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## 博 士 論 文

具延滯時間之藕合神經網路的動態

Collective Dynamics for Neural Networks

Coupled with Delays

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中 華 民 國 九 十 九 年 一 月

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## 摘　　要

本論文發展了一個新的方法學去探討具時間延遲之藕合神經網路系統的全局與局部動態行為。特別地，我們將這個方法實際運用到幾個神經網路模型上。其中所討論的全局動態行為包括了系統的全局同步化、抗同步化與全局收斂性。藉著觀察一些一維非自治方程式的幾何意義，並利用迭代推論來控制這些方程式；配合進一步的迭代討論進而確立以上提到的那些全局動態行為。所開發的方法可以同時用來推導出與延遲時間有關也可無關的結果。論文中所研究的局部動態行為包括了平衡點的穩定性和Hopf-分岔分析。我們的研究建立了一套異於線性穩定方法與李雅普諾夫函數方法的非典型方法來分析平衡點的穩定性；並且也可研究平穩點的吸引盆。通過觀察線性系統之特徵方程式的幾何結構並搭配Hopf-分岔理論，可推導出具體且容易驗證的條件來確保時間延遲所導致之同步週期解或異步週期解的存在性。 我們選取了三種類型的神經網路模型來實際應用這種方法學並以處理一些現有文獻中受到關心的議題。第一個模型是一般形式的Hopfield-type 神經網路。我們確立此類神經網路在擁有$3^n$個平衡點時的全局收斂性與平衡點的穩定性。我們的方法建立了一套有系統的方式來解決具相加形式之神經網路的收斂性與平衡點穩定性問題。而其餘兩個模型分別具最鄰近藕合形式的神經網路與具兩個環狀結構子網路的藕合神經網路。對於這樣的兩個模型，我們研究了系統的全局收斂性、全局同步化、平穩點的穩定性與延遲時間誘發的振盪與非同步。其所得到的結果可以映證或呼應一些現有文獻的數值模擬結果或推測。我們也在文中描敘幾個數值模擬，以佐證所獲得之理論。

# Collective Dynamics for Neural Networks Coupled with Delays

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## ABSTRACT

We are interested in the collective dynamics for coupled systems with delays. To study global and local dynamics for some coupled systems, a new methodology is developed in this thesis. In particular, we implement this approach in several neural networks. Herein the global dynamics include global synchronization, anti-phase motion, global convergence (to single equilibrium or multiple equilibria) of the networks. Unfolded from the geometric structures of several associated non-autonomous scalar equations, an iteration scheme is designed to control the dynamics of these equations. Further iteration arguments are then developed to establish previous mentioned global dynamics for the networks. The approach we develop can be used to derive both delay-dependent and delay-independent criteria.The local dynamics we investigate include stability of the equilibria and Hopf bifurcation. Our studies establish the stability of equilibria by a nonstandard approach, as compared to the linearization with computation of the characteristic roots, and the Lyapunov function approach. Moreover, the basins of attraction for the stable quilibria can be investigated.Via a geometrical observation on the characteristic equation of the linearized system, the delayed Hopf bifurcation theory is adopted to guarantee the existence of delay-induced synchronous or asynchronous oscillations. The present approach is general and can be applied to several neural network models. To respond to some research issues in the literature, we investigate three neural network models.The first model is a general Hopfield-type neural network with delays. We establish the convergent dynamics and stability of the equilibria for such a networks with $3^n$ equilibria. Our study provides a systematic approach to investigate multistability and convergence of dynamics for additive-type neural networks. The latter two ones are the network with nearest-neighbor coupling and the network comprising two sub-networks with loop structure. For such two models, we investigate the synchronization, delay-induced oscillations and delay-induced asynchrony, the convergence of dynamics and stability of equilibria. Moreover, our results for the latter two models provide theoretical support to some numerical findings, and answer or respond to some conjectures in the existing literature. A number of numerical simulations are presented to illustrate our theory.

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# 目　　錄

# Chapter 1

# Introduction

There have been increasingly intensive studies on nervous systems and neuronal models in the past few decades. Quantitative description on ion conduction and the associated electrophysiology lead to the celebrated Hodgkin-Huxley model and the reduced Morris-Lecar model which were largely employed to simulate neuronal activities at different levels of details (Ermentrout and Kopell 1998, Jirsa and McIntosh 2007, Gloveli et al. 2009).

The collective behavior for a population of neurons, not intrinsic to any individual neuron, is very rich, engrossing and believed to play a key role in neural information processing. For example, coherent rhythms are ubiquitous in the nervous systems. Such rhythms play important roles in various cognitive activities. In the gamma range of frequencies in the cortex, the coherent rhythms are also important in the creation of cell assemblies (Karbowski and Kopell 2000). The phenomenon of collective synchronization is believed to be responsible for self-organization in nature (Kuramoto 1984). It is a common and elementary phenomenon in many biological and physical systems (Peskin 1975, Strogatz and Stewart 1993, Chang and Juang 2008). In many regions of brain, synchronization activity has been observed and implicated as a correlate of behavior and cognition (White et al. 1998). It is known that synchronization encourages the strengthening of mutual connections among neurons. Synchronous oscillations have been observed in the visual cortex (Gray et al. 1989). It is possible that synchronous behaviors can occur without rhythms. However, in visual cortex, neurons with a distance apart synchronize their activity only in presence of rhythms (Karbowski and Kopell 2000).

Due to propagation of action potentials along the axon, to be more realistic, the neuronal systems should incorporate time delays. The introduction of delays into the models can lead to the emergence of completely new behaviour that is not possible on the absence of delays. The coherent rhythms of neurons across distances are able to synchronize under conduction delays. The question of how cells can synchronize in spite of delays thus becomes an important issue (Ermentrout and Kopell 1998, Kopell et al. 2000). There are situations that synchronization occurs because of delay, especially when there are inhibitory synaptic connections in the network (Wang and Buzsaki 1998, White et al. 1998). Crook et al. (1997) studied a continuum model of the cortex, with excitatory coupling and distance dependent delays, and found for small enough delay the synchronous oscillation is stable, but for larger delays this oscillation loses stability to a travelling wave.

An artificial neural network, often just called a “neural network”, is an information processing paradigm to mimic the way biological nervous systems process information. It consists of an interconnected group of artificial neurons. The theories for these networks were developed and applied considerably to various computational tasks including memory storage and pattern recognition (Hopfield 1984, Chua 1998, Zhou et al. 2004). One of the classical and best understood examples of neural networks is the Hopfield-type neural network (Hopfield 1984), which is able to store certain memories or patterns in a manner rather similar to the brain. In (Hopfield 1984), Hopfield has shown how an ensemble of simple processing units can have fairly complex collective behavior. The Hopfield-type neural network with delays was introduced by Marcus and Westervelt (1989). Later, more modelings in neural network took delays into account (Buric and Todorovic 2003, Campbell 2006, Lu et al. 2009, Roska and Chua 1992, Wu 2001). Thereafter, delayed neural networks have been extensively studied, see for example (Campbell 2004, 2005, Faria 2000, Guo 2005, Guo and Huang 2003, Huang and Wu 2003, Shih and Tseng 2008, Wu et al. 1999). Indeed, delay can modify the collective dynamics for neural networks; for example, it can induce oscillation or change the stability of the stationary solution (Campbell 2006). Delay can also induce synchronization (Marti and Masoller 2003, Fatihcan and Jurgen 2004), desynchrony (Campbell et al. 2006), stability (Hsu and Yang 2007), and oscillation death (Atay 2003).

Recently, there were some investigations on coupled neural networks which comprise sub-networks of neurons (Campbell 2004, Song et al. 2009-1 , Song et al. 2009-2) The real network architecture can be extremely complicated. A large neural network may consist of billions of neurons, and sub-collections of neurons often assemble through inner connections. The rich dynamics arising from the interaction of simple networks have been a source of interest for scientists modeling the collective behavior of real-life systems. Coupled neural networks can exhibit a variety of interesting behaviors which are very different from their behaviors in isolation qualitatively. For example, oscillations may arise as a result of the coupling between sub-networks in a population of neurons. It was reported that certain sub-networks interactions such as pathological synchronization is related to Parkinsons disease and epilepsy (Grosse et al. 2002). The investigation of the dynamics under the properties within the sub-network and the connections among the sub-networks is thus rather appealing. In a population of neurons, internal delays within sub-networks and transmission delays among sub-networks may be of different time scale and need to be modeled separately, as remarked in (Campbell et al. 2004, Campbell 2006).

For neural network models, it is appealing to investigate how the collective dynamics are determined by the connection strength, nonlinear coupling functions, the size of the network and transmission delays. In this thesis, we shall perform investigation in these directions on three neural network models; namely, Hopfield-type neural network with delays, the network with nearest-neighbor coupling and the network comprising two sub- networks with loop structure (The detailed description of theses models are arranged in Chapter 2). Herein, the dynamics of the networks we are interested in include global synchronization, anti-phase motion, convergent dynamics for monostable or multistable networks, delayed-induced oscillations and delay-induced asynchrony. Our study on the first model provides a systematic approach to investigate multistability and convergence of dynamics for additive-type neural networks. Our results for the latter two models provide theoretical support to some numerical findings, and answer or respond to some conjectures in the existing literature.

For convergent dynamics of multistable neural networks, a common approach is the monotone theory. By such a theory, it can be established that the system admits generic quasiconvergence. Herein, generic quasiconvergence for a system is referred to that almost every solution tends to the set of stationary solutions, not to a single one. Moreover, the monotone theory can not be applied, if the system fails to generate an eventually strongly monotone semiflow. The existing studies on synchronization for delayed neural networks mostly adopted the approach of Lyapunov function (Campbell 2006, Campbell et al. 2006). One of the conclusions therein is that if the coupling strength is small enough, the system can achieve global synchronization in spite of delays. However, such a synchronization may reduce to the situation that all components converge asymptotically to the same synchronous equilibrium point. The majority of investigations on delayed-induced oscillation, delayed-induced stability, oscillation death, and spatio-temporal patterns are based on linearized stability analysis, Hopf-bifurcation theorem, and equivariant (group-symmetric) bifurcation theory. In this thesis, we develop a new methodology to study global and local dynamics for several forms of artificial neural networks coupled with delays. By iteration arguments, we establish the dynamics of global synchronization, anti-phase motion, convergence to single equilibrium and multiple equilibria for the coupled networks. Via a geometrical observation on the characteristic equation of the linearized system, criteria for the existence of synchronous or asynchronous periodic solutions can be derived. Moreover, the stability of equilibria and basins of attraction for the stable equilibria can also be investigated. The presented arguments for confirming stability of equilibria are nonstandard in delayed equations, as compared to the ones of linearization with computing the characteristic roots, and the Lyapunov function approach, employed in (Belair et al. 1996, Campbell et al. 2004, Shayer et al. 2000, Song et al. 2005).

The thesis is organized as follows. In Chapter 2, we first introduce some realistic neuronal models in Section 2.1. The description of three forms of (artificial) neural networks considered in this thesis and their respective research motivation and questions are given in Section 2.2. In Chapter 3, we study several scalar equations with time-dependent input, which provides a basis for investigating the global collective dynamics of the coupled neural network. The derived results for three neural network models are arranged in Chapter 4-6 respectively.

# Chapter 2

# Mathematical Models on Neurons and Neural Networks

We first introduce some biological neuronal models in Section 2.1. In Section 2.2, we shall introduce three neural network models considered in this thesis and their respective research motivation and questions.

## 2.1 Neuronal models

One of the most important and widely-used models of neurons is the Hodgkin-Huxley model. Such a model was published in 1952 to describe the generation of action potentials in the squid giant axon [27]. Therein the squid axon carries three major current: voltage-gated persistent $K^+$ current with four activation gates (resulting in the term $n^4$ in the equation below, where $n$ is the activation variable for $K^+$); voltage-gated transient $Na^+$ current with three activation gates and one inactivation gate (the term $m^3h$ below); and Ohmic leak current $I_L$, which is carried mostly by $Cl^-$ ions.

$$\begin{aligned}
C\dot{V} &= I - g_K n^4 (V - E_K) - g_{Na} m^3 h (V - E_{Na}) - g_L (V - E_L) \\
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V)m \\
\dot{h} &= \alpha_h(V)(1-h) - \beta_h(V)h \\
\dot{n} &= \alpha_n(V)(1-n) - \beta_n(V)n.
\end{aligned}$$

Here, $V$ represents the membrance potential and each term in the first equation represents a separate current; $\alpha_m$, $\alpha_h$ and $\alpha_n$ are voltage dependent steady-state functions [38].

A reduced model from Hodgkin-Huxley model were introduced by FitzHugh and Nagumo in 1961 [26] and 1962 [56]. To describe "regenerative self-excitation" by a nonlinear positive-feedback membrane voltage and recovery by a linear negative-feedback gate voltage, they developed the model described by

$$\frac{dV}{dt} = V - V^3 - w + I_{\text{ext}}$$
$$\tau\frac{dw}{dt} = V - a - bw$$

where $V$ represents the membrane potential, and the variable $w$ is a recovery variable and $I_{\text{ext}}$ is the magnitude of stimulus current.

In 1981 Morris and Lecar combined Hodgkin-Huxley and FitzHugh-Nagumo into a voltage-gated calcium channel model with a delayed-rectifier potassium channel, represented by [54]:

$$C\dot{v} = I - g_{\text{L}}(v - v_{\text{L}}) - g_{\text{Ca}}\beta(v)(v - v_{\text{Ca}}) - g_{\text{K}}n(v - v_K)$$
$$\dot{n} = \tau(v)(\alpha(v) - n),$$

where the functions $\alpha(v)$, $\beta(v)$, $\tau(v)$ are given by

$$\beta(v) = (1/2)[1 + \tanh(v - v_1)/v_2],$$
$$\alpha(v) = (1/2)[1 + \tanh(v - v_3)/v_4],$$
$$\tau(v) = \phi\cosh((v - v_3)/(2v_4));$$

$v_{\text{K}}$, $v_{\text{Ca}}$ and $v_{\text{L}}$ are the reversal potentials of potassium, calcium and leakage currents respectively; $g_{\text{K}}$, $g_{\text{Ca}}$ and $g_{\text{L}}$ are corresponding maximal specific conductance, and $C$ is the specific membrane capacitance; $v_i$, $i = 1, \cdots, 4$ are constants.

A network of $n$ neurons is usually modelled by

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}_i(\mathbf{x}_i(t)) + \Sigma_{i=1}^{n}\mathbf{f}_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t)), i = 1, \cdots, n. \tag{2.1}$$

The variable $\mathbf{x}_i$ represents all the variables describing the physical state of the cell body of the $i$th neuron in the network, e.g., in Hodgkin-Huxley model, $\mathbf{x}_i = (V_i, m_i, n_i, h_i)$. The function $\mathbf{F}_i$ represents the intrinsic dynamics of the $i$th neuron. The function $\mathbf{f}_{ij}$ is the coupling function and represents the input to the $i$th neuron from the $j$th neuron. If the $j$th neuron is connected to the $i$th via a chemical synapse, then the coupling function is given by $\mathbf{f}_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t)) = c_{ij}h_{ij}^{\text{pre}}(\mathbf{x}_j(t))h_{ij}^{\text{post}}(\mathbf{x}_i(t))$. This is called synaptic coupling. Here $h_{ij}^{\text{pre}}$ is a sigmoidal function. $h_{ij}^{\text{post}}$ is typically a linear function. If the neurons are connected via a gap junction, then the coupling function is $\mathbf{f}_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t)) = C_{ij}(\mathbf{x}_i(t) - \mathbf{x}_j(t))$ , where $C_{ij}$ is the matrix of coupling coefficients. This is called gap junctional or diffusive coupling.

If considering the effect of the action potential when it reaches the end of the axon, then a common approach is to include a time delay in the coupling term. Assume that $\tau_{ij} > 0$ represents the time taken for the action potential to propagate along the axon connecting neuron $j$, then the general coupling term becomes

$$\mathbf{f}_{ij}(\mathbf{x}_i(t), \mathbf{x}_j(t - \tau_{ij})). \tag{2.2}$$

The analysis of the behavior of coupled Hodgkin-Huxley neurons [44, 45], coupled FitzHugh-Nagumo neurons [5, 6] and coupled Morris-Lecar neurons [40, 70] have for some time past been the subjects of many papers.

In this thesis, we shall present our approach and results via three neural network models. The first model is the Hopfield-type neural network with delays and the second one is the neural network with nearest-neighbor coupling. Both of such two forms are as (2.1) which comprise each $\mathbf{x}_i$ being a scalar variable $x_i$. The third model is the network comprising two sub-networks with loop structure; namely, as (2.1) with $n = 2$ and $\mathbf{x}_i$ being a multi-dimensional variable. We expect the approach developed in this thesis can be applied to more realistic and general neuronal models. We shall pursue this issue in future studies. Indeed, our approach can be successfully applied to establish the synchronization of coupled FitzHugh-Nagumo neuronal network (as the scale of the network is 2 or 3) in our latest research.

## 2.2 Neural networks

In the following three subsections, we introduce three neural network models considered in this thesis and their corresponding research motivation respectively. The detailed results and justification for these models are arranged in Chapter 4-6.

### 2.2.1 Hopfield-type Network

In Chapter 4, we consider the Hopfield-type neural network with delays as follows:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + J_i, \tag{2.3}$$

where $i = 1, 2, \cdots, n$, $\mu_i > 0$; $\alpha_{ij}$, $\beta_{ij} \in \mathbb{R}$ denote the instantaneous feedback and delayed feedback connection strength from the $i$th to the $j$th unit; the time-dependent lags $\tau_{ij}(t) \geq 0$ are bounded continuous functions defined on $[t_0, +\infty)$, for some $t_0 \in \mathbb{R}$; $J_i \in \mathbb{R}$ correspond to the external bias; $g_i : \mathbb{R} \to \mathbb{R}$ in the following class:

$$\text{Class } \mathcal{A} : \begin{cases} g_i \in C^2, \ \lim_{\xi \to +\infty} g_i(\xi) = v_i \in \mathbb{R}, \ \lim_{\xi \to -\infty} g_i(\xi) = u_i \in \mathbb{R} \\ \exists\ \sigma_i \in \mathbb{R}, g_i^{'}(\sigma_i) > g_i^{'}(\xi) > 0, \text{ for } \xi \neq \sigma_i, \text{ and } g_i^{''}(\xi) \cdot \xi < 0, \text{ for } \xi \neq \sigma_i. \end{cases}$$

These are bounded smooth sigmoidal functions and the commonly adopted ones are $g_i(\xi) = \tanh \xi$, and $g_i(\xi) = 1/[1 + e^{-\xi/\varepsilon_i}]$ with $\varepsilon_i > 0$. Without loss of generality, we set $\sigma_i = 0$, for all $i$, throughout this thesis. We denote the bounds for the activation functions, the slopes of the activation functions and the time lags by

$$\rho_i := \max\{|u_i|, |v_i|\}, \qquad L_i := g_i'(0) \geq g_i'(\xi), \text{ for all } \xi \in \mathbb{R} \tag{2.4}$$

$$\tau := \max_{1 \leq i,j \leq n} \{\tau_{ij}\}, \qquad \tau_{ij}(t) \leq \tau_{ij}, \text{ for all } t \in [t_0, +\infty). \tag{2.5}$$

“Multistability”, a notion to describe coexistence of multiple stable equilibria or cycles, is essential in several applications of neural networks, including pattern recognition and associative memory storage [35, 18, 25, 33]. Recently, a systematic methodology on existence of multiple stationary solutions for the Hopfield neural network with or without delays has been reported in [15]. More precisely, the structure of single-neuron equation is employed to construct the existence of $3^n$ equilibria, $2^n$ positively invariant sets and basins of attraction for $2^n$, among these $3^n$, stable equilibria. However, there was no theoretical methodology to capture behaviors for solutions lying outside or crossing these basins, hence the global dynamical picture. In this thesis, we develop a new treatment to conclude the convergence of dynamics for (2.3). Under this formulation, certain componentwise dynamical properties are derived and a subsequent iteration scheme is designed to confirm that every solution of the system converges to one of the equilibria as time tends to infinity. With this formulation, we justify that there exist exactly $2^n$ stable equilibria and exactly $(3^n - 2^n)$ unstable equilibria for (2.3). The conclusion for this existence of exact number of stable and unstable equilibria is new due to distinct treatment. The presented arguments for confirming stability of equilibria are nonstandard in delayed equations, as compared to the linearization with computation of the characteristic roots, and the Lyapunov function approach, employed in [2, 7, 60, 63].

### 2.2.2 Neural network with nearest-neighbor coupling

In Chapter 5, we consider a special form of (2.3), which comprises a ring of identical elements with nearest-neighbor coupling, under a transmission delay. The individual element is determined by a scalar equation with a linear decay and nonlinear delayed feedback. This network is then modelled by the system of nonlinear delayed differential equations

$$\dot{x}_i(t) = -\mu x_i(t) + \alpha g_I(x_i(t-\tau_I)) + \beta[g_T(x_{i-1}(t-\tau_T)) + g_T(x_{i+1}(t-\tau_T))], \quad (2.6)$$

where $i$ (mod $N$); $N$ is the scale of the coupled network; $\mu \geq 0$ means self-decay rate; $\alpha$ and $\beta$ are respectively the synaptic strength of *self-feedback* and (nearest-neighbor) *coupling* with corresponding delays $\tau_I \geq 0$ and $\tau_T \geq 0$; $g_I$ and $g_T$ are the activation functions of class $\mathcal{A}$ with $g(0) = 0$. We say that the self-feedback (resp. coupling) is *inhibitory* if $\alpha < 0$ (resp. $\beta < 0$) and *excitatory* if $\alpha > 0$ (resp. $\beta > 0$), and the self-feedback (resp. coupling) strength is strong/weak if the magnitude of $\alpha$ (resp. $\beta$) is large/small. To simplify the presentation, we shall consider system (2.6) with $g_I = g_T =: g$, $-1 < g(\xi) < 1$ and $g'(0) = 1$. We can also treat the case of $g_I \neq g_T$; basically there is no qualitative difference between the cases $g_I = g_T$ and $g_I \neq g_T$. We shall focus on the effect from scale of the network ($N$), self-decay ($\mu$), self-feedback strength ($\alpha$), coupling strength ($\beta$), delays ($\tau_I$, $\tau_T$), and the characteristic of $g$ upon synchrony, convergent dynamics and oscillation of (2.6).

Herein, setting $\tau_{\max} := \{\tau_I, \tau_T\}$, we say that the coupled network (2.6) attains global synchronization (in-phase) if all components of the network tend to be identical, namely

$$x_i(t) - x_{i+1}(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \cdots, N-1,$$

for solution $(x_1(t), \cdots, x_N(t))$ of (2.6), starting from arbitrary initial condition $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^N)$ at $t = t_0$. We denote by $(2.6)_0$ system (2.6) with odd activation functions $g$ in class $\mathcal{A}$, ie., $g$ also satisfies $g(-\xi) = -g(\xi)$, for all $\xi \in \mathbb{R}$. As an evidence of asynchrony, we also consider solutions in the form of standing wave for system $(2.6)_0$; for example, if $N = 3$, the solutions $(x_1(t), x_2(t), x_3(t))$, with two of the components in opposite sign, and the other equal to zero; i.e.,

$$x_i(t) = -x_j(t), \; x_k(t) \equiv 0,$$

and $(i, j, k) = (1, 2, 3)$ or its permutation, cf. [3].

There is a large amount of literature on synchronization in artificial neural networks. Some of these works consider systems with time delays [8, 73, 75, 78, 79]. One of the conclusions therein is that if the coupling strength is small enough, the system can achieve global synchronization in spite of delays. However, such a synchronization may reduce to the situation that every solution converges asymptotically to the same synchronous equilibrium point. A common approach for such a conclusion is the method of Lyapunov function. To elucidate the coherent rhythms in a neural network, it is important to explore the existence of nontrivial oscillation when the network is globally synchronized as well as asynchronous oscillation which are induced by delays or scale of the network [14]. Existence of nontrivial oscillations induced by delays has been reported for network models similar to (2.6) [9, 8, 31, 32, 37]. The common approach adopted in these works is the bifurcation theory. Most of these results focus on the emergence and the stability of spatio-temporal patterns or oscillations bifurcated from the trivial solution. However, from the criteria derived therein, the effect of self-feedback and coupling strength upon emergence of synchronous or asynchronous oscillation induced by delays is not apparent. In this thesis, via a geometrical observation on the characteristic equation of the linearized system, certain criteria for the existence of synchronous periodic solutions and standing wave solutions are derived. The criteria illuminate the dependence of the synchronous or asynchronous oscillations induced by $\tau_I$ or $\tau_T$ on parameters $\alpha$ and $\beta$.

There are some conjectures on synchrony of (2.6) under delay-dependent or scale-dependent criteria. Campbell et al. [8] considered system (2.6) with $\mu = 1$ and $N = 3$ and conjectured that if $|\beta| < |1-\alpha|$ and $0 \le \tau_I < \tau_S^{(1)}$ for some $\tau_S^{(1)}$, or $|\beta| < |1-\alpha|/2$ and $0 \le \tau_I < \tau_S^{(2)}$ for some $\tau_S^{(2)}$, then (2.6) can be synchronized for all $\tau_T \ge 0$. In [75], it was conjectured that when the scale of the network $N$ is odd, (2.6) can be synchronized if $|\alpha| + 2|\cos((N-1)\pi/N)||\beta| < 1$. We shall respond and provide evidence to these conjectures.

Recently, some new analytical methodologies have been developed to study multistability in Hopfield-tpe neural networks [10, 15, 16, 17, 61, 76]. Most of these studies on "Multistability" are associated with the multistability induced by strong excitatory self-feedback. The investigations therein can be extended to establish the coexistence of $3^n$ synchronous and asynchronous equilibria with $2^n$ among them being stable for system (2.6), if the self-feedback is excitatory and its strength $\alpha$ is sufficiently stronger than the coupling strength $\beta$. On the other hand, there exists a different type of multistability which comprises 3 synchronous equilibria for network (2.6) and neural networks of similar type [31, 60, 73]. Let us call this the second type of multistability for system (2.6). To conclude the convergent dynamics for the networks admitting the second type of multistability, a common approach is the monotone dynamics theory. For example, applying such a theory, [31] established the "generic" convergence to 3 synchronous equilibria for a unidirectional excitatory ring of four identical neurons. Therein, the "excitatory coupling" is crucial for the network to generate an eventually strongly monotone semiflow. Wu et al. [73] conjectured that the generic dynamics for system (2.6) with $N = 3$, $\mu = 1$ and $\tau_I = \tau_T$ are convergence to two stable synchronous equilibria if $|\alpha - \beta| < 1$ and $\alpha + 2\beta > 1$. This conjecture was not resolved if $\alpha < 0$ or $\beta < 0$, due to that the standard ordering in that region is invalid. In this thesis, an iteration approach is developed to overcome the restriction from the monotone dynamics approach and establishes the "global" convergence for the network which admits the multistability of the second type. According to our results, roughly speaking, the second type of multistability for system (2.6) can be generated by "strong excitatory coupling".

Motivated by the above-mentioned unsolved problems in the literature and an attempt to elucidate more complete dynamical scenario for system (2.6), the aims of Chapter 5 for (2.6) are to

(i) derive criteria (may depend on $\mu$, $\alpha$, $\beta$, $\tau_I$, $\tau_T$, $N$, $g$) for synchronization of system (2.6),

(ii) describe and distinguish the differences between synchrony induced by various combinations of $\alpha$ and $\beta$,

(iii) distinguish the differences between synchrony for (2.6) with and without delays,

(iv) establish the convergence of dynamics for (2.6) which admits multistability induced by "strong excitatory coupling"; and distinguish the difference between the multistability induced by "strong excitatory self-feedback" and the one by"strong excitatory coupling",

(v) establish the existence of nontrivial oscillation and asynchrony induced by delays, and illustrate the asynchrony induced by the network scale via numerical evidence.

### 2.2.3 System comprising two sub- networks with loop structure

In Chapter 6, we consider a neural network that consists of a pair of one-way loops each with $K$ neurons and two-way coupling between a single neuron of each loop. Each loop has the form:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \alpha_i g(x_{i-1}(t-\tau_I)),\ i = 1, 2, \cdots, K \pmod{K}. \tag{2.7}$$

Herein, $x_i$ represents the normalized voltage of neuron $i$; $\tau_I \geq 0$ is the internal time delay; $g$ is an activation function of class $\mathcal{A}$ with $g(0) = 0$. The coupled $K$-loops considered is of the following form:

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \alpha_i g(x_{i-1}(t-\tau_I)),\ i = 1, 2 \cdots, K-1 \text{ (mod K)}, \\ \dot{x}_K(t) = -\mu_K x_K(t) + \alpha_K g(x_{K-1}(t-\tau_I)) + cg(y_K(t-\tau_T)), \\ \dot{y}_i(t) = -\mu_i y_i(t) + \alpha_i g(y_{i-1}(t-\tau_I)),\ i = 1, 2 \cdots, K-1 \text{ (mod K)}, \\ \dot{y}_K(t) = -\mu_K y_K(t) + \alpha_K g(y_{K-1}(t-\tau_I)) + cg(x_K(t-\tau_T)), \end{cases} \tag{2.8}$$

where $\tau_T \geq 0$ is transmission time delay between two coupled loops. We also set $\tau_{\max} := \max\{\tau_I, \tau_T\}$. The interaction between two loops is inhibitory if $c < 0$, and excitatory if $c > 0$. For simplicity of presentation, we set $\mu_i = 1$, $\alpha_i = 1$ for all $i$, $-1 < g(\xi) < 1$ and focus on the effect of dynamics for (2.8) from the gain of response function ($L$), the coupling strength between two loops ($c$), the internal delay ($\tau_I$) and the transmission delay ($\tau_T$).

The work of Campbell et al. [7] studied a neural network with two coupled loops, each with three neurons. The model considered therein, while similar to (2.8), allows asymmetric coupling between two loops, but is without internal delay. The investigation therein focusses on the existence of equilibria and their stability for the isolated loop and coupled loops, as well as bifurcations at the trivial equilibrium. Song et al. [64] studied a neural network which consists of two sub-networks each with two neurons. The system is again similar to (2.8), but with internal delay identical to transmission delay. Song et al. [65] considered bidirectional coupling between two sub-networks but without internal delays. These two works studied Hopf bifurcation at the trivial solution (the origin) in some parameter region and that when the coupling delay increases the spatio-temporal patterns of bifurcating periodic solutions alternate from in-phase to anti-phase for positive coupling strength. The results in these works basically adopt local analysis and depict local behaviors for the system. Our approach can also be adapted to treat these models.

In Chapter 6, we consider global dynamics for the coupled system (2.8). We say that the two coupled loops attain global synchronization (in-phase) if the corresponding components of two loops tend to be identical, namely

$$x_i(t) - y_i(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \cdots, K,$$

for solution $(x_1(t), \cdots, x_K(t), y_1(t), \cdots, y_K(t))$ of (2.8), starting from arbitrary initial condition $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^{2K})$ at $t = t_0$. We also consider anti-phase for the two loops, which means

$$x_i(t) + y_i(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \cdots, K,$$

for every solution $(x_1(t), \cdots, x_K(t), y_1(t), \cdots, y_K(t))$ of (2.8). Via the methodology developed in this thesis, we establish the dynamics of global synchronization, anti-phase motion, convergence to single equilibrium and multiple equilibria for coupled network (2.8); that is, various synchronous and asymptotic phases can be concluded with our treatments. The approach we develop can be used to derive both delay-dependent and delay-independent criteria. Our studies also extend to stability and basins of attraction for the nontrivial equilibria. In addition, we establish the criteria for Hopf bifurcation so that under these criteria and the conditions for synchronization, system (2.8) admits synchronous periodic solutions. While the internal delay plays the role of inducing oscillation, the magnitude of transmission delay affects synchronization. Our computation on the existence of bifurcated periodic orbits indicates the parameter range for oscillation and thus illuminates the delineation for the whole dynamical scenario. With this approach, we are able to provide theoretical support to some numerical findings in [7].

# Chapter 3

# Preliminaries

In this chapter, we shall consider two types of scalar equations in Section 3.1 and 3.2 respectively. The first one is in the form of difference equations deduced from (2.6) or (2.8) in considering synchronization, and the second one is for consideration of convergent dynamics, for the multi-dimensional system (2.3), (2.6) or (2.8) .

## 3.1 Scalar equation for synchronization

In this section, we introduce the scalar equation associated with the synchronization for (2.8). Let $x(t)$ and $y(t)$ be $C^1$ scalar functions which are eventually attracted by some closed and bounded interval $\mathcal{Q}$; namely, $x(t)$ and $y(t)$ remain in $\mathcal{Q}$, for all time $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. Let $w(t)$ be a bounded continuous function defined for $t \geq t_0$. Assume that $z(t) = x(t) - y(t)$ satisfies the following scalar function:

$$\dot{z}(t) = -z(t) - \beta[g(x(t-\tau)) - g(y(t-\tau))] + w(t), \ \ t \geq t_0, \tag{3.1}$$

where $\beta \in \mathbb{R}$, $\tau \geq 0$, and $g$ is an activation of class $\mathcal{A}$ with $-1 < g(\xi) < 1$, $g(0) = 0$, $g'(\xi) \leq L := g'(0)$. We set

$$\check{L} := \min\{g^{'}(\xi) : \xi \in \mathcal{Q}\}. \tag{3.2}$$

We present the basic formations and propositions in Section 3.1.1. Some extensions form (3.1) to other scalar equations are given in Section 3.1.2. The proofs for the lemma and theorems in the Section 3.1.1 and 3.1.2 are given in Section 3.1.3.

### 3.1.1 Formulations and properties

The main result (Theorem 3.1.4) in Section 3.1.1 asserts that there exist a bounded and closed interval containing zero to which every solution of (3.1) converges, under some $\tau$-dependent conditions. A variant of this formulation leads to the same conclusion under a $\tau$-independent condition. First, let us introduce the $\tau$-dependent result. Below, we shall, iteratively, define two kinds of scalar functions which depict the upper and lower bounds for the dynamics of (3.1) respectively as time proceeds. Such iterative construction aims at capturing the asymptotical behaviors for the solutions of (3.1). The construction of upper and lower functions depends on the sign of $\beta$. We shall demonstrate the formulation for the $\tau$-dependent result and $\beta > 0$ case to present our main idea, and provide the results for the other cases without detailed proofs.

For $T \geq t_0$, we denote

$$|w|^{\max}(T) := \sup\{|w(t)| : t \geq T\}.$$

We then define

$$\hat{h}(\xi) := \begin{cases} -\xi + 2\beta + |w|^{\max}(t_0) & \text{if } \xi \geq 0, \\ -(1+\beta L)\xi + 2\beta + |w|^{\max}(t_0) & \text{if } \xi < 0, \end{cases}$$
$$\check{h}(\xi) := \begin{cases} -(1+\beta L)\xi - 2\beta - |w|^{\max}(t_0) & \text{if } \xi \geq 0, \\ -\xi - 2\beta - |w|^{\max}(t_0) & \text{if } \xi < 0. \end{cases}$$

It can be seen that $\hat{h}(\xi) > \check{h}(\xi)$ and $\hat{h}(\xi) = -\check{h}(-\xi)$. The decreasing and piecewise linear functions $\hat{h}$ and $\check{h}$ have unique zeros at $\hat{A}^h$ and $\check{A}^h$ respectively, where $\hat{A}^h = 2\beta + |w|^{\max}(t_0) \geq 0$ and $\check{A}^h = -\hat{A}^h \leq 0$, cf. Fig. 3.1. Notably, $\hat{h}$ and $\check{h}$ depict preliminary upper and lower bounds respectively, for the dynamics of (3.1) with $\beta > 0$. That is,

$$\check{h}(z(t)) < \dot{z}(t) < \hat{h}(z(t)), \text{ for all } t \geq t_0, \tag{3.3}$$

for arbitrary solution $z(t)$ of (3.1). This leads to the following proposition. Herein, $\mathcal{Q}$ and $\tilde{t}_0$ have to be provided a priori as introducing the scalar equation (3.1).

**Proposition 3.1.1** Assume that $\beta > 0$, If $z(t)$ satisfies (3.1), then (3.3) holds. Subsequently, there exists some $T_{x,y} \geq \tilde{t}_0 + \tau$ such that $z(t)$ belongs to $[\check{A}^h, \hat{A}^h]$ for all $t \geq T_{x,y} - \tau$. Moreover,

$$\check{h}(\hat{A}^h) < \dot{z}(t) < \hat{h}(\check{A}^h), \text{ for all } t \geq T_{x,y} - \tau.$$

**Remark 3.1.1** Note that if $\beta > 0$, then $0 \geq \check{h}(\hat{A}^h) = -(2+\beta L)(2\beta + |w|^{\max}(t_0))$, $(2+\beta L)(2\beta + |w|^{\max}(t_0)) = \hat{h}(\check{A}^h) \geq 0$; in addition, $x(t)$ and $y(t)$ lie in $\mathcal{Q}$ for all $t \geq T_{x,y} - \tau$, where $T_{x,y}$ is given in Proposition 3.1.1.

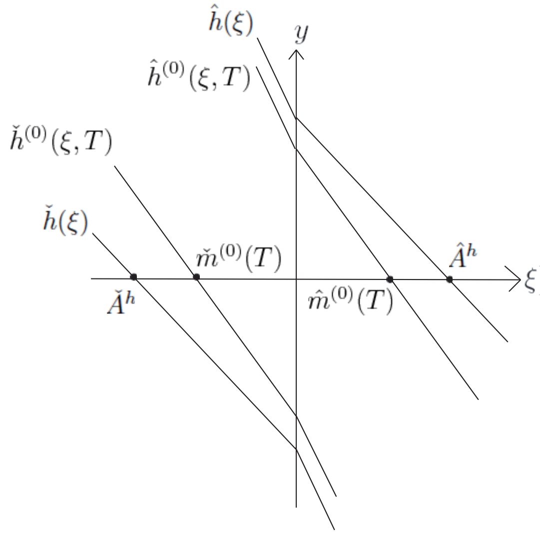



Figure 3.1: Configurations of functions $\hat{h}$, $\check{h}$, $\hat{h}^{(0)}(\cdot,T)$ and $\check{h}^{(0)}(\cdot,T)$

Now, for each $T \geq t_0$, we introduce the following functions:

$$\hat{h}^{(0)}(\xi,T) = \begin{cases} -(1+\beta\check{L})\xi + \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta L)\xi + \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$$

$$\check{h}^{(0)}(\xi,T) = \begin{cases} -(1+\beta L)\xi + \tau\beta L\check{h}(\hat{A}^h) - |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta\check{L})\xi + \tau\beta L\check{h}(\hat{A}^h) - |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$$

where $\check{L}$ is defined in (3.2). The idea for formulation of $\check{h}^{(0)}(\cdot,T)$ and $\hat{h}^{(0)}(\cdot,T)$ will be revealed in the following discussions. Notably, $\check{h}^{(0)}(\xi,T) = -\hat{h}^{(0)}(-\xi,T)$. We consider the following condition for (3.1).

Condition (H1a): $\beta > 0$ and $\tau < 2/[L(2+\beta L)(2\beta + |w|^{\max}(t_0))]$.

Under condition (H1a), a direct computation yields that, for $T \geq t_0$,

$$\check{h}(\xi) < \check{h}^{(0)}(\xi,T) < \hat{h}^{(0)}(\xi,T) < \hat{h}(\xi), \text{ for all } \xi \in \mathbb{R}. \tag{3.4}$$

Herein, $\check{h}^{(0)}(\cdot,T)$ and $\hat{h}^{(0)}(\cdot,T)$ depict the lower and upper bounds for the dynamics of (3.1), which are more precise than $\check{h}(\cdot)$ and $\hat{h}(\cdot)$ respectively as time gets larger. Let $\check{m}^{(0)}(T)$ (resp. $\hat{m}^{(0)}(T)$) be the unique solution of $\check{h}^{(0)}(\cdot,T) = 0$ (resp. $\hat{h}^{(0)}(\cdot,T) = 0$) lying in interval $[\check{A}^h, \hat{A}^h]$, as depicted in Fig. 3.1. Notably, $\check{m}^{(0)}(T) = -\hat{m}^{(0)}(T) \leq 0$.

It follows from Proposition 3.1.1 and Remark 3.1.1 that if $\beta > 0$, for each $T \geq T_{x,y}$,

$$\check{h}^{(0)}(z(t),T) < \dot{z}(t) < \hat{h}^{(0)}(z(t),T), \text{ for all } t \geq T. \tag{3.5}$$

Let us verify (3.5) and explain the formulation of $\check{h}^{(0)}, \hat{h}^{(0)}$. First, $x(t-\tau)$ and $y(t-\tau) \in \mathcal{Q}$ for all $t \geq T \geq T_{x,y}$; therefore, for $t \geq T \geq T_{x,y}$, $\dot{z}(t) = -z(t) - \beta g'(\zeta)z(t-\tau) + w(t) = -z(t) - \beta g'(\zeta)[z(t) - \dot{z}(s)\tau] + w(t)$, for some $\zeta \in \mathcal{Q}$ and $s \geq t-\tau \geq T_{x,y} - \tau$. Notice that $\check{h}(\hat{A}^h) < \dot{z}(s) < \hat{h}(\check{A}^h)$, $\check{h}(\hat{A}^h) \leq 0$ and $\hat{h}(\check{A}^h) \geq 0$. If $z(t) \geq 0$, then $\dot{z}(t) < -z(t) - \beta\check{L}z(t) + \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T) =: \hat{h}^{(0)}(z(t), T)$. If $z(t) < 0$, then $\dot{z}(t) < -z(t) - \beta Lz(t) + \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T) =: \hat{h}^{(0)}(z(t), T)$. Hence, the right-hand inequality of (3.5) is verified. The left-hand inequality can be treated similarly. We thus derive the following proposition.

**Proposition 3.1.2.** For $T \geq T_{x,y}$, inequality (3.5) holds under condition (H1a). Consequently, $z(t)$ eventually enters and stays afterward in $[\check{m}^{(0)}(T), \hat{m}^{(0)}(T)] = [-\hat{m}^{(0)}(T), \hat{m}^{(0)}(T)]$.

Similar to the construction of $\check{h}^{(0)}(\cdot, T)$ and $\hat{h}^{(0)}(\cdot, T)$, we shall define the following functions iteratively. For $k \in \mathbb{N}$ and $T \geq t_0$

$$\hat{h}^{(k)}(\xi, T) := \begin{cases} -(1+\beta\check{L})\xi + \tau\beta L\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta L)\xi + \tau\beta L\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$$

$$\check{h}^{(k)}(\xi, T) := \begin{cases} -(1+\beta L)\xi + \tau\beta L\check{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) - |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta\check{L})\xi + \tau\beta L\check{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) - |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$$

where $\check{m}^{(k)}(T)$ (resp. $\hat{m}^{(k)}(T)$) is the unique solution of $\check{h}^{(k)}(\cdot, T) = 0$ (resp. $\hat{h}^{(k)}(\cdot, T) = 0$). Notice that $\check{h}^{(k)}(\xi, T) = -\hat{h}^{(k)}(-\xi, T)$ and $\check{m}^{(k)}(T) = -\hat{m}^{(k)}(T) \leq 0$. Let us define

$$|w|^{\max}(\infty) := \lim_{T\to\infty} |w|^{\max}(T).$$

It shall be shown that the successively defined $\check{h}^{(k)}$ and $\hat{h}^{(k)}$ control the dynamics of (3.1) more precisely as $k$ and $T$ increase. We summarize the properties for the above-defined terms in the following lemma and theorems. Their proofs will be deferred until Section 3.1.3.

**Lemma 3.1.3.** Assume that condition (H1a) holds. Then, for each $T \geq t_0$, the sequences $\{\check{m}^{(k)}(T)\}_{k\geq 0}$, $\{\hat{m}^{(k)}(T)\}_{k\geq 0}$ can be defined iteratively. Moreover, (i) for any fixed $k \in \mathbb{N} \cup \{0\}$, $\hat{m}^{(k)}(T)$ is decreasing and $\check{m}^{(k)}(T)$ is increasing with respect to $T \geq t_0$; (ii) for any $T \geq t_0$, there exists $m(T) \geq 0$, such that $\hat{m}^{(k)}(T) \to m(T)$ decreasingly, and $\check{m}^{(k)}(T) \to -m(T)$ increasingly, as $k \to \infty$; (iii) there exists $m_p \geq 0$, such that $m(T) \to m_p$ decreasingly, as $T \to \infty$; (iv) $0 \leq m(T) = |w|^{\max}(T)/[(1+$

$\beta\check{L}) - \tau\beta L(2 + \beta L + \beta\check{L})]$, for any $T \geq t_0$; (v) $\cap_{T\geq t_0}[-m(T), m(T)] = [-m_p, m_p]$, and

$$0 \leq m_p \leq \frac{|w|^{\max}(\infty)}{(1 + \beta\check{L}) - \tau\beta L(2 + \beta L + \beta\check{L})}. \tag{3.6}$$

Following Lemma 3.1.3, it can be shown that if $z(t)$ satisfies (3.1), then for each $T \geq T_{x,y}$ and $k \in \mathbf{N}$, $z(t)$ converges to $[-\hat{m}^{(k)}(T), \hat{m}^{(k)}(T)]$ as $t \to \infty$. Moreover, $[-\hat{m}^{(k)}(T), \hat{m}^{(k)}(T)]$ shrinks to $[-m_p, m_p]$, as $k \to \infty$, $T \to \infty$. Accordingly, $z(t)$ converges to interval $[-m_p, m_p]$, as $t \to \infty$.

**Theorem 3.1.4.** If $z(t)$ satisfies (3.1), then $z(t)$ converges to interval $[-m_p, m_p]$, as $t \to \infty$, under condition (H1a).

For the case of $\beta < 0$, we can also derive analogous result.

**Theorem 3.1.5.** If $z(t)$ satisfies (3.1), then $z(t)$ converges to an interval $[-m_q, m_q]$ under condition (H2a): $-1/L < \beta < 0$, $\tau < 2(1+\beta L)/[L(2+\beta L)(2|\beta|+|w|^{\max}(t_0))]$; moreover,

$$0 \leq m_q \leq |w|^{\max}(\infty)/\{(1 + \beta L) + \tau\beta L(2 + \beta L + \beta\check{L})\}.$$

The assumptions and conclusions in the previous two theorems are both $\tau$-dependent. Indeed, via similar arguments, we can derive a $\tau$-independent conclusion as follows:

**Theorem 3.1.6.** If $z(t)$ satisfies (3.1), then $z(t)$ converges to an interval $[-m_r, m_r]$, as $t \to \infty$, under condition (H3a): $|\beta| < 1/L - |w|^{\max}(t_0)/2$; moreover,

$$0 \leq m_r \leq |w|^{\max}(\infty)/(1 - |\beta|L).$$

All results in this subsection can be extended to the following more general scalar equation:

$$\dot{z}(t) = -z(t) - \beta[g(x(t-\tau) - g(y(t-\tau)))] + w(t) + v(t), \tag{3.7}$$

where $v$ is a continuous function with $v(t) \to 0$, as $t \to \infty$. The form of (3.7) includes the following special one

$$\dot{x}(t) = -x(t) - \beta g(x(t-\tau)) + w(t) + v(t), \tag{3.8}$$

where $x(t)$ remains in $\mathcal{Q}$, for all time after some $\tilde{t}_0 \geq t_0$. Notably, (3.8) is like $y = 0$ in (3.7). Since the results in Theorems 3.1.4-3.1.6 concern the asymptotical behavior of all solutions to the equation, it is straightforward to conclude the following corollary.

**Corollary 3.1.7.** Every solution of (3.7) or (3.8) converges to $[-m_p, m_p]$ (resp. $[-m_q, m_q]$, $[-m_r, m_r]$) under condition (H1a) (resp. (H2a), (H3a)).

Let us also consider the following equation:

$$\dot{x}(t) = -x(t) + w(t), \tag{3.9}$$

which is a special form of (3.8) with $\beta = 0$, and $v(t) = 0$, for $t \geq t_0$. The following corollary is obvious.

**Corollary 3.1.8.** Every solution of (3.9) converges to an interval $[-m_s, m_s]$; moreover,

$$0 \leq m_s \leq |w|^{\max}(\infty).$$

### 3.1.2 Some extensions

The arguments for (3.1) can be extended to the form as follows, which is associated with the synchronization for (2.6). Let $x(t)$ and $y(t)$ be $C^1$ scalar functions which are eventually attracted by some closed and bounded interval $\mathcal{Q}$; namely, $x(t)$ and $y(t)$ remain in $\mathcal{Q}$, for all time $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. Let $w(t)$ be a bounded continuous function defined for $t \geq t_0$. Assume that $z(t) = x(t) - y(t)$ satisfies the following scalar function:

$$\dot{z}(t) = -\mu z(t) - \Sigma_{i=1}^{2}\gamma_i[g(x(t-\tau_i)) - g(y(t-\tau_i))] + w(t), \; t \geq t_0, \tag{3.10}$$

where $\mu > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\tau_1, \tau_2 \geq 0$, and $g$ is an activation function in class $\mathcal{A}$ with $-1 < g(\xi) < 1$, $g(0) = 0$, $g'(\xi) \leq L := g'(0) = 1$. We denote

$$\tau := \max\{\tau_1, \tau_2\}. \tag{3.11}$$

Now, let us introduce some notations. For $\gamma \in \mathbb{R}$, set

$$\hat{\gamma} := \begin{cases} \gamma, & \gamma \geq 0, \\ \gamma\check{L}, & \gamma < 0, \end{cases} \quad \check{\gamma} := \begin{cases} \gamma\check{L}, & \gamma \geq 0, \\ \gamma, & \gamma < 0. \end{cases} \tag{3.12}$$

Obviously, $\widehat{(-\gamma)} = -\check{\gamma}$, $\widecheck{(-\gamma)} = -\hat{\gamma}$, and $\hat{\gamma} \geq \check{\gamma}$, as we have assumed $L = 1$.

**Theorem 3.1.9.** If $z(t)$ satisfies (3.10), then $z(t)$ converges to interval $[-n_p, n_p]$, as $t \to \infty$, under condition (H1b): $\Sigma_{i=1}^{2}\check{\gamma}_i \geq 0$, $\Sigma_{i=1}^{2}(\tau_i|\gamma_i|) \leq 2\Sigma_{i=1}^{2}|\gamma_i|/[(2 + \Sigma_{i=1}^{2}\hat{\gamma}_i/\mu)(2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0))]$; moreover,

$$0 \leq n_p \leq \frac{|w|^{\max}(\infty)}{\mu + \Sigma_{i=1}^{2}\check{\gamma}_i - \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)(2\mu + \Sigma_{i=1}^{2}\hat{\gamma}_i + \Sigma_{i=1}^{2}\check{\gamma}_i)}.$$

**Theorem 3.1.10.** If $z(t)$ satisfies (3.10), then $z(t)$ converges to an interval $[-n_q, n_q]$, as $t \to \infty$ under condition (H2b): $\gamma_1 \geq 0$ and $\tau_1\gamma_1(2+\gamma_1/\mu)(2\Sigma_{i=1}^2|\gamma_i|+|w|^{\max}(t_0)) \leq 2\Sigma_{i=1}^2|\gamma_i|(1-|\gamma_2|/\mu) - |\gamma_2||w|^{\max}(t_0)/\mu$; moreover,

$$0 \leq n_q \leq \frac{|w|^{\max}(\infty)}{\mu + \gamma_1\check{L} - |\gamma_2| - \tau_1\gamma_1(2\mu + \gamma_1 + \gamma_1\check{L})}.$$

**Theorem 3.1.11.** If $z(t)$ satisfies (3.10), then $z(t)$ converges to an interval $[-n_r, n_r]$, as $t \to \infty$, under condition (H3b): $\Sigma_{i=1}^2|\gamma_i| < \mu - |w|^{\max}(t_0)/2$; moreover,

$$0 \leq n_r \leq |w|^{\max}(\infty)/(\mu - \Sigma_{i=1}^2|\gamma_i|).$$

### 3.1.3 Proofs for lemma and theorems

We provide the proofs for Lemma 3.1.3, Theorems 3.1.4-3.1.6, 3.19-3.1.11 in this subsection.

**Proof of Lemma 3.1.3.** The labelling in the proof corresponds to the one in the statement of Lemma 3.1.3.

(i) Let us show that for any $T \geq t_0$, $\check{m}^{(k)}(T)$ and $\hat{m}^{(k)}(T)$ are well-defined for all $k \in \mathbb{N} \cup \{0\}$. First, let us claim that the following inequalities

$$\check{h}^{(k-1)}(\xi, T) \leq \check{h}^{(k)}(\xi, T) \leq \hat{h}^{(k)}(\xi, T) \leq \hat{h}^{(k-1)}(\xi, T), \tag{3.13}$$

hold for all $\xi \in \mathbb{R}$, $k \in \mathbb{N}$. To justify that (3.13) holds as $k = 1$, by the definitions of $\check{h}^{(0)}$, $\check{h}^{(1)}$, $\hat{h}^{(1)}$ and $\hat{h}^{(0)}$, it suffices to verify the following three inequalities:

$$\begin{aligned}
\hat{h}(\check{A}^h) &\geq \hat{h}^{(0)}(\check{m}^{(0)}(T), T),\\
\hat{h}^{(0)}(\check{m}^{(0)}(T), T) &\geq \check{h}^{(0)}(\hat{m}^{(0)}(T), T),\\
\check{h}^{(0)}(\hat{m}^{(0)}(T), T) &\geq \check{h}(\hat{A}^h).
\end{aligned}$$

It is obvious that all these three inequalities hold under condition (H1a), cf. Fig. 3.1. Assume that (3.13) holds for $k = 1, \cdots, j-1$, where $j \in \mathbb{N}$. To show that it also holds for $k = j$, by definitions of $\check{h}^{(j-1)}$, $\check{h}^{(j)}$, $\hat{h}^{(j)}$ and $\hat{h}^{(j-1)}$, it suffices to justify the following three inequalities:

$$\begin{aligned}
\hat{h}^{(j-2)}(\check{m}^{(j-2)}(T), T) &\geq \hat{h}^{(j-1)}(\check{m}^{(j-1)}(T), T),\\
\hat{h}^{(j-1)}(\check{m}^{(j-1)}(T), T) &\geq \check{h}^{(j-1)}(\hat{m}^{(j-1)}(T), T),\\
\check{h}^{(j-1)}(\hat{m}^{(j-1)}(T), T) &\geq \check{h}^{(j-2)}(\hat{m}^{(j-2)}(T), T).
\end{aligned}$$

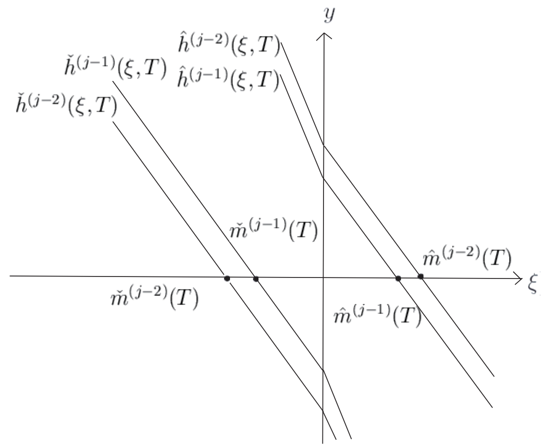



Figure 3.2: Configurations for functions $\check{h}^{(j-1)}(\cdot,T)$, $\check{h}^{(j-2)}(\cdot,T)$, $\hat{h}^{(j-2)}(\cdot,T)$ and $\hat{h}^{(j-1)}(\cdot,T)$ for fixed $T \geq t_0$.

Note that $\check{h}^{(j-2)}(\xi,T) \leq \check{h}^{(j-1)}(\xi,T) \leq \hat{h}^{(j-1)}(\xi,T) \leq \hat{h}^{(j-2)}(\xi,T)$ for $\xi \in \mathbb{R}$, since (3.13) holds for $k = j-1$. It is obvious that all these three inequalities hold, cf. Fig. 3.2. Hence (3.13) hold for all $k \in \mathbb{N}$; subsequently, $\check{h}^{(0)}(\xi,T) \leq \check{h}^{(k)}(\xi,T) \leq \hat{h}^{(k)}(\xi,T) \leq \hat{h}^{(0)}(\xi,T)$ for all $\xi \in \mathbb{R}$, and $k \in \mathbb{N}\cup\{0\}$. Note that $\check{h}^{(k)}(\xi,T)$ and $\hat{h}^{(k)}(\xi,T)$ are vertical shift of $\check{h}^{(0)}(\xi,T)$ and $\hat{h}^{(0)}(\xi,T)$ respectively. Accordingly, both $\check{m}^{(k)}(T)$ and $\hat{m}^{(k)}(T)$ are well defined for all $k \in \mathbb{N}\cup\{0\}$ under condition (H1a). Moreover, it is a straightforward result that $\hat{m}^{(j)}(T_1) \leq \hat{m}^{(j)}(T_2)$ and $\check{m}^{(j)}(T_1) \geq \check{m}^{(j)}(T_2)$, for any $T_1 > T_2 \geq t_0$, since that $\hat{h}^{(j)}(\cdot,T_1) \leq \hat{h}^{(j)}(\cdot,T_2)$ and $\check{h}^{(j)}(\cdot,T_1) \geq \check{h}^{(j)}(\cdot,T_2)$. Thus, for each $k \in \mathbb{N}\cup\{0\}$, $\check{m}^{(k)}(T)$ increases and $\hat{m}^{(k)}(T)$ decreases, with respect to $T$.

(ii) By (3.13), it can be shown that, for each $T \geq t_0$,

$$\hat{m}^{(k+1)}(T) \leq \hat{m}^{(k)}(T),\ \check{m}^{(k+1)}(T) \geq \check{m}^{(k)}(T),\ \text{for all } k \geq 0.$$

Moreover, $\hat{m}^{(k)}(T) \geq 0$ and $\check{m}^{(k)}(T) = -\hat{m}^{(k)}(T)$, for $i \in \mathbb{N}$. It follows that, for any $T \geq t_0$, there exist some $m(T) \geq 0$ such that $\lim_{k\to\infty} \hat{m}^{(k)}(T) = m(T) \geq 0$, and $\lim_{k\to\infty} \check{m}^{(k)}(T) = -m(T) \leq 0$.

(iii) For each $k \in \mathbb{N}\cup\{0\}$, it has been shown that $\hat{m}^{(k)}(T_2) \geq \hat{m}^{(k)}(T_1)$, if $T_1 > T_2 \geq t_0$; subsequently, $\lim_{k\to\infty} \hat{m}^{(k)}(T_2) \geq \lim_{k\to\infty} \hat{m}^{(k)}(T_1)$, i.e. $m(T_2) \geq m(T_1)$. Therefore, there exists some $m_p \in \mathbb{R}$ such that $m(T) \to m_p$ decreasingly as $T \to \infty$, since $m(T)$ is bounded below for all $T \geq t_0$.

(iv) It is obvious that for all $T \geq t_0$, $m(T) \geq 0$. Next, we justify that $m(T) \leq$

$|w|^{\max}(T)/[(1+\beta\check{L})-\tau\beta L(2+\beta L+\beta\check{L})]$, for any $T \geq t_0$. For any fixed $T \geq t_0$, it is not difficult to observe that $\{\hat{h}^{(k)}(\cdot,T)|_{[\check{A}^h,\hat{A}^h]}\}_{k\geq 0}$ are uniformly bounded and equicontinuous. Moreover $\hat{h}^{(k)}(\xi,T)$ decreases with respect to $k$. By Ascoli-Azela Theorem, for any fixed $T \geq t_0$, there exists some continuous function $\hat{h}^{(\infty)}(\cdot,T)$ defined on $[\check{A}^h,\hat{A}^h]$ such that

$$\hat{h}^{(k)}(\cdot,T) \downarrow \hat{h}^{(\infty)}(\cdot,T) \text{ uniformly on } [\check{A}^h,\hat{A}^h], \text{ as } k \to \infty.$$

Recall that $\check{h}^{(k)}(\xi,T) = -\hat{h}^{(k)}(-\xi,T)$. It follows that

$$\check{h}^{(k)}(\cdot,T) \uparrow \check{h}^{(\infty)}(\cdot,T) \text{ uniformly on } [\check{A}^h,\hat{A}^h],$$

where $\check{h}^{(\infty)}(\xi,T) = -\hat{h}^{(\infty)}(-\xi,T)$. It is obvious that, for any fixed $T \geq t_0$

$$\check{h}^{(\infty)}(\xi,T) \leq \hat{h}^{(\infty)}(\xi,T), \text{ for all } \xi \in [\check{A}^h,\hat{A}^h].$$

We summarize the properties of $\check{h}^{(\infty)}(\cdot,T)$ and $\hat{h}^{(\infty)}(\cdot,T)$ as follows:

(P1): $\hat{h}^{(k)}(\hat{m}^{(k)}(T),T) \to \hat{h}^{(\infty)}(m(T),T)$, $\hat{h}^{(k)}(\check{m}^{(k)}(T),T) \to \hat{h}^{(\infty)}(-m(T),T)$ as $k \to \infty$,

(P2): $\hat{h}^{(\infty)}(\xi,T) = \begin{cases} -(1+\beta\check{L})\xi + \tau\beta L\hat{h}^{(\infty)}(-m(T),T) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta L)\xi + \tau\beta L\hat{h}^{(\infty)}(-m(T),T) + |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$

(P3): $\hat{h}^{(\infty)}(m(T),T) = 0$.

Below, let us justify these properties one by one. The first result in (P1) holds since

$$\begin{aligned} & |\hat{h}^{(k)}(\hat{m}^{(k)}(T),T) - \hat{h}^{(\infty)}(m(T),T)| \\ \leq\ & |\hat{h}^{(k)}(\hat{m}^{(k)}(T),T) - \hat{h}^{(\infty)}(\hat{m}^{(k)}(T),T)| + |\hat{h}^{(\infty)}(\hat{m}^{(k)}(T),T) - \hat{h}^{(\infty)}(m(T),T)|, \end{aligned}$$

and both $|\hat{h}^{(k)}(\hat{m}^{(k)}(T),T)-\hat{h}^{(\infty)}(\hat{m}^{(k)}(T),T)|$ and $|\hat{h}^{(\infty)}(\hat{m}^{(k)}(T),T)-\hat{h}^{(\infty)}(m(T),T)|$ converge to 0 as $k \to \infty$. The remaining part in (P1) can be justified similarly. If $\xi \geq 0$, then

$$\begin{aligned} \hat{h}^{(\infty)}(\xi,T) &= \lim_{k\to\infty} \hat{h}^{(k)}(\xi,T) \\ &= \lim_{k\to\infty} \{-(1+\beta\check{L})\xi + \tau\beta L\hat{h}^{(k-1)}(\check{m}^{(k-1)(}T),T) + |w|^{\max}(T)\} \\ &= -(1+\beta\check{L})\xi + \tau\beta L \lim_{k\to\infty}\{\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T)\} + |w|^{\max}(T) \\ &= -(1+\beta\check{L})\xi + \tau\beta L\hat{h}^{(\infty)}(-m(T),T) + |w|^{\max}(T). \end{aligned}$$

Similar argument can be applied to the case $\xi < 0$. Hence, property (P2) follows. Next, (P3) follows from property (P1) since $\hat{h}^{(k)}(\hat{m}^{(k)}(T),T) = 0$. Due to properties

(P2) and (P3), for each $T$, $\hat{h}^{(\infty)}(\cdot,T)$ is a strictly decreasing function and has a unique zero at $m(T)$. As $\check{h}^{(\infty)}(\xi,T)=-\hat{h}^{(\infty)}(-\xi,T)$, $-m(T)$ is the unique zero of $\check{h}^{(\infty)}(\cdot,T)=0$. Since $\hat{h}^{(\infty)}(-m(T),T)=(1+\beta L)m(T)+\tau\beta L\hat{h}^{(\infty)}(-m(T),T)+|w|^{\max}(T)$, we derive

$$0\leq \hat{h}^{(\infty)}(-m(T),T)=\frac{(1+\beta L)m(T)+|w|^{\max}(T)}{1-\tau\beta L}.$$

Consequently, for $\xi\geq 0$,

$$\hat{h}^{(\infty)}(\xi,T)=-(1+\beta\check{L})\xi+\frac{\tau\beta L[(1+\beta L)m(T)+|w|^{\max}(T)]}{1-\tau\beta L}+|w|^{\max}(T). \quad (3.14)$$

With the help of (3.14) and $\hat{h}^{(\infty)}(m(T),T)=0$, it can be derived that

$$m(T)=|w|^{\max}(T)/[(1+\beta\check{L})-\tau\beta L(2+\beta L+\beta\check{L})]. \quad (3.15)$$

(v) It is obvious that $\cap_{T\geq t_0}[-m(T),m(T)]=[-m_p,m_p]$. In addition, $m_p\leq m(T)$ for any $T\geq t_0$. With $m(T)$ given in (3.15), we thus obtain

$$0\leq m_p\leq |w|^{\max}(\infty)/\{(1+\beta\check{L})-\tau\beta L(2+\beta L+\beta\check{L})\}.$$

**Proof of Theorem 3.1.4.** Let $z(t)$ be a solution to (3.1). The following claim will lead to that for each $T\geq T_{x,y}$, $z(t)$ converges to $[-m(T),m(T)]$ as $t\to\infty$. Subsequently $z(t)$ converges to $[-m_p,m_p]$ as $t\to\infty$. To complete the proof, it suffices to prove the claim.

Claim: For arbitrarily fixed $T\geq T_{x,y}$, and any fixed $n\in\mathbb{N}$, there exist some increasing sequence $\{T_k\}_{k=0}^n$ with $T_{k+1}\geq T_k+\tau$, for $k=0,1,\cdots,n-1$ and $T_0\geq T+\tau$, such that

$$\begin{cases}\check{h}^{(k)}(z(t),T)<\dot{z}(t)<\hat{h}^{(k)}(z(t),T), \text{ for } t\geq T_k+\tau,\ k=0,1,\cdots,n-1;\\ z(t)\in[\check{m}^{(k)}(T),\hat{m}^{(k)}(T)], \text{ for } t\geq T_{k+1},\ k=0,1,\cdots,n-1.\end{cases} \quad (3.16)$$

Let us justify the claim. First, it is easy to show that $\check{h}^{(0)}(z(t),T)<\dot{z}(t)<\hat{h}^{(0)}(z(t),T)$ for all $t\geq T_0+\tau:=(T+\tau)+\tau>T$ due to Proposition 3.1.2. Accordingly, there exists some $T_1\geq T_0+\tau$, such that $z(t)\in[\check{m}^{(0)}(T),\hat{m}^{(0)}(T)]$ for all $t\geq T_1$. Thus, (3.16) holds for $n=1$. Now, we assume that (3.16) holds for $n=\ell-1\geq 1$. We shall show that (3.16) holds for $n=\ell$. For any $t\geq T_{\ell-1}+\tau$, $g(x(t-\tau))-g(y(t-\tau))=g'(\zeta)z(t-\tau)$, for some $\zeta\in\mathcal{Q}$. $z(t-\tau)=z(t)-\dot{z}(s)\tau$ for some $s\in(t-\tau,t)$. Therefore, $\dot{z}(t)=-z(t)-\beta g'(\zeta)z(t-\tau)+w(t)=-z(t)-\beta g'(\zeta)[z(t)-\dot{z}(s)\tau]+w(t)$. Notice that $\check{h}^{(\ell-2)}(\hat{m}^{(\ell-2)}(T),T) < \dot{z}(s) < \hat{h}^{(\ell-2)}(\check{m}^{(\ell-2)}(T),T)$; $\check{h}^{(\ell-2)}(\hat{m}^{(\ell-2)}(T),T) \leq 0$ and $\hat{h}^{(\ell-2)}(\check{m}^{(\ell-2)}(T),T) \geq 0$ since that $s \geq T_{\ell-1} \geq T_{\ell-2}+\tau$. If $z(t) \geq 0$, then $\dot{z}(t) < -z(t) - \beta\check{L}z(t) + \tau\beta L\hat{h}^{(\ell-2)}(\check{m}^{(\ell-2)}(T),T) + |w|^{\max}(T) = \hat{h}^{(\ell-1)}(z(t),T)$. If $z(t) < 0$, then $\dot{z}(t) < -z(t) - \beta Lz(t) + \tau\beta L\hat{h}^{(\ell-2)}(\check{m}^{(\ell-2)}(T),T) + |w|^{\max}(T) = \hat{h}^{(\ell-1)}(z(t),T)$. Similarly, we can show that $\dot{z}(t) > \check{h}^{(\ell-1)}(z(t),T)$. Accordingly, there exists some $T_\ell \geq T_{\ell-1}+\tau$ such that $z(t) \in [\check{m}^{(\ell-1)}(T), \hat{m}^{(\ell-1)}(T)]$, for $t \geq T_\ell$. The claim is thus justified.

**Proof of Theorem 3.1.5.** We recompose the upper and lower functions in the formulation for the case of $\beta > 0$ as follows:

$$\hat{h}(\xi) := \begin{cases} -(1+\beta L)\xi + 2|\beta| + |w|^{\max}(t_0), & \text{if } \xi \geq 0, \\ -\xi + 2|\beta| + |w|^{\max}(t_0), & \text{if } \xi < 0; \end{cases}$$

$$\hat{h}^{(0)}(\xi,T) := \begin{cases} -(1+\beta L)\xi - \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta\check{L})\xi - \tau\beta L\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi < 0; \end{cases}$$

$$\hat{h}^{(k)}(\xi,T) := \begin{cases} -(1+\beta L)\xi - \tau\beta L\hat{h}^{(k-1)}(\check{m}^{(k-1)}T),T) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(1+\beta\check{L})\xi - \tau\beta L\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T) + |w|^{\max}(T), & \text{for } \xi < 0; \end{cases}$$

$$\check{h}(\xi) := -\hat{h}(-\xi),\ \check{h}^{(0)}(\xi,T) := -\hat{h}^{(0)}(-\xi,T),\ \check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T).$$

Then the proof of the theorem is similar to the one for Theorem 3.1.4.

**Proof of Theorem 3.1.6.** We recompose the upper and lower functions in the formulation for Theorem 3.1.4 as follows:

$$\hat{h}(\xi) := -\xi + 2|\beta| + |w|^{\max}(t_0),$$
$$\hat{h}^{(0)}(\xi,T) := -\xi + |\beta|L\hat{A}^h + |w|^{\max}(T),$$
$$\hat{h}^{(k)}(\xi,T) := -\xi + |\beta|L\hat{m}^{(k-1)}(T) + |w|^{\max}(T),$$
$$\check{h}(\xi) := -\hat{h}(-\xi),\ \check{h}^{(0)}(\xi,T) := -\hat{h}^{(0)}(-\xi,T),\ \check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T).$$

Then the proof follows from similar process as Theorem 3.1.4.

**Proof of Theorem 3.1.9.** The proof resembles the one for Theorem 3.1.4 by recomposing the upper and lower formulation:

$$\hat{h}(\xi) := \begin{cases} -\mu\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0) & \text{if } \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^{2}\hat{\gamma}_i)\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0) & \text{if } \xi < 0, \end{cases}$$

$$\hat{h}^{(0)}(\xi, T) := \begin{cases} -(\mu + \Sigma_{i=1}^{2}\check{\gamma}_i)\xi + \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^{2}\hat{\gamma}_i)\xi + \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T), & \text{for } \xi < 0, \end{cases}$$

$$\hat{h}^{(k)}(\xi, T) := \begin{cases} -(\mu + \Sigma_{i=1}^{2}\check{\gamma}_i)\xi + \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |w|^{\max}(T), & \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^{2}\hat{\gamma}_i)\xi + \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |w|^{\max}(T), & \xi < 0. \end{cases}$$

$$\check{h}(\xi) := -\hat{h}(-\xi),\ \check{h}^{(0)}(\xi, T) := -\hat{h}^{(0)}(-\xi, T),\ \check{h}^{(k)}(\xi, T) := -\hat{h}^{(k)}(-\xi, T).$$

**Proof of Theorem 3.1.10.** The proof resembles the one for Theorem 3.1.4 by recomposing the upper and lower formulation:

$$\hat{h}(\xi) := \begin{cases} -\mu\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0) & \xi \geq 0, \\ -(\mu + \gamma_1)\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0) & \xi < 0; \end{cases}$$

$$\hat{h}^{(0)}(\xi, T) := \begin{cases} -(\mu + \gamma_1\check{L})\xi + \tau_1\gamma_1\hat{h}(\check{A}^h) + |\gamma_2|\hat{A}^h + |w|^{\max}(T), & \xi \geq 0, \\ -(\mu + \gamma_1)\xi + \tau_1\gamma_1\hat{h}(\check{A}^h) + |\gamma_2|\hat{A}^h + |w|^{\max}(T), & \xi < 0; \end{cases}$$

$$\hat{h}^{(k)}(\xi, T) := \begin{cases} -(\mu + \gamma_1\check{L})\xi + \tau_1\gamma_1\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |\gamma_2|\hat{m}^{(k-1)}(T) + |w|^{\max}(T), & \xi \geq 0, \\ -(\mu + \gamma_1)\xi + \tau_1\gamma_1\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T), T) + |\gamma_2|\hat{m}^{(k-1)}(T) + |w|^{\max}(T), & \xi < 0; \end{cases}$$

$$\check{h}(\xi) := -\hat{h}(-\xi),\ \check{h}^{(0)}(\xi, T) := -\hat{h}^{(0)}(-\xi, T),\ \check{h}^{(k)}(\xi, T) := -\hat{h}^{(k)}(-\xi, T).$$

**Proof of Theorem 3.1.11.** The proof resembles the one for Theorem 3.1.4 by recomposing the formulation for upper and lower functions:

$$\hat{h}(\xi) := -\mu\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0),$$
$$\hat{h}^{(0)}(\xi, T) := -\mu\xi + \Sigma_{i=1}^{2}|\gamma_i|\hat{A}^h + |w|^{\max}(T),$$
$$\hat{h}^{(k)}(\xi, T) := -\mu\xi + \Sigma_{i=1}^{2}|\gamma_i|\hat{m}^{(k-1)}(T) + |w|^{\max}(T),$$
$$\check{h}(\xi) := -\hat{h}(-\xi),\ \check{h}^{(0)}(\xi, T) := -\hat{h}^{(0)}(-\xi, T),\ \check{f}^{(k)}(\xi, T) := -\hat{h}^{(k)}(-\xi, T).$$

## 3.2 Scalar equation for convergence

In this section, we introduce the scalar equation associated with the convergence for (2.3). We consider the following scalar equation with time-dependