admissible pattern, where $\mathbf{T}_n = [t_{n;i,j}]$ is a $2^n \times 2^n$ matrix with $t_{n;i,j} \in \{0,1\}$. $\mathbf{T}_n(\mathcal{B})$ represents the set of patterns that are $\mathcal{B}$-admissible and x-periodic of period n with height 2. The trace operator $\mathbf{T}_n$ can be used to construct (doubly) periodic $\mathcal{B}$-admissible patterns. Indeed, for $k \geq 1$ and $0 \leq l \leq n-1$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr(\mathbf{T}_n^k R_n^l), \tag{1.10}$$

where R_n is a $2^n \times 2^n$ rotational matrix defined by

$$\begin{cases} R_{n;i,2i-1} = 1 \text{ and } R_{n;2^{n-1}+i,2i} = 1 & \text{for } 1 \leq i \leq 2^{n-1}, \\ R_{n;i,j} = 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{R}_n = \sum_{l=0}^{n-1} R_n^l$; now based on (1.10), $\zeta_n(s)$ becomes

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr(\mathbf{T}_n^k\mathbf{R}_n)s^{nk}\right), \tag{1.11}$$

which is a generalization of (1.2).

To elucidate the method used to study (1.11), $\mathbf{T}_n$ is firstly assumed to be symmetric. Then $\mathbf{T}_n$ can be expressed in Jordan canonical form as

$$\mathbf{T}_n = \mathbf{UJU}^t \tag{1.12}$$

where the eigen-matrix $\mathbf{U} = (U_1, ..., U_N)$ is an $N \times N$ matrix which consists of linearly independent (column) eigenvectors U_j, $1 \leq j \leq N$ and $N \equiv 2^n$. Jordan matrix $\mathbf{J} = diag(\lambda_j)$ is a diagonal $N \times N$ matrix, which comprises eigenvalues λ_j, $1 \leq j \leq N$. Now,

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr(\mathbf{T}_n^k\mathbf{R}_n)s^{nk} \\ = &\frac{1}{n}tr(\mathbf{U}(\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{J}^k s^{nk})\mathbf{U}^t\mathbf{R}_n) \\ = &\sum_{j=1}^{N}\frac{1}{n}\left|\mathbf{R}_n \circ U_jU_j^t\right| \log(1-\lambda_j s^n)^{-1} \end{aligned} \tag{1.13}$$

can be proven, where $\circ$ is a Hadamard product: if $A = [a_{i,j}]_{M\times M}$ and $B = [b_{i,j}]_{M\times M}$, then $A \circ B = [a_{i,j}b_{i,j}]_{M\times M}$.

Evaluating the coefficients $|\mathbf{R}_n \circ U_jU_j^t|$ of $\log(1-\lambda_j s^n)^{-1}$ is important. Now, the R_n-symmetry of $\mathbf{T}_n$ is crucial. Indeed, let U be an eigenvector of $\mathbf{T}_n$ with eigenvalue λ, then R_n^lU is also eigenvector of $\mathbf{T}_n$ for all $0 \leq l \leq n-1$. Notably, $R_n^n = I_{2^n}$, where I_m is the $m \times m$ identity matrix.

U is called R_n-symmetric, if $R_n^lU = U$ for all $0 \leq l \leq n-1$. And U is called anti-symmetric if $\sum_{l=0}^{n-1} R_n^lU = 0$. Additionally, for any given eigenvalue λ, the associated eigenspace E_λ can be proven to be spanned by symmetric eigenvectors $\overline{U}_j$, $1 \leq j \leq p_\lambda$, and anti-symmetric eigenvectors U_j', $1 \leq j \leq q_\lambda$: $E_\lambda = \{\overline{U}_1, \cdots, \overline{U}_{p_\lambda}, U_1', \cdots, U_{q_\lambda}'\}$, where $p_\lambda + q_\lambda = dim(E_\lambda)$ and p_λ or q_λ can be zero.

Therefore, for each eigenvalue λ of $\mathbf{T}_n$,

$$\chi(\lambda) \equiv \frac{1}{n}\sum_{\lambda_j=\lambda} |\mathbf{R}_n \circ U_j U_j^t| = p_\lambda \tag{1.14}$$

is the number of linearly independent symmetric eigenvectors of $\mathbf{T}_n$ with respect to λ, a non-negative integer. Hence, choosing eigen-matrix $\mathbf{U}$ in (1.12), which consists of symmetric and anti-symmetric eigenvectors, yields

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_n)} (1-\lambda s^n)^{-\chi(\lambda)} \tag{1.15}$$

as a rational function, as in (1.4).

To further study $\chi(\lambda)$ in (1.14), the reduced trace operator τ_n is introduced as follows. From the rotational matrix R_n, for $1 \le i \le 2^n$, the equivalent class $C_n(i)$ of i is defined as $C_n(i) = \left\{ j \middle| \left(R_n^l\right)_{i,j} = 1 \text{ for some } 1 \le l \le n \right\}$. The index set $\mathcal{I}_n$ of n is defined by $\mathcal{I}_n = \left\{ i \middle| 1 \le i \le 2^n, i \le j \text{ for all } j \in C_n(i) \right\}$ and χ_n is the cardinal number of $\mathcal{I}_n$. Indeed, χ_n is the number of necklaces that can be made from n beads of two colors when the necklaces can be rotated but not turned over. Furthermore,

$$\chi_n = \frac{1}{n}\sum_{d|n} \phi(d) 2^{n/d}, \tag{1.16}$$

where $\phi(d)$ is the Euler totient function.

Then, the reduced trace operator $\tau_n = [\tau_{n;i,j}]$ of $\mathbf{T}_n$ is a $\chi_n \times \chi_n$ matrix that is defined by

$$\tau_{n;i,j} = \sum_{k\in C_n(j)} t_{n;i,k} \tag{1.17}$$

for each $i,j \in \mathcal{I}_n$. $\lambda \in \Sigma(\mathbf{T}_n)$ with $\chi(\lambda) \ge 1$ can be verified if and only if $\lambda \in \Sigma(\tau_n)$. Moreover, $\chi(\lambda)$ is the algebraic multiplicity of τ_n with eigenvalue λ. Therefore,

$$\zeta_n(s) = \left(\det\left(I - s^n \tau_n\right)\right)^{-1}, \tag{1.18}$$

a similar formula as in (1.3). Hence, the zeta function $\zeta(s)$ is obtained as

$$\zeta(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n \tau_n\right)\right)^{-1}, \tag{1.19}$$

which is an infinite product of rational functions. Equation (1.19) generalizes (1.3) and is a solution to Lind's Problem 7.2. Furthermore, according to (1.19), the coefficients of Taylor series expansion for $\zeta(s)$ at $s = 0$ are integers, as obtained by Lind [36].

As presented elsewhere [6], an another trace operator $\widehat{\mathbf{T}}_n$ is $\mathcal{B}$-admissible and y-periodic of period n with width 2 along the x-axis. Indeed, $\mathcal{L}_2$ can be parameterized as another Hermite normal form, and n-th order zeta function $\widehat{\zeta}_n(s)$ is defined by

$$\widehat{\zeta}_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right) s^{nk}\right), \tag{1.20}$$

and the zeta function $\widehat{\zeta}(s)$ is defined by

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty}\widehat{\zeta}_n(s). \tag{1.21}$$

Therefore,

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty}\prod_{\lambda\in\Sigma(\widehat{\mathbf{T}}_n)}(1-\lambda s^n)^{-\widehat{\chi}(\lambda)} \tag{1.22}$$

$$= \prod_{n=1}^{\infty}\left(\det\left(I - s^n\widehat{\tau}_n\right)\right)^{-1}. \tag{1.23}$$

The construction of the zeta functions ζ and $\widehat{\zeta}$ in rectangular coordinates can be extended to an inclined coordinates system. Indeed, let the unimodular transformation γ be an element of the unimodular group $GL_2(\mathbb{Z})$: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c and d are integers and $ad - bc = \pm 1$. The lattice L_γ is defined by

$$L_\gamma \equiv \begin{pmatrix} n & l \\ 0 & k \end{pmatrix}_\gamma \mathbb{Z}^2 = \begin{pmatrix} na & la+kc \\ nb & lb+kd \end{pmatrix}\mathbb{Z}^2. \tag{1.24}$$

The n-th order zeta function of $\zeta^0_{\mathcal{B}}(s)$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) s^{nk}\right), \tag{1.25}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is given by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{n=1}^{\infty}\zeta_{\mathcal{B};\gamma;n}(s). \tag{1.26}$$

The n-th order rotational matrix $R_{\gamma;n}$, trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ and reduced trace operator $\tau_{\gamma;n}(\mathcal{B})$ can also be introduced and

$$\zeta_{\mathcal{B};\gamma;n}(s) = \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}. \tag{1.27}$$

Therefore, the zeta function $\zeta_{\mathcal{B};\gamma}$ is given by

$$\zeta_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty}(\det(I - s^n\tau_{\gamma;n}))^{-1}. \tag{1.28}$$

Since the iterated sum in (1.25) and (1.26) is a rearrangement of $\zeta^0_{\mathcal{B}}(s)$,

$$\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s) \tag{1.29}$$

for $|s| < \exp(-g(\mathcal{B}))$. The identity (1.29) yields a family of identities when $\zeta_{\mathcal{B};\gamma}$ is expressed as a Taylor series expansion at the origin $s = 0$. The further applications of these identities in number theory will appear elsewhere.

Note that, one may consider the zeta functions $\zeta^r_{\mathcal{B}}$, which only involves $\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}$-periodic patterns, defined by

$$\zeta^r_{\mathcal{B}} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}\right)s^{nk}\right). \tag{1.30}$$

However, in general, for $n \geq 1$, $\chi(\lambda)$ is not an integer in (1.15) for $\zeta^r_{\mathcal{B};n}$ and $\zeta^r_{\mathcal{B};n}$ is not a rational function. Therefore, $\zeta^r_{\mathcal{B}}$ is not an infinite product of rational functions and may lose some important properties such as $GL_2(\mathbb{Z})$ invariant.

The thermodynamic zeta function raised by Ruelle [45] with weight function $\theta : \mathbb{X} \to (0, \infty)$ was defined by Lind [36] as

$$\zeta^0_{\alpha,\theta}(s) = \exp\left(\sum_{L\in\mathcal{L}_d}\left\{\sum_{x\in fix_L(\alpha)}\prod_{\mathbf{k}\in\mathbb{Z}^d/L}\theta\left(\alpha^{\mathbf{k}}x\right)\right\}\frac{s^{[L]}}{[L]}\right), \tag{1.31}$$

where $fix_L(\alpha)$ is the set of points fixed by $\alpha^{\mathbf{n}}$ for all $\mathbf{n} \in L$.

For the Ising model, where α is a shift of finite type given by $\mathcal{B}$ and the weight function θ is a potential with finite range, the previous arguments apply. Indeed, the trace operator $\mathbf{T}_{Ising;n}(\mathcal{B})$ and reduced trace operator $\tau_{Ising;n}(\mathcal{B})$ can be defined, and the zeta function is

$$\zeta_{Ising;\mathcal{B}}(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n \tau_{Ising;n}\right)\right)^{-1}. \tag{1.32}$$

Equation (1.32) is a solution of Lind's Problem 7.5. Furthermore, the relations of critical phenomenon in phase transition with the zeta functions will be investigated later.

Notably, the methods herein also apply to sofic shifts. The results will appear elsewhere.

It is clear that in many situations the three-dimensional problems are more related to our real

world phenomena. Now, the zeta functions of d-dimensional shifts of finite type are studied for

$d \geq 3$, and the previous results of $\mathbb{Z}^2$ are extended. For simplicity, only the zeta functions for three-dimensional shifts of finite type are introduced and the general case is studied in Subsection 3.2.

Let $\mathbb{Z}_{m\times m\times m}$ be the $m \times m \times m$ cubic lattice in $\mathbb{Z}^3$ and $\mathcal{S}$ be the finite set of symbols (alphabets or colors). $\mathcal{S}^{\mathbb{Z}_{m\times m\times m}}$ is the set of all local patterns on $\mathbb{Z}_{m\times m\times m}$. Denote $\mathcal{B} \subset \mathcal{S}^{\mathbb{Z}_{m\times m\times m}}$ a basic set of admissible local patterns and $\mathcal{P}(\mathcal{B})$ the set of all periodic patterns that are generated by $\mathcal{B}$ on $\mathbb{Z}^3$.

The Hermite normal form [39] can be used to parameterize $\mathcal{L}_3$ as

$$\mathcal{L}_3 = \left\{ \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3 : a_i \geq 1, 1 \leq i \leq 3, 0 \leq b_{ij} \leq a_i - 1, i+1 \leq j \leq 3 \right\}.$$

Given a basic set $\mathcal{B}$. Let $L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3 \in \mathcal{L}_3$, denote $\mathcal{P}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$ the set of all L-periodic patterns that are generated by $\mathcal{B}$ on $\mathbb{Z}^3$ and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$ the number of $\mathcal{P}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$. Then, the zeta function in (1.1) is

$$\zeta^0_{\mathcal{B}} = \exp\left(\sum_{i=1}^{3}\sum_{a_i=1}^{\infty}\sum_{j=i+1}^{3}\sum_{b_{ij}=0}^{a_i-1}\frac{1}{a_1a_2a_3}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)s^{a_1a_2a_3}\right). \tag{1.33}$$

Similar to (1.8) and (1.9), the $(a_1, a_2; b_{12})$-th zeta function is defined by

$$\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1a_2}\sum_{a_3=1}^{\infty}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\frac{1}{a_3}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)s^{a_1a_2a_3}\right) \tag{1.34}$$

and the zeta function $\zeta_{\mathcal{B}}(s)$ is given by

$$\zeta_{\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s). \tag{1.35}$$

The trace operator $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$ and rotational matrices $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ are introduced. After the rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ is demonstrated the reduced trace operator $\tau_{a_1,a_2;b_{12}}(\mathcal{B})$ can be defined. Finally, as in (1.18), $\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s)$ can be represented as a rational function:

$$\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s) = \left(\det\left(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}\right)\right)^{-1}. \tag{1.36}$$

Hence,

$$\zeta_{\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\left(\det\left(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}\right)\right)^{-1} \tag{1.37}$$

is a reciprocal of an infinite product of polynomials. Here, we show (1.36) by using a simpler and more straightforward method than that for (1.18). However, the proof of (1.18) is also valid for $d \geq 3$.

Additionally, for any $\gamma \in GL_3(\mathbb{Z})$, the zeta function can also be represented in γ-coordinates. Therefore, a family of zeta functions exists that have the same integer coefficients in their Taylor series expansions at $s = 0$.

As in (1.32), the thermodynamic zeta function for the three-dimensional Ising model with finite range interactions can also be represented as a reciprocal of an infinite product of polynomials. The three-dimensional model can be applied to study three-dimensional phase-transitions problems. The further results need to be investigated.

Some references that are related to our work are listed here. Zeta functions and related topics [1; 5; 11; 20; 22; 23; 24; 30; 31; 36; 37; 38; 40; 41; 42; 44; 45; 47]; patterns generation problems and lattice dynamical systems [2; 3; 4; 6; 7; 8; 12; 13; 14; 15; 16; 17; 18; 19; 25; 26; 28; 29; 34; 35], and phase-transitions in statistical physics [9; 10; 32; 33; 43] have all been covered elsewhere.

The rest of this dissertation is organized as follows. In Section 2, the trace operator $\mathbf{T}_n(\mathcal{B})$ and rotational matrix R_n are introduced to accommodate the periodic patterns. Based on the rotational symmetry of the trace operator, the reduced trace operator $\tau_n(\mathcal{B})$ is defined. Therefore, the rationality of $\zeta_{\mathcal{B};n}$ is obtained. The results also hold when inclined coordinates are used for any unimodular transformation $\gamma \in GL_2(\mathbb{Z})$. The meromorphic extension of zeta function is studied. The zeta function of the solution set of equations on $\mathbb{Z}^2$ with numbers from a finite field is also investigated. Finally, the method is applied to thermodynamic zeta function for the square Ising model with a finite range interactions.

In Section 3, the three-dimensional case is studied first. The trace operator $\mathbf{T}_{a_1,a_2;b_{12}}$ and rotational matrices $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ are introduced to study periodic patterns. The rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ induces the reduced trace operator $\tau_{a_1,a_2;b_{12}}$ and then the rationality of

$\zeta_{\mathcal{B};a_1,a_2;b_{12}}$ is obtained. The results hold for any inclined coordinates, determined by unimodular transformation in $GL_3(\mathbb{Z})$. Finally, the d-dimensional cases, $d \geq 4$, and thermodynamic zeta functions for the three-dimensional Ising model with finite range interactions are studied.

2 Zeta functions for two-dimensional shifts of finite type

In this section, zeta functions for two-dimensional shifts of finite type are studied.

2.1 Periodic patterns

This subsection first reviews the ordering matrices of local patterns and trace operators [4; 6]. It then derives rotational matrices R_n and $\mathbf{R}_n$, and studies their properties.

The R_n-symmetry of the trace operator is also discussed. Finally, some properties of periodic patterns in $\mathbb{Z}^2$ are investigated. In particular, the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is proven to be $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic.

For clarity, two symbols on the 2×2 lattice $\mathbb{Z}_{2\times 2}$ are initially examined. Subsection 2.3 addresses more general situations.

2.1.1 Ordering matrices and Trace operators

For given positive integers N_1 and N_2, the rectangular lattice $\mathbb{Z}_{N_1\times N_2}$ is defined by

$$\mathbb{Z}_{N_1\times N_2} = \{(n_1,n_2)|0 \le n_1 \le N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1\}.$$

In particular, $\mathbb{Z}_{2\times 2} = \{(0,0),(1,0),(0,1),(1,1)\}$. Define the set of all global patterns on $\mathbb{Z}^2$ with two symbols $\{0,1\}$ by

$$\Sigma_2^2 = \{0,1\}^{\mathbb{Z}^2} = \left\{U | U : \mathbb{Z}^2 \to \{0,1\}\right\}.$$

Here, $\mathbb{Z}^2 = \{(n_1,n_2)|n_1,n_2 \in \mathbb{Z}\}$, the set of all planar lattice points (vertices). The set of all local patterns on $\mathbb{Z}_{N_1\times N_2}$ is defined by

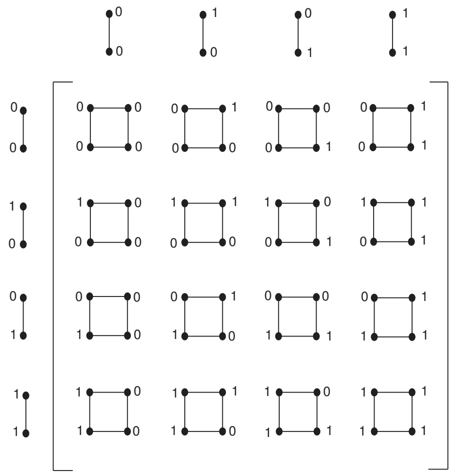
$$\Sigma_{N_1\times N_2} = \{U|_{\mathbb{Z}_{N_1\times N_2}} : U \in \Sigma_2^2\}.$$

Now, for any given $\mathcal{B} \subset \Sigma_{2\times 2}$, $\mathcal{B}$ is called a basic set of admissible local patterns. In short, $\mathcal{B}$ is a basic set. An $N_1 \times N_2$ pattern U is called $\mathcal{B}$-admissible if for any vertex (lattice point) (n_1,n_2) with $0 \le n_1 \le N_1 - 2$ and $0 \le n_2 \le N_2 - 2$, there exists a 2×2 admissible pattern $(\beta_{k_1,k_2})_{0\le k_1,k_2\le 1} \in \mathcal{B}$ such that

$$U_{n_1+k_1,n_2+k_2} = \beta_{k_1,k_2},$$

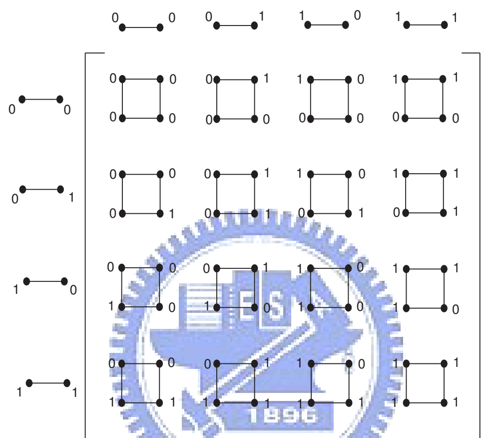
for $0 \le k_1, k_2 \le 1$. Denote by $\Sigma_{N_1\times N_2}(\mathcal{B})$ the set of all $\mathcal{B}$-admissible patterns on $\mathbb{Z}_{N_1\times N_2}$. As presented elsewhere [4], the ordering matrices $\mathbf{X}_{2\times 2}$ and $\mathbf{Y}_{2\times 2}$ are introduced to arrange systematically all local patterns in $\Sigma_{2\times 2}$.

Indeed, the horizontal ordering matrix $\mathbf{X}_{2\times 2} = [x_{p,q}]_{4\times 4}$ is defined by

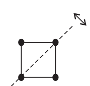


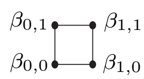
. (2.1)

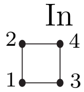
The vertical ordering matrix $\mathbf{Y}_{2\times 2} = [y_{i,j}]_{4\times 4}$ is defined by

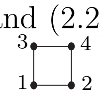


. (2.2)

It is clear that the local pattern $y_{i,j}$ in $\mathbf{Y}_{2\times 2}$ is the reflection $\frac{\pi}{4}$ of $x_{i,j}$ in $\mathbf{X}_{2\times 2}$, i.e,  . The reflection can be represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in $GL_2(\mathbb{Z})$ with determinant -1.

In (2.1) and (2.2), the orders of the pattern 

, $\beta_{i,j} \in \{0, 1\}$, are given by 

and 

respectively. $\mathbf{X}_{2\times 2}$ and $\mathbf{Y}_{2\times 2}$ are clearly related as follows.

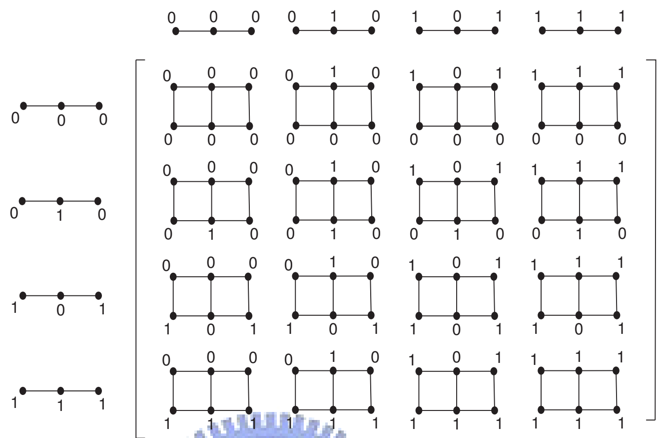
$$\mathbf{X}_{2\times 2} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{2,1} & y_{2,2} \\ y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{4,1} & y_{4,2} \\ y_{3,3} & y_{3,4} & y_{4,3} & y_{4,4} \end{bmatrix} \tag{2.3}$$

and

$$\mathbf{Y}_{2\times 2} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} \\ x_{1,3} & x_{1,4} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{4,1} & x_{4,2} \\ x_{3,3} & x_{3,4} & x_{4,3} & x_{4,4} \end{bmatrix}. \tag{2.4}$$

The set $\mathbf{C}_{2\times 2} = [c_{i,j}]$, which consists of all x-periodic patterns of period 2 with height 2 can be constructed from $\mathbf{Y}_{2\times 2}$ as follows.

$\mathbf{C}_{2\times 2} =$



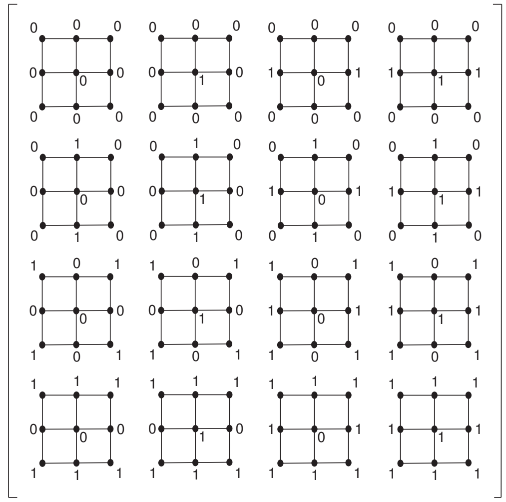
(2.5)

The patterns in $\mathbf{C}_{2\times 2}$ are expressed as elements in $\Sigma_{3\times 2}$ and are understood to be extendable periodically in the x-direction to all of $\mathbb{Z}_{\infty\times 2}$. Notably,

$$\begin{cases} c_{1,2} \cong c_{1,3}, \quad c_{2,1} \cong c_{3,1}, \quad c_{2,2} \cong c_{3,3}, \\ c_{2,3} \cong c_{3,2}, \quad c_{2,4} \cong c_{3,4}, \quad c_{4,2} \cong c_{4,3}, \end{cases} \tag{2.6}$$

where $c_{i,j} \cong c_{i',j'}$ means that $c_{i',j'}$ is an x-translation by one step from $c_{i,j}$. Later, the translation invariance property (2.6) will be shown to imply R_2-symmetry of the trace operator $\mathbf{T}_2$.

Finally, $\mathbf{P}_{2\times 2}$ denotes the set of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$-periodic patterns, which can be recorded from $\mathbf{C}_{2\times 2}$ or $\mathbf{Y}_{2\times 2}$ as an element in $\Sigma_{3\times 3}$ as follows.

$$\mathbf{P}_{2\times 2} = $$  . (2.7)

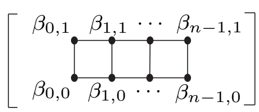
Notably, the upper two rows from the top of each pattern in $\mathbf{P}_{2\times 2}$ is $\mathbf{C}^t_{2\times 2}$, where $\mathbf{C}^t_{2\times 2}$ is the transpose of $\mathbf{C}_{2\times 2}$.

Therefore, $\mathbf{P}_{2\times 2}$ can be regarded as a "Hadamard type product $\bullet$" of $\mathbf{C}_{2\times 2}$ with $\mathbf{C}^t_{2\times 2}$, given by the following construction.

$$\mathbf{P}_{2\times 2} = \mathbf{C}_{2\times 2} \bullet \mathbf{C}^t_{2\times 2}; \tag{2.8}$$

the lower two rows of each pattern in $\mathbf{P}_{2\times 2}$ come from $\mathbf{C}_{2\times 2}$, and the upper two rows come from $\mathbf{C}^t_{2\times 2}$; they are glued together by the middle row. Equation (2.8) is the prototype for constructing doubly periodic patterns of $\mathbb{Z}^2$ from x-periodic patterns. Later, this idea will be generalized to all doubly periodic patterns.

The y-ordering matrices of patterns in $\Sigma_{n\times 2}$, $n \geq 2$, can be ordered analogously by

$$\mathbf{Y}_{n\times 2} = [y_{n;i,j}] = $$  $_{2^n \times 2^n}$, (2.9)

where

$$\begin{cases} i = \psi(\beta_{0,0}\beta_{1,0}\cdots\beta_{n-1,0}), \\ j = \psi(\beta_{0,1}\beta_{1,1}\cdots\beta_{n-1,1}), \end{cases} \tag{2.10}$$

and the n-th order counting function $\psi \equiv \psi_n : \{0,1\}^{\mathbb{Z}_n} \to \{j | 1 \leq j \leq 2^n\}$ is defined by

$$\psi(\beta_0\beta_1\cdots\beta_{n-1}) = 1 + \sum_{j=0}^{n-1} \beta_j 2^{(n-1-j)}. \tag{2.11}$$

The recursive formulas for generating $\mathbf{Y}_{n\times 2}$ from $\mathbf{Y}_{2\times 2}$, taken from another investigation [4], is as follows.

Let

$$\mathbf{Y}_{n\times 2} = \begin{bmatrix} Y_{n\times 2;1} & Y_{n\times 2;2} \\ Y_{n\times 2;3} & Y_{n\times 2;4} \end{bmatrix}, \tag{2.12}$$

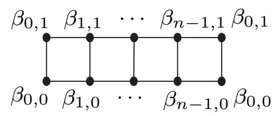
where $Y_{n\times 2;i}$ is a $2^{n-1} \times 2^{n-1}$ matrix of patterns. Then,

$$\mathbf{Y}_{(n+1)\times 2} = \begin{bmatrix} x_{1,1}Y_{n\times 2;1} & x_{1,2}Y_{n\times 2;2} & x_{2,1}Y_{n\times 2;1} & x_{2,2}Y_{n\times 2;2} \\ x_{1,3}Y_{n\times 2;3} & x_{1,4}Y_{n\times 2;4} & x_{2,3}Y_{n\times 2;3} & x_{2,4}Y_{n\times 2;4} \\ x_{3,1}Y_{n\times 2;1} & x_{3,2}Y_{n\times 2;2} & x_{4,1}Y_{n\times 2;1} & x_{4,2}Y_{n\times 2;2} \\ x_{3,3}Y_{n\times 2;3} & x_{3,4}Y_{n\times 2;4} & x_{4,3}Y_{n\times 2;3} & x_{4,4}Y_{n\times 2;4} \end{bmatrix} \tag{2.13}$$

is a $2^{n+1} \times 2^{n+1}$ matrix.

Hence, x-periodic patterns of period n with height 2 can be expressed in $\Sigma_{(n+1)\times 2}$, and recorded as an element in $\mathbf{C}_{n\times 2}$ by

$$\mathbf{C}_{n\times 2} = \Big[\quad \Big]_{2^n \times 2^n}, \tag{2.14}$$



where $\beta_{i,j} \in \{0, 1\}$.

Now, given any basic set $\mathcal{B}$, define the associated horizontal and vertical transition matrices
$\mathbf{H}_2 = \mathbf{H}_2(\mathcal{B}) = [a_{p,q}]$ and $\mathbf{V}_2 = \mathbf{V}_2(\mathcal{B}) = [b_{i,j}]$ by

$$a_{p,q} = \begin{cases} 1 & \text{if} \quad x_{p,q} \in \mathcal{B}, \\ 0 & \text{if} \quad x_{p,q} \notin \mathcal{B}, \end{cases} \quad \text{and} \quad b_{i,j} = \begin{cases} 1 & \text{if} \quad y_{i,j} \in \mathcal{B}, \\ 0 & \text{if} \quad y_{i,j} \notin \mathcal{B}, \end{cases} \tag{2.15}$$

respectively. Then,

$$\mathbf{H}_2 = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{2,1} & b_{2,2} \\ b_{1,3} & b_{1,4} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{4,1} & b_{4,2} \\ b_{3,3} & b_{3,4} & b_{4,3} & b_{4,4} \end{bmatrix}, \tag{2.16}$$

and

$$\mathbf{V}_2 = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{2,1} & a_{2,2} \\ a_{1,3} & a_{1,4} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{4,1} & a_{4,2} \\ a_{3,3} & a_{3,4} & a_{4,3} & a_{4,4} \end{bmatrix}. \tag{2.17}$$

The associated column matrices $\widetilde{\mathbf{H}}_2$ of $\mathbf{H}_2$ and $\widetilde{\mathbf{V}}_2$ of $\mathbf{V}_2$ are defined as

$$\widetilde{\mathbf{H}}_2 = \begin{bmatrix} a_{1,1} & a_{2,1} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{4,1} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{1,4} & a_{2,4} \\ a_{3,3} & a_{4,3} & a_{3,4} & a_{4,4} \end{bmatrix} \tag{2.18}$$

and

$$\widetilde{\mathbf{V}}_2 = \begin{bmatrix} b_{1,1} & b_{2,1} & b_{2,1} & b_{2,2} \\ b_{3,1} & b_{4,1} & b_{3,2} & b_{4,2} \\ b_{1,3} & b_{2,3} & b_{1,4} & b_{2,4} \\ b_{3,3} & b_{4,3} & b_{3,4} & b_{4,4} \end{bmatrix}, \tag{2.19}$$

respectively.

The trace operators $\mathbf{T}_2 = \mathbf{T}_2(\mathcal{B})$ and $\widehat{\mathbf{T}}_2 = \widehat{\mathbf{T}}_2(\mathcal{B})$ which were introduced in [6] are defined as

$$\mathbf{T}_2 = \mathbf{V}_2 \circ \widetilde{\mathbf{H}}_2 \quad and \quad \widehat{\mathbf{T}}_2 = \mathbf{H}_2 \circ \widetilde{\mathbf{V}}_2, \tag{2.20}$$

where $\circ$ is the Hadamard product: if $A = [\alpha_{i,j}]_{p\times p}$ and $B = [\beta_{i,j}]_{p\times p}$, then $A \circ B = [\alpha_{i,j}\beta_{i,j}]_{p\times p}$. More precisely,

$$\mathbf{T}_2 = [t_{i,j}]_{2^2\times 2^2} = \begin{bmatrix} a_{1,1}a_{1,1} & a_{1,2}a_{2,1} & a_{2,1}a_{1,2} & a_{2,2}a_{2,2} \\ a_{1,3}a_{3,1} & a_{1,4}a_{4,1} & a_{2,3}a_{3,2} & a_{2,4}a_{4,2} \\ a_{3,1}a_{1,3} & a_{3,2}a_{2,3} & a_{4,1}a_{1,4} & a_{4,2}a_{2,4} \\ a_{3,3}a_{3,3} & a_{3,4}a_{4,3} & a_{4,3}a_{3,4} & a_{4,4}a_{4,4} \end{bmatrix} \tag{2.21}$$

and

$$\widehat{\mathbf{T}}_2 = \left[\widehat{t}_{i,j}\right]_{2^2\times 2^2} = \begin{bmatrix} b_{1,1}b_{1,1} & b_{1,2}b_{2,1} & b_{2,1}b_{1,2} & b_{2,2}b_{2,2} \\ b_{1,3}b_{3,1} & b_{1,4}b_{4,1} & b_{2,3}b_{3,2} & b_{2,4}b_{4,2} \\ b_{3,1}b_{1,3} & b_{3,2}b_{2,3} & b_{4,1}b_{1,4} & b_{4,2}b_{,24} \\ b_{3,3}b_{3,3} & b_{3,4}b_{4,3} & b_{4,3}b_{3,4} & b_{4,4}b_{4,4} \end{bmatrix}. \tag{2.22}$$

From (2.5), (2.17) and (2.21), clearly

$$t_{i,j} = \begin{cases} 1 & \text{if} \quad c_{i,j} \text{ is } \mathcal{B}\text{-admissible}, \\ 0 & \text{if} \quad c_{i,j} \text{ is not } \mathcal{B}\text{-admissible}, \end{cases} \tag{2.23}$$

where $c_{i,j} \in \mathbf{C}_{2\times 2}$.

Therefore, $\mathbf{T}_2$ is the transition matrix of the $\mathcal{B}$-admissible and x-periodic patterns of period 2 with height 2. Similarly, $\widehat{\mathbf{T}}_2$ is the transition matrix of $\mathcal{B}$-admissible and y-periodic patterns of period 2 with width 2.

The translation invariance property (2.6) of $\mathbf{C}_{2\times 2}$ implies the following symmetry of $\mathbf{T}_2$;

$$\begin{cases} t_{1,2} = t_{1,3}, \quad t_{2,1} = t_{3,1}, \quad t_{2,2} = t_{3,3}, \\ t_{2,3} = t_{3,2}, \quad t_{2,4} = t_{3,4}, \quad t_{4,2} = t_{4,3}. \end{cases} \tag{2.24}$$

The symmetry of (2.6) or (2.24) can also be identified as the rotational symmetry of a cylinder since elements in $\mathbf{C}_{2\times 2}$ can be regarded as cylindrical patterns.

The recursive formulas of $\mathbf{Y}_{n\times 2}$ can also be applied to $\mathbf{V}_n$. Indeed, if

$$\mathbf{V}_n = \begin{bmatrix} V_{n;1} & V_{n;2} \\ V_{n;3} & V_{n;4} \end{bmatrix}_{2^n \times 2^n},$$

where $V_{n;j}$ is a $2^{n-1} \times 2^{n-1}$ matrix, then

$$\mathbf{V}_{n+1} = \begin{bmatrix} a_{1,1}V_{n;1} & a_{1,2}V_{n;2} & a_{2,1}V_{n;1} & a_{2,2}V_{n;2} \\ a_{1,3}V_{n;3} & a_{1,4}V_{n;4} & a_{2,3}V_{n;3} & a_{2,4}V_{n;4} \\ a_{3,1}V_{n;1} & a_{3,2}V_{n;2} & a_{4,1}V_{n;1} & a_{4,2}V_{n;2} \\ a_{3,3}V_{n;3} & a_{3,4}V_{n;4} & a_{4,3}V_{n;3} & a_{4,4}V_{n;4} \end{bmatrix} \tag{2.25}$$

and

$$\mathbf{T}_n = \mathbf{V}_n \circ \begin{bmatrix} E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{3,1} & a_{4,1} \end{bmatrix} & E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,2} & a_{2,2} \\ a_{3,2} & a_{4,2} \end{bmatrix} \\ E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,3} & a_{2,3} \\ a_{3,3} & a_{4,3} \end{bmatrix} & E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,4} & a_{2,4} \\ a_{3,4} & a_{4,4} \end{bmatrix} \end{bmatrix}, \tag{2.26}$$

where $\otimes$ is the Kroncker (tensor) product and E_j is the $j \times j$ full matrix.

Now, $\mathbf{T}_n$ represents the transition matrix of $\mathcal{B}$-admissible x-periodic patterns of period n with height 2. Similarly, $\widehat{\mathbf{T}}_n$ represents the transition matrix of $\mathcal{B}$-admissible y-periodic patterns of period n with width 2.

2.1.2 Rotational matrices

In this subsection, the rotational matrices R_n and the invariant property of $\mathbf{C}_{n\times 2}$ under R_n are investigated and the R_n-symmetry of $\mathbf{T}_n$ is then proven.

The shift of any n-sequence $\overline{\beta} = (\beta_0\beta_1\cdots\beta_{n-2}\beta_{n-1})$, $n \geq 2$, $\beta_j \in \{0,1\}$, is defined by

$$\sigma((\beta_0\beta_1\cdots\beta_{n-2}\beta_{n-1})) \equiv \sigma_n((\beta_0\beta_1\cdots\beta_{n-2}\beta_{n-1})) = (\beta_1\beta_2\cdots\beta_{n-1}\beta_0). \tag{2.27}$$

The subscript of σ_n is omitted for brevity. Notably, the shift (to the left) of any one-dimensional periodic sequence $(\beta_0\beta_1\cdots\beta_{n-1}\beta_0\cdots)$ of period n becomes $(\beta_1\beta_2\cdots\beta_{n-1}\beta_0\beta_1\cdots)$.

The $2^n \times 2^n$ rotational matrix $R_n = [R_{n;i,j}]$, $R_{n;i,j} \in \{0,1\}$, is defined by

$$R_{n;i,j} = 1 \quad \text{if and only if}$$

$$i = \psi(\beta_0\beta_1\cdots\beta_{n-1}) \quad \text{and} \quad j = \psi(\sigma(\beta_0\beta_1\cdots\beta_{n-1})) = \psi(\beta_1\beta_2\cdots\beta_{n-1}\beta_0). \tag{2.28}$$

From (2.28), for convenience, denote by

$$j = \sigma(i). \tag{2.29}$$

Clearly, R_n is a permutation matrix: each row and column of R_n has one and only one element with a value of 1. Indeed, R_n can be written explicitly as follows, the proof is omitted.

Lemma 2.1

$$\begin{cases} R_{n;i,2i-1} = 1 \text{ and } R_{n;2^{n-1}+i,2i} = 1 & \text{for } 1 \leq i \leq 2^{n-1}, \\ R_{n;i,j} = 0 & \text{otherwise,} \end{cases} \tag{2.30}$$

or equivalently,

$$\sigma(i) \equiv \sigma_n(i) = \begin{cases} 2i-1 & \textit{for} \quad 1 \le i \le 2^{n-1}, \\ 2(i-2^{n-1}) & \textit{for} \quad 1+2^{n-1} \le i \le 2^n. \end{cases} \tag{2.31}$$

Furthermore, $R_n^n = I_{2^n}$ *and for any* $1 \le j \le n-1$,

$$(R_n^j)_{i,\sigma^j(i)} = 1. \tag{2.32}$$

The equivalent class $C_n(i)$ of i is defined by

$$\begin{aligned} C_n(i) = & \ \{\sigma^j(i) | 0 \le j \le n-1\} \\ = & \ \left\{ j \middle| \left(R_n^l\right)_{i,j} = 1 \text{ for some } 1 \le l \le n \right\}. \end{aligned} \tag{2.33}$$

Clearly, either $C_n(i) = C_n(j)$ or $C_n(i) \cap C_n(j) = \emptyset$. Let i be the smallest element in its equivalent class, and the index set $\mathcal{I}_n$ of n is defined by

$$\begin{aligned} \mathcal{I}_n = & \ \{i | 1 \le i \le 2^n, i \le \sigma^q(i), 1 \le q \le n-1\} \\ = & \ \left\{ i \middle| 1 \le i \le 2^n, i \le j \text{ for all } j \in C_n(i) \right\}. \end{aligned} \tag{2.34}$$

Therefore, for each $n \ge 1$, $\{j | 1 \le j \le 2^n\} = \bigcup_{i \in \mathcal{I}_n} C_n(i)$. The cardinal number of $\mathcal{I}_n$ is denoted by χ_n. Notably, χ_n can be identified as the number of necklaces that can be made from n beads of two colors, when the necklaces can be rotated but not turned over [48]. Moreover, χ_n is expressed as

$$\chi_n = \frac{1}{n} \sum_{d|n} \phi(d) 2^{n/d}, \tag{2.35}$$

where $\phi(n)$ is the Euler totient function, which counts the numbers smaller or equal to n and prime relative to n,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{2.36}$$

For $n = 2$ and 3, R_n and $C_n(i)$ are as follows.

Example 2.2 R_n, $\mathcal{I}_n$ *and* $C_n(i)$ *for* $n = 2$ *and* 3,

$$\text{(i) } R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{cases} C_2(1) = \{1\}, & 1 \to 1, \\ C_2(2) = C_2(3) = \{2,3\}, & 2 \to 3 \to 2, \\ C_2(4) = \{4\}, & 4 \to 4, \\ \mathcal{I}_2 = \{1,2,4\}. \end{cases}$$

$$\text{(ii) For } R_3, \begin{cases} 1 \to 1, \\ 2 \to 3 \to 5 \to 2, \\ 4 \to 7 \to 6 \to 4, \\ 8 \to 8, \\ \mathcal{I}_3 = \{1,2,4,8\}. \end{cases}$$

The following proposition shows the permutation character of R_n and the proof is omitted.

Proposition 2.3 *Let* $\mathbf{M} = [M_{i,j}]_{2^n \times 2^n}$ *be a matrix where* $M_{i,j}$ *denotes a number or a pattern or a set of patterns. Then,*

$$(R_n\mathbf{M})_{i,j} = M_{\sigma(i),j} \text{ and } (\mathbf{M}R_n)_{i,j} = M_{i,\sigma^{-1}(j)}. \tag{2.37}$$

Furthermore, for any $l \geq 1$,

$$(R_n^l\mathbf{M})_{i,j} = M_{\sigma^l(i),j} \text{ and } (\mathbf{M}R_n^l)_{i,j} = M_{i,\sigma^{-l}(j)}. \tag{2.38}$$

In the following, x-periodic patterns of period n with height $k \geq 1$ are studied. More notation is required.

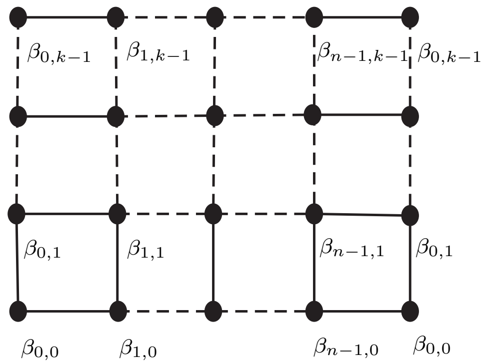
Definition 2.4

(i) For any $n \geq 1$, *let* $(\beta_0\beta_1\cdots\beta_{n-1})^\infty$ *be a periodic sequence of period n, denoted by* $\overline{\beta} = (\beta_0\cdots\beta_{n-1})$. $\sigma(\overline{\beta}) = \sigma((\beta_0\beta_1\cdots\beta_{n-1})) = (\beta_1\beta_2\cdots\beta_{n-1}\beta_0)$. *For any fixed* $n \geq 1$ *and any* $j \geq 0$, *denote by* $\overline{\beta}_j = (\beta_{0,j}\beta_{1,j}\cdots\beta_{n-1,j})$ *a periodic sequence of period n.*

(ii) For fixed $n \geq 1$ and any $k \geq 1$, denote by

$$
\begin{aligned}
&[\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}] \\
&= (\beta_{0,0}\beta_{1,0}\cdots\beta_{n-1,0})^{\infty} \oplus \cdots \oplus (\beta_{0,k-1}\beta_{1,k-1}\cdots\beta_{n-1,k-1})^{\infty}
\end{aligned}
$$

=



,

a x-periodic pattern of period n with height k.

(iii) A Hadamard type product $\bullet$ of patterns is defined as follows.

$$[\overline{\beta}_0\overline{\beta}_1] \bullet [\overline{\beta}_1\overline{\beta}_2] = [\overline{\beta}_0\overline{\beta}_1\overline{\beta}_2]$$

and

$$[\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}] = [\overline{\beta}_0\overline{\beta}_1] \bullet [\overline{\beta}_1\overline{\beta}_2] \bullet \cdots \bullet [\overline{\beta}_{k-2}\overline{\beta}_{k-1}].$$

(iv) A $2^n \times 2^n$ ordering matrix $\mathbf{C}_{n\times k} = [C_{n\times k;i,j}]$ of x-periodic patterns of period n with height $k \geq 2$ is defined by

$$C_{n\times k;i,j} = \{[\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}] | \psi(\overline{\beta}_0) = i \text{ and } \psi(\overline{\beta}_{k-1}) = j\}.$$

(v) For $n \geq 1$ and $k \geq 2$, denote by $\mathbf{D}_{n,k} = [D_{n,k;i,j}]$ the ordering matrix of patterns, which consists of a first row $\overline{\beta}_0$ and the k-th row $\overline{\beta}_{k-1}$ of $\mathbf{C}_{n\times k}$:

$$D_{n,k;i,j} = \{[\overline{\beta}_0\overline{\beta}_{k-1}] | [\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}] \in C_{n\times k;i,j}\}.$$

Some remarks should be made.

Remark 2.5

(1) For any $n \geq 1$, the length of $\overline{\beta}$ in (i) and $\overline{\beta}_j$ in (ii) depends on n. For simplicity, these dependencies are omitted.

(2) The product $\bullet$ defined in (iii) applies only when the top row of the first pattern is identical to the first row of the second pattern.

(3) In (iv), when $k = 2$, (2.14) applies.

(4) $C_{n\times k;i,j}$ is a set of patterns with the same first and k-th rows. $\mathbf{D}_{n,k}$ is exactly $\mathbf{C}_{n\times 2}$, but, importantly, in $\mathbf{C}_{n\times k}$, all patterns in the entry $C_{n\times k;i,j}$ have the same top and first rows, which can be used to construct y-periodic patterns with a shift in the (k+1)-th row.

In the following lemma, R_n is used to shift the first row in $\mathbf{D}^t_{n,k}$.

Lemma 2.6 *Let $i = \psi(\overline{\beta}_0)$ and $j = \psi(\overline{\beta}_{k-1})$. Then*

(i) $(R_n\mathbf{D}^t_{n,k})_{i,j} = [\overline{\beta}_{k-1}\sigma(\overline{\beta}_0)]$,

(ii) $(\mathbf{C}_{n\times k} \bullet R_n\mathbf{D}^t_{n,k})_{i,j} = [\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}\sigma(\overline{\beta}_0)]$.

Proof. (i) follows easily from Proposition 2.3 and part (v) of Definition 2.4. From parts (i) and (iii) of Definition 2.4, a product in (ii) is legitimate since the top row of $\mathbf{C}_{n\times k}$ and the first row of $R_nD^t_{n,k}$ are $\overline{\beta}_{k-1}$, and (ii) follows from (i). ∎

Furthermore, the following result shows that the patterns in $\mathbf{C}_{n\times k} \bullet R^l_n\mathbf{D}^t_{n,k}$ are the same as the patterns in $diag(\mathbf{C}_{n\times(k+1)}R^{n-l}_n)$ where $diag(\mathbf{M})$ is the diagonal part of $\mathbf{M}$, such that $diag(\mathbf{M}) = I \circ \mathbf{M}$. They are important in constructing y-periodic patterns.

Proposition 2.7 *For any $n \geq 2$, $k \geq 1$ and $0 \leq l \leq n$,*

$$\begin{aligned} \textit{patterns in } \mathbf{C}_{n\times k} \bullet R^l_n\mathbf{D}^t_{n,k} &= \textit{patterns in } diag(\mathbf{C}_{n\times(k+1)}R^{n-l}_n) \\ &= \{[\overline{\beta}_0\cdots\overline{\beta}_{k-1}\sigma^l(\overline{\beta}_0)] | [\overline{\beta}_0\cdots\overline{\beta}_{k-1}] \in \mathbf{C}_{n\times k}\}. \end{aligned}$$

Proof. By (2.38), for any $0 \leq l \leq n-1$, $1 \leq i,j \leq 2^{n+1}$,

$$(\mathbf{C}_{n\times(k+1)}R^{n-l}_n)_{i,j} = \{[\overline{\beta}_0\cdots\overline{\beta}_{k-1}\sigma^{l-n}(\overline{\beta}_k)] : \psi(\overline{\beta}_0) = i \text{ and } \psi(\overline{\beta}_k) = j\}.$$

Since $\psi(\overline{\beta}_k) = \psi(\overline{\beta}_0) = i$ implies $\overline{\beta}_k = \overline{\beta}_0$,

$$(\mathbf{C}_{n\times(k+1)}R^{n-l}_n)_{i,i} = \{[\overline{\beta}_0\cdots\overline{\beta}_{k-1}\sigma^{l-n}(\overline{\beta}_0)] : \psi(\overline{\beta}_0) = i\}.$$

However, for any $1 \leq i,j \leq 2^n$, part (ii) of Lemma 2.6 implies

$$(\mathbf{C}_{n\times k} \bullet R^l_nD^t_{n,k})_{i,j} = [\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}\sigma^l(\overline{\beta}_0)].$$

Now, for any $0 \leq l \leq n-1$ and $\overline{\beta} = (\beta_0\cdots\beta_{n-1})$,

$$\sigma^l(\overline{\beta}) = \sigma^{l-n}(\overline{\beta}).$$

The proof is complete. ■

The rotational symmetry of $\mathbf{T}_n$ is determined by studying $\mathbf{C}_{n\times 2}$ in more detail. Given a basic admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, $\mathbf{T}_n$ is defined by (2.26). Let $[\overline{\beta}_0\overline{\beta}_1] \in \mathbf{C}_{n\times 2}$, for $0 \le j \le n-1$, denote

$$p_j = 2\beta_{j,0} + \beta_{j,1} + 1,$$

then the associated entry in $\mathbf{T}_n$ is

$$\mathbf{T}_n([\overline{\beta}_0\overline{\beta}_1]) \equiv a_{p_0,p_1}a_{p_1,p_2}\cdots a_{p_{n-1},p_0}. \tag{2.39}$$

$[\overline{\beta}_0\overline{\beta}_1]$ is $\mathcal{B}$-admissible if and only if $a_{p_j,p_{j+1}} = 1$ for all $0 \le j \le n-1$, where $p_n = p_0$.

Theorem 2.8 *For any $n \ge 2$, the trace operator $\mathbf{T}_n = [t_{n;i,j}]_{2^n\times 2^n}$ has the following R_n-symmetry:*

$$t_{n;\sigma^l(i),\sigma^l(j)} = t_{n;i,j} \tag{2.40}$$

for all $1 \le i, j \le 2^n$ and $0 \le l \le n-1$.

Proof. Given $[\overline{\beta}_0\overline{\beta}_1] \in \mathbf{C}_{n\times 2}$, all $[\sigma^l(\overline{\beta}_0)\sigma^l(\overline{\beta}_1)]$, $0 \le l \le n-1$, represent similar x-periodic patterns. The entry of $[\sigma^l(\overline{\beta}_0)\sigma^l(\overline{\beta}_1)]$ in $\mathbf{T}_n$ is

$$\mathbf{T}_n([\sigma^l(\overline{\beta}_0)\sigma^l(\overline{\beta}_1)]) = a_{p_l,p_{l+1}}a_{p_{l+1},p_{l+2}}\cdots a_{p_{n-1},p_0}a_{p_0,p_1}\cdots a_{p_{l-1},p_l}. \tag{2.41}$$

Comparing (2.39) with (2.41) clearly reveals that

$$\mathbf{T}_n([\overline{\beta}_0\overline{\beta}_1]) = \mathbf{T}_n([\sigma^l(\overline{\beta}_0)\sigma^l(\overline{\beta}_1)]) \tag{2.42}$$

for all $0 \le l \le n-1$. Additionally, if $\mathbf{T}_n = [t_{n;i,j}]$ with $i = \psi(\overline{\beta}_0)$ and $j = \psi(\overline{\beta}_1)$, then (2.42) implies

$$t_{n;\sigma^l(i),\sigma^l(j)} = t_{n;i,j} \text{ for all } 0 \le l \le n-1.$$

The proof is complete. ■

Proposition 2.7 and Theorem 2.8 yield the following theorem.

Theorem 2.9 *For any $n \ge 2$ and $k \ge 2$, $0 \le l \le n-1$,*

$$|\mathbf{T}_n^{k-1} \circ R_n^l \mathbf{T}_n^t| = tr(\mathbf{T}_n^k R_n^{n-l}) \tag{2.43}$$

and

$$|\mathbf{T}_n^{k-1} \circ \mathbf{R}_n \mathbf{T}_n^t| = tr(\mathbf{T}_n^k \mathbf{R}_n), \tag{2.44}$$

where

$$\mathbf{R}_n = \sum_{l=0}^{n-1} R_n^l. \tag{2.45}$$

Proof. From Proposition 2.7, (2.39) and the properties of $\mathbf{T}_n$, (2.43) follows. Equations (2.43) and (2.45) yield (2.44). The proof is complete. ■

2.1.3 Periodic patterns

This subsection studies in detail (double) periodic patterns in $\mathbb{Z}^2$. Indeed, consider a lattice L with Hermite normal form,

$$L = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2, \tag{2.46}$$

where $n \geq 1$, $k \geq 1$ and $0 \leq l \leq n-1$. A pattern $U = (\beta_{i,j})_{i,j \in \mathbb{Z}}$ is called L-periodic if every $i, j \in \mathbb{Z}$

$$\beta_{i+np+lq,j+kq} = \beta_{i,j} \tag{2.47}$$

for all $p, q \in \mathbb{Z}$.

The periodicity of $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$ and $\begin{bmatrix} n & 0 \\ 0 & k' \end{bmatrix}$ are closely related as follows.

Proposition 2.10 *For any $n \geq 2$, $k \geq 1$ and $0 \leq l \leq n-1$, $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic patterns are $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic where (n, l) is the greatest common divisor (GCD) of n and l.*

Proof. By (2.47), the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is easily identified as $\begin{bmatrix} n & l \cdot m \\ 0 & k \cdot m \end{bmatrix}$-periodic for all $m \in \mathbb{N}$. By taking $m = \frac{n}{(n,l)}$, the result holds. ■

Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, defined on square lattice $\mathbb{Z}_{2\times 2}$, the periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2}$. Let $\mathbb{Z}_{2\times 2}((i,j))$ be the square lattice with the left-bottom vertex (i,j):

$$\mathbb{Z}_{2\times 2}((i,j)) = \{(i,j),(i+1,j),(i,j+1),(i+1,j+1)\}.$$

Now, the admissibility is demonstrated to have to be verified on finite square lattices.

Proposition 2.11 *An L-periodic pattern U is $\mathcal{B}$-admissible if and only if*

$$U\big|_{\mathbb{Z}_{2\times 2}((i,j))} \in \mathcal{B} \tag{2.48}$$

for any $0 \leq i \leq n-1$ and $0 \leq j \leq k-1$.

Proof. The proof follows easily from (2.47). The details are left to the reader. ■

According to Proposition 2.11, the admissibility of U is determined by

$$(\beta_{i,j})_{0\leq i\leq n, 0\leq j\leq k},$$

and $(\beta_{i,j})_{0\leq i\leq n, 0\leq j\leq k}$ with the periodic property (2.47). Therefore, the following theorem can be obtained.

Theorem 2.12 *Given a basic admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, an L-periodic pattern U is $\mathcal{B}$-admissible if and only if*

$$[\overline{\beta}_0\overline{\beta}_1\cdots\overline{\beta}_{k-1}] \text{ and } [\overline{\beta}_{k-1}\sigma^{n-l}(\overline{\beta}_0)] \text{ are } \mathcal{B}\text{-admissible.} \tag{2.49}$$

Proposition 2.7 and Theorem 2.12 yield the following main results.

Theorem 2.13 *For $n \geq 1$, $0 \leq l \leq n-1$ and $k \geq 1$, denote by $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ the cardinal number of the set of $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and $\mathcal{B}$-admissible patterns. For $n \geq 2$, $0 \leq l \leq n-1$ and $k \geq 2$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}_n^k R_n^l\right) = |\mathbf{T}_n^{k-1} \circ R_n^{n-l}\mathbf{T}_n^t| \tag{2.50}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) = |\mathbf{T}_n^{k-1} \circ \mathbf{R}_n\mathbf{T}_n^t|. \tag{2.51}$$

For $n \geq 2$ and $0 \leq l \leq n-1$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & 1 \end{bmatrix}\right) = tr(\mathbf{T}_n R_n^l) = |diag(\mathbf{T}_n) \circ R_n^{n-l}\mathbf{T}_n^t| \tag{2.52}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & 1 \end{bmatrix}\right) = tr(\mathbf{T}_n \mathbf{R}_n) = |diag(\mathbf{T}_n) \circ \mathbf{R}_n\mathbf{T}_n^t|. \tag{2.53}$$

Furthermore, let

$$\mathbf{T}_1 = \begin{bmatrix} a_{1,1}a_{1,1} & a_{2,2}a_{2,2} \\ a_{3,3}a_{3,3} & a_{4,4}a_{4,4} \end{bmatrix} \text{ and } R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{2.54}$$

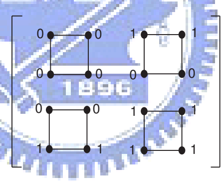
then

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\right) = tr(\mathbf{T}_1^k). \tag{2.55}$$

Proof. By Proposition 2.7, Theorem 2.12 and the construction of $\mathbf{T}_n$, the results (2.50) to (2.53) hold for $n \geq 2$, $0 \leq l \leq n-1$ and $k \geq 1$.

For $n = 1$, define

$$\mathbf{C}_{1\times 2} =$$



, (2.56)

which is the collection of x-periodic patterns of period 1 with height 2. Then, $\mathcal{B}$-admissible patterns of $\mathbf{C}_{1\times 2}$ are represented by $\mathbf{T}_1$ as defined in (2.54). Theorem 2.12 and the construction of $\mathbf{T}_1$ easily yields (2.55). The proof is complete. ∎

The n-th order zeta function $\zeta_n(s)$ can now be obtained as follows.

Theorem 2.14 *For $n \geq 1$,*

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr(\mathbf{T}_n^k\mathbf{R}_n)s^{kn}\right). \tag{2.57}$$

Proof. The results follow from Theorem 2.13. ∎

2.2 Rationality of ζ_n

This subsection proves that ζ_n is a rational function, as specified by (1.18). To elucidate the method, the symmetric $\mathbf{T}_n$ is considered initially. For any $n \geq 1$, let λ_j be an eigenvalue of $\mathbf{T}_n$: $\mathbf{T}_n U_j = \lambda_j U_j$, $1 \leq j \leq N$ and $N \equiv 2^n$. If $\mathbf{T}_n$ is symmetric, then the Jordan form of $\mathbf{T}_n$ [27] is

$$\mathbf{T}_n = \mathbf{U}\mathbf{J}\mathbf{U}^t, \tag{2.58}$$

where

$$\mathbf{U}^t = \mathbf{U}^{-1}. \tag{2.59}$$

The eigen-matrix $\mathbf{U}$ in (2.58) is defined by

$$\mathbf{U} = [U_1, U_2, \cdots, U_N]_{N\times N} = [u_{i,j}]_{N\times N}, \tag{2.60}$$

where $U_j = (u_{1,j}, u_{2,j}, \cdots, u_{N,j})^t$ is the j-th (column) eigenvector, and

$$\mathbf{J} = diag(\lambda_1, \lambda_2, \cdots, \lambda_N). \tag{2.61}$$

Moreover, λ_j can be arranged such that $\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$. Equation (2.59) implies

$$\sum_{p=1}^{N} u_{i,p}u_{j,p} = \delta_{i,j} \quad \text{and} \quad \sum_{q=1}^{N} u_{q,i}u_{q,j} = \delta_{i,j}. \tag{2.62}$$

Now, Theorem 2.15 will be proven.

Theorem 2.15 *Assume $\mathbf{T}_n$ is symmetric; then*

$$\frac{1}{n}\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = \frac{1}{n}tr\left(\mathbf{T}_n^k\mathbf{R}_n\right) = \sum_{\lambda\in\Sigma(\mathbf{T}_n)}\chi(\lambda)\lambda^k, \tag{2.63}$$

where $\Sigma(\mathbf{T}_n)$ is the spectrum of $\mathbf{T}_n$,

$$\chi(\lambda) = \sum_{\lambda_j=\lambda}\chi(\lambda_j) \tag{2.64}$$

and

$$\chi(\lambda_j) = \tfrac{1}{n}|\mathbf{R}_n \circ U_j U_j^t| = \tfrac{1}{n}\sum_{i\in\mathcal{I}_n}\tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)^2, \tag{2.65}$$

where $\omega_{n,i}$ *is the cardinal number of* $C_n(i)$*. Moreover,*

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_n)} (1-\lambda s^n)^{-\chi(\lambda)}. \tag{2.66}$$

Proof. Clearly,

$$\begin{aligned} & tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) \\ = \;& tr\left(\mathbf{U} diag(\lambda_j)\mathbf{U}^t \mathbf{R}_n\right) \\ = \;& \sum_{j=1}^{N}\left\{\sum_{i=1}^{N} u_{i,j}\sum_{p=1}^{N} u_{p,j}\left(\sum_{l=1}^{n-1} R^l_{n;p,i}\right)\right\}\lambda_j. \end{aligned}$$

For each j, $1\le j\le N$,

$$\begin{aligned} & \sum_{i=1}^{N} u_{i,j}\left(\sum_{p=1}^{N} u_{p,j}\sum_{l=0}^{n-1} R^l_{n;p,i}\right) \\ = \;& \sum_{i=1}^{N} u_{i,j}\left(\sum_{l=0}^{n-1} u_{\sigma^{-l}(i),j}\right) \\ = \;& \sum_{i\in\mathcal{I}_n}\tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)\left(\sum_{l=0}^{n-1} u_{\sigma^{-l}(i),j}\right) \\ = \;& \sum_{i\in\mathcal{I}_n}\tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)^2. \end{aligned}$$

The following is easily verified;

$$|\mathbf{R}_n \circ U_j U_j^t| = \sum_{i\in\mathcal{I}_n}\frac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)^2. \tag{2.67}$$

Then, (2.63)∼(2.65) follow. From [21],

$$\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{J}^k s^{kn} = diag\left(\log(1-\lambda_j s^n)^{-1}\right). \tag{2.68}$$

Therefore, (2.66) holds. The proof is complete. ∎

We now extend Theorem 2.15 to general $\mathbf{T}_n$. In this case, the Jordan form for $\mathbf{T}_n$ is

$$\mathbf{T}_n = \mathbf{UJU}^{-1}, \tag{2.69}$$

where $\mathbf{U}$ is given as (2.60) and U_j, $1 \leq j \leq N$, is an eigenvector or generalized eigenvector [21; 27]. Denote by

$$\mathbf{U}^{-1} = [w_{i,j}] = [W_1; W_2; \cdots ; W_N]_{N \times N} \tag{2.70}$$

with $W_i = (w_{i,1}, w_{i,2}, \cdots , w_{i,N})$, the i-th row vector.

$$\mathbf{J} = diag(J_1, J_2, \cdots , J_Q), \tag{2.71}$$

where J_q is the Jordan block, $1 \leq q \leq Q$:

$$J_q = \begin{bmatrix} \lambda_q & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_q & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_q & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_q \end{bmatrix}_{M_q \times M_q} , \tag{2.72}$$

$M_q \geq 1$.

As is well-known [21], for any Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{M \times M} \tag{2.73}$$

and

$$\log(I - tJ) = \begin{bmatrix} \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \cdots & \mu_{1,M} \\ 0 & \mu_{2,2} & \mu_{2,3} & \cdots & \mu_{2,M} \\ & & \ddots & & \vdots \\ & & & & \\ 0 & 0 & \cdots & 0 & \mu_{M,M} \end{bmatrix}, \tag{2.74}$$

where

$$\mu_{i,i+j-1} = \mu_{1,j} \quad \text{for } 1 \le j \le M \text{ and } 1 \le i \le M+1-j, \tag{2.75}$$

and

$$\mu_{i,j} = 0 \quad \text{if } i > j. \tag{2.76}$$

In particular, $1 \le i \le M$,

$$\mu_{i,i} = \log(1-\lambda t). \tag{2.77}$$

Therefore,

$$\begin{aligned} &\sum_{k=1}^{\infty} \tfrac{1}{k} \mathbf{J}^k s^{kn} \\ &= -\log(I - s^n \mathbf{J}) \\ &= -diag\left(\log(I - s^n J_1), \cdots, \log(I - s^n J_Q)\right) \\ &= -[\mu_{i,j}]_{N\times N}, \end{aligned} \tag{2.78}$$

where

$$\log(I - s^n J_q) = \begin{bmatrix} \mu_{q;1,1} & \mu_{q;1,2} & \mu_{q;1,3} & \cdots & \mu_{q;1,M_q} \\ 0 & \mu_{q;2,2} & \mu_{q;2,3} & \cdots & \mu_{q;2,M_q} \\ & & \ddots & & \vdots \\ & & & & \\ 0 & 0 & \cdots & 0 & \mu_{q;M_q,M_q} \end{bmatrix} \tag{2.79}$$

and

$$\mu_{q;i,i} = \log(1-\lambda_q s^n),\ 1 \le q \le Q. \tag{2.80}$$

Now, Theorem 2.15 is generalized for general $\mathbf{T}_n$.

Lemma 2.16 *For $n \ge 1$, in (2.69) and (2.70) the generalized eigen-matrix is denoted by*

$$\mathbf{U} = [U_{1,1}\cdots U_{1,M_1};\cdots;U_{q,1}\cdots U_{q,M_q};\cdots;U_{Q,1}\cdots U_{Q,M_Q}]_{N\times N},$$

and its inverse is denoted by

$$\mathbf{U}^{-1} = [W_{1,1};\cdots;W_{1,M_1};\cdots;W_{q,1};\cdots;W_{q,M_q};\cdots;W_{Q,1};\cdots;W_{Q,M_Q}]_{N\times N}.$$

Then,

$$\zeta_n(s) = \prod_{q=1}^{Q} \prod_{1\le i\le j\le M_q} \exp\left(-\chi_{q;i,j}\mu_{q;i,j}\right), \tag{2.81}$$

where

$$\begin{aligned} \chi_{q;i,j} &= \tfrac{1}{n}|\mathbf{R}_n \circ U_{q,i}W_{q,j}| \\ &= \tfrac{1}{n}\sum_{p\in\mathcal{I}_n} \tfrac{\omega_{n,p}}{n} \left(\sum_{l=0}^{n-1} u_{q;\sigma^l(p),i}\right)\left(\sum_{l=0}^{n-1} w_{q;j,\sigma^l(p)}\right). \end{aligned} \tag{2.82}$$

In particular, if

$$\chi_{q;i,j} = 0 \textit{ for all } i \ne j, \tag{2.83}$$

then

$$\begin{aligned} \zeta_n(s) &= \prod_{q=1}^{Q}(1-\lambda_q s^n)^{-\chi_q} \\ &= \prod_{\lambda\in\Sigma(\mathbf{T}_n)}(1-\lambda s^n)^{-\chi(\lambda)}, \end{aligned} \tag{2.84}$$

where

$$\chi_q = \frac{1}{n}\sum_{i=1}^{M_q} |\mathbf{R}_n \circ U_{q;i}W_{q;i}| \tag{2.85}$$

and

$$\chi(\lambda) = \sum_{\lambda_q=\lambda} \chi_q. \tag{2.86}$$

Proof. From (2.69) and (2.78),

$$\zeta_n(s) = \exp\left(\frac{1}{n} tr\left(\mathbf{U}\left(-diag(\log(I-s^nJ_1),\cdots,\log(I-s^nJ_Q))\right)\mathbf{U}^{-1}\mathbf{R}_n\right)\right).$$

Now,

$$tr\left(\mathbf{U}diag(\log(I-s^nJ_1),\cdots,\log(I-s^nJ_Q))\mathbf{U}^{-1}\mathbf{R}_n\right)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{r=1}^{N}\sum_{p=1}^{N}u_{p,i}\mu_{i,j}w_{j,r}\left(\sum_{l=0}^{n-1}R^l_{n;r,p}\right)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p\in\mathcal{I}_n}\frac{\omega_{n,p}}{n}\left(\sum_{l=0}^{n-1}u_{\sigma^l(p),i}\right)\left(\sum_{l=0}^{n-1}w_{j,\sigma^{-l}(p)}\right)\mu_{i,j}.$$

Therefore, (2.81) follows. Clearly, if (2.83) holds, then (2.84) holds. The proof is complete. ■

In the rest of the section, (2.83) is proven and $\chi(\lambda)$ is shown to be a nonnegative integer. Therefore, ζ_n is a rational function. Some of the symmetry properties of the eigenvectors associated with the R_n-symmetry of $\mathbf{T}_n$ are investigated first.

Lemma 2.17 *For $n \geq 1$, if U is an eigenvector, then R_n^lU is also an eigenvector for any $0 \leq l \leq n-1$. Furthermore, if U is a generalized eigenvector, then R_n^lU is also a generalized eigenvector for any $0 \leq l \leq n-1$.*

Based on Lemma 2.17, the equivalent class $\mathcal{R}(U)$ of eigenvector U is introduced by R_n.

Definition 2.18 *For any $N \times 1$ column vector U,*

$$\mathcal{R}(U) = \left\{R_n^lU|0 \leq l \leq n-1\right\}. \tag{2.87}$$

U is called (R_n-) symmetric if $\mathcal{R}(U) = \{U\}$, such meaning that $u_j = u_i$ for all $j \in C_n(i)$ or

$$R_n^lU = U \tag{2.88}$$

for all $0 \leq l \leq n-1$. U is called (R_n-) anti-symmetric if $\sum_{l=0}^{n-1} R_n^lU = 0$, such meaning

$$\sum_{l=0}^{n-1}U_{\sigma^l(i)} = 0 \tag{2.89}$$

for all $i \in \mathcal{I}_n$.

For a symmetric eigenvector U, the following property is observed.

Lemma 2.19 *Let* $U = (u_1, u_2, \cdots, u_N)^t$ *and* $W = (w_1, w_2, \cdots, w_N)$,

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i \in \mathcal{I}_n} \frac{1}{\omega_{n,i}} \left(\sum_{j \in C_n(i)} u_j \right) \left(\sum_{j \in C_n(i)} w_j \right). \tag{2.90}$$

Furthermore, if U *is symmetric, then*

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = WU = \sum_{j=1}^{N} u_j w_j, \tag{2.91}$$

and if U *is anti-symmetric, then*

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = 0. \tag{2.92}$$

Proof. Clearly,

$$\sum_{l=0}^{n-1} u_{\sigma^l(i)} = \frac{n}{\omega_{n,i}} \sum_{j \in C_n(i)} u_j \quad \text{and} \quad \sum_{l=0}^{n-1} w_{\sigma^l(i)} = \frac{n}{\omega_{n,i}} \sum_{j \in C_n(i)} w_j. \tag{2.93}$$

Therefore, substituting (2.93) into (2.82) yields

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i \in \mathcal{I}_n} \frac{1}{\omega_{n,i}} \left(\sum_{j \in C_n(i)} u_j \right) \left(\sum_{j \in C_n(i)} w_j \right).$$

If U is symmetric, then

$$\sum_{j \in C_n(i)} u_j = \omega_{n,i} u_i.$$

Hence,

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i \in \mathcal{I}_n} \left(\sum_{j \in C_n(i)} u_i w_j \right) = \sum_{j=1}^{N} u_j w_j = WU.$$

The proof is complete. ∎

The following orthogonal matrix Q_n is very useful in finding symmetric and anti-symmetric eigenvectors of $\mathbf{T}_n$, the details of proof is omitted.

Lemma 2.20 *For* $n \geq 2$, *the* $n \times n$ *matrix* $Q_n =$

$$
\begin{bmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\
\sqrt{\frac{n-1}{n}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} \\
0 & \sqrt{\frac{n-2}{n-1}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} \\
 & & & \vdots & & \\
0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix} \tag{2.94}
$$

is orthogonal.

In the following lemma, when Q_n is used, $\mathcal{R}(U)$ can be expressed by symmetric and anti-symmetric eigenvectors.

Lemma 2.21 *For $n \geq 2$, given eigenvector U, define*

$$\overline{U}_1 = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} R_n^l U \tag{2.95}$$

and, $2 \leq j \leq n$,

$$\overline{U}_j = \sqrt{\frac{n-j+1}{n-j+2}} R_n^{j-2} U - \frac{1}{\sqrt{n-j+1}\sqrt{n-j+2}} \sum_{k=j-1}^{n-1} R_n^k U. \tag{2.96}$$

If $\mathcal{R}(U)$ has rank κ, for some κ, $1 \leq \kappa \leq n$,

(i) then $\{\overline{U}_j\}_{j=1}^n$ also has rank κ;

(ii) if $\overline{U}_1 \neq 0$, then $\overline{U}_1$ is symmetric, and for each j, $2 \leq j \leq n$, $\overline{U}_j$ is anti-symmetric.

Proof. Clearly,

$$\left(\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_n\right)^t = Q_n \left(U, R_n U, \cdots, R_n^j U, \cdots, R_n^{n-1} U\right)^t.$$

Since Q_n is orthogonal, (i) holds.

Since $R_n(\overline{U}_1) = \overline{U}_1$, $\overline{U}_1$ is symmetric. For $2 \leq j \leq n$ and $i \in \mathcal{I}_n$,

$$\begin{aligned}\sum_{l=0}^{n-1}(\overline{U}_j)_{\sigma^l(i)} &= \sqrt{\tfrac{n-j+1}{n-j+2}}\left(\sum_{l=0}^{n-1}(R_n^{j-2}U)_{\sigma^l(i)} - \tfrac{1}{n-j+1}\sum_{k=j-1}^{n-1}\sum_{l=0}^{n-1}(R_n^k U)_{\sigma^l(i)}\right)\\ &= \sqrt{\tfrac{n-j+1}{n-j+2}}\left(\sum_{l=0}^{n-1}u_{\sigma^l(i)} - \tfrac{1}{n-j+1}\sum_{k=j-1}^{n-1}\sum_{l=0}^{n-1}u_{\sigma^l(i)}\right)\\ &= 0.\end{aligned}$$

Therefore, $\overline{U}_j$ is anti-symmetric for any $2 \le j \le n$. The proof is complete. ■

The main result can now be proven.

Theorem 2.22 *For* $n \ge 1$,

$$\frac{1}{n}tr\left(\mathbf{T}_n^k\mathbf{R}_n\right) = \sum_{\lambda\in\Sigma(\mathbf{T}_n)}\chi(\lambda)\lambda^k \tag{2.97}$$

and

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_n)}(1-\lambda s^n)^{-\chi(\lambda)}, \tag{2.98}$$

where $\chi(\lambda)$ *is the number of linearly independent symmetric eigenvectors and generalized eigenvectors of* $\mathbf{T}_n$ *with eigenvalue* λ.

Proof. The case of symmetric $\mathbf{T}_n$ is considered first. Let E_λ be the eigenspace of $\mathbf{T}_n$ with eigenvalue λ. By Lemma 2.21, E_λ is spanned by linearly independent symmetric eigenvectors $\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_p$ and anti-symmetric eigenvectors $U'_1, U'_2, \cdots, U'_{p'}$, where $p + p' = dim(E_\lambda)$ and p or p' may be zero.

Now,

$$\begin{aligned}\chi(\lambda) &= \tfrac{1}{n}\left(\sum_{j=1}^{p}|\mathbf{R}_n\circ\overline{U}_j\overline{U}_j^t| + \sum_{j=1}^{p'}|\mathbf{R}_n\circ U'_j(U'_j)^t|\right)\\ &= p,\end{aligned} \tag{2.99}$$

which is the number of linearly independent symmetric eigenvectors of $\mathbf{T}_n$ with eigenvalue λ.

For general $\mathbf{T}_n$, in Jordan canonical form (2.69) and (2.71), $\mathbf{U}$ can be decomposed into

$$\mathbf{U} = E_{\lambda_1}\oplus E_{\lambda_2}\oplus\cdots\oplus E_{\lambda_Q}.$$

Each E_{λ_j} is spanned by symmetric eigenvectors and generalized eigenvectors $\overline{U}_{j,1}, \overline{U}_{j,2}, \cdots, \overline{U}_{j,p_j}$ and anti-symmetric eigenvectors and generalized eigenvectors $U'_{j,1}, U'_{j,2}, \cdots, U'_{j,p'_j}$, and $p_j + p'_j = dim(E_{\lambda_j})$.

The inverse matrix is $\mathbf{U}^{-1} =$

$$\left[\overline{W}_{1,1}; \cdots; \overline{W}_{1,p_1}; W'_{1,1}; \cdots; W'_{1,p'_1}; \cdots; \overline{W}_{Q,1}; \cdots; \overline{W}_{Q,p_Q}; W'_{Q,1}; \cdots; W'_{Q,p'_Q}\right].$$

Lemma 2.19 implies

$$\tfrac{1}{n}|\mathbf{R}_n \circ \overline{U}_{j,i}\overline{W}_{j',k}| = \delta_{jj'}\delta_{ik} \quad \text{and} \quad \tfrac{1}{n}|\mathbf{R}_n \circ U'_{j,i}W'_{j',k}| = 0.$$

Therefore,

$$\begin{aligned}\chi(\lambda_j) &= p_j \\ &= \text{the number of linearly independent symmetric eigenvectors and} \\ &\quad \text{generalized eigenvectors of } \mathbf{T}_n \text{ with eigenvalue } \lambda_j.\end{aligned}$$

The result follows. The proof is complete. ∎

Now, the reduced trace operator τ_n of $\mathbf{T}_n$ is recalled as in (1.17).

Definition 2.23 *For $n \geq 1$, $\mathbf{T}_n = [t_{n;i,j}]$. For each $i, j \in \mathcal{I}_n$, define*

$$\tau_{n;i,j} = \sum_{k \in C_n(j)} t_{n;i,k} \tag{2.100}$$

and denote the reduced trace operator of $\mathbf{T}_n$ by $\tau_n = [\tau_{n;i,j}]$, which is a $\chi_n \times \chi_n$ matrix.

The following theorem indicates that τ_n is more effective in computing the eigenvalues with rotationally symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_n$. See also Examples 2.54 and 2.55.

Theorem 2.24 *$\lambda \in \Sigma(\mathbf{T}_n)$ with $\chi(\lambda) \geq 1$ if and only if $\lambda \in \Sigma(\tau_n)$. Moreover, $\chi(\lambda)$ is the algebraic multiplicity of τ_n with eigenvalue λ. Furthermore,*

$$\frac{1}{n}\sum_{l=0}^{n-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = \sum_{\lambda \in \Sigma(\tau_n)} \chi(\lambda)\lambda^k = tr(\tau_n^k), \tag{2.101}$$

and

$$\zeta_n(s) = \exp\left(\sum_{k=1}^{\infty} \frac{tr(\tau_n^k)}{k} s^{nk}\right). \tag{2.102}$$

Proof. Let $\lambda \in \Sigma(\mathbf{T}_n)$ be an eigenvalue with rotationally symmetric eigenvector $U = (u_1, u_2, \cdots, u_{2^n})^t$, where $u_i = u_j$ for any $i \in \mathcal{I}_n$ and $j \in C_n(i)$. Define $V = (u_1, \cdots, u_i, \cdots, u_{2^n})^t$ for $i \in \mathcal{I}_n$. Then, clearly, $\mathbf{T}_n U = \lambda U$ implies $\tau_n V = \lambda V$.

On the other hand, if $\tau_n V = \lambda V$ and $V = (v_1, \cdots, v_i, \cdots, v_{2^n})^t$, then V can be extended to U, a 2^n-vector, by $u_j = v_i$ for $i \in \mathcal{I}_n$ and $j \in C_n(i)$. Then, $\mathbf{T}_n U = \lambda U$ and U is rotationally symmetric. The arguments also hold for a generalized eigenvector.

Finally, (2.101) follows from (2.51) and (2.97), and (2.102) follows from (1.8) and (2.101). The proof is complete. ■

Remark 2.25 *According to Theorem 2.24, the following is easily verified;*

$$\sum_{\lambda \in \Sigma(\mathbf{T}_n)} \chi(\lambda) = \sum_{\lambda \in \Sigma(\tau_n)} \chi(\lambda) = \chi_n. \tag{2.103}$$

Theorem 2.24 yields the following result.

Theorem 2.26 *For $n \geq 1$,*

$$\zeta_n(s) = \left(\det\left(I - s^n \tau_n\right)\right)^{-1} \tag{2.104}$$

$$= \prod_{\lambda \in \Sigma(\tau_n)} \left(1 - \lambda s^n\right)^{-\chi_n(\lambda)}, \tag{2.105}$$

where $\chi_n(\lambda)$ is the algebraic multiplicity of $\lambda \in \Sigma(\tau_n)$ and

$$\zeta(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n \tau_n\right)\right)^{-1} \tag{2.106}$$

$$= \prod_{n=1}^{\infty} \prod_{\lambda \in \Sigma(\tau_n)} \left(1 - \lambda s^n\right)^{-\chi_n(\lambda)}. \tag{2.107}$$

2.3 More symbols on larger lattice

This subsection extends the results found in previous sections to any finite number of symbols
$p \geq 2$ on any finite square lattice $\mathbb{Z}_{m \times m}$, $m \geq 2$. The results are outlined here and the details are left to the reader. The proofs of the theorems are omitted for brevity.

For fixed positive integers $p \geq 2$ and $m \geq 2$, the set of symbols is denoted by
$\mathcal{S}_p = \{0, 1, 2, \cdots, p-1\}$ and the basic square lattice is $\mathbb{Z}_{m \times m}$. We need the following notations.

For any fixed $n \geq m$, such as in (2.14), the x-periodic patterns of period n with height m can be recorded as $C_{n\times m;i,j}$ in $\mathbf{C}_{n\times m}$ by $C_{n\times m;i,j} =$

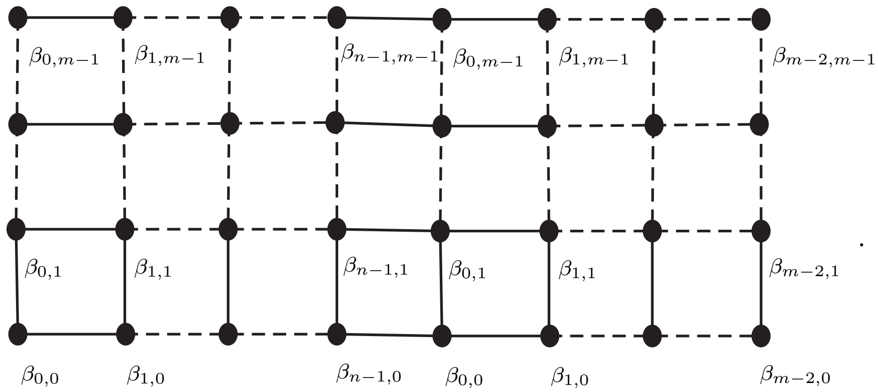


Fig 2.1.

Similarly, when $1 \leq n \leq m-1$, $\mathbf{C}_{n\times m} = [C_{n\times m;i,j}]$ can also be defined as an $(n+m-1)\times m$ pattern in Fig 2.1.

Then, for any $n \geq 1$, the associated trace operator $\mathbf{T}_{n\times m} = [t_{n\times m;i,j}]$ can be defined by

$$t_{n\times m;i,j} = 1 \text{ if and only if } C_{n\times m;i,j} \text{ is } \mathcal{B}\text{-admissible}. \tag{2.108}$$

Now, for any $n \geq 1$, the corresponding rotational matrix $R_{n\times(m-1)}$ which is a zero-one $p^{n(m-1)} \times p^{n(m-1)}$ matrix is defined by

$$R_{n\times(m-1);i,j} = 1 \text{ if and only if}$$

$$j = \sigma(i), \tag{2.109}$$

where i is given by $1 \leq i \leq p^{n(m-1)}$ and $1 \leq \sigma(i) \leq p^{n(m-1)}$ is represented by

$$\sigma(i) = \psi\left([\sigma(\overline{\beta}_0)\sigma(\overline{\beta}_1)\cdots\sigma(\overline{\beta}_{m-2})]\right). \tag{2.110}$$

The explicit expression for $R_{n\times(m-1)}$, like (2.31), can also be obtained and the result is omitted here.

As (2.33) and (2.34), the equivalent class $C_{n\times(m-1)}(i)$ of i is defined by

$$C_{n\times(m-1)}(i) = \{\sigma^j(i)|0 \le j \le n-1\}$$
$$= \left\{ j \middle| \left(R^l_{n\times(m-1)}\right)_{i,j} = 1 \text{ for some } 1 \le l \le n \right\}, \tag{2.111}$$

and the index set $\mathcal{I}_{n\times(m-1)}$ of n is defined by

$$\mathcal{I}_{n\times(m-1)} = \{i|1 \le i \le p^{n(m-1)}, i \le \sigma^q(i), 1 \le q \le n-1\}$$
$$= \left\{ i \middle| 1 \le i \le p^{n(m-1)}, i \le j \text{ for all } j \in C_{n\times(m-1)}(i) \right\}. \tag{2.112}$$

The cardinal number of $\mathcal{I}_{n\times(m-1)}$ is denoted by $\chi_{n\times(m-1)}$ and $\chi_{n\times(m-1)}$ is equal to the number of necklaces that can be made from 2^{m-1} colors, when the necklaces can be rotated but not turned over [48]. $\chi_{n\times(m-1)}$ is expressed as

$$\chi_{n\times(m-1)} = \frac{1}{n}\sum_{d|n}\phi(d)\left(2^{m-1}\right)^{n/d}. \tag{2.113}$$

Like Proposition 2.3, $R_{n\times(m-1)}$ has the permutation properties. Now, define

$$\mathbf{R}_{n\times(m-1)} = \sum_{l=0}^{n-1} R^l_{n\times(m-1)}. \tag{2.114}$$

A similar result to Theorem 2.13 can now be obtained for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

Theorem 2.27 *For $n \ge 1$, $k \ge 1$ and $0 \le l \le n-1$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m} R^l_{n\times(m-1)}\right) \tag{2.115}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m}\mathbf{R}_{n\times(m-1)}\right). \tag{2.116}$$

As in (1.8), the n-th order zeta function is given by

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)s^{kn}\right). \tag{2.117}$$

From Theorem 2.27, the following theorem is obtained.

Theorem 2.28 *For any $n \geq 1$,*

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr\left(\mathbf{T}_{n\times m}^k \mathbf{R}_{n\times(m-1)}\right)s^{nk}\right). \tag{2.118}$$

The proof that $\zeta_n(s)$ is a rational function depends on the fact that $\mathbf{T}_{n\times m}$ is also $R_{n\times(m-1)}$-symmetric, which is stated as follows.

Proposition 2.29 *For any $n \geq 1$,*

$$t_{n\times m;\sigma(i),\sigma(j)} = t_{n\times m;i,j} \tag{2.119}$$

for any $1 \leq i, j \leq p^{n(m-1)}$.

Then the reduced trace operator $\tau_{n\times m}$ of $\mathbf{T}_{n\times m}$ is defined as follows.

Definition 2.30 *For $n \geq 1$, the reduced trace operator $\tau_{n\times m} = [\tau_{n\times m;i,j}]$ of $\mathbf{T}_{n\times m}$ is a $\chi_{n\times(m-1)} \times \chi_{n\times(m-1)}$ matrix defined by*

$$\tau_{n\times m;i,j} = \sum_{k\in C_{n\times(m-1)}(j)} t_{n\times m;i,k} \tag{2.120}$$

for each $i, j \in \mathcal{I}_{n\times(m-1)}$.

The notion of symmetric and anti-symmetric eigenvectors of $\mathbf{T}_{n\times m}$ can also be defined as in Definition 2.18. Now, the main result can be obtained.

Theorem 2.31 *For any $n \geq 1$,*

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_{n\times m})} (1-\lambda s^n)^{-\chi(\lambda)} \tag{2.121}$$

$$= \left(\det\left(I - s^n\tau_{n\times m}\right)\right)^{-1}, \tag{2.122}$$

where $\chi(\lambda)$ is the number of linearly independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{n\times m}$ with eigenvalue λ. The zeta function is

$$\zeta(s) = \prod_{n=1}^{\infty}\left(\det\left(I - s^n\tau_{n\times m}\right)\right)^{-1}. \tag{2.123}$$

2.4 Zeta functions presented in inclined coordinates

This subsection will present the zeta function with respect to the inclined coordinates, as determined by applying unimodular transformations in $GL_2(\mathbb{Z})$. $\mathbb{Z}^2$ is known to be invariant with respect to unimodular transformation. Indeed, Lind [36] proved that $\zeta^0_{\mathcal{B};\gamma} = \zeta^0_{\mathcal{B}}$ for any $\gamma \in GL_2(\mathbb{Z})$: the zeta function is independent of a choice of basis for $\mathbb{Z}^2$. This section presents the constructions of the trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ and the reduced trace operator $\tau_{\gamma;n}(\mathcal{B})$, then determines $\zeta_{\mathcal{B};\gamma;n}$ and $\zeta_{\mathcal{B};\gamma}$. Finally, $\zeta_{\mathcal{B};\gamma}$ is obtained as

$$\zeta_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty} (\det(I - s^n \tau_{\gamma;n}(\mathcal{B})))^{-1}. \tag{2.124}$$

As mentioned in (1.29), $\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s)$ in $|s| < \exp(-g(\mathcal{B}))$, for any $\gamma \in GL_2(\mathbb{Z})$, which yields a family of identities when $\zeta_{\mathcal{B};\gamma}$ is expressed as Taylor series at the origin $s = 0$. Furthermore, for some $\mathcal{B} \subset \Sigma_{2\times 2}$, we may find a $\gamma \in GL_2(\mathbb{Z})$ such that $\zeta_{\mathcal{B};\gamma}$ offers a better description of poles and natural boundary of $\zeta^0_{\mathcal{B}}$ when $\zeta_{\mathcal{B}}$ and $\widehat{\zeta}_{\mathcal{B}}$ fail to do so, see Example 2.56.

For simplicity, only $\mathcal{B} \subset \Sigma_{2\times 2}$ with two symbols are considered. The general cases can be treated analogously.

We begin with the study in the modular group $SL_2(\mathbb{Z})$. The results also hold for any $\gamma \in GL_2(\mathbb{Z})$ with $\det \gamma = -1$.

Recall the modular group

$$SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| \; a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ is called a unimodular transformation. Then,

$$\mathbb{Z}^2 = \{p(a, c) + q(b, d) | p, q \in \mathbb{Z}\} \tag{2.125}$$

holds, here $\mathbb{Z}^2$ is the set of lattice points (vertices).

Consider the set of all finite-index subgroups $\mathcal{L}_2$ of $\mathbb{Z}^2$ by

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2 \Big| \; a_{11}a_{22} - a_{12}a_{21} \geq 1, a_{ij} \in \mathbb{Z}, 1 \leq i, j \leq 2 \right\},$$

here $\mathbb{Z}^2 = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} | n_1, n_2 \in \mathbb{Z} \right\}$. An equivalent relation $\sim$ exists in $\mathcal{L}_2$. Two sublattices $L = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2$ and $L' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \mathbb{Z}^2$ are equivalent if L and L' determine the same sublattice of $\mathbb{Z}^2$: $L' = L$.

The following result states the existence of unique Hermite normal upper (or lower) triangular forms within each equivalent class in $\mathcal{L}_2$.

Proposition 2.32 *For each* $L = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$, *there is a unique* $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$, $n, k \geq 1$ *and* $0 \leq l \leq n-1$, *and* $\begin{bmatrix} k_1 & 0 \\ l_1 & n_1 \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$, $n_1, k_1 \geq 1$ *and* $0 \leq l_1 \leq n_1 - 1$, *such that they are equivalent, where*

$$nk = n_1 k_1 = a_{11}a_{22} - a_{12}a_{21}. \tag{2.126}$$

The proof can be found elsewhere [39].

For a given $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, the lattice points in γ-coordinates are

$$(1,0)_\gamma = (a,b) \quad \text{and} \quad (0,1)_\gamma = (c,d),$$

and the unit vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Notably, when $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, standard rectangular coordinates are used and the subscript γ is omitted.

The parallelogram with respect to γ is defined by

$$M_\gamma = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma = \begin{bmatrix} na & la + kc \\ nb & lb + kd \end{bmatrix}.$$

Let $L_\gamma = M_\gamma \mathbb{Z}^2$. Then,

$$L_\gamma = \gamma^t L \tag{2.127}$$

is easily verified.

The Hermite normal form in Proposition 2.32 indicates the existence and uniqueness of $0 \le l_j \le n_j - 1$, $1 \le k_j$ for $j = 1, 2$, such that

$$L_\gamma = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 = \begin{bmatrix} n_1 & l_1 \\ 0 & k_1 \end{bmatrix} \mathbb{Z}^2 = \begin{bmatrix} k_2 & 0 \\ l_2 & n_2 \end{bmatrix} \mathbb{Z}^2 \tag{2.128}$$

with $n_1 k_1 = n_2 k_2 = nk$. Therefore, the n-th order zeta function of $\zeta^0_{\mathcal{B}}(s)$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \frac{1}{k} \Gamma_{\mathcal{B}} \left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma \right) s^{nk} \right) \tag{2.129}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{n=1}^{\infty} \zeta_{\mathcal{B};\gamma;n}(s). \tag{2.130}$$

Since (2.128) holds, the iterated sum in (2.129) and (2.130) is a rearrangement of $\zeta^0_{\mathcal{B}}(s)$. Therefore,

$$\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s) \tag{2.131}$$

for $|s| < \exp(-g(\mathcal{B}))$. See Proposition 2.44 (i) and another work [36].

The main purpose of this subsection is to establish results that are similar to Theorems 2.22, 2.26 and 2.31:

$$\zeta_{\mathcal{B};\gamma;n}(s) = \prod_{\lambda \in \Sigma(\mathbf{T}_{\gamma;n})} (1 - \lambda s^n)^{-\chi_{\gamma;n}} \tag{2.132}$$

$$= \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}, \tag{2.133}$$

where $\mathbf{T}_{\gamma;n}$ is the trace operator with respect to γ and $\tau_{\gamma;n}$ is the associated reduced trace operator of $\mathbf{T}_{\gamma;n}$. The following introduces cylindrical matrix and rotational symmetrical operator $R_{\gamma;n}$. The proofs of the results are omitted.

In the following, a unimodular transformation γ is given and fixed. Let $\mathbb{Z}_{\gamma;n\times m}$ be the $n \times m$ lattice with one side in the $\gamma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} a \\ b \end{pmatrix}$ direction and the other side in the $\gamma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} c \\ d \end{pmatrix}$ direction. The total number of lattice points on $\mathbb{Z}_{\gamma;n\times m}$ is $n \cdot m$. The ordering matrix $\mathbf{Y}_{\gamma;n\times m} = [y_{\gamma;n\times m;i,j}]$ of local patterns $[\beta_{\gamma;\alpha_1,\alpha_2}]_{0:n-1,0:m-1}$ is defined on $\mathbb{Z}_{\gamma;n\times m}$. On $\mathbb{Z}_{\gamma;2\times 2}$ and $\mathbb{Z}_{\gamma;n\times 2}$, $\mathbf{Y}_{\gamma;2\times 2}$ is arranged as in (2.2) and $\mathbf{Y}_{\gamma;n\times 2}$ is defined recursively as in (2.12) and (2.13), except that the horizontal is now in the γ_1 direction and the vertical is in the γ_2 direction.

The γ_1-periodic patterns of period n with height m on $\mathbb{Z}_{\gamma;(n+1)\times m}$ can be recorded as $C_{\gamma;n\times m;i,j}$ in a cylindrical matrix $\mathbf{C}_{\gamma;n\times m}$. The shift operator σ_γ is defined to shift one step to the left in the γ_1 direction.

Since the admissible local pattern $\mathcal{B}$ is given on square lattice $\mathbb{Z}_{2\times 2}$, the periodic patterns in γ-coordinates that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2}$. Let $\mathbb{Z}_{2\times 2}\left((i,j)_\gamma\right)$ be the square lattice with the left-bottom vertex $(i,j)_\gamma = (i',j')$:

$$\mathbb{Z}_{2\times 2}\left((i,j)_\gamma\right) = \left\{(i',j'),(i'+1,j'),(i',j'+1),(i'+1,j'+1)\right\}.$$

Now, the admissibility is demonstrated to have to be verified on finite square lattices as follows.

Proposition 2.33 *Given* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *and* $n \geq 1$, $k \geq 1$ *and* $0 \leq l \leq n-1$. *An* $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma$*-periodic pattern* U *is* $\mathcal{B}$*-admissible if and only if*

$$U\big|_{\mathbb{Z}_{2\times 2}((\xi,\eta)_\gamma)} \in \mathcal{B} \tag{2.134}$$

for any $0 \leq \xi \leq n-1$ *and* $0 \leq \eta \leq k-1$.

For a given basic set $\mathcal{B} \subset \{0,1\}^{\mathbb{Z}_{2\times 2}}$, the definition of trace operator $\mathbf{T}_{\gamma;n}$ of $\mathcal{B}$ has to be justified, since $\mathcal{B}$ is given in a 2×2 square lattice in the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ directions and $\mathbf{T}_{\gamma;n}$ is defined in the $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ directions.

For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, the height $h(\gamma)$ of γ is

$$h = h(\gamma) = |a| + |b|, \tag{2.135}$$

and the width $w(\gamma)$ of γ is

$$w = w(\gamma) = |c| + |d|. \tag{2.136}$$

The following lemma determines that the first square lattice that occurs in a parallelogram in the γ-coordinates.

Lemma 2.34 *For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, there exists exactly one square lattice that is determined by a parallelogram with vertices $(0,0)_\gamma$, $(w,0)_\gamma$, $(0,h)_\gamma$ and $(w,h)_\gamma$. The square lattice has either vertices $(0,h)_\gamma$ and $(w,0)_\gamma$ or vertices $(0,0)_\gamma$ and $(w,h)_\gamma$.*

The lemma shows that the existence of the parallelogram contains exactly $n \cdot k$ square lattices, as follows.

Proposition 2.35 *Given $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, for any $n \geq 1$ and $k \geq 1$, exactly $n \cdot k$ square lattices have pairs of vertices that lie on the parallelogram that is determined by $(0,0)_\gamma$, $(w+n-1,0)_\gamma$, $(0,h+k-1)_\gamma$ and $(w+n-1,h+k-1)_\gamma$.*

For a given $\mathcal{B}$, $\gamma \in SL_2(\mathbb{Z})$ and $n \geq 1$, the trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ acts exactly on n square lattices which lie in the γ_1-direction.

Therefore, consider $\mathbb{Z}_{\gamma;n+w,h+1}$. From Proposition 2.35, n square lattices have pairs of vertices on $\mathbb{Z}_{\gamma;n+w,h+1}$. The γ_1-periodic patterns with period n and height $h+1$ are denoted by $\mathbf{C}_{\gamma;n+w,h+1}$.

The trace operator $\mathbf{T}_{\gamma;n} = \mathbf{T}_{\gamma;n}(\mathcal{B}) = [t_{\gamma;n;i,j}]$, associated with $\mathcal{B}$, is defined by

$$t_{\gamma;n;i,j} = 1 \quad \text{if and only if} \quad \text{the pattern in } \mathbf{C}_{\gamma;n+w,h+1;i,j} \text{ is } \mathcal{B}\text{-admissible.} \tag{2.137}$$

As in another study [6], a recursive formula exists for $\mathbf{T}_{\gamma;n+1}$ in terms of $\mathbf{C}_{\gamma;n+w+1,h+1;i,j}$, $\mathcal{B}$ and γ.

A similar result as in Proposition 2.7 can be obtained; the detailed proof is omitted.

Proposition 2.36 *For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, $n \geq 1$ and $k \geq 1$, $\left(\mathbf{T}^k_{\gamma;n}\right)_{i,j}$ is the number of $\mathcal{B}$-admissible patterns of the form*

$$\begin{aligned} &\left[\overline{\beta}_{\gamma;0}\overline{\beta}_{\gamma;1}\cdots\overline{\beta}_{\gamma;h+k-1}\right] \\ =\; &\left[\overline{\beta}_{\gamma;0}\cdots\overline{\beta}_{\gamma;h-1}\right] \bullet \left[\overline{\beta}_{\gamma;1}\cdots\overline{\beta}_{\gamma;h}\right] \bullet\cdots\bullet \left[\overline{\beta}_{\gamma;k}\cdots\overline{\beta}_{\gamma;h+k-1}\right], \end{aligned}$$

where

$$i = \psi\left(\left[\overline{\beta}_{\gamma;0} \cdots \overline{\beta}_{\gamma;h-1}\right]\right) \tag{2.138}$$

and

$$j = \psi\left(\left[\overline{\beta}_{\gamma;k} \cdots \overline{\beta}_{\gamma;k+h-1}\right]\right). \tag{2.139}$$

Now, for any $n \geq 1$, the associated rotational matrix $R_{\gamma;n}$ which is a zero-one $2^{nh} \times 2^{nh}$ matrix is defined by

$$R_{\gamma;n;i,j} = 1 \quad \text{if and only if} \quad j = \sigma_\gamma(i), \tag{2.140}$$

where $1 \leq i \leq 2^{nh}$ is given by (2.138) and $1 \leq \sigma_\gamma(i) \leq 2^{nh}$ is defined by

$$\sigma_\gamma(i) = \psi\left(\left[\sigma_\gamma(\overline{\beta}_{\gamma;0})\sigma_\gamma(\overline{\beta}_{\gamma;1}) \cdots \sigma_\gamma(\overline{\beta}_{\gamma;h-1})\right]\right). \tag{2.141}$$

The equivalent class $C_{\gamma;n}(i)$, the index set $\mathcal{I}_{\gamma;n}$ and the cardinal number $\chi_{\gamma;n}$ of $\mathcal{I}_{\gamma;n}$ can be defined as similar to (2.111)$\sim$(2.113) and are omitted here. Now, the reduced trace operator is defined as follows.

Definition 2.37 *For $n \geq 1$, the reduced trace operator $\tau_{\gamma;n} = [\tau_{\gamma;n;i,j}]$ of $\mathbf{T}_{\gamma;n}$ is a $\chi_{\gamma;n} \times \chi_{\gamma;n}$ matrix defined by*

$$\tau_{\gamma;n;i,j} = \sum_{k \in C_{\gamma;n}(j)} t_{\gamma;n;i,k} \tag{2.142}$$

for each $i, j \in \mathcal{I}_{\gamma;n}$.

Now, define

$$\mathbf{R}_{\gamma;n} = \sum_{l=0}^{n-1} R^l_{\gamma;n}. \tag{2.143}$$

It is easy to verify that all results also hold for any $\gamma \in GL_2(\mathbb{Z})$ with $\det\gamma = -1$. The main results as in Theorem 2.13 are then obtained.

Theorem 2.38 *Given any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$. Then, for any $n \geq 1$, $k \geq 1$ and $0 \leq l \leq n-1$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) = tr\left(\mathbf{T}^k_{\gamma;n} R^l_{\gamma;n}\right) \tag{2.144}$$

and

$$\sum_{l=0}^{n-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_{\gamma}\right) = tr\left(\mathbf{T}_{\gamma;n}^{k}\mathbf{R}_{\gamma;n}\right). \tag{2.145}$$

Moreover,

$$\zeta_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k} tr\left(\mathbf{T}_{\gamma;n}^{k}\mathbf{R}_{\gamma;n}\right) s^{nk}\right). \tag{2.146}$$

Finally, by the argument as in Subsections 2.2 and 2.3, the rationality of the n-th order zeta function $\zeta_{\mathcal{B};\gamma;n}$ is established, as in Theorems 2.22, 2.26 and 2.31.

Theorem 2.39 *For any* $\mathcal{B} \subset \Sigma_{2\times 2}$ *and* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$,

$$\zeta_{\mathcal{B};\gamma;n}(s) = \prod_{\lambda \in \Sigma(\mathbf{T}_{\gamma;n}(\mathcal{B}))} (1 - \lambda s^n)^{-\chi_{\gamma;n}(\lambda)} \tag{2.147}$$

$$= \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}, \tag{2.148}$$

where the exponent $\chi_{\gamma;n}(\lambda)$ *is the number of linearly independent* $R_{\gamma;n}$*-symmetric eigenvectors of* $\mathbf{T}_{\gamma;n}(\mathcal{B})$ *with respect to eigenvalue* λ. *The zeta function of* $\mathcal{B}$ *with respect to* γ*-coordinates is*

$$\zeta_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}. \tag{2.149}$$

An immediate consequence of (2.149) is the following result, see Proposition 2.44 and [36].

Corollary 2.40 *For any* $\mathcal{B} \subset \Sigma_{2\times 2}$ *and* $\gamma \in GL_2(\mathbb{Z})$, *the Taylor series for* $\zeta_{\mathcal{B};\gamma}$ *at* $s = 0$ *has integer coefficients.*

Proof. Since $\tau_{\gamma;n}$ has integer entries for any $n \geq 1$. The result follows. ■

We now briefly investigate the zeta functions presented in the lower Hermite normal form. For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$ and $n \geq 1$, define

$$\widehat{\zeta}_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_{\gamma}\right) s^{nk}\right), \tag{2.150}$$

where $\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_\gamma = \begin{bmatrix} ka+lc & nc \\ kb+ld & nd \end{bmatrix}$ and

$$\widehat{\zeta}_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty} \widehat{\zeta}_{\mathcal{B};\gamma;n}(s). \tag{2.151}$$

Denote by

$$\widehat{\gamma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.152}$$

the reflection $\frac{\pi}{4}$ with respect to the diagonal axis $y = x$. Then we have the following results.

Theorem 2.41 *For any $\gamma \in GL_2(\mathbb{Z})$,*

$$\widehat{\zeta}_{\mathcal{B};\gamma;n} = \zeta_{\mathcal{B};\widehat{\gamma}\gamma;n} \tag{2.153}$$

and

$$\widehat{\zeta}_{\mathcal{B};\gamma} = \zeta_{\mathcal{B};\widehat{\gamma}\gamma}. \tag{2.154}$$

In particular,

$$\widehat{\zeta}_{\mathcal{B}} = \zeta_{\mathcal{B};\widehat{\gamma}}. \tag{2.155}$$

Proof. For any $n \geq 1, k \geq 1$ and $0 \leq l \leq n-1$, and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$, denote by the lattices

$$\widehat{L} = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix} \mathbb{Z}^2 \text{ and } \widehat{L}_\gamma = \widehat{M}_\gamma \mathbb{Z}^2, \tag{2.156}$$

where the parallelogram $\widehat{M}_\gamma$ is defined by

$$\widehat{M}_\gamma = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_\gamma. \tag{2.157}$$

As in (2.127), it is easy to verify

$$\widehat{L}_\gamma = \gamma^t \widehat{L}, \tag{2.158}$$

and

$$\widehat{L} = L_{\widehat{\gamma}}, \quad L = \widehat{L}_{\widehat{\gamma}} \quad \text{and} \quad \widehat{L}_\gamma = L_{\widehat{\gamma}\gamma}, \quad L_\gamma = \widehat{L}_{\widehat{\gamma}\gamma}. \tag{2.159}$$

Therefore,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right) = \Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_{\widehat{\gamma}}\right). \tag{2.160}$$

Hence, (2.160) implies

$$\widehat{\zeta}_{\mathcal{B};n} = \zeta_{\mathcal{B};\widehat{\gamma};n}$$

and

$$\widehat{\zeta}_{\mathcal{B}} = \zeta_{\mathcal{B};\widehat{\gamma}}.$$

Therefore, (2.153) and (2.154) follow. The proof is complete. ■

Remark 2.42 *From Theorem 2.41, for any $\mathcal{B} \subset \Sigma_{2\times 2}$ there is a family of zeta functions $\{\zeta_{\mathcal{B};\gamma} | \gamma \in GL_2(\mathbb{Z})\} = \left\{\widehat{\zeta}_{\mathcal{B};\gamma} | \gamma \in GL_2(\mathbb{Z})\right\}$. In computation, it is much easier to study $\zeta_{\mathcal{B}}$ and $\widehat{\zeta}_{\mathcal{B}}$, i.e., the rectangular zeta functions. However, for certain $\mathcal{B}$, some other $\gamma \in GL_2(\mathbb{Z})$ may give a better description, see Example 2.56.*

Remark 2.43 *For any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$, $\zeta_{\mathcal{B};\gamma}$ in (2.149), which is an infinite product of rational function, is a rearrangement of $\zeta^0_{\mathcal{B}}$ in (1.6), which is a triple series. In deriving the rationality of $\zeta_{\mathcal{B};\gamma;n}$, the basic formula used is the power series*

$$\sum_{k=1}^{\infty} \frac{t^k}{k} = -\log(1-t). \tag{2.161}$$

The other rearrangements of $\zeta^0_{\mathcal{B}}$ may not have the form as in (2.149). For example, for any $m \geq 1$, denote by

$$f_{\mathcal{B};m}(s) = \exp\left(\sum_{n|m}\sum_{l=0}^{n-1} \frac{1}{m}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & \frac{m}{n} \end{bmatrix}\right) s^m\right) \tag{2.162}$$

and

$$f_{\mathcal{B}}(s) = \prod_{m=1}^{\infty} f_{\mathcal{B};m}(s). \tag{2.163}$$

In general, $f_{\mathcal{B};m}(s)$ is not a rational function of the form as in (1.3). It is also not clear how to identify the poles or natural boundary of $f_{\mathcal{B}}(s)$ from (2.162) and (2.163), see Subsection 2.5.

2.5 Analyticity and meromorphic extensions of zeta functions

This subsection studies the analyticity and meromorphisms of zeta functions obtained in the previous sections. Possible applications to number theory are also indicated. For simplicity, only $\mathcal{B} \subset \Sigma_{2\times 2}$ is considered. The general cases can be treated analogously.

2.5.1 Analyticity of zeta functions

Recall the analyticity results of Lind [36]. Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, the analytic region found by Lind is related to quantity $g(\mathcal{B})$, which specifies the growth rate of admissible periodic patterns. Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$,

$$\begin{aligned} g(\mathcal{B}) &\equiv \limsup_{[L]\to\infty} \tfrac{1}{[L]} \log \Gamma_{\mathcal{B}}(L) \\ &= \lim_{n\to\infty} \sup_{[L]\geq n} \frac{\log \Gamma_{\mathcal{B}}(L)}{[L]}. \end{aligned} \tag{2.164}$$

Recall the results of Lind [36] that are related to analyticity of zeta functions.

Proposition 2.44 *According to Lind, [36]*

(i) The zeta function

$$\zeta_{\mathcal{B}}^{0}(s) = \exp\left(\sum_{L\in\mathcal{L}_2} \frac{\Gamma_{\mathcal{B}}(L)}{[L]} s^{[L]}\right) \tag{2.165}$$

has radius of convergence $\exp(-g(\mathcal{B}))$ *and is analytic in* $|s| < \exp(-g(\mathcal{B}))$.

(ii) $\zeta_{\mathcal{B}}^{0}$ *satisfies the product formula,*

$$\zeta_{\mathcal{B}}^{0}(s) = \prod_{\alpha} \pi_2(s^{|\alpha|}), \tag{2.166}$$

where the product is taken over all admissible periodic patterns α *with respect to* $\mathcal{B}$*, and*

$$\pi_2(s) = \sum_{n=1}^{\infty} P(n) s^n, \tag{2.167}$$

where $P(n)$ *is the partition function.*

(iii) The Taylor series for $\zeta_{\mathcal{B}}^{0}(s)$ *has integer coefficients.*

Now, Propositions 2.44 and 2.32 and Theorem 2.41 imply

Theorem 2.45 *For any admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$,*

$$\zeta^0_{\mathcal{B}}(s) = \zeta_{\mathcal{B};\gamma}(s) = \widehat{\zeta}_{\mathcal{B};\gamma}(s) \tag{2.168}$$

for $|s| < \exp(-g(\mathcal{B}))$. Moreover, $\zeta_{\mathcal{B};\gamma}$ and $\widehat{\zeta}_{\mathcal{B};\gamma}$ have the same (integer) coefficients in their Taylor series expansions around $s = 0$.

Proof. Since

$$\sum_{L\in\mathcal{L}_2} \frac{\Gamma_{\mathcal{B}}(L)}{[L]} s^{[L]}$$

is absolutely convergent in $|s| < \exp(-g(\mathcal{B}))$, for each $\gamma \in GL_2(\mathbb{Z})$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_{\gamma}\right) s^{nk}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_{\gamma}\right) s^{nk}$$

are absolutely convergent in $|s| < \exp(-g(\mathcal{B}))$. Hence (2.168) holds. From (2.168) and Proposition 2.44 (iii) or Corollary 2.40, $\zeta_{\mathcal{B};\gamma}$ and $\widehat{\zeta}_{\mathcal{B};\gamma}$ have the same (integer) coefficients in their Taylor series expansions around $s = 0$. The proof is complete. ■

Remark 2.46 *The set of identities of (2.168) may lead some interesting results in number theory. See examples in next subsection.*

The rest of subsection discusses the meromorphicity of zeta function $\zeta_{\mathcal{B};\gamma}$. We need the following notations.

Definition 2.47

(i) Given any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$. The meromorphic domain $\mathcal{M}_{\mathcal{B};\gamma}$ of $\zeta_{\mathcal{B};\gamma}$is defined by

$$\mathcal{M}_{\mathcal{B};\gamma} = \{s \in \mathbb{C} | \zeta_{\mathcal{B};\gamma}(s) \text{ is meromorphic at } s\}. \tag{2.169}$$

(ii) The pole set $\mathcal{P}_{\mathcal{B};\gamma}$ of $\zeta_{\mathcal{B};\gamma}$ is defined by

$$\begin{aligned}\mathcal{P}_{\mathcal{B};\gamma} &= \{s \in \mathbb{C} | 1 - \lambda s^n = 0, \text{ where } \lambda \in \Sigma(\mathbf{T}_{\gamma;n}(\mathcal{B})), \chi_{\gamma;n}(\lambda) \geq 1 \text{ and } n \geq 1\} \\ &= \{s \in \mathbb{C} | 1 - \lambda s^n = 0, \text{ where } \lambda \in \Sigma(\tau_{\mathcal{B};\gamma;n}) \text{ and } n \geq 1\}.\end{aligned} \tag{2.170}$$

(iii) $\zeta_{\mathcal{B};\gamma}$ has a natural boundary $\partial\mathcal{M}_{\mathcal{B};\gamma}$ if every point in $\partial\mathcal{M}_{\mathcal{B};\gamma}$ is singular.

Remark 2.48 *$\zeta_{\mathcal{B};\gamma}$ has a natural boundary if*

$$\overline{\mathcal{P}}_{\mathcal{B};\gamma} \supseteq \partial\mathcal{M}_{\mathcal{B};\gamma}. \tag{2.171}$$

In studying the infinite products $\zeta_{\mathcal{B};\gamma}(s)$, the associated infinite series

$$\xi_{\mathcal{B};\gamma}(s) \equiv \sum_{n=1}^{\infty} \left(\sum_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} \lambda\chi_{\gamma;n}(\lambda) \right) s^n \tag{2.172}$$

is useful. Denote by

$$\lambda^*_{\mathcal{B};\gamma} \equiv \limsup_{n\to\infty} \left(\sum_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} |\lambda|\chi_{\gamma;n}(\lambda) \right)^{\frac{1}{n}}. \tag{2.173}$$

Let

$$S^*_{\mathcal{B};\gamma} \equiv \left(\lambda^*_{\mathcal{B};\gamma}\right)^{-1}. \tag{2.174}$$

Therefore, $\xi_{\mathcal{B};\gamma}$ absolutely converges for $|s| < S^*_{\mathcal{B};\gamma}$.

Furthermore, the reciprocal of $\zeta_{\mathcal{B};\gamma}$,

$$\zeta^{-1}_{\mathcal{B};\gamma} \equiv \prod_{n=1}^{\infty} \prod_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} (1-\lambda s^n)^{\chi_{\gamma;n}(\lambda)} \tag{2.175}$$

is absolutely convergent in $|s| < S^*_{\mathcal{B};\gamma}$. The similar notations can also be introduced to $\widehat{\zeta}_{\mathcal{B};\gamma}$, the details are omitted here.

Accordingly, zeta functions $\zeta_{\mathcal{B};\gamma}$ have the following meromorphic property.

Theorem 2.49 *Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$. Then zeta function $\zeta_{\mathcal{B};\gamma}$ is meromorphic in $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$ and may have poles in $\mathcal{P}_{\mathcal{B};\gamma} \cap \{s \in \mathbb{C} \mid |s| < \mathcal{S}^*_{\mathcal{B};\gamma}\}$, i.e., $\{s \in \mathbb{C} \mid |s| < \mathcal{S}^*_{\mathcal{B};\gamma}\} \subset \mathcal{M}_{\mathcal{B};\gamma}$.*

Proof. For each $s \notin \mathcal{P}_{\mathcal{B};\gamma}$ and $|s| < S^*_{\mathcal{B};\gamma}$, $\zeta_{\mathcal{B};\gamma}$ is convergent and has an isolated pole in $\mathcal{P}_{\mathcal{B};\gamma}$ when $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$, and then is meromorphic in $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$. The proof is complete. ∎

Theorem 2.50 *Given admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$. For any γ and γ' in $GL_2(\mathbb{Z})$, the zeta functions $\zeta_{\mathcal{B};\gamma} = \zeta_{\mathcal{B};\gamma'}$ in $|s| < \min(\mathcal{S}^*_{\mathcal{B};\gamma}, \mathcal{S}^*_{\mathcal{B};\gamma'})$.*

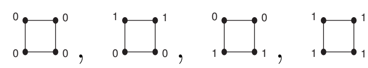
Proof. Since $\zeta_{\mathcal{B};\gamma}$ and $\zeta_{\mathcal{B};\gamma'}$ are meromorphic functions and are equal to $\zeta^0_{\mathcal{B}}$ on $|s| < \exp(-g(\mathcal{B}))$, by uniqueness theorem of meromorphic functions [46], they are equal on $|s| < \min(\mathcal{S}^*_{\mathcal{B};\gamma}, \mathcal{S}^*_{\mathcal{B};\gamma'})$. ∎

Remark 2.51 *Given $\mathcal{B} \subset \Sigma_{2\times 2}$, can we find a $\gamma \in GL_2(\mathbb{Z})$ such that $\zeta_{\mathcal{B};\gamma}$ is the maximum meromorphic extension of $\zeta^0_{\mathcal{B}}$, i.e, for any meromorphic extension $\zeta'_{\mathcal{B}}$ of $\zeta^0_{\mathcal{B}}$, $\zeta_{\mathcal{B};\gamma}$ is a meromorphic extension of $\zeta'_{\mathcal{B}}$? In particular, for any $\gamma' \in GL_2(\mathbb{Z})$, $\mathcal{M}_{\mathcal{B};\gamma'} \subseteq \mathcal{M}_{\mathcal{B};\gamma}$? Furthermore, is there $\gamma \in GL_2(\mathbb{Z})$ such that $\zeta_{\mathcal{B};\gamma}$ admits a natural boundary? These two problems are closely related. The complete answers are not clear. See examples studied in Subsection 2.5.2 and Subsection 2.6.*

2.5.2 EXAMPLES

This subsection presents some examples to elucidate the methods described above.

Example 2.52 Consider

$\mathcal{B} = \{$  $\}.$ (2.176)

Clearly,

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{V}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2.177}$$

First, $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right)$ are computed directly as follow:

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = 2^k \quad \text{for any} \quad 0 \le l \le n-1 \tag{2.178}$$

and

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right) = 2^{(n,l)} \quad \text{for any} \quad 1 \le l \le n-1, \tag{2.179}$$

where (n, l) is the greatest common divisor of n and l, are easily verified.

Consequently, for any $n \ge 1$,

$$\zeta_n(s) \;= \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{n2^k}{k}s^{kn}\right) = (1-2s^n)^{-1}$$

and the zeta function $\zeta(s) = \prod_{n=1}^{\infty}(1-2s^n)^{-1}$ with $S^* = 1$, which was obtained by Lind in [36].

However, (2.179) implies

$$\begin{aligned}\widehat{\zeta}_n(s) &= \exp\left(\left(\frac{1}{n}\sum_{l=1}^{n}2^{(n,l)}\right)\sum_{k=1}^{\infty}\frac{s^{kn}}{k}\right) \\ &= (1-s^n)^{-\widehat{\chi}_n},\end{aligned} \tag{2.180}$$

and the zeta function $\widehat{\zeta}(s) = \prod_{n=1}^{\infty}(1-s^n)^{-\widehat{\chi}_n}$, where

$$\widehat{\chi}_n = \frac{1}{n}\sum_{l=1}^{n}2^{(n,l)}. \tag{2.181}$$

Now, it is easy to check that $\lim_{n\to\infty}(\widehat{\chi}_n)^{\frac{1}{n}} = 2$. Therefore, $\widehat{S}^* = \frac{1}{2}$ as in (2.173) and (2.174) for $\widehat{\zeta}(s)$.

Theorem 2.45 implies that the zeta function $\zeta^0_{\mathcal{B}}(s)$ of $\mathcal{B}$ given by (2.176) is

$$\zeta^0_{\mathcal{B}}(s) = \prod_{n=1}^{\infty}(1-2s^n)^{-1} = \prod_{n=1}^{\infty}(1-s^n)^{-\widehat{\chi}_n} \tag{2.182}$$

in $|s| < \frac{1}{2}$. The natural boundary of (2.182) is $|s| = 1$ and ζ has poles

$$\left\{2^{-\frac{1}{n}}e^{2\pi ij/n} : 0 \le j \le n-1, n \ge 1\right\},$$

as described elsewhere [36].

However, (2.177) implies

$$\mathbf{T}_2 = \mathbf{V}_2 \quad \text{and} \quad \widehat{\mathbf{T}}_2 = \mathbf{H}_2.$$

Furthermore,

$$\mathbf{T}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2^n \times 2^n} \tag{2.183}$$

and

$$\widehat{\mathbf{T}}_n = I_{2^n}, \tag{2.184}$$

where I_{2^n} is the $2^n \times 2^n$ identity matrix. Therefore,

$$\widehat{\zeta}_n(s) = (1 - s^n)^{-\chi_n}, \tag{2.185}$$

where χ_n is the cardinal number of $\mathcal{I}_n$. Now, (2.181) and (2.185) imply $\widehat{\chi}_n = \chi_n$, i.e.,

$$\chi_n = \frac{1}{n}\sum_{l=1}^{n} 2^{(n,l)}. \tag{2.186}$$

Note that (2.186) also follows from the identity (2.182).

Moreover, (2.184) implies

$$\widehat{\zeta}_n(s) = \exp\left(\frac{1}{n} tr(\mathbf{R}_n) \log(1 - s^n)^{-1}\right).$$

Therefore, (2.185) implies

$$\frac{1}{n} tr(\mathbf{R}_n) = \chi_n. \tag{2.187}$$

Hence,

$$tr(\mathbf{R}_n) = \sum_{l=1}^{n} 2^{(n,l)}. \tag{2.188}$$

The following example can also be solved explicitly and is helpful in elucidating the natural boundary and location of the poles of the zeta function.

Example 2.53 Consider

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.189}$$

Then,

$$\mathbf{V}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = G \otimes G, \tag{2.190}$$

where

$$G=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.191}$$

is the one-dimensional golden-mean matrix, which has eigenvalues

$$g=\tfrac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{g}=\tfrac{1-\sqrt{5}}{2}=-g^{-1}. \tag{2.192}$$

Now,

$$\widetilde{\mathbf{H}}_2=\mathbf{V}_2 \quad \text{and} \quad \widetilde{\mathbf{V}}_2=\mathbf{H}_2. \tag{2.193}$$

Then,

$$\mathbf{T}_2=\mathbf{V}_2\circ\widetilde{\mathbf{H}}_2=\mathbf{V}_2=G\otimes G$$

can be verified, and for any $n\geq 2$,

$$\mathbf{T}_n=\underbrace{G\otimes G\otimes\cdots\otimes G\otimes G}_{n-1 \text{ times } \otimes}=\overset{n-1}{\otimes}G, \tag{2.194}$$

which is the $n-1$ times Kronecker product of G.

The spectrum of $\mathbf{T}_n$ is

$$\Sigma(\mathbf{T}_n)=\{g^{n-j}\overline{g}^j|0\leq j\leq n\}, \tag{2.195}$$

which has n+1 members. The number of linearly independent symmetric eigenvectors of $g^{n-j}\overline{g}^j$ is

$$\begin{aligned}\chi_{n,j} =&\ \chi(g^{n-j}\overline{g}^j) \\ =&\sum_{\substack{i\in\mathcal{I}_n \\ \lambda_{n,i}=g^{n-j}\overline{g}^j}} 1.\end{aligned} \tag{2.196}$$

Clearly, $\chi_{n,0}=\chi_{n,n}=1$. Furthermore for any $1\leq j\leq n-1$, by Burnside's Lemma,

$$\chi_{n,j}=\frac{1}{n}\sum_{d|(j,n-j)}\phi((j,n-j)/d)C_{jd/(j,n-j)}^{nd/(j,n-j)}, \tag{2.197}$$

where ϕ is the Euler totient function (2.36). The detailed proof of (2.197) is omitted for brevity.

Therefore,

$$\zeta_n(s) = \prod_{j=0}^{n} \left(1 - g^{n-j}\overline{g}^j s^n\right)^{-\chi_{n,j}} \tag{2.198}$$

and

$$\zeta(s) = \prod_{n=1}^{\infty} \zeta_n(s). \tag{2.199}$$

From (2.197),

$$\limsup_{n\to\infty} \max_{0\le j\le n} \left(\left|g^{n-j}\overline{g}^j \chi_{n,j}\right|\right)^{\frac{1}{n}} = 2,$$

which implies $S^* = \frac{1}{2}$ in (2.174).

Now, consider $\widehat{\mathbf{T}}_n$ and the associated zeta function $\widehat{\zeta}(s)$. Clearly,

$$\widehat{\mathbf{T}}_2 = \mathbf{H}_2 \circ \widetilde{V}_2 = \mathbf{H}_2.$$

To study higher-order $\widehat{\mathbf{T}}_n$, $n \ge 3$, the recursive formula of $\mathbf{H}_n$ must be obtained. Let

$$\mathbf{H}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix}. \tag{2.200}$$

Then,

$$\mathbf{H}_{n+1} = \begin{bmatrix} H_{n;1} & H_{n;2} & H_{n;1} & 0 \\ H_{n;3} & H_{n;4} & H_{n;3} & 0 \\ H_{n;1} & H_{n;2} & H_{n;1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, for $n \ge 2$,

$$\widehat{\mathbf{T}}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix} \circ \begin{bmatrix} \overset{n-2}{\bigotimes} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \overset{n-2}{\bigotimes} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ \overset{n-2}{\bigotimes} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \overset{n-2}{\bigotimes} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{2.201}$$

The remaining matrix of $\widehat{\mathbf{T}}_n$ can be verified to be a full matrix $E_{\widehat{r}_n}$ after the zero rows and columns have been deleted, where $\widehat{r}_n$ is the sum of entries in the first row of $\widehat{\mathbf{T}}_n$. Hence, the maximum eigenvalue $\widehat{\lambda}_n$ of $\widehat{\mathbf{T}}_n$ equals $\widehat{r}_n$, the other eigenvalues are zeros.

From (2.201), it is easy to verify

$$\widehat{\lambda}_{n+1} = \widehat{\lambda}_n + \widehat{\lambda}_{n-1} \tag{2.202}$$

with $\widehat{\lambda}_2 = 3$ and $\widehat{\lambda}_3 = 4$. Therefore,

$$\widehat{\zeta}_n(s) = (1 - \widehat{\lambda}_n s^n)^{-1} \tag{2.203}$$

and

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty}(1 - \widehat{\lambda}_n s^n)^{-1}. \tag{2.204}$$

Now, $\widehat{\lambda}_n$ and g^n must be compared. Let

$$g^n = \alpha_n g + \beta_n$$

with $\alpha_2 = \beta_2 = 1$. Then, $\alpha_{n+1} = \alpha_n + \beta_n$ and $\beta_{n+1} = \alpha_n$. That

$$\widehat{\lambda}_n = \alpha_n + 2\beta_n$$

can be verified and

$$\widehat{\lambda}_{n+1} - g^{n+1} = -\left(\frac{(\sqrt{5}-1)\alpha_{n+1} + 2\beta_n}{(\sqrt{5}-1)\alpha_n + 2\beta_{n-1}}\right)(\widehat{\lambda}_n - g^n). \tag{2.205}$$

Equation (2.205) implies

$$\widehat{\lambda}_{2n}^{-\frac{1}{2n}} < g^{-1} < \widehat{\lambda}_{2n+1}^{-\frac{1}{2n+1}}. \tag{2.206}$$

Equation (2.206) implies that the meromorphic extension $\widehat{\zeta}$ of $\zeta^0_{\mathcal{B}}$ satisfies $\widehat{S}^* = g^{-1}$ and has poles on $\left\{\widehat{\lambda}_{2n}^{-\frac{1}{2n}} e^{\pi i j/n} : 0 \leq j \leq 2n-1, n \geq 1\right\}$ with the natural boundary $|s| = g^{-1}$. Furthermore, (2.199) and (2.204) lead an identity involving $\chi_{n,j}$ and g, the detail is omitted.

2.6 Equations on $\mathbb{Z}^2$ with numbers in a finite field

This subsection briefly discusses the equations on $\mathbb{Z}^2$ with numbers in a finite field, see [31; 36; 47]. The problems can be studied by applying the methods that were developed in the previous subsections. Lind [36] considered the following example.

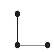
Example 2.54 Consider $F_2 = \{0, 1\}$ and

$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{2.207}$$

In this case, $\mathbb{X}$ is a compact group with coordinate-wise operations, and it is invariant under the natural $\mathbb{Z}^2$-shift action σ.

The equation

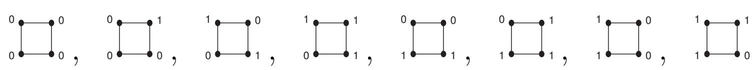
$$x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \tag{2.208}$$

can be interpreted as a pattern generation problem on L-shape lattices: . Indeed, the solutions of (2.208) are given by

$$\mathcal{B}(\mathbb{L}) = \left\{ \quad \right\}, \tag{2.209}$$

which consists of all even patterns on L-shape lattices. $\mathcal{B}(\mathbb{L})$ can be extended to $\mathbb{Z}_{2\times 2}$ as

$$\mathcal{B} = \left\{ \quad \right\}. \tag{2.210}$$



That

$$\Sigma(\mathcal{B}) = \mathbb{X} \tag{2.211}$$

can be easily verified.

Therefore,

$$\mathbf{H}_2 = \mathbf{H}_2(\mathcal{B}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \mathbf{V}_2 \tag{2.212}$$

and

$$\widetilde{\mathbf{H}}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \widetilde{\mathbf{V}}_2. \tag{2.213}$$

According to (2.54),

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \zeta_1(s) = \tfrac{1}{1-s}.$$

For $\mathbf{T}_2 = \mathbf{V}_2 \circ \widetilde{\mathbf{H}}_2$, (2.212) and (2.213) imply

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tau_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.214}$$

with

$$\zeta_2(s) = \frac{1}{1-s^2}.$$

In general, for any $n \geq 1$, induction can be used to show that each row of $\mathbf{T}_n$ has exactly a single 1 and each column has either two 1s or all 0s. Therefore, the eigenvalue λ of $\mathbf{T}_n$ is $|\lambda| = 1$ or $\lambda = 0$. With a rotationally symmetric eigenvector, $\mathbf{T}_n$ generates the graph with equivalent classes $C_n(i)$ as vertices and has $m(n)$ disjoint cycles; each cycle has period $p_{n,k} \geq 1$, $1 \leq k \leq m(n)$. In computing, it is more efficient to compute $\lambda \in \Sigma(\tau_n)$ with algebraic multiplicity $\chi(\lambda)$.

The following can be demonstrated

$$\zeta_n(s) = \prod_{k=1}^{m(n)} \frac{1}{(1-\rho_{n,k}s^n)\cdots(1-\rho_{n,k}^{p_{n,k}-1}s^n)\,(1-s^n)} = \prod_{k=1}^{m(n)} \frac{1}{1-s^{np_{n,k}}}, \tag{2.215}$$

where $\rho_{n,k} = e^{\frac{2\pi i}{p_{n,k}}}$. Hence,

$$\zeta(s) = \prod_{n=1}^{\infty} \prod_{k=1}^{m(n)} \frac{1}{1-s^{np_{n,k}}}. \tag{2.216}$$

For $n = 1$ to 20, the numbers and periods of cycles are listed in Table 2.1.

n	1	2	3	4	5		6		7		8
p	1	1	1	1	1	3	1	2	1	7	1
q	1	1	2	1	1	1	2	1	3	1	1

9		10			11		12			13	
1	7	1	3	6	1	31	1	2	4	1	63
2	4	1	1	4	1	3	2	1	5	1	5

14				15			16	17		
1	2	7	14	1	3	15	1	1	5	15
3	4	1	20	4	4	72	1	1	3	256

18				19		20			
1	2	7	14	1	511	1	3	6	12
2	1	4	259	1	27	1	1	4	272

p : the period of cycle.

$q = q(p)$: the number of cycles with period p.

Table 2.1.

From Table 2.1, ζ_n can be written for $1 \leq n \leq 20$. For example,

$$\zeta_{14} = \frac{1}{(1-s^{14})^3(1-s^{28})^4(1-s^{98})(1-s^{196})^{20}}.$$

Up to $n = 20$, the Taylor expansion of (2.216) at $s = 0$, which recovers Lind's result [36] (p.438), is

$$\zeta_{\mathcal{B}}(s) = 1 + s + 2s^2 + 4s^3 + 6s^4 + 9s^5 + 16s^6 + 24s^7 + 35s^8 + 54s^9 \tag{2.217}$$
$$+78s^{10} + 110s^{11} + 162s^{12} + 226s^{13} + 317s^{14} + 446s^{15} + 612s^{16}$$
$$+834s^{17} + 1146s^{18} + 1543s^{19} + 2071s^{20} + \cdots.$$

Further investigation is needed to understand τ_n and $p_{n,k}$ for large n. The results will appear elsewhere.

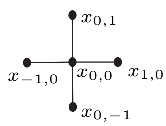
Lind [36] showed that the zeta function ζ^0 defined by (2.207) is analytic in $|s| < 1$. By (2.216), all poles of ζ appear on $|s| = 1$. Therefore, ζ is analytic in $|s| < 1$ with natural boundary $|s| = 1$.

In the following example, the harmonic patterns on square-cross lattice $\mathbb{L}$: , which were studied by Ledrappier [31], are investigated.

Example 2.55 Let $F_2 = \{0, 1\}$ and

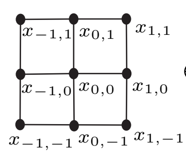
$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} = x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1} \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{2.218}$$

As in Example 2.54, the basic set on $\mathbb{L}$ is

$$\mathcal{B}(\mathbb{L}) = \Bigg\{$$


$$\in F_2^{\mathbb{L}} : x_{0,0} + x_{-1,0} + x_{0,-1} + x_{1,0} + x_{0,1} = 0 \Bigg\}, \tag{2.219}$$

which consists of all even patterns on a square-cross lattice. $\mathcal{B}(\mathbb{L})$ can be extended to $\mathbb{Z}_{3\times 3}$ as

$$\mathcal{B} = \Bigg\{$$


$$\in F_2^{\mathbb{Z}_{3\times 3}} : x_{0,0} + x_{-1,0} + x_{0,-1} + x_{1,0} + x_{0,1} = 0 \Bigg\}. \tag{2.220}$$

Then, that

$$\Sigma(\mathcal{B}) = \mathbb{X} \tag{2.221}$$

can be easily verified.

Now, by (2.108), the associated trace operator $\mathbf{T}_{n\times 3}(\mathcal{B})$ can be constructed for $n \geq 1$. Furthermore, the rotational matrix $R_{n\times 2}$ is defined by (2.109). The number $\chi_{n\times 2}$ of the equivalent classes of $R_{n\times 2}$ can be shown to be the number of n-bead necklaces with four colors. The formulae for $\chi_{n\times 2}$, $n \geq 1$, is given by (2.113) with $m = 3$.

As in Example 2.54, the reduced trace operator $\tau_{n\times 3}$ of $\mathbf{T}_{n\times 3}$ is more convenient for computing the n-th order zeta function ζ_n. The definition and results of the reduced trace operator for more symbols on larger lattices are similar to Definition 2.23 and Theorem 2.26.

By the same argument as in Example 2.54, let the graph generated by $\mathbf{T}_{n\times 3}$ have $m(n)$ disjoint cycles, each of period $p_{n,k} \geq 1$, for $1 \leq k \leq m(n)$. Then, the n-th order zeta function can be represented as

$$\zeta_n(s) = \prod_{k=1}^{m(n)} \frac{1}{1 - s^{np_{n,k}}}. \tag{2.222}$$

Hence,

$$\zeta(s) = \prod_{n=1}^{\infty} \prod_{k=1}^{m(n)} \frac{1}{1 - s^{np_{n,k}}}. \tag{2.223}$$

Table 2.2 presents the numbers and periods of cycles of $\mathbf{T}_{n\times 3}$. For brevity, only $n = 1$ to 9 are listed.

n	1		2		3				4			5			
p	1	3	1	3	1	2	3	6	1	3	6	1	3	5	15
q	1	1	1	3	2	2	2	2	1	7	8	7	7	9	9

6						7			8			
1	2	3	4	6	12	1	3	9	1	3	6	12
2	6	6	8	10	48	1	1	260	1	7	88	640

9							
1	2	3	6	7	14	21	42
2	2	2	2	260	390	260	390

p : the period of cycle.

$q = q(p)$: the number of cycles with period p.

Table 2.2.

Up to $n = 16$, the Taylor expansion of (2.223) at $s = 0$ is

$$\zeta_{\mathcal{B}}(s) = 1 + s + 2s^2 + 5s^3 + 7s^4 + 17s^5 + 32s^6 + 46s^7 + 84s^8 + 140s^9 \tag{2.224}$$

$$+229s^{10} + 384s^{11} + 615s^{12} + 938s^{13} + 1483s^{14} + 2353s^{15} + 3563s^{16} + \cdots .$$

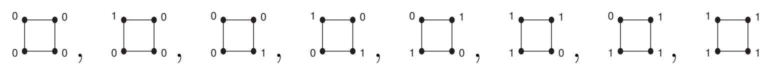
The analyticity and the natural boundary of the zeta function in (2.223) need further investigation. The results will appear elsewhere.

In the following example, we study the equation on the diagonal lattice $\mathbb{L}$: ⋰ and show that the rectangular zeta function $\zeta = \widehat{\zeta}$ fails to describe poles and natural boundary of ζ^0 but ζ_γ works well with $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Example 2.56 Let $F_2 = \{0, 1\}$ and

$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j+1} = 0 \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{2.225}$$

It is clear that the solutions of $x_{i,j} + x_{i+1,j+1} = 0 \bmod 2$ are given by

$\mathcal{B} = \{$  $\}.$ (2.226)

Now,

$$\mathbf{H}_2 = \mathbf{V}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{2.227}$$

and

$$\widetilde{\mathbf{H}}_2 = \widetilde{\mathbf{V}}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{2.228}$$

It is easy to verify

$$\mathbf{T}_1 = \widehat{\mathbf{T}}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R_1^t \tag{2.229}$$

and

$$\mathbf{T}_2 = \widehat{\mathbf{T}}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R_2^t. \tag{2.230}$$

Furthermore, for $n \geq 3$, we show that

$$\mathbf{T}_n = \widehat{\mathbf{T}}_n = R_n^t. \tag{2.231}$$

Indeed, by the recursive formula of $\mathbf{V}_n$, it can be verified that $\mathbf{V}_{n;i,j} = 1$ if and only if

$$\begin{cases} i = 2j-1 \text{ and } 2j & \text{for } 1 \leq j \leq 2^{n-1}, \\ i = 2(i-2^{n-1})-1 \text{ and } 2(i-2^{n-1}) & \text{for } 2^{n-1}+1 \leq j \leq 2^n. \end{cases} \tag{2.232}$$

Therefore, by applying (2.26), $\mathrm{T}_n = [t_{n;i,j}]$ with $t_{n;i,j} = 1$ if and only if

$$\begin{cases} i = 2j-1 & \text{for } 1 \leq j \leq 2^{n-1}, \\ i = 2(i-2^{n-1}) & \text{for } 2^{n-1}+1 \leq j \leq 2^n. \end{cases} \tag{2.233}$$

Hence,

$$\mathbf{T}_n = R_n^t. \tag{2.234}$$

Therefore,

$$\zeta_n(s) = \frac{1}{(1-s^n)^{\chi_n}}, \tag{2.235}$$

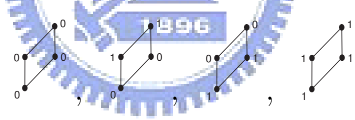
where χ_n is the cardinal number of $\mathcal{I}_n$, and

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{(1-s^n)^{\chi_n}}. \tag{2.236}$$

As in Example 2.52, $\lim_{n\to\infty} \chi_n^{\frac{1}{n}} = 2$ and then $S^* = \frac{1}{2}$.

On the other hand, consider

$$\mathcal{B}' = \left\{ \quad \right\}. \tag{2.237}$$



Then,

$$\Sigma(\mathcal{B}') = \Sigma(\mathcal{B}). \tag{2.238}$$

In particular,

$$\Gamma_{\mathcal{B}'}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) = 2^k. \tag{2.239}$$

Therefore, as in Example 2.52,

$$\zeta_{\gamma;n} = \frac{1}{1-2s^n}.$$

We can also use the construction of $\mathbf{T}_{\gamma;n}$ in Subsection 2.4 to study $\zeta_{\gamma;n}$. Indeed, it is easy to see that

$$\mathbf{T}_{\gamma;1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{2.240}$$

Therefore,

$$\zeta_{\gamma;1} = \frac{1}{1-2s}. \tag{2.241}$$

Furthermore, for any $n \geq 2$, after deleting the zero columns and rows of $\mathbf{T}_{\gamma;n}$, $\mathbf{T}_{\gamma;n}$ is reduced to $\mathbf{T}_{\gamma;1}$. Therefore,

$$\zeta_{\gamma;n} = \frac{1}{1-2s^n}. \tag{2.242}$$

Hence,

$$\zeta_{\gamma} = \prod_{n=1}^{\infty} \frac{1}{1-2s^n}. \tag{2.243}$$

Then, ζ_γ has natural boundary with $|s| = 1$ and has poles

$$\left\{ 2^{-\frac{1}{n}} e^{2\pi i j/n} : 0 \leq j \leq n-1, n \geq 1 \right\}.$$

Motivated by Examples 2.54∼2.56, given a finite field F and a set of finite lattice points $\mathbb{L} \subset \mathbb{Z}^2$, consider the equation

$$\sum_{(i,j)\in\mathbb{L}} x_{i,j} = 0 \quad \text{in } F. \tag{2.244}$$

Then, denote the solution set of (2.244) on $\mathbb{Z}^2$ by

$$\mathbb{X}(\mathbb{L}) = \left\{ x \in F^{\mathbb{Z}^2} : \sum_{(i,j)\in\mathbb{L}} x_{i+k,j+l} = 0, (k,l) \in \mathbb{Z}^2 \right\}. \tag{2.245}$$

Denoted by

$$\mathcal{B}(\mathbb{L}) = \left\{ x : \mathbb{L} \to F : \sum_{(i,j)\in\mathbb{L}} x_{i,j} = 0 \right\}, \tag{2.246}$$

$\mathcal{B}(\mathbb{L}) \subset F^{\mathbb{L}}$ is the set of admissible local patterns.

Let $\mathbb{Z}_{m\times m}$ be the smallest rectangular lattice that contains $\mathbb{L}$. Let $\mathcal{B}$ be the set of all admissible patterns on $\mathbb{Z}_{m\times m}$ that can be generated from $\mathcal{B}(\mathbb{L})$. Then, the following can be easily verified;

$$\mathbb{X}(\mathbb{L}) = \Sigma(\mathcal{B}). \tag{2.247}$$

The results presented in previous subsections apply to $\Sigma(\mathcal{B})$ and then to $\mathbb{X}(\mathbb{L})$. The above method can also be applied to any finite set of equations defined on $\mathbb{L}$ with numbers in F, since the solution set $\mathcal{B}(\mathbb{L}) \subset F^{\mathbb{L}}$ and can be extended to a unique admissible set $\mathcal{B} \subseteq F^{\mathbb{Z}_{m\times m}}$.

2.7 Square lattice Ising model with finite range interaction

This subsection extends the results presented in previous sections to the thermodynamic zeta function for a square lattice Ising model with finite range interaction, see Ruelle [45] and Lind [36]. For simplicity, the square lattice Ising model with nearest neighbor interaction is considered.

The square lattice Ising model with external field $\mathcal{H}$, the coupling constant $\mathcal{J}$ in the horizontal direction, and the coupling constant $\mathcal{J}'$ in the vertical direction is now considered. Each site (i,j) of the square lattice $\mathbb{Z}^2$ has a spin $u_{i,j}$ with two possible values, $+1$ or -1. First, assume that the state space is $\{+1,-1\}^{\mathbb{Z}^2}$. Given a state $U = \{u_{i,j}\}_{i,j\in\mathbb{Z}}$ in $\{+1,-1\}^{\mathbb{Z}^2}$, denoted by $U_{m\times n} = U\big|_{\mathbb{Z}_{m\times n}} = \{u_{i,j}\}_{0\leq i\leq m-1, 0\leq j\leq n-1}$.

Define the Hamiltonian (energy) $\mathcal{E}(U_{m\times n})$ for $U_{m\times n}$ by

$$\mathcal{E}(U_{m\times n}) = -\mathcal{J} \sum_{\substack{0\leq i\leq m-2\\ 0\leq j\leq n-1}} u_{i,j}u_{i+1,j} - \mathcal{J}' \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-2}} u_{i,j}u_{i,j+1} - \mathcal{H} \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-1}} u_{i,j}. \tag{2.248}$$

Therefore, the partition function $\mathcal{Z}_{m\times n}$ is defined by

$$\mathcal{Z}_{m\times n} = \sum_{\substack{U_{m\times n}\\ \in\{+1,-1\}^{\mathbb{Z}_{m\times n}}}} \exp\left[\mathbf{K} \sum_{\substack{0\leq i\leq m-2\\ 0\leq j\leq n-1}} u_{i,j}u_{i+1,j} + \mathbf{L} \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-2}} u_{i,j}u_{i,j+1} + \mathbf{h} \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-1}} u_{i,j}\right], \tag{2.249}$$

where $\mathbf{K}=\mathcal{J}/k_BT$,$\mathbf{L}=\mathcal{J}'/k_BT$, $\mathbf{h}=\mathcal{H}/k_BT$, k_B is Boltzmann's constant and T is the temperature.

To the thermodynamic zeta function, given $L=\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\mathbb{Z}^2\in\mathcal{L}_2$, the partition function for the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic states is defined by

$$\mathcal{Z}_L=\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)=$$

$$\sum_{U\in fix_L\left(\{+1,-1\}^{\mathbb{Z}^2}\right)}\exp\left[\mathbf{K}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}}u_{i,j}u_{i+1,j}+\mathbf{L}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}}u_{i,j}u_{i,j+1}+\mathbf{h}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}}u_{i,j}\right]. \tag{2.250}$$

Then, as in (1.31), the thermodynamic zeta function for the square lattice Ising model with nearest neighbor interaction can be defined by

$$\begin{aligned}\zeta^0(s)\equiv\zeta^0_{Ising}(s)&\equiv\exp\left(\sum_{L\in\mathcal{L}_2}\mathcal{Z}_L\frac{s^{[L]}}{[L]}\right)\\&=\exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)s^{nk}\right).\end{aligned} \tag{2.251}$$

To simplify the notation, the subscript Ising is omitted in this subsection whenever such omission will not cause confusion.

As (1.8) and (1.9), for any $n\ge 1$, define the n-th order thermodynamic zeta function $\zeta_{Ising;n}(s)$ as

$$\zeta_n(s)\equiv\zeta_{Ising;n}(s)\equiv\exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)s^{nk}\right); \tag{2.252}$$

the thermodynamic zeta function $\zeta_{Ising}(s)$ is given by

$$\zeta(s)\equiv\zeta_{Ising}(s)\equiv\prod_{n=1}^{\infty}\zeta_n(s). \tag{2.253}$$

Since the discussion of $\zeta_n(s)$ is similar to that in Subsections 2.1 and 2.2, only the parts of the arguments that differ are emphasized. The results are outlined here and the details are left to the reader.

According to the spin $u_{i,j} \in \{+1, -1\}$ for $i, j \in \mathbb{Z}$, replacing all the symbols "0" in (2.1) and (2.2) with the symbol "-1" yields the ordering matrices $\mathbf{X}_{Ising;2\times 2}$ and $\mathbf{Y}_{Ising;2\times 2}$.

The ordering matrix $\mathbf{X}_{Ising;n\times 2}$, $\mathbf{Y}_{Ising;n\times 2}$ and the cylindrical ordering matrix $\mathbf{C}_{Ising;n\times 2}$ can be obtained in the same way. The recursive formulae for generating $\mathbf{Y}_{Ising;n\times 2}$ form $\mathbf{Y}_{Ising;2\times 2}$ are as in (2.13).

Given $L \in \mathcal{L}_2$, (2.250) yields

$$\mathcal{Z}_L = \sum_{U \in fix_L\left(\{+1,-1\}^{\mathbb{Z}^2}\right)} \prod_{\substack{0\leq i\leq n-1\\ 0\leq j\leq k-1}} \exp\left[u_{i,j}\left(\mathbf{K}u_{i+1,j} + \mathbf{L}u_{i,j+1} + \mathbf{h}\right)\right]. \tag{2.254}$$

Based on (2.254), the associated horizontal transition matrix $\mathbf{H}_{Ising;2} = [a_{I;i,j}]_{4\times 4}$ and the vertical transition matrix $\mathbf{V}_{Ising;2} = [b_{I;i,j}]_{4\times 4}$ are defined as

$$\mathbf{H}_{Ising;2} = \begin{bmatrix} e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} \\ e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} \end{bmatrix} = [a_{I;i,j}]_{4\times 4}, \tag{2.255}$$

and

$$\mathbf{V}_{Ising;2} = \begin{bmatrix} e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} \\ e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} \end{bmatrix} = [b_{I;i,j}]_{4\times 4}, \tag{2.256}$$

respectively. Similar to (2.18) and (2.19), the associated column matrices $\widetilde{\mathbf{H}}_{Ising;2}$ of $\mathbf{H}_{Ising;2}$ and $\widetilde{\mathbf{V}}_{Ising;2}$ of $\mathbf{V}_{Ising;2}$ are defined as

$$\widetilde{\mathbf{H}}_{Ising;2} = \begin{bmatrix} a_{I;1,1} & a_{I;2,1} & a_{I;1,2} & a_{I;2,2} \\ a_{I;3,1} & a_{I;4,1} & a_{I;3,2} & a_{I;4,2} \\ a_{I;1,3} & a_{I;2,3} & a_{I;1,4} & a_{I;2,4} \\ a_{I;3,3} & a_{I;4,3} & a_{I;3,4} & a_{I;4,4} \end{bmatrix} \tag{2.257}$$

and

$$\widetilde{\mathbf{V}}_{Ising;2} = \begin{bmatrix} b_{I;1,1} & b_{I;2,1} & b_{I;1,2} & b_{I;2,2} \\ b_{I;3,1} & b_{I;4,1} & b_{I;3,2} & b_{I;4,2} \\ b_{I;1,3} & b_{I;2,3} & b_{I;1,4} & b_{I;2,4} \\ b_{I;3,3} & b_{I;4,3} & b_{I;3,4} & b_{I;4,4} \end{bmatrix}. \tag{2.258}$$

Therefore, the trace operators $\mathbf{T}_{Ising;2}$ and $\widehat{\mathbf{T}}_{Ising;2}$ are defined as

$$\mathbf{T}_{Ising;2} = \mathbf{V}_{Ising;2} \circ \widetilde{\mathbf{H}}_{Ising;2} \quad \text{and} \quad \widehat{\mathbf{T}}_{Ising;2} = \mathbf{H}_{Ising;2} \circ \widetilde{\mathbf{V}}_{Ising;2}. \tag{2.259}$$

The recursive formulas for $\mathbf{T}_{Ising;n}$ and $\widehat{\mathbf{T}}_{Ising;n}$ are similar to (2.26). Constructing $\mathbf{T}_{Ising;2}$ and the rotational matrix R_n yield a similar result to that of Theorem 2.13 for $\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

Theorem 2.57 *Given* $n \geq 2, 0 \leq l \leq n-1, k \geq 1,$

$$\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{Ising;n} R^l_n\right). \tag{2.260}$$

Furthermore, let

$$\mathbf{T}_{Ising;1} = \begin{bmatrix} a_{I;1,1}a_{I;1,1} & a_{I;2,2}a_{I;2,2} \\ a_{I;3,3}a_{I;3,3} & a_{I;4,4}a_{I;4,4} \end{bmatrix};$$

then

$$\mathcal{Z}\left(\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{Ising;1}\right) \quad \text{for } k \geq 1.$$

From Theorem 2.57, the n-th order thermodynamic zeta function $\zeta_{Ising;n}$ can now be obtained as follows.

Theorem 2.58 *For any $n \geq 1$,*

$$\zeta_{Ising;n} = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty} tr\left(\mathbf{T}_{Ising;n}^{k}\mathbf{R}_n\right) s^{nk}\right). \tag{2.261}$$

The R_n-symmetric property of $\mathbf{T}_{Ising;n}$ is essential to the rationality of n-th order thermodynamic zeta function $\zeta_{Ising;n}$.

Proposition 2.59 *For any $n \geq 1$,*

$$\mathbf{T}_{Ising;n;\sigma^l(i),\sigma^l(j)} = \mathbf{T}_{Ising;n;i,j} \tag{2.262}$$

for all $1 \leq i, j \leq 2^n$ and $0 \leq l \leq n-1$.

Similarly, the associated reduced trace operator $\tau_{Ising;n}$ can be defined as in (2.100). Finally, by the arguments presented in Subsection 2.2, the rationality of the n-th order thermodynamic zeta function $\zeta_{Isng;n}$ is established as follows.

Theorem 2.60 *For $n \geq 1$,*

$$\zeta_{Ising;n}(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_{Ising;n})} (1-\lambda s^n)^{-\chi(\lambda)} \tag{2.263}$$

$$= \left(\det\left(I - s^n \tau_{Ising;n}\right)\right)^{-1}, \tag{2.264}$$

where $\chi(\lambda)$ is the number of linear independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{Ising;n}$ with eigenvalue λ. Furthermore,

$$\zeta_{Ising}(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n \tau_{Ising;n}\right)\right)^{-1}. \tag{2.265}$$

The state space $\{+1,-1\}^{\mathbb{Z}^2}$ is extended to the shift of finite type given by $\mathcal{B} \subseteq \{+1,-1\}^{\mathbb{Z}_{2\times 2}}$. Given $\mathcal{B} \subseteq \{+1,-1\}^{\mathbb{Z}_{2\times 2}}$ and $L = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\mathbb{Z}^2 \in \mathcal{L}_2$, the partition function for $\mathcal{B}$ with $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic patterns is defined as

$$\mathcal{Z}_L(\mathcal{B}) = \mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) =$$

$$\sum_{U\in fix_L(\Sigma(\mathcal{B}))} \exp\left[\mathbf{K}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}u_{i+1,j} + \mathbf{L}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}u_{i,j+1} + \mathbf{h}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}\right], \qquad (2.266)$$

where $u_{n,j} = u_{0,j}$, $0 \le j \le k-1$ and $u_{i,k} = u_{i,0}$, $0 \le i \le n-1$. Hence, the thermodynamic zeta function is defined by

$$\zeta^0_{Ising;\mathcal{B}}(s) \equiv \exp\left(\sum_{L\in\mathcal{L}_2} \mathcal{Z}_L(\mathcal{B})\frac{s^{[L]}}{[L]}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}\right) s^{nk}\right). \qquad (2.267)$$

Similar to (2.252) and (2.253), for any $n \ge 1$, the n-th order thermodynamic zeta function $\zeta_{Ising;\mathcal{B};n}(s)$ is defined as

$$\zeta_{Ising;\mathcal{B};n}(s) \equiv \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}\right) s^{nk}\right) \qquad (2.268)$$

and the thermodynamic zeta function $\zeta_{Ising;\mathcal{B}}(s)$ is given by

$$\zeta_{Ising;\mathcal{B}}(s) \equiv \prod_{n=1}^{\infty}\zeta_{Ising;\mathcal{B};n}(s). \qquad (2.269)$$

Equations (2.15), (2.255) and (2.256) are combined to define the associated horizontal transition matrix and vertical transition matrix as follows.

$$\mathbf{H}_{Ising;2}(\mathcal{B}) = \mathbf{H}_{Ising;2} \circ \mathbf{H}_2(\mathcal{B}) \qquad (2.270)$$

and

$$\mathbf{V}_{Ising;2}(\mathcal{B}) = \mathbf{V}_{Ising;2} \circ \mathbf{V}_2(\mathcal{B}). \qquad (2.271)$$

Therefore, the trace operator $\mathbf{T}_{Ising;n}(\mathcal{B})$ and the associated reduced trace operator $\tau_{Ising;n}(\mathcal{B})$ can be defined for all $n \ge 1$ as above. Since all arguments for $\zeta_{Ising;\mathcal{B};n}$ are similar to those above; the final result is as follows.

Theorem 2.61 *For $n \geq 1$,*

$$\zeta_{Ising;\mathcal{B};n}(s) = \prod_{\lambda \in \Sigma(\mathbf{T}_{Ising;n}(\mathcal{B}))} (1 - \lambda s^n)^{-\chi(\lambda)} \tag{2.272}$$

$$= \left[\det\left(I - s^n \tau_{Ising;n}(\mathcal{B})\right)\right]^{-1}, \tag{2.273}$$

where $\chi(\lambda)$ is the number of linear independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{Ising;n}(\mathcal{B})$ with eigenvalue λ. Moreover,

$$\zeta_{Ising;\mathcal{B}}(s) = \prod_{n=1}^{\infty} \left[\det\left(I - s^n \tau_{Ising;n}(\mathcal{B})\right)\right]^{-1}. \tag{2.274}$$

Remark 2.62 *The results in this subsection hold for models with finite range interaction.*

3 Zeta functions for higher-dimensional shifts of finite type

This section studies the zeta functions for d-dimensional shifts of finite type, $d \geq 3$.

3.1 Three-dimensional shifts of finite type

In this subsection, the zeta functions for three-dimensional shifts of finite type are investigated.

3.1.1 Periodic patterns, trace operator and rotational matrices

This subsection studies the properties of the periodic patterns and derives trace operator and rotational matrices. Furthermore, $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$ can be expressed in terms of the trace of the products of the trace operator and rotational matrices .

For clarity, two symbols on $2 \times 2 \times 2$ lattice $\mathbb{Z}_{2\times 2\times 2}$ are examined first. For given positive integers N_1, N_2 and N_3, the rectangular lattice $\mathbb{Z}_{N_1\times N_2\times N_3}$ is defined by

$$\mathbb{Z}_{N_1\times N_2\times N_3} = \left\{(n_1, n_2, n_3) : 0 \leq n_i \leq N_i - 1, 1 \leq i \leq 3\right\}.$$

In particular,

$$\mathbb{Z}_{2\times 2\times 2} = \left\{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\right\}.$$

Define the set of all global patterns on $\mathbb{Z}^3$ with two symbols $\{0,1\}$ by

$$\Sigma_2^3 = \{0,1\}^{\mathbb{Z}^3} = \left\{U \mid U : \mathbb{Z}^3 \to \{0,1\}\right\}.$$

Here, $\mathbb{Z}^3 = \{(n_1,n_2,n_3) : n_1,n_2,n_3 \in \mathbb{Z}\}$, the set of all three-dimensional lattice points (vertices). The set of all local patterns on $\mathbb{Z}_{N_1\times N_2\times N_3}$ is defined by

$$\Sigma_{N_1\times N_2\times N_3} = \{U|_{\mathbb{Z}_{N_1\times N_2\times N_3}} : U \in \Sigma_2^3\},$$

and a local pattern of a global pattern U on $\mathbb{Z}_{N_1\times N_2\times N_3}$ is denoted by

$$U_{N_1\times N_2\times N_3} \equiv U|_{\mathbb{Z}_{N_1\times N_2\times N_3}} = \left(u_{\alpha_1,\alpha_2,\alpha_3}\right)_{0\leq\alpha_i\leq N_i-1, 1\leq i\leq 3},$$

where $u_{\alpha_1,\alpha_2,\alpha_3} \in \{0,1\}$. To simplify the notation, the subscripts of $U_{N_1\times N_2\times N_3}$ and $\left(u_{\alpha_1,\alpha_2,\alpha_3}\right)_{0\leq\alpha_i\leq N_i-1,1\leq i\leq 3}$ are omitted whenever such omission will not cause confusion.

Now, for any given $\mathcal{B} \subset \Sigma_{2\times2\times2}$, $\mathcal{B}$ is called a basic set of admissible local patterns. In short, $\mathcal{B}$ is a basic set. A local pattern $U_{N_1\times N_2\times N_3} = (u_{\alpha_1,\alpha_2,\alpha_3})$ is called $\mathcal{B}$-admissible if for any vertex (lattice point) (n_1,n_2,n_3) with $0 \leq n_i \leq N_i - 2$, $1 \leq i \leq 3$, there exist a $2\times2\times2$ admissible local pattern $(\beta_{k_1,k_2,k_3})_{0\leq k_1,k_2,k_3\leq 1} \in \mathcal{B}$ such that

$$u_{n_1+k_1,n_2+k_2,n_3+k_3} = \beta_{k_1,k_2,k_3}$$

for $0 \leq k_1,k_2,k_3 \leq 1$.

Given a lattice $L \in \mathcal{L}_3$ with Hermite normal form,

$$L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3, \qquad (3.1)$$

where $a_i \geq 1$ for $1 \leq i \leq 3$ and $0 \leq b_{ij} \leq a_i - 1$ for $i+1 \leq j \leq 3$. A global pattern $U = \left(u_{\alpha_1,\alpha_2,\alpha_3}\right)_{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}}$ is called L-periodic or $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}$-periodic if for every $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}$

$$u_{\alpha_1+a_1p+b_{12}q+b_{13}r,\alpha_2+a_2q+b_{23}r,\alpha_3+a_3r} = u_{\alpha_1,\alpha_2,\alpha_3} \qquad (3.2)$$

for all $p, q, r \in \mathbb{Z}$.

The periodicity of $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}$ and $\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2' & 0 \\ 0 & 0 & a_3' \end{bmatrix}$ are closely related as follows.

Proposition 3.1 *For $a_i \geq 1$, $1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$, let*

$$s_1 = \frac{a_1}{(a_1, b_{12})} \quad \text{and} \quad s_2 = \left[\frac{a_1}{(a_1, b_{13})}, \frac{s_1 a_2}{(s_1 a_2, b_{23})} \right],$$

where (m, n) is the greatest common divisor of m and n and $[p, q]$ is the least common multiple of p and q. Then, $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}$-periodic patterns are $\begin{bmatrix} a_1 & 0 & 0 \\ 0 & s_1 a_2 & 0 \\ 0 & 0 & s_2 a_3 \end{bmatrix}$-periodic.

Proof. By (3.2), the $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}$-periodic pattern is easily identified as $\begin{bmatrix} a_1 & m_1 b_{12} & m_2 b_{13} \\ 0 & m_1 a_2 & m_2 b_{23} \\ 0 & 0 & m_2 a_3 \end{bmatrix}$-periodic for all $m_1, m_2 \in \mathbb{N}$. By taking $m_1 = s_1$ and $m_2 = s_2$, the result holds. ■

Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$, defined on cubic lattice $\mathbb{Z}_{2\times2\times2}$, the L-periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times2\times2}$. For $n_1, n_2, n_3 \in \mathbb{Z}$, let $\mathbb{Z}_{2\times2\times2}((n_1, n_2, n_3))$ be the cubic lattice with the smallest vertex (n_1, n_2, n_3):

$$\mathbb{Z}_{2\times2\times2}((n_1, n_2, n_3)) = \{(n_1 + k_1, n_2 + k_2, n_3 + k_3) : 0 \leq k_1, k_2, k_3 \leq 1\}.$$

Now, the admissibility of L-periodic patterns is demonstrated to be verified on finite cubic lattices.

Proposition 3.2 *An L-periodic pattern U is B-admissible if and only if*

$$U|_{\mathbb{Z}_{2\times2\times2}((\alpha_1, \alpha_2, \alpha_3))} \in \mathcal{B}$$

for $0 \leq \alpha_i \leq a_i - 1$, $1 \leq i \leq 3$.

Proof. Sine $\mathcal{B} \subset \Sigma_{2\times2\times2}$, it is sufficient to prove

$$\left\{U \mid_{\mathbb{Z}_{2\times2\times2}((\alpha_1,\alpha_2,\alpha_3))}: \alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}\right\}$$

$$= \left\{U \mid_{\mathbb{Z}_{2\times2\times2}((\alpha_1,\alpha_2,\alpha_3))}: 0 \le \alpha_i \le a_i - 1, 1 \le i \le 3\right\}.$$

The proof follows easily from (3.2). The details are left to the reader. ■

According to Proposition 3.2, the admissibility of an L-periodic pattern U is determined by $U \mid_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}} = (u_{\alpha_1,\alpha_2,\alpha_3})$ and $U \mid_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}}$ has the periodic property that is given by (3.2), which can be divided into two parts:

$$\begin{cases} u_{a_1,\alpha_2,\alpha_3} = u_{0,\alpha_2,\alpha_3} \\ u_{\alpha_1,a_2,\alpha_3} = u_{[\alpha_1-b_{12}]_{a_1},0,\alpha_3} \end{cases} \tag{3.3}$$

for $0 \le \alpha_i \le a_i$, $1 \le i \le 3$, where $[m]_n \equiv m \ (mod\ n)$;

$$u_{\alpha_1,\alpha_2,a_3} = \begin{cases} u_{[\alpha_1-b_{12}-b_{13}]_{a_1},0,0} & \text{if} \quad \alpha_2 - b_{23} = a_2 \\ u_{[\alpha_1-b_{13}]_{a_1},\alpha_2-b_{23},0} & \text{if} \quad 0 \le \alpha_2 - b_{23} \le a_2 - 1 \\ u_{[\alpha_1+b_{12}-b_{13}]_{a_1},\alpha_2-b_{23}+a_2,0} & \text{if} \quad -a_2 + 1 \le \alpha_2 - b_{23} \le -1 \end{cases} \tag{3.4}$$

for $0 \le \alpha_1 \le a_1$, $0 \le \alpha_2 \le a_2$.

Notably, $(u_{\alpha_1,\alpha_2,\alpha_3})_{0\le\alpha_1\le a_1,0\le\alpha_2\le a_2,\alpha_3}$ has the same structure (3.3) for all $0 \le \alpha_3 \le a_3$, which fact is useful in constructing the cylindrical ordering matrix. Then, the set of all local patterns in $\Sigma_{a_1+1,a_2+1,a_3+1}$ that satisfy the periodic property (3.3) is denoted by $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$. However, (3.4) is important in allowing patterns in $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$ to become L-periodic and it will be used to define the rotational matrices later.

Now, the counting function for $U_{n_1\times n_2\times n_3} = (u_{\alpha_1,\alpha_2,\alpha_3})$ in $\Sigma_{n_1\times n_2\times n_3}$, $n_1, n_2, n_3 \ge 1$, is defined by

$$\psi\left(U_{n_1\times n_2\times n_3}\right) = 1 + \sum_{\alpha_1=0}^{n_1-1}\sum_{\alpha_2=0}^{n_2-1}\sum_{\alpha_3=0}^{n_3-1} u_{\alpha_1,\alpha_2,\alpha_3} 2^{n_1n_2(n_3-1-\alpha_3)+n_1(n_2-1-\alpha_2)+n_1-1-\alpha_1}. \tag{3.5}$$

Similar to (3.5), the counting function $\overline{\psi}$ for patterns $\overline{U}$ in $\mathbb{P}_{n_1,n_2;l;1}$, $0 \le l \le n_1 - 1$, is defined by

$$\overline{\psi}\left(\overline{U}\right) \equiv \psi\left(\overline{U} \mid_{\mathbb{Z}_{n_1\times n_2\times 1}}\right). \tag{3.6}$$

Notably, $\overline{\psi}$ is bijective from $\mathbb{P}_{n_1,n_2;l;1}$ to $\{i \mid 1 \le i \le 2^{n_1 n_2}\}$.

Given $n_1, n_2 \ge 1$, $0 \le l \le n_1 - 1$, $h \ge 1$, a local pattern $\overline{U}$ in $\mathbb{P}_{n_1,n_2;l;h}$ can be represented as

$$\overline{U} = \overline{U}_0 \oplus_z \overline{U}_1 \oplus_z \cdots \oplus_z \overline{U}_{h-1}, \tag{3.7}$$

where $\overline{U}_i \in \mathbb{P}_{n_1,n_2;l;1}$, $0 \le i \le h-1$, and $\overline{U}' \oplus_z \overline{U}''$ means that $\overline{U}''$ is put on the top (in the z-direction) of $\overline{U}'$. Therefore, the cylindrical ordering matrix $\mathbb{C}_{n_1,n_2;l;h} = [C_{n_1,n_2;l;h;i,j}]_{2^{n_1 n_2} \times 2^{n_1 n_2}}$ of patterns in $\mathbb{P}_{n_1,n_2;l;h}$ is defined by

$$C_{n_1,n_2;l;h;i,j} = \left\{ \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{h-1} \mid \overline{\psi}(\overline{U}_0) = i \text{ and } \overline{\psi}(\overline{U}_{h-1}) = j \right\}. \tag{3.8}$$

In particular, for $h = 2$, $\mathbb{C}_{n_1,n_2;l;2}$ can be applied to construct the associated trace operator. Notably the set $\mathbb{C}_{n_1,n_2;l;2;i,j}$ contains exactly one pattern.

Now, given $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the associated trace operator $\mathbf{T}_{n_1,n_2;l}(\mathcal{B}) = [t_{n_1,n_2;l;i,j}]$, with $t_{n_1,n_2;l;i,j} \in \{0,1\}$, can be defined by

$$t_{n_1,n_2;l;i,j} = 1 \quad \text{if and only if} \quad \text{the pattern in } C_{n_1,n_2;l;2;i,j} \text{ is } \mathcal{B}\text{-admissible}. \tag{3.9}$$

Remark 3.3 *Given* $L' = \begin{bmatrix} a_1 & b_{12} & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3$, *(3.3) and (3.4) easily verify that*

$$\begin{aligned} &\left\{ U|_{\mathbb{Z}_{a_1+1,a_2+1,a_3+1}} : U \textit{ is } L'\textit{-periodic} \right\} \\ = \; &\left\{ \overline{U} = \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{a_3} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \overline{U}_0 = \overline{U}_{a_3} \right\}. \end{aligned} \tag{3.10}$$

Furthermore, given $\mathcal{B} \subset \Sigma_{2\times2\times2}$, *from Proposition 3.2 and the construction of the transition matrix* $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right) = tr\left(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B}) \right). \tag{3.11}$$

The shift maps and the related rotational matrices are considered below for general

$$L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3.$$

Let $n_1, n_2 \geq 1$, $0 \leq l \leq n_1 - 1$; the shift (to the left) in the x-direction of any pattern $\overline{U} = (u_{\alpha_1,\alpha_2,0})$ in $\mathbb{P}_{n_1,n_2;l;1}$, $u_{\alpha_1,\alpha_2,0} \in \{0,1\}$, is defined by

$$\sigma_{x;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0})) = \left(u^{(1)}_{\alpha_1,\alpha_2,0}\right)_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

where

$$u^{(1)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1+1-l]_{n_1},0,0} & \text{if } \alpha_2 = n_2, \\ u_{[\alpha_1+1]_{n_1},\alpha_2,0} & \text{if } 0 \leq \alpha_2 \leq n_2 - 1. \end{cases} \tag{3.12}$$

Similarly, the shift (to the below) in the y-direction is defined by

$$\sigma_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0})) = \left(u^{(2)}_{\alpha_1,\alpha_2,0}\right)_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

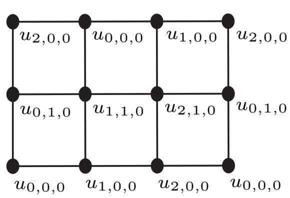
where

$$u^{(2)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1-l]_{n_1},\alpha_2+1-n_2,0} & \text{if } \alpha_2 + 1 \geq n_2, \\ u_{[\alpha_1]_{n_1},\alpha_2+1,0} & \text{if } 0 \leq \alpha_2 + 1 \leq n_2 - 1. \end{cases} \tag{3.13}$$

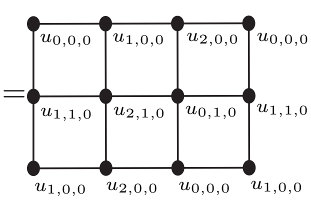
Notably, $\sigma_{x;n_1,n_2;l}$ and $\sigma_{y;n_1,n_2;l}$ are automorphisms on $\mathbb{P}_{n_1,n_2;l;1}$.

The following example illustrates $\sigma_{x;n_1,n_2;l}$ and $\sigma_{y;n_1,n_2;l}$.

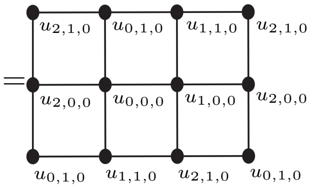
Example 3.4 *Let*

$\overline{U} = (u_{\alpha_1,\alpha_2,0}) \equiv$  $\in \mathbb{P}_{3,2;1;1}$

be a local pattern that lies on the plane $\{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{Z}\}$. *Now, consider* $\sigma_{x;3,2;1}$ *and* $\sigma_{y;3,2;1}$ *which are acting on* $\overline{U}$. *Then it is easy to see*

$\sigma_{x;3,2;1}\left(\overline{U}\right) =$ 

and

$$\sigma_{y;3,2;1}\left(\overline{U}\right) =$$ 

.

Moreover, both $\sigma_{x;3,2;1}\left(\overline{U}\right)$ *and* $\sigma_{y;3,2;1}\left(\overline{U}\right)$ *are also belong to* $\mathbb{P}_{3,2;1;1}$.

From (3.12) and (3.13), for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$, the following can be straightforwardly verified;

$$\sigma^{r_1}_{x;n_1,n_2;l}\left(\sigma^{r_2}_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))\right) = \left(u^{(3)}_{\alpha_1,\alpha_2,0}\right)_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

where

$$u^{(3)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1+r_1-l]_{n_1},\alpha_2+r_2-n_2,0} & \text{if} \quad n_2 \leq \alpha_2 + r_2 \leq 2n_2 - 1, \\ u_{[\alpha_1+r_1]_{n_1},\alpha_2+r_2,0} & \text{if} \quad 0 \leq \alpha_2 + r_2 \leq n_2 - 1. \end{cases} \tag{3.14}$$

Furthermore,

$$\sigma_{y;n_1,n_2;l} \circ \sigma_{x;n_1,n_2;l} = \sigma_{x;n_1,n_2;l} \circ \sigma_{y;n_1,n_2;l} \tag{3.15}$$

and

$$\sigma^{n_1}_{x;n_1,n_2;l} = \sigma^{l}_{x;n_1,n_2;l}\left(\sigma^{n_2}_{y;n_1,n_2;l}\right) = \text{ identity map.} \tag{3.16}$$

Hence,

$$\sigma^{-1}_{x;n_1,n_2;l} \equiv \sigma^{n_1-1}_{x;n_1,n_2;l} \quad \text{and} \quad \sigma^{-1}_{y;n_1,n_2;l} \equiv \sigma^{l}_{x;n_1,n_2;l}\left(\sigma^{n_2-1}_{y;n_1,n_2;l}\right). \tag{3.17}$$

Therefore, for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$,

$$\sigma^{-r_1}_{x;n_1,n_2;l}\left(\sigma^{-r_2}_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))\right) = \left(u^{(4)}_{\alpha_1,\alpha_2,0}\right)_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

where

$$u^{(4)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1-r_1-l]_{n_1},0,0} & \text{if} \quad \alpha_2 - r_2 = n_2, \\ u_{[\alpha_1-r_1]_{n_1},\alpha_2-r_2,0} & \text{if} \quad 0 \leq \alpha_2 - r_2 \leq n_2 - 1, \\ u_{[\alpha_1-r_1+l]_{n_1},\alpha_2-r_2+n_2,0} & \text{if} \quad -n_2 + 1 \leq \alpha_2 - r_2 \leq -1. \end{cases} \tag{3.18}$$

Now, the two rotational matrices $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$ are defined as follows.

Definition 3.5 *The* $2^{n_1 n_2} \times 2^{n_1 n_2}$ *x-rotational matrix* $R_{x;n_1,n_2;l} = [R_{x;n_1,n_2;l;i,j}]$, $R_{x;n_1,n_2;l;i,j} \in \{0,1\}$, *is defined by*

$$R_{x;n_1,n_2;l;i,j} = 1 \quad \text{if and only if} \quad i = \overline{\psi}(\overline{U}) \text{ and } j = \overline{\psi}(\sigma_{x;n_1,n_2;l}(\overline{U})), \tag{3.19}$$

where $\overline{U} \in \mathbb{P}_{n_1,n_2;l;1}$. *From (3.19), for convenience, denote by*

$$j = \sigma_x(i). \tag{3.20}$$

Similarly, the $2^{n_1 n_2} \times 2^{n_1 n_2}$ *y-rotational matrix* $R_{y;n_1,n_2;l} = [R_{y;n_1,n_2;l;i,j}]$, $R_{y;n_1,n_2;l;i,j} \in \{0,1\}$, *is defined by*

$$R_{y;n_1,n_2;l;i,j} = 1 \quad \text{if and only if} \quad i = \overline{\psi}(\overline{U}) \text{ and } j = \overline{\psi}(\sigma_{y;n_1,n_2;l}(\overline{U})), \tag{3.21}$$

where $\overline{U} \in \mathbb{P}_{n_1,n_2;l;1}$. *From (3.21), for convenience, denote by*

$$j = \sigma_y(i). \tag{3.22}$$

Obviously, $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$ are permutation matrices. By (3.16), $R^{n_1}_{x;n_1,n_2;l} = R^{l}_{x;n_1,n_2;l} R^{n_2}_{y;n_1,n_2;l} = I_{2^{n_1 n_2}}$, where I_n is the $n \times n$ identity matrix.

Example 3.6 *Let* $n_1 = 2$, $n_2 = 1$ *and* $l = 1$,

$$R_{x;2,1;1} = R_{y;2,1;1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$R^2_{x;2,1;1} = R_{x;2,1;1} R_{y;2,1;1} = I_4 \quad \text{but} \quad R_{y;2,1;1} \neq I_4.$$

The following proposition shows the permutation characters of $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$.

Proposition 3.7 *Let* $\mathbb{M} = [M_{i,j}]_{2^{n_1 n_2} \times 2^{n_1 n_2}}$ *be a matrix where* $M_{i,j}$ *denotes a number or a pattern or a set of patterns. Then*

$$(\mathbb{M}R_{x;n_1,n_2;l})_{i,j} = M_{i,\sigma_x^{-1}(j)} \quad and \quad (\mathbb{M}R_{y;n_1,n_2;l})_{i,j} = M_{i,\sigma_y^{-1}(j)}. \tag{3.23}$$

Furthermore, for any $r \geq 1$

$$\left(\mathbb{M}R^r_{x;n_1,n_2;l}\right)_{i,j} = M_{i,\sigma_x^{-r}(j)} \quad and \quad \left(\mathbb{M}R^r_{y;n_1,n_2;l}\right)_{i,j} = M_{i,\sigma_y^{-r}(j)}. \tag{3.24}$$

Proof. For any $1 \leq i,j \leq 2^{n_1 n_2}$, by (3.20),

$$\begin{aligned}(\mathbb{M}R_{x;n_1,n_2;l})_{i,j} &= \sum_q M_{i,q} R_{x;n_1,n_2;l;q,j} \\ &= M_{i,\sigma_x^{-1}(j)} R_{x;n_1,n_2;l;\sigma_x^{-1}(j),j} \\ &= M_{i,\sigma_x^{-1}(j)}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathbb{M}R_{y;n_1,n_2;l})_{i,j} &= \sum_q M_{i,q} R_{y;n_1,n_2;l;q,j} \\ &= M_{i,\sigma_y^{-1}(j)} R_{y;n_1,n_2;l;\sigma_y^{-1}(j),j} \\ &= M_{i,\sigma_y^{-1}(j)}.\end{aligned}$$

Applying (3.23) r times yields (3.24). The proof is complete. ■

Now, the following lemma can be obtained.

Lemma 3.8 *Given* $L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3$,

$$\left\{ U\mid_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}} : U \text{ is } L\text{-periodic} \right\}$$

$$= \left\{ \overline{U} = \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{a_3} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \overline{U}_{a_3} = \sigma^{-b_{13}}_{x;a_1,a_2;b_{12}} \left(\sigma^{-b_{23}}_{y;a_1,a_2;b_{12}}(\overline{U}_0)\right) \right\} \tag{3.25}$$

Proof. From (3.4) and (3.18),

$$\left\{ \overline{U} = \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{a_3} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \overline{U}_{a_3} = \sigma^{-b_{13}}_{x;a_1,a_2;b_{12}} \left(\sigma^{-b_{23}}_{y;a_1,a_2;b_{12}}(\overline{U}_0)\right) \right\}$$

$$= \left\{ \overline{U} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \overline{U} \text{ satisfies } (3.4) \right\}.$$

Then, by the construction of $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$, the last set is equal to

$$\{U \in \Sigma_{a_1+1,a_2+1,a_3+1} : U \text{ satisfies } (3.3) \text{ and } (3.4)\}$$

$$= \{U \in \Sigma_{a_1+1,a_2+1,a_3+1} : U \text{ satisfies } (3.2)\}.$$

Therefore, (3.25) follows. The proof is complete. ■

Proposition 3.2, 3.7 and Lemma 3.8 yield the following main results for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$.

Theorem 3.9 *Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For $a_i \geq 1, 1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = tr\left(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B}) R^{b_{13}}_{x;a_1,a_2;b_{12}} R^{b_{23}}_{y;a_1,a_2;b_{12}}\right). \tag{3.26}$$

Furthermore,

$$\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = tr\left(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B}) \mathbf{R}_{a_1,a_2;b_{12}}\right), \tag{3.27}$$

where

$$\mathbf{R}_{a_1,a_2;b_{12}} = \sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} R^{b_{13}}_{x;a_1,a_2;b_{12}} R^{b_{23}}_{y;a_1,a_2;b_{12}}. \tag{3.28}$$

Proof. From Proposition 3.2, Lemma 3.8 and the construction of $\mathbb{C}_{a_1,a_2;b_{12};a_3+1}$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$$

$$= \sum_{i=1}^{2^{a_1 a_2}} \sharp\left\{\overline{U} \in C_{a_1,a_2;b_{12};a_3+1;i,j} : \overline{U} \text{ is } \mathcal{B}\text{-admissible and } j = \sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(i)\right)\right\},$$

where $\sharp S$ is the cardinal number of set S.

Then, Proposition 3.7 and the construction of $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$, $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ easily yield (3.26). Equation (3.27) holds from (3.26) and (3.28). The proof is complete. ■

The $(a_1, a_2; b_{12})$-th zeta function $\zeta_{a_1,a_2;b_{12}}(s)$ can now be obtained as follows.

Theorem 3.10 *Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$. For $a_i \geq 1, 1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,*

$$\zeta_{a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1 a_2}\sum_{a_3=1}^{\infty}\frac{1}{a_3} tr\left(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B})\mathbf{R}_{a_1,a_2;b_{12}}\right) s^{a_1 a_2 a_3}\right). \tag{3.29}$$

Proof. The results follow from Theorem 3.9. ■

3.1.2 Rationality of $\zeta_{a_1,a_2;b_{12}}$

This subsection proves that $\zeta_{a_1,a_2;b_{12}}$ is a rational function. First, the rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ is introduced.

Theorem 3.11 *Given $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$. Denote by $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B}) = [t_{a_1,a_2;b_{12};i,j}]$. For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$,*

$$t_{a_1,a_2;b_{12};\sigma_x^{-1}(i),\sigma_x^{-1}(j)} = t_{a_1,a_2;b_{12};i,j} \tag{3.30}$$

and

$$t_{a_1,a_2;b_{12};\sigma_y^{-1}(i),\sigma_y^{-1}(j)} = t_{a_1,a_2;b_{12};i,j} \tag{3.31}$$

for all $1 \leq i, j \leq 2^{a_1 a_2}$. Furthermore,

$$t_{a_1,a_2;b_{12};\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(i)\right),\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(j)\right)} = t_{a_1,a_2;b_{12};i,j} \tag{3.32}$$

for all $1 \leq i, j \leq 2^{a_1 a_2}$, $-a_1 + 1 \leq r_1 \leq a_1 - 1$ and $-a_2 + 1 \leq r_2 \leq a_2 - 1$.

Proof. The proof of (3.31) is similar to that of (3.30) and omitted. We now prove (3.30).

Given $1 \leq i, j \leq 2^{a_1 a_2}$, $C_{a_1,a_2;b_{12};2;i,j}$ and $C_{a_1,a_2;b_{12};2;\sigma_x^{-1}(i),\sigma_x^{-1}(j)}$ contain only one pattern respectively. Let

$$\overline{U} = \overline{U}_0 \oplus_z \overline{U}_1 = (u_{\alpha_1,\alpha_2,\alpha_3}) \in C_{a_1,a_2;b_{12};2;i,j}$$

with $\overline{\psi}(\overline{U}_0) = i$ and $\overline{\psi}(\overline{U}_1) = j$, and

$$\overline{U}' = \overline{U}_0' \oplus_z \overline{U}_1' = (u'_{\alpha_1,\alpha_2,\alpha_3}) \in C_{a_1,a_2;b_{12};2;\sigma_x^{-1}(i),\sigma_x^{-1}(j)}$$

with $\overline{\psi}(\overline{U}_0') = \sigma_x^{-1}(i)$ and $\overline{\psi}(\overline{U}_1') = \sigma_x^{-1}(j)$. Since $\mathcal{B} \subset \Sigma_{2\times2\times2}$ and (3.9), to prove (3.30) is equal to prove

$$\begin{aligned} & \{(u_{n_1+k_1,n_2+k_2,k_3})_{0\leq k_1,k_2,k_3\leq 1} : 0 \leq n_1 \leq a_1-1, 0 \leq n_2 \leq a_2-1\} \\ = \; & \{(u'_{n_1+k_1,n_2+k_2,k_3})_{0\leq k_1,k_2,k_3\leq 1} : 0 \leq n_1 \leq a_1-1, 0 \leq n_2 \leq a_2-1\}. \end{aligned} \tag{3.33}$$

Since $\overline{\psi}(\overline{U}_0) = i$ and $\overline{\psi}(\overline{U}_0') = \sigma_x^{-1}(i)$, by (3.18),

$$u'_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1-1-b_{12}]_{a_1},0,0} & \text{if} \quad \alpha_2 = a_2, \\ u_{[\alpha_1-1]_{a_1},\alpha_2,0} & \text{if} \quad 0 \leq \alpha_2 \leq a_2-1. \end{cases}$$

Similarly, from $\overline{\psi}(\overline{U}_1) = j$ and $\overline{\psi}(\overline{U}_1') = \sigma_x^{-1}(j)$,

$$u'_{\alpha_1,\alpha_2,1} = \begin{cases} u_{[\alpha_1-1-b_{12}]_{a_1},0,1} & \text{if} \quad \alpha_2 = a_2, \\ u_{[\alpha_1-1]_{a_1},\alpha_2,1} & \text{if} \quad 0 \leq \alpha_2 \leq a_2-1. \end{cases}$$

Then, (3.33) is directly obtained.

Therefore, (3.30) and (3.31) hold. For $0 \leq r_1 \leq a_1 - 1$ and $0 \leq r_2 \leq a_2 - 1$, by applying (3.31) r_2 times and (3.30) r_1 times, (3.32) holds. From (3.15), (3.16) and (3.17), (3.32) follows. The proof is complete. ■

To study the rationality of $\zeta_{a_1,a_2;b_{12}}$, we need more definitions and properties about the two shifts in (3.20) and (3.22) as follows.

Given $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, for $1 \leq i \leq 2^{a_1a_2}$, the equivalent class $\mathcal{C}_{a_1,a_2;b_{12}}(i)$ of i is defined by

$$\mathcal{C}_{a_1,a_2;b_{12}}(i) \equiv \left\{\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(i)\right) : 0 \leq r_1 \leq a_1-1, 0 \leq r_2 \leq a_2-1\right\}. \tag{3.34}$$

Clearly,

$$\text{either } \mathcal{C}_{a_1,a_2;b_{12}}(i) = \mathcal{C}_{a_1,a_2;b_{12}}(j) \text{ or } \mathcal{C}_{a_1,a_2;b_{12}}(i) \cap \mathcal{C}_{a_1,a_2;b_{12}}(j) = \emptyset. \tag{3.35}$$

The cardinal number of $\mathcal{C}_{a_1,a_2;b_{12}}(i)$ is denoted by $\omega_{a_1,a_2;b_{12};i}$. Let i be the smallest element in its equivalent class, and the index set $\mathcal{I}_{a_1,a_2;b_{12}}$ is defined by

$$\mathcal{I}_{a_1,a_2;b_{12}} = \left\{ i : 1 \le i \le 2^{a_1 a_2}, i \le j \text{ for all } j \in \mathcal{C}_{a_1,a_2;b_{12}}(i) \right\}. \tag{3.36}$$

Therefore,

$$\{ j : 1 \le j \le 2^{a_1 a_2} \} = \bigcup_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \mathcal{C}_{a_1,a_2;b_{12}}(i). \tag{3.37}$$

The cardinal number of $\mathcal{I}_{a_1,a_2;b_{12}}$ is denoted by $\chi_{a_1,a_2;b_{12}}$.

The following example illustrates $\mathcal{C}_{2,2;j}(i)$.

Example 3.12

$$\begin{cases} \mathcal{C}_{2,2;0}(1) = \{1\} \\ \mathcal{C}_{2,2;0}(2) = \{2,3,5,9\} \\ \mathcal{C}_{2,2;0}(4) = \{4,13\} \\ \mathcal{C}_{2,2;0}(6) = \{6,11\} \\ \mathcal{C}_{2,2;0}(7) = \{7,10\} \\ \mathcal{C}_{2,2;0}(8) = \{8,12,14,15\} \\ \mathcal{C}_{2,2;0}(16) = \{16\} \\ \mathcal{I}_{2,2;0} = \{1,2,4,6,7,8,16\} \end{cases} \quad \begin{cases} \mathcal{C}_{2,2;1}(1) = \{1\} \\ \mathcal{C}_{2,2;1}(2) = \{2,3,5,9\} \\ \mathcal{C}_{2,2;1}(4) = \{4,13\} \\ \mathcal{C}_{2,2;1}(6) = \{6,7,10,11\} \\ \mathcal{C}_{2,2;1}(8) = \{8,12,14,15\} \\ \mathcal{C}_{2,2;1}(16) = \{16\} \\ \mathcal{I}_{2,2;1} = \{1,2,4,6,8,16\} \end{cases}$$

The equivalent classes are invariant under the two shift maps. Therefore, the following proposition is directly obtained and the proof is omitted.

Proposition 3.13 *Given* $a_1, a_2 \ge 1$ *and* $0 \le b_{12} \le a_1 - 1$. *Let* $N \equiv 2^{a_1 a_2}$ *and* $V = (v_1, v_2, \cdots, v_N)^t$, *for* $1 \le i \le N$,

$$\sum_{r_1=0}^{a_1-1} \sum_{r_2=0}^{a_2-1} v_{\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(i)\right)} = \frac{a_1 a_2}{\omega_{a_1,a_2;b_{12};i}} \sum_{j \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} v_j. \tag{3.38}$$

For the rationality of $\zeta_{a_1,a_2;b_{12}}$, the reduced trace operator $\tau_{a_1,a_2;b_{12}}$ of $\mathbf{T}_{a_1,a_2;b_{12}}$ is introduced as follows.

Definition 3.14 *For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the reduced trace operator $\tau_{a_1,a_2;b_{12}} = [\tau_{a_1,a_2;b_{12};i,j}]$ of $\mathbf{T}_{a_1,a_2;b_{12}} = [t_{a_1,a_2;b_{12};i,j}]$ is a $\chi_{a_1,a_2;b_{12}} \times \chi_{a_1,a_2;b_{12}}$ matrix and is defined by*

$$\tau_{a_1,a_2;b_{12};i,j} = \sum_{k \in \mathcal{C}_{a_1,a_2;b_{12}}(j)} t_{a_1,a_2;b_{12};i,k} \tag{3.39}$$

for each $i, j \in \mathcal{I}_{a_1,a_2;b_{12}}$.

The following theorem expresses the average of $\Gamma_{\mathcal{B}}$ in terms of the trace of the reduced trace operator τ and plays a crucial role in proving the rationality of $\zeta_{a_1,a_2;b_{12}}$. The proof here is simpler and more straightforward than the proofs in Subsection 2.2 for $d = 2$.

Theorem 3.15 *Given $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$. For $a_i \geq 1$, $1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,*

$$\begin{aligned} \frac{1}{a_1 a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) &= tr\left(\tau^{a_3}_{a_1,a_2;b_{12}}\right) \\ &= \sum_{\lambda \in \Sigma(\tau_{a_1,a_2;b_{12}})} \chi_{a_1,a_2;b_{12}}(\lambda)\lambda^{a_3}, \end{aligned} \tag{3.40}$$

where $\Sigma(\tau_{a_1,a_2;b_{12}})$ is the spectrum of $\tau_{a_1,a_2;b_{12}}$ and $\chi_{a_1,a_2;b_{12}}(\lambda)$ is the algebraic multiplicity of $\tau_{a_1,a_2;b_{12}}$ with eigenvalue λ.

Proof. For simplicity, let $N = 2^{a_1 a_2}$ and $\mathbf{T}_{a_1,a_2;b_{12}} = [t_{i,j}]$. From Proposition 3.7 and Theorem 3.9,

$$\begin{aligned} &\frac{1}{a_1 a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) \\ = &\frac{1}{a_1 a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} \mathrm{tr}\left(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}} R^{b_{13}}_{x;a_1,a_2;b_{12}} R^{b_{23}}_{y;a_1,a_2;b_{12}}\right) \\ = &\frac{1}{a_1 a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{i=1}^{N}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{i,k_1}t_{k_1,k_2}\cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(i)\right)}. \end{aligned}$$

Now, by Eq. (3.37), the last sum becomes

$$\frac{1}{a_1a_2}\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}}\sum_{q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{q,k_1}t_{k_1,k_2}\cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(q)\right)}. \tag{3.41}$$

Fixed $q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)$, there exist $0\le r_1\le a_1-1$ and $0\le r_2\le a_2-1$ such that $q=\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(i)\right)$. Then, by Theorem 3.11,

$$\begin{aligned}
&\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{q,k_1}t_{k_1,k_2}\cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(q)\right)}\\
=&\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{\sigma_x^{r_1}\left(\sigma_y^{r_2}(q)\right),\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_1)\right)}t_{\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_1)\right),\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_2)\right)}\\
&\qquad\qquad\cdots t_{\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_{a_3-1})\right),\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}\left(\sigma_x^{r_1}\left(\sigma_y^{r_2}(q)\right)\right)\right)}\\
=&\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{i,\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_1)\right)}t_{\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_1)\right),\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_2)\right)}\cdots t_{\sigma_x^{r_1}\left(\sigma_y^{r_2}(k_{a_3-1})\right),\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(i)\right)}.
\end{aligned}$$

Since $\left\{\sigma_x^{r_1}\left(\sigma_y^{r_2}(m)\right):1\le m\le N\right\}=\{m:1\le m\le N\}$, the last sum becomes

$$\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{i,k_1}t_{k_1,k_2}\cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(i)\right)} \tag{3.42}$$

Therefore, Eq. (3.41) is equal to

$$\frac{1}{a_1a_2}\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}}\omega_{a_1,a_2;b_{12};i}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} t_{i,k_1}t_{k_1,k_2}\cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}\left(\sigma_y^{-b_{23}}(i)\right)}. \tag{3.43}$$

According to Proposition 3.13, Eq. (3.43) is equal to

$$\begin{aligned}
&\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}}\sum_{j=1}^{a_3-1}\sum_{k_j=1}^{N} t_{i,k_1}\cdots t_{k_{a_3-2},k_{a_3-1}}\left(\sum_{q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q}\right)\\
=&\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}}\sum_{j=1}^{a_3-1}\sum_{k_j\in\mathcal{I}_{a_1,a_2;b_{12}}}\sum_{q_j\in\mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1}\cdots t_{q_{a_3-2},q_{a_3-1}}\left(\sum_{q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q}\right).
\end{aligned} \tag{3.44}$$

For any $q_{a_3-1} \in \mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})$, there exist $0 \leq r_1 \leq a_1 - 1$ and $0 \leq r_2 \leq a_2 - 1$ such that

$$q_{a_3-1} = \sigma_x^{-r_1}\left(\sigma_y^{-r_2}(k_{a_3-1})\right).$$

Then, by Theorem 3.11,

$$\begin{aligned}
\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q} &= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{\sigma_x^{-r_1}\left(\sigma_y^{-r_2}(k_{a_3-1})\right),q} \\
&= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},\sigma_x^{r_1}\left(\sigma_y^{r_2}(q)\right)} \\
&= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{a_3-1} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1} \cdots t_{q_{a_3-2},q_{a_3-1}} \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q} \right) \\
= &\sum_{j=1}^{a_3-2} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1} \cdots \left(\sum_{q_{a_3-1} \in \mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})} t_{q_{a_3-2},q_{a_3-1}} \right) \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q} \right) \\
= &\sum_{j=1}^{a_3-2} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1} \cdots \left(\sum_{q_{a_3-1} \in \mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})} t_{k_{a_3-2},q_{a_3-1}} \right) \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q} \right) \\
&\vdots \\
= &\left(\sum_{q_1 \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{i,q_1} \right) \left[\prod_{j=2}^{a_3-1} \left(\sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{k_{j-1},q_j} \right) \right] \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q} \right) \\
= &\tau_{a_1,a_2;b_{12};i,k_1} \tau_{a_1,a_2;b_{12};k_1,k_2} \cdots \tau_{a_1,a_2;b_{12};k_{a_3-1},i}
\end{aligned}$$

Finally, (3.44) is equal to

$$\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}}\sum_{j=1}^{a_3-1}\sum_{k_j\in\mathcal{I}_{a_1,a_2;b_{12}}}\tau_{a_1,a_2;b_{12};i,k_1}\tau_{a_1,a_2;b_{12};k_1,k_2}\cdots\tau_{a_1,a_2;b_{12};k_{a_3-1},i}$$

$$= \operatorname{tr}\left(\tau^{a_3}_{a_1,a_2;b_{12}}\right)$$

$$= \sum_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})}\chi_{a_1,a_2;b_{12}}(\lambda)\lambda^{a_3}.$$

The proof is complete. ■

Therefore, the rationality of $\zeta_{a_1,a_2;b_{12}}$ and ζ can be obtained as follows.

Theorem 3.16 *For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$,*

$$\begin{aligned}\zeta_{a_1,a_2;b_{12}}(s) &= \left(\det\left(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}\right)\right)^{-1}\\ &= \prod_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})}\left(1-\lambda s^{a_1a_2}\right)^{-\chi_{a_1,a_2;b_{12}}(\lambda)},\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\zeta(s) &= \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\left(\det\left(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}\right)\right)^{-1}\\ &= \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\prod_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})}\left(1-\lambda s^{a_1a_2}\right)^{-\chi_{a_1,a_2;b_{12}}(\lambda)}.\end{aligned} \tag{3.46}$$

Proof. By using the power series

$$-\log(1-t) = \sum_{n=1}^{\infty}\frac{t^n}{n}, \tag{3.47}$$

equation (3.45) follows from (1.34) and Theorem 3.15. Equation (3.46) follows form (1.35) and (3.45). ■

The following example is used to demonstrate the application of the above result.

Example 3.17 *Consider*

$$\mathcal{B} = \left\{U_{2\times2\times2} = (u_{\alpha_1,\alpha_2,\alpha_3}) \in \Sigma_{2\times2\times2} : u_{0,0,j} = u_{1,0,j} = u_{0,1,j} = u_{1,1,j} \text{ for } j = 0, 1\right\}.$$

Clearly, the set $\mathcal{P}(\mathcal{B})$ of all $\mathcal{B}$-admissible and periodic patterns is

$$\left\{U=(u_{\alpha_1,\alpha_2,\alpha_3})\in\Sigma_2^3 : u_{i,j,k}=u_{0,0,k} \text{ for all } i,j,k\in\mathbb{Z}\right\}.$$

Then, it is easy to verify that

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)=2^{a_3}$$

for $a_i\geq 1$, $1\leq i\leq 3$, $0\leq b_{ij}\leq a_i-1$, $i+1\leq j\leq 3$. *Therefore,*

$$\zeta_{a_1,a_2;b_{12}}(s)=\left(1-2s^{a_1a_2}\right)^{-1} \tag{3.48}$$

and

$$\zeta(s)=\prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\left(1-2s^{a_1a_2}\right)^{-a_1}. \tag{3.49}$$

However, (3.48) and (3.49) can be obtained from (3.45) and (3.46). The trace operator

$$\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})=\mathbf{T}_{a_1,a_2;0}(\mathcal{B})=\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2^{a_1a_2}\times 2^{a_1a_2}}.$$

Since $\mathcal{C}_{a_1,a_2;b_{12}}(1)=\{1\}$ *and* $\mathcal{C}_{a_1,a_2;b_{12}}(2^{a_1a_2})=\{2^{a_1a_2}\}$, *the reduced trace operator*

$$\tau_{a_1,a_2;b_{12}}(\mathcal{B})=\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{\chi_{a_1,a_2;b_{12}}\times\chi_{a_1,a_2;b_{12}}}.$$

Therefore,

$$\begin{aligned}\zeta_{a_1,a_2;b_{12}}(s) &= \left(\det\left(I-s^{a_1a_2}\tau_{a_1,a_2;b_{12}}\right)\right)^{-1} \\ &= \left(1-2s^{a_1a_2}\right)^{-1}\end{aligned}$$

and

$$\zeta(s)=\prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\left(1-2s^{a_1a_2}\right)^{-a_1}.$$

Equations (3.48) and (3.49) are recovered.

3.1.3 Zeta functions in inclined coordinates

This subsection presents the zeta function with respect to inclined coordinates, determined by applying the unimodular transformations in $GL_3(\mathbb{Z})$. $\mathbb{Z}^3$ is known to be invariant under the unimodular transformation in $GL_3(\mathbb{Z})$. Indeed, Lind [36] proved that the zeta function $\zeta^0_{\mathcal{B}}$ is independent of a choice of basis for $\mathbb{Z}^3$. Recall that

$$GL_d(\mathbb{Z})=\left\{\gamma=[\gamma_{ij}]_{1\leq i,j\leq d}:\gamma_{ij}\in\mathbb{Z}\text{ for }1\leq i,j\leq d\text{ and }|\det(\gamma)|=1\right\}.$$

This subsection presents the construction of the trace operator $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B})$ and the reduced trace operator $\tau_{\gamma;a_1,a_2;b_{12}}(\mathcal{B})$, and then determines $\zeta_{\gamma;a_1,a_2;b_{12}}$ and $\zeta_{\mathcal{B};\gamma}$. Finally, $\zeta_{\mathcal{B};\gamma}$ is obtained as

$$\zeta_{\mathcal{B};\gamma}(s)=\prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\left(\det\left(I-s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}\right)\right)^{-1}. \quad (3.50)$$

For simplicity, only $\mathcal{B}\subset\Sigma_{2\times2\times2}$ with two symbols are considered. The general cases can be treated analogously.

For a given $\gamma=\begin{bmatrix}\gamma_{11}&\gamma_{12}&\gamma_{13}\\\gamma_{21}&\gamma_{22}&\gamma_{23}\\\gamma_{31}&\gamma_{32}&\gamma_{33}\end{bmatrix}\in GL_3(\mathbb{Z})$, the lattice points in γ-coordinates are

$$(1,0,0)_\gamma=(\gamma_{11},\gamma_{12},\gamma_{13}),\ \ (0,1,0)_\gamma=(\gamma_{21},\gamma_{22},\gamma_{23}),\ \ (0,0,1)_\gamma=(\gamma_{31},\gamma_{32},\gamma_{33}),$$

and the unit vectors are

$$\begin{pmatrix}1\\0\\0\end{pmatrix}_\gamma=\begin{pmatrix}\gamma_{11}\\\gamma_{12}\\\gamma_{13}\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix}_\gamma=\begin{pmatrix}\gamma_{21}\\\gamma_{22}\\\gamma_{23}\end{pmatrix}\ \text{and}\ \begin{pmatrix}0\\0\\1\end{pmatrix}_\gamma=\begin{pmatrix}\gamma_{31}\\\gamma_{32}\\\gamma_{33}\end{pmatrix}.$$

Notably, when $\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, standard rectangular coordinates are used and the subscript γ is omitted.

The matrix M_γ is defined by

$$M_\gamma = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma = \gamma^t \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}.$$

Let $L_\gamma = M_\gamma \mathbb{Z}^3$. Then,

$$L_\gamma = \gamma^t \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3 \tag{3.51}$$

is easily verified.

A global pattern $U_\gamma = \left(u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}\right)_{\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}}$ is called L_γ-periodic or $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma$-periodic if for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$,

$$u_{(\alpha_1+a_1p+b_{12}q+b_{13}r,\alpha_2+a_2q+b_{23}r,\alpha_3+a_3r)_\gamma} = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} \tag{3.52}$$

for all $p, q, r \in \mathbb{Z}$. Therefore, the $(a_1, a_2; b_{12})$-th zeta function of $\zeta^0_{\mathcal{B}}(s)$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1a_2} \sum_{a_3=1}^{\infty} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \frac{1}{a_3} \Gamma_{\mathcal{B}} \left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma \right) s^{a_1a_2a_3} \right) \tag{3.53}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{b_{12}=0}^{a_1-1} \zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}. \tag{3.54}$$

The following introduces the cylindrical ordering matrix, the trace operator and the rotational matrices. The proofs of the results as in previous subsections are omitted.

Fix a $\gamma \in GL_3(\mathbb{Z})$. Let $\mathbb{Z}_{\gamma;n_1\times n_2\times n_3}$ be the $n_1 \times n_2 \times n_3$ lattice with the basis

$$\gamma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{pmatrix} \text{ and } \gamma_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{pmatrix}.$$

The total number of lattice points on $\mathbb{Z}_{\gamma;n_1\times n_2\times n_3}$ is $n_1 \cdot n_2 \cdot n_3$.

Since the basic set $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$, the L_γ-periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2\times 2}$. Let $(n_1, n_2, n_3)_\gamma = (m_1, m_2, m_3)$,

$$\mathbb{Z}_{2\times 2\times 2}\left((n_1, n_2, n_3)_\gamma\right) = \left\{(m_1 + k_1, m_2 + k_2, m_3 + k_3) : 0 \leq k_1, k_2, k_3 \leq 1\right\}.$$

Now, the admissibility is demonstrated to have to be verified on finite lattice as follows.

Proposition 3.18 *Given* $\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \in GL_3(\mathbb{Z})$. *An* L_γ*-periodic pattern* U *is* $\mathcal{B}$*-admissible if and only if*

$$U\mid_{\mathbb{Z}_{2\times 2\times 2}}\left((\alpha_1, \alpha_2, \alpha_3)_\gamma\right) \in \mathcal{B}$$

for $0 \leq \alpha_i \leq a_i - 1, 1 \leq i \leq 3$.

For $a_1, a_2, a_3 \geq 1$, it is easy to verify that there exist positive integers $\widehat{a}_1(\gamma)$, $\widehat{a}_2(\gamma)$ and $\widehat{a}_3(\gamma)$ such that

$$\bigcup_{i=1}^{3}\bigcup_{\alpha_i=0}^{a_i-1}\mathbb{Z}_{2\times 2\times 2}\left((\xi_1+\alpha_1, \xi_2+\alpha_2, \xi_3+\alpha_3)_\gamma\right) \subseteq \mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}$$

for some $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$.

According to Proposition 3.18, the admissibility of an L_γ-periodic pattern U is determined by $U\mid_{\mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}} = \left(u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}\right)_{0\leq\alpha_i\leq\widehat{a}_i-1, 1\leq i\leq 3}$ and $U\mid_{\mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}}$ has the the periodic condition that is given by (3.52), which can be divided into two parts: (i) for $0 \leq \alpha_i \leq \widehat{a}_i - 1$, $1 \leq i \leq 3$ and $p, q \in \mathbb{Z}$, if $0 \leq \alpha_1 + a_1 p + b_{12} q \leq \widehat{a}_1 - 1$ and $0 \leq \alpha_2 + a_2 q \leq \widehat{a}_2 - 1$,

$$u_{(\alpha_1+a_1p+b_{12}q, \alpha_2+a_2q, \alpha_3)_\gamma} = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}; \tag{3.55}$$

(ii) for $0 \le \alpha_i \le \widehat{a}_i - 1$, $1 \le i \le 3$, $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}\backslash\{0\}$, if $0 \le \alpha_1 + a_1 p + b_{12} q + b_{13} r \le \widehat{a}_1 - 1$, $0 \le \alpha_2 + a_2 q + b_{23} r \le \widehat{a}_2 - 1$ and $0 \le \alpha_3 + a_3 r \le \widehat{a}_3 - 1$,

$$u_{(\alpha_1 + a_1 p + b_{12} q + b_{13} r, \alpha_2 + a_2 q + b_{23} r, \alpha_3 + a_3 r)_\gamma} = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}. \tag{3.56}$$

Then, for $h \ge 1$, the set of all local patterns on $\mathbb{Z}_{\gamma;\widehat{a}_1 \times \widehat{a}_2 \times h}$ that satisfy (3.55) with $0 \le \alpha_3 \le h-1$ is denoted by $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$.

Similar to (3.6), the counting function $\overline{\psi}_\gamma$ for patterns $\overline{U}_\gamma$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ is defined by

$$\overline{\psi}_\gamma\left(\overline{U}_\gamma\right) = 1 + \sum_{\alpha_1=0}^{a_1-1} \sum_{\alpha_2=0}^{a_2-1} \sum_{\alpha_3=0}^{h-1} u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} 2^{a_1 a_2 (h-1-\alpha_3) + a_1(a_2-1-\alpha_2) + a_1 - 1 - \alpha_1}.$$

A local pattern $\overline{U}_\gamma$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ can be represented as

$$\overline{U}_\gamma = \overline{U}_{\gamma;0} \oplus_{\gamma_3} \overline{U}_{\gamma;1} \oplus_{\gamma_3} \cdots \oplus_{\gamma_3} \overline{U}_{\gamma;h-1},$$

where $\overline{U}_{\gamma;i} \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};1}$, $0 \le i \le h-1$, and $\overline{U}'_\gamma \oplus_z \overline{U}''_\gamma$ means that $\overline{U}''_\gamma$ is put on the top (in the γ_3-direction) of $\overline{U}'_\gamma$. For $0 \le i \le j \le h-1$, let $\overline{U}_{\gamma;i:j} = \overline{U}_{\gamma;i} \oplus_{\gamma_3} \cdots \oplus_{\gamma_3} \overline{U}_{\gamma;j}$. Therefore, for $h \ge \widehat{a}_3$, the cylindrical ordering matrix $\mathbb{C}_{\gamma;a_1,a_2;b_{12};h} = [C_{\gamma;a_1,a_2;b_{12};h;i,j}]_{2^{a_1a_2(h-1)} \times 2^{a_1a_2(h-1)}}$ of patterns in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ is defined by

$$C_{\gamma;a_1,a_2;b_{12};h;i,j} = \left\{\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};h} : \overline{\psi}_\gamma(\overline{U}_{\gamma;0:\widehat{a}_3-2}) = i \text{ and } \overline{\psi}_\gamma(\overline{U}_{\gamma;h-\widehat{a}_3+1:h-1}) = j\right\}.$$

In particular, for $h = \widehat{a}_3$, $\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3}$ can be used to construct the associated trace operator. Notably the set $\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j}$ either contains exactly one pattern or is an empty set.

Now, given $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the associated trace operator $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) = [t_{\gamma;a_1,a_2;b_{12};i,j}]$, with $t_{\gamma;a_1,a_2;b_{12};i,j} \in \{0,1\}$, can be defined by $t_{\gamma;a_1,a_2;b_{12};i,j} = 1$ if and only if

$$\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j} \ne \emptyset \text{ and the pattern in } \mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j} \text{ is } \mathcal{B}\text{-admissible.} \tag{3.57}$$

Now, the shift (to the left) in the γ_1-direction of any pattern $\overline{U}_\gamma = (u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$, $u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} \in \{0,1\}$, is defined by

$$\sigma_{\gamma_1;a_1,a_2;b_{12}}\left((u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})\right) = \left(u^{(1)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}\right)_{0\le\alpha_1\le\widehat{a}_1-1, 0\le\alpha_2\le\widehat{a}_2-1, 0\le\alpha_3\le\widehat{a}_3-2}$$

where

$$u^{(1)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} = \begin{cases} u_{(\alpha_1+1,\alpha_2,\alpha_3)_\gamma} & \text{if } 0 \leq \alpha_1 \leq \widehat{a}_1 - 2, \\ u_{([\alpha_1+1]_{a_1},\alpha_2,\alpha_3)_\gamma} & \text{if } \alpha_1 = \widehat{a}_1 - 1. \end{cases} \tag{3.58}$$

Similarly, the shift (to the below) in the γ_2-direction is defined by

$$\sigma_{\gamma_2;a_1,a_2;b_{12}}\left((u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})\right) = \left(u^{(2)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}\right)_{0\leq\alpha_1\leq\widehat{a}_1-1,0\leq\alpha_2\leq\widehat{a}_2-1,0\leq\alpha_3\leq\widehat{a}_3-2},$$

where

$$u^{(2)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} = \begin{cases} u_{(\alpha_1,\alpha_2+1,\alpha_3)_\gamma} & \text{if } 0 \leq \alpha_2 \leq \widehat{a}_2 - 2, \\ u_{([\alpha_1-b_{12}]_{a_1},\alpha_2+1-a_2,\alpha_3)_\gamma} & \text{if } \alpha_2 = \widehat{a}_2 - 1. \end{cases} \tag{3.59}$$

Notably, $\sigma_{\gamma_1;a_1,a_2;b_{12}}$ and $\sigma_{\gamma_2;a_1,a_2;b_{12}}$ are automorphism on $\mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. Furthermore,

$$\sigma_{\gamma_2;a_1,a_2;b_{12}} \circ \sigma_{\gamma_1;a_1,a_2;b_{12}} = \sigma_{\gamma_1;a_1,a_2;b_{12}} \circ \sigma_{\gamma_2;a_1,a_2;b_{12}}$$

and

$$\sigma^{a_1}_{\gamma_1;a_1,a_2;b_{12}} = \sigma^{b_{12}}_{\gamma_1;a_1,a_2;b_{12}}\left(\sigma^{a_2}_{\gamma_2;a_1,a_2;b_{12}}\right) = \text{ identity map.}$$

Now, the rotational matrices with respect to γ is defined as follows.

Definition 3.19 *The $2^{a_1a_2(\widehat{a}_3-1)} \times 2^{n_1n_2(\widehat{a}_3-1)}$ γ_1-rotational matrix $R_{\gamma_1;a_1,a_2;b_{12}} = [R_{\gamma_1;a_1,a_2;b_{12};i,j}]$, $R_{\gamma_1;a_1,a_2;b_{12};i,j} \in \{0,1\}$, is defined by*

$$R_{\gamma_1;a_1,a_2;b_{12};i,j} = 1 \quad \textit{if and only if} \quad i = \overline{\psi}_\gamma(\overline{U}_\gamma) \textit{ and } j = \overline{\psi}_\gamma(\sigma_{\gamma_1;a_1,a_2;b_{12}}(\overline{U}_\gamma)), \tag{3.60}$$

where $\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. From (3.60), for convenience, denote by

$$j = \sigma_{\gamma_1}(i). \tag{3.61}$$

Similarly, the $2^{a_1a_2(\widehat{a}_3-1)} \times 2^{n_1n_2(\widehat{a}_3-1)}$ γ_2-rotational matrix $R_{\gamma_2;a_1,a_2;b_{12}} = [R_{\gamma_2;a_1,a_2;b_{12};i,j}]$, $R_{\gamma_2;a_1,a_2;b_{12};i,j} \in \{0,1\}$, is defined by

$$R_{\gamma_2;a_1,a_2;b_{12};i,j} = 1 \quad \textit{if and only if} \quad i = \overline{\psi}_\gamma(\overline{U}_\gamma) \textit{ and } j = \overline{\psi}_\gamma(\sigma_{\gamma_2;a_1,a_2;b_{12}}(\overline{U}_\gamma)), \tag{3.62}$$

where $\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. *From (3.62), for convenience, denote by*

$$j = \sigma_{\gamma_2}(i). \tag{3.63}$$

Moreover,

$$\mathbf{R}_{\gamma;a_1,a_2;b_{12}} = \sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} R^{b_{13}}_{\gamma_1;a_1,a_2;b_{12}} R^{b_{23}}_{\gamma_2;a_1,a_2;b_{12}}. \tag{3.64}$$

The main results for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right)$ as in Theorem 3.9 and 3.10 are obtained as follows and the proofs are omitted.

Theorem 3.20 *Given a basic set* $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$. *For* $a_i \geq 1, 1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right) = tr\left(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) R^{b_{13}}_{\gamma_1;a_1,a_2;b_{12}} R^{b_{23}}_{\gamma_2;a_1,a_2;b_{12}}\right) \tag{3.65}$$

and

$$\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right) = tr\left(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) \mathbf{R}_{\gamma;a_1,a_2;b_{12}}\right). \tag{3.66}$$

Furthermore,

$$\zeta_{\gamma;a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1 a_2}\sum_{a_3=1}^{\infty}\frac{1}{a_3} tr\left(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) \mathbf{R}_{\gamma;a_1,a_2;b_{12}}\right) s^{a_1 a_2 a_3}\right). \tag{3.67}$$

The equivalent class $\mathcal{C}_{\gamma;a_1,a_2;b_{12}}(i)$, the cardinal number $\omega_{\gamma;a_1,a_2;b_{12};i}$ of $\mathcal{C}_{\gamma;a_1,a_2;b_{12}}(i)$, the index set $\mathcal{I}_{\gamma;a_1,a_2;b_{12}}$ and the cardinal number of $\chi_{\gamma;a_1,a_2;b_{12}}$ can be defined as in Subsection 3.1.2 and are omitted here.

Definition 3.21 *For* $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, *the reduced trace operator* $\tau_{\gamma;a_1,a_2;b_{12}} = [\tau_{\gamma;a_1,a_2;b_{12};i,j}]$ *of* $\mathbf{T}_{\gamma;a_1,a_2;b_{12}} = [t_{\gamma;a_1,a_2;b_{12};i,j}]$ *is a* $\chi_{\gamma;a_1,a_2;b_{12}} \times \chi_{\gamma;a_1,a_2;b_{12}}$ *matrix defined by*

$$\tau_{\gamma;a_1,a_2;b_{12};i,j} = \sum_{k \in \mathcal{C}_{\gamma;a_1,a_2;b_{12}}(j)} t_{\gamma;a_1,a_2;b_{12};i,k} \tag{3.68}$$

for each $i, j \in \mathcal{I}_{\gamma;a_1,a_2;b_{12}}$.

By the argument as in Subsection 3.1.2, the rotational symmetry of $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ can be obtained and then yields the rationality of the $(a_1, a_2; b_{12})$-th zeta function $\zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}$. The results are stated as follows.

Theorem 3.22 *Given* $\mathcal{B} \subset \Sigma_{2\times2\times2}$ *and* $\gamma \in GL_3(\mathbb{Z})$*. For* $a_i \geq 1$, $1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,

$$\begin{aligned} \frac{1}{a_1 a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_{\gamma}\right) &= tr\left(\tau^{a_3}_{\gamma;a_1,a_2;b_{12}}\right) \\ &= \sum_{\lambda \in \Sigma(\tau_{\gamma;a_1,a_2;b_{12}})} \chi_{\gamma;a_1,a_2;b_{12}}(\lambda)\lambda^{a_3}. \end{aligned} \tag{3.69}$$

where $\Sigma(\tau_{\gamma;a_1,a_2;b_{12}})$ *is the spectrum of* $\tau_{\gamma;a_1,a_2;b_{12}}$ *and* $\chi_{\gamma;a_1,a_2;b_{12}}(\lambda)$ *is the algebraic multiplicity of* $\tau_{\gamma;a_1,a_2;b_{12}}$ *with eigenvalue* λ*. Moreover,*

$$\begin{aligned} \zeta_{\gamma;a_1,a_2;b_{12}}(s) &= \left(\det\left(I - s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}\right)\right)^{-1} \\ &= \prod_{\lambda \in \Sigma(\tau_{\gamma;a_1,a_2;b_{12}})} (1 - \lambda s^{a_1a_2})^{-\chi_{\gamma;a_1,a_2;b_{12}}(\lambda)}, \end{aligned} \tag{3.70}$$

and

$$\zeta_{\gamma}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\left(\det\left(I - s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}\right)\right)^{-1}. \tag{3.71}$$

Corollary 3.23 *For any* $\mathcal{B} \subset \Sigma_{2\times2\times2}$ *and* $\gamma \in GL_3(\mathbb{Z})$*, the Taylor series expansions for* $\zeta_{\mathcal{B};\gamma}$ *at* $s = 0$ *has integer coefficients.*

Proof. Since $\tau_{\gamma;a_1,a_2;b_{12}}$ has integer entries for any $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the result follows. ■

Now, that $\zeta_{\mathcal{B};\gamma}$ are meromorphic extensions of $\zeta^0_{\mathcal{B}}$ is obtained as follows.

Theorem 3.24 *Given $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For any $\gamma \in GL_3(\mathbb{Z})$,*

$$\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s) \tag{3.72}$$

for $|s| < \exp(-g(\mathcal{B}))$, where

$$g(\mathcal{B}) = \limsup_{[L]\to\infty} \frac{1}{[L]} \log \Gamma_{\mathcal{B}}(L). \tag{3.73}$$

Moreover, $\zeta_{\mathcal{B};\gamma}$ has the same (integer) coefficients in its Taylor series expansions at $s = 0$, for all $\gamma \in GL_3(\mathbb{Z})$.

Proof. By Lind [36], $\zeta^0_{\mathcal{B}}$ has radius of convergence $\exp(-g(\mathcal{B}))$ and is analytic in $|s| < \exp(-g(\mathcal{B}))$. Since $\zeta_{\mathcal{B};\gamma}$ is a rearrangement of $\zeta^0_{\mathcal{B}}$, (3.72) holds. From Lind [36] or Corollary 3.23, $\zeta_{\mathcal{B};\gamma}$ has the same integer coefficients in its Taylor series expansions at $s = 0$. The proof is complete. ■

Remark 3.25 *From Theorem 3.22, for any $\mathcal{B} \subset \Sigma_{2\times2\times2}$, there exists a family of zeta functions $\{\zeta_{\mathcal{B};\gamma} : \gamma \in GL_3(\mathbb{Z})\}$. For certain $\mathcal{B}$, the other $\gamma \in GL_3(\mathbb{Z})$ may give a different description to $\zeta_{\mathcal{B}}$; see Example 3.17 and the following Example 3.26. Those different descriptions of $\zeta^0_{\mathcal{B}}$ may be useful in studying zeta functions.*

Example 3.26 *Consider the basic set $\mathcal{B}$ in Example 3.17 and $\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is easy to verify that*

$$\mathbf{T}_{\gamma;a_1,a_2;b_{12}} = \mathbf{T}_{\gamma;a_1,a_2;0}$$

for $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$. Moreover, after the zero columns and rows of $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ (or $\tau_{\gamma;a_1,a_2;b_{12}}$) were deleted, $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ ($\tau_{\gamma;a_1,a_2;b_{12}}$) is reduced to $\mathbf{T}_{\gamma;1,a_2;0}$ ($\tau_{\gamma;1,a_2;0}$). Clearly

$$\mathbf{T}_{\gamma;1,a_2;0} = I_{2^{a_2}}$$

and

$$\tau_{\gamma;1,a_2;0} = I_{\chi_{a_2}},$$

where

$$\chi_n = \frac{1}{n}\sum_{d|n}\phi(d)2^{n/d},$$

and $\phi(d)$ is the Euler totient function.

Hence,

$$\zeta_{\gamma;a_1,a_2;b_{12}} = (1 - s^{a_1a_2})^{-\chi_{a_2}} \tag{3.74}$$

and

$$\zeta_\gamma = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty} (1 - s^{a_1a_2})^{-a_1\chi_{a_2}}. \tag{3.75}$$

It can be proved that $g(\mathcal{B}) = \log 2$. Therefore, from Example 3.17 and Theorem 3.24,

$$\prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty} (1 - s^{a_1a_2})^{-a_1\chi_{a_2}} = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty} (1 - 2s^{a_1a_2})^{-a_1} \tag{3.76}$$

for $|s| < \frac{1}{2}$, and they have the same integer coefficients in their Taylor series expansions at $s = 0$.

3.2 Further results

This subsection briefly describes the results for $\mathbb{Z}^d$, $d \geq 4$, and more symbols on larger lattice. The thermodynamic zeta function for the three-dimensional Ising model with finite range interactions is also studied.

3.2.1 Higher-dimensional shifts of finite type

This subsection consider the zeta functions for shifts of finite type on $\mathbb{Z}^d$, $d \geq 4$. Only brief statements are made here.

As in [36], $\mathcal{L}_d$ can be parameterized by using Hermite normal form [39]:

$$\mathcal{L}_d = \left\{ \begin{bmatrix} a_1 & b_{12} & \cdots & b_{1d} \\ 0 & a_2 & \cdots & b_{2d} \\ & \vdots & & \\ 0 & 0 & \cdots & a_d \end{bmatrix} \mathbb{Z}^d : a_i \geq 1, 1 \leq i \leq d, 0 \leq b_{ij} \leq a_i - 1, i+1 \leq j \leq d \right\}. \tag{3.77}$$

Let the lattice $\mathbb{L}_d = \{(n_1, n_2, \cdots, n_d) : 0 \leq n_i \leq 1, 1 \leq i \leq d\}$. Fixed a basic set $\mathcal{B} \subset \{0,1\}^{\mathbb{L}_d}$. For $a_i \geq 1$, $1 \leq i \leq d-1$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq d-1$, the (a_i, b_{ij})-th zeta function is defined by

$$
\begin{aligned}
&\zeta_{\mathcal{B};(a_i,b_{ij})}(s) \\
\equiv \quad & \exp\left(\frac{1}{a_1 \cdots a_{d-1}} \sum_{a_d=1}^{\infty} \sum_{i=1}^{d-1} \sum_{b_{id}=0}^{a_i-1} \frac{1}{a_d} \Gamma_{\mathcal{B}} \left(\begin{bmatrix} a_1 & b_{12} & b_{13} & \cdots & b_{1d} \\ 0 & a_2 & b_{23} & \cdots & b_{2d} \\ 0 & 0 & a_3 & \cdots & b_{3d} \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & a_d \end{bmatrix} \right) s^{a_1 \cdots a_d} \right)
\end{aligned}
\tag{3.78}
$$

and

$$
\zeta_{\mathcal{B}}(s) \equiv \prod_{i=1}^{d-1} \prod_{a_i=1}^{\infty} \prod_{j=i+1}^{d-1} \prod_{b_{ij}=0}^{a_i-1} \zeta_{\mathcal{B};(a_i,b_{ij})}(s). \tag{3.79}
$$

As in Subsections 3.1.1 and 3.1.2, the cylindrical ordering matrix, the trace operator, the rotational matrices and the reduced trace operator can be defined. The method in Subsections 3.1.1 and 3.1.2 can also be applied to verify that $\zeta_{\mathcal{B};(a_i,b_{ij})}$ is a rational function. Therefore, $\zeta_{\mathcal{B}}$ is an infinite product of rational functions. Furthermore, given any $\gamma \in GL_d(\mathbb{Z})$, the result also holds in γ-coordinates. Hence, a family of zeta functions exists with the same integer coefficients in their Taylor series expansions at $s = 0$, and yields a family of identities in number theory.

3.2.2 More symbols on larger lattice

This subsection extends the results of the previous sections and subsections to any finite number of symbols and any finite lattice. For simplicity, only the zeta functions for three-dimensional shifts of finite type are discussed. Given a set of symbols $\mathcal{S}_p = \{0, 1, \cdots, p-1\}$, $p \geq 2$, a set of finite lattice points $\mathbb{L} \subset \mathbb{Z}^3$ and a basic set $\mathcal{B}(\mathbb{L}) \subset \mathcal{S}_p^{\mathbb{L}}$. Let $\mathbb{Z}_{m \times m \times m}$ be the smallest cubic lattice that contains $\mathbb{L}$ and $\mathcal{B}(\mathbb{Z}_{m \times m \times m})$ be the set of all admissible patterns that are generated by $\mathcal{B}(\mathbb{L})$. Then, it is easy to verify that

$$\mathcal{P}\left(\mathcal{B}(\mathbb{Z}_{m\times m\times m})\right)=\mathcal{P}\left(\mathcal{B}(\mathbb{L})\right).$$

Therefore, only $\mathcal{B}\subset\mathcal{S}_p^{\mathbb{Z}_{m\times m\times m}}$, for $m\geq 2$, need to be considered. The definitions of cylindrical ordering matrix and the rotational matrices must be adjusted and the details are omitted here. Then, the associated trace operator and reduced trace operator can also be defined. Hence, by the arguments similar to those made in Subsections 3.1.1, 3.1.2 and 3.1.3, the results for $\mathcal{B}\subset\mathcal{S}_p^{\mathbb{Z}_{m\times m\times m}}$ also hold.

3.2.3 Three-dimensional Ising model with finite range interactions

This subsection will extend the results to the $\mathbb{Z}^3$ lattice Ising model with finite range interactions. For simplicity, only the case of the nearest neighbor interactions is considered. Let the $\mathbb{Z}^3$ lattice Ising model with external field $\mathcal{H}$, the coupling constant $\mathcal{J}_1$ in the x-direction, the coupling constant $\mathcal{J}_2$ in the y-direction and the coupling constant $\mathcal{J}_3$ in the z-direction. Each site $(\alpha_1,\alpha_2,\alpha_3)$ of $\mathbb{Z}^3$ lattice has a spin $u_{\alpha_1,\alpha_2,\alpha_3}$ with two possible values, $+1$ or -1. Assume that the state space is given by $\mathcal{B}\subset\{0,1\}^{\mathbb{Z}_{2\times 2\times 2}}$. Given a state $U=(u_{\alpha_1,\alpha_2,\alpha_3})\in\{0,1\}^{\mathbb{Z}^3}$, denote by $U_{n_1\times n_2\times n_3}=U\mid_{\mathbb{Z}_{n_1\times n_2\times n_3}}=(u_{\alpha_1,\alpha_2,\alpha_3})_{0\leq\alpha_i\leq n_i-1,1\leq i\leq 3}$.

Now, the Hamiltonian (energy) $\mathcal{E}\left(U_{n_1\times n_2\times n_3}\right)$ is defined by

$$\begin{aligned}
&\mathcal{E}\left(U_{n_1\times n_2\times n_3}\right)\\
=\;& -\mathcal{J}_1\sum_{\substack{0\leq\alpha_1\leq n_1-2\\0\leq\alpha_3\leq n_2-1\\0\leq\alpha_3\leq n_3-1}}u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1+1,\alpha_2,\alpha_3}-\mathcal{J}_2\sum_{\substack{0\leq\alpha_1\leq n_1-1\\0\leq\alpha_2\leq n_2-2\\0\leq\alpha_3\leq n_3-1}}u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1,\alpha_2+1,\alpha_3} \qquad (3.80)\\
&-\mathcal{J}_3\sum_{\substack{0\leq\alpha_1\leq n_1-1\\0\leq\alpha_2\leq n_2-1\\0\leq\alpha_3\leq n_3-2}}u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1,\alpha_2,\alpha_3+1}-\mathcal{H}\sum_{\substack{0\leq\alpha_1\leq n_1-1\\0\leq\alpha_2\leq n_2-1\\0\leq\alpha_3\leq n_3-1}}u_{\alpha_1,\alpha_2,\alpha_3}.
\end{aligned}$$

Given $L=\begin{bmatrix}a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{bmatrix}\mathbb{Z}^3\in\mathcal{L}_3$, the set of all $\mathcal{B}$-admissible and L-periodic patterns is denoted by $\mathcal{P}_{\mathcal{B}}(L)$. Then, the partition function for $\mathcal{B}$ with L-periodic patterns is defined as

$$\mathcal{Z}_{\mathcal{B}}(L) = \mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) =$$

$$\sum_{U\in\mathcal{P}_{\mathcal{B}}(L)} \exp\left[\sum_{\substack{0\leq\alpha_1\leq n_1-1\\0\leq\alpha_2\leq n_2-1\\0\leq\alpha_3\leq n_3-1}} u_{\alpha_1,\alpha_2,\alpha_3}\left(\mathbf{K}_1 u_{\alpha_1+1,\alpha_2,\alpha_3} + \mathbf{K}_2 u_{\alpha_1,\alpha_2+1,\alpha_3} + \mathbf{K}_3 u_{\alpha_1,\alpha_2,\alpha_3+1} + \mathbf{h}\right)\right], \tag{3.81}$$

where $\mathbf{K}_i = \mathcal{J}_i/k_B T$, $1 \leq i \leq 3$, k_B is Boltzmann's constant and T is the temperature. Therefore, the thermodynamic zeta function is defined by

$$\zeta^0_{Ising;\mathcal{B}}(s) \equiv \exp\left(\sum_{L\in\mathcal{L}_3} \mathcal{Z}_{\mathcal{B}}(L)\frac{s^{[L]}}{[L]}\right). \tag{3.82}$$

As (1.34) and (1.35), for any $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the $(a_1, a_2; b_{12})$-th thermodynamic zeta function $\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}(s)$ is defined as

$$\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}(s) \equiv \exp\left(\frac{1}{a_1 a_2}\sum_{a_3=1}^{\infty}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\frac{1}{a_3}\mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) s^{a_1 a_2 a_3}\right) \tag{3.83}$$

and the thermodynamic zeta function $\zeta_{Ising;\mathcal{B}}(s)$ is given by

$$\zeta_{Ising;\mathcal{B}}(s) \equiv \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}(s). \tag{3.84}$$

Since the spin $u_{\alpha_1,\alpha_2,\alpha_3} \in \{+1,-1\}$, the cylindrical ordering matrix $\mathbb{C}_{Ising;a_1,a_2;b_{12};h} = [C_{Ising;a_1,a_2;b_{12};h;i,j}]$ is obtained by replacing all symbols "0" in $\mathbb{C}_{a_1,a_2;b_{12};h}$ with the symbols "-1". Notably, exactly one patterns exists in $C_{Ising;a_1,a_2;b_{12};2;i,j}$ and the pattern is given by $U_{Ising;i,j} \equiv U_{Ising;a_1,a_2;b_{12};2;i,j} = (u_{\alpha_1,\alpha_2,\alpha_3})$. Define

$$\mathcal{Z}_B\left(U_{Ising;i,j}\right) \equiv \exp\left[\sum_{\substack{0\le\alpha_1\le a_1-1\\0\le\alpha_2\le a_2-1}} u_{\alpha_1,\alpha_2,0}\left(\mathbf{K}_1 u_{\alpha_1+1,\alpha_2,0}+\mathbf{K}_2 u_{\alpha_1,\alpha_2+1,0}+\mathbf{K}_3 u_{\alpha_1,\alpha_2,1}+\mathbf{h}\right)\right]. \tag{3.85}$$

Then, the trace operator $\mathbf{T}_{Ising;a_1,a_2;b_{12}} = [t_{Ising;a_1,a_2;b_{12};i,j}]$ is defined by

$$\begin{cases} t_{Ising;a_1,a_2;b_{12};i,j} = 0 & \text{if} \quad U_{Ising;i,j} \text{ is not } \mathcal{B}\text{-admissible}, \\ t_{Ising;a_1,a_2;b_{12};i,j} = \mathcal{Z}_B\left(U_{Ising;i,j}\right) & \text{if} \quad U_{Ising;i,j} \text{ is } \mathcal{B}\text{-admissible}. \end{cases} \tag{3.86}$$

Therefore, the associated reduced operator $\tau_{Ising;a_1,a_2;b_{12}}$ can be defined as in Definition 3.14. Since all arguments for the rationality of $\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}$ are similar to those in Subsections 3.1.1 and 3.1.2, only the final result is stated, as follows.

Theorem 3.27 *For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$,*

$$\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}(s) = \left(\det\left(I - s^{a_1 a_2}\tau_{Ising;a_1,a_2;b_{12}}\right)\right)^{-1} \tag{3.87}$$

and

$$\zeta_{Ising;\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\left(\det\left(I - s^{a_1 a_2}\tau_{Ising;a_1,a_2;b_{12}}\right)\right)^{-1}. \tag{3.88}$$

Notably, this result also holds in γ-coordinates for $\gamma \in GL_3(\mathbb{Z})$.

References

[1] M. Artin and B. Mazur, *On periodic points*, Annals Math. **81** (1965), 82-99.

[2] J.C. Ban, K.P. Chien and S.S. Lin, *Spatial disorder of CNN with asymmetric output function*, International J. of Bifurcation and Chaos **11**, (2001a), 2085-2095.

[3] J.C. Ban, J.C. Hsu and S.S. Lin, *Spatial disoder of cellular neural network with biased term*, International J. of Bifurcation and Chaos **12**, (2002), 525-534.

[4] J.C. Ban and S.S. Lin, *Patterns generation and transition matrices in multi-dimensional lattice models*, Discrete Contin. Dyn. Syst. **13**, (2005), no. 3, 637-658.

[5] J.C. Ban, W.G. Hu, S.S. Lin and Y.H. Lin, *Zeta functions for two-dimensional shifts of finite type*, preprint.

[6] J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns generation and spatial entropy in two dimensional lattice models*, Asian J. Math. **11**, (2007), no. 3, 497-534.

[7] J.C. Ban, S.S. Lin and Y.H. Lin, *Three-dimensional cellular neural nerworks and patterns generation problems*, International J. of Bifurcation and Chaos **18**, (2008), no. 14, 957-984.

[8] J.C. Ban, S.S. Lin and C.W. Shih, *Exact number of mosaic patterns in one-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **11**, (2001b), 1645-1653.

[9] R. J. Baxter, *Eight-vertex model in lattice statistics.*, Phys. Rev. Lett. **26**, (1971) 832-833.

[10] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.

[11] R. Bowen and O. Lanford, *Zeta functions of restrictions of the shift transformation*, Proc. AMS Symp. Pure Math. **14** (1970), 43-49.

[12] S.N. Chow and J. Mallet-Paret, *Pattern formation and spatial chaos in lattice dynamical systems II*, IEEE Trans. Circuits Syst. **42**, (1995), 752-756.

[13] S.N. Chow, J. Mallet-Paret and E.S. Van Vleck, *Dynamics of lattice differential equations*, International J. of Bifurcation and Chaos **6**, (1996a), 1605-1621.

[14] S.N. Chow, J. Mallet-Paret and E.S. Van Vleck, *Pattern formation and spatial chaos in spatially discrete evolution equations*, Random Comput. Dynam. **4**, (1996) 109-178.

[15] L.O. Chua, K.R. Crounse, M. Hasler and P. Thiran, *Pattern formation properties of autonomous cellular neural networks*, IEEE Trans. Circuits Systems **42**, (1995) 757-774.

[16] L.O. Chua and T. Roska, *The CNN paradigm*, IEEE Trans. Circuits Systems **40**, (1993) 147-156.

[17] L.O. Chua and L. Yang, *Cellular neural networks: Theory*, IEEE Trans. Circuits Systems **35**, (1988) 1257-1272.

[18] L.O. Chua and L. Yang, *Cellular neural networks: Applications*, IEEE Trans. Circuits Systems **35**, (1988) 1273-1290.

[19] L.O. Chua, *CNN: A paradigm for complexity* World Scientific Series on Nonlinear Science, Series A,31 (World Scietific, Singapore, 1998).

[20] D. Fried, *Finitely presented dynamical systems*, Ergodic Th. and Dynam. Syst. **7** (1987), 489-507.

[21] F.R. Gantmacher, *The theory of matrices* (2 vols. Chelsea, New York, 1959)

[22] G. H. Hardy and S. Ramanujan, *Asympototic formulae for the distribution of integers of various types*, Proc. Royal Soc. **A 95**, 144-155.

[23] G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers*, Fifth Ed., Oxford, 1988.

[24] W.G. Hu and S.S. Lin, *Zeta functions for higher-dimensional shifts of finite type*, International J. of Bifurcation and Chaos, to appear.

[25] M. Itoh and L.O. Chua, *Equivalent CNN cell models and patterns*, International J. of Bifurcation and Chaos **13**, (2003), 1055-1161.

[26] M. Itoh and L.O. Chua, *Image processing and self-organizing CNN*, International J. of Bifurcation and Chaos **15**, (2005), 2939-2958.

[27] R.A. Horn and C.R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, 1990)

[28] J. Juang and S.S. Lin, *Cellular Neural Networks: Mosaic pattern and spatial chaos*, SIAM J. Appl. Math. **60**, (2000) 891-915.

[29] J. Juang, S.S. Lin, W.W. Lin and S.F. Shieh, *The spatial entropy of two-dimensional subshifts of finite type*, International J. of Bifurcation and Chaos **10**, (2000) 2845-2852.

[30] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Th. and Dynam. Syst. **9** (1989), 691-735.

[31] F. Ledrappier, *Un champ markovien peut être d'entropie nulle et mélangeant*, C. R. Acad. Sc. Paris Ser. A **287** (1978), 561-562.

[32] E.H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18**, (1967) 692-694.

[33] E.H. Lieb, *The Residual Entropy of Square Ice*, Phys. Rev. **162**, (1967) 162-172.

[34] S.S. Lin and T.S. Yang, *Spatial entropy of one-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **10**, (2000) 2129-2140.

[35] S.S. Lin and T.S. Yang, *On the spatial entropy and patterns of two-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **12**, (2002) 115-128.

[36] D. Lind, *A zeta function for $\mathbb{Z}^d$-actions*, London Math. Soc. Lecture Note Series **228**, Cambridge Univ. Press, (1996) 433-450.

[37] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, (1995).

[38] D. Lind, K. Schmidt, and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Inventiones Math. **101** (1990), 593-629.

[39] C. MacDuffie, *The Theory of Matrices*, Chelsea, New York, 1956.

[40] A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215-220.

[41] N.G. Markley and M.E. Paul, *Maximal measures and entropy for Z^ν subshifts of finite type.* Classical mechanics and dynamical systems (Medford, Mass., 1979) 135-157, Lecture Notes in Pure and Appl. Math., **70**, Dekker, New York, (1981).

[42] N.G. Markley and M.E. Paul, *Matrix subshifts for Z^ν symbolic dynamics*, Proc. London Math. Soc., (3) **43**, (1981) 251-272.

[43] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition.*, Phys. Rev. **65**, (1944) 117-149.

[44] M. Pollicott and K. Schmidt, *Ergodic theory of $\mathbb{Z}^d$ actions*, London Math. Soc. Lecture Note Series. **228**. Cambridge University Press, 1996.

[45] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math and its Appl, Vol 5. Addison-Wesley, Reading, Massachussetts, 1978.

[46] S. Saks and A. Zygmund; tr. by E. J. Scott, *Analytic functions*, Elsevier Science Ltd, 1952.

[47] K. Schmidt, *Dynamical systems of algebraic origin*, Basel, Birkhäuser Verlag, 1995.

[48] N. J. A. Sloane and S. Plouffe, *The encyclopedia of integer sequences*, Academic Press, 1995.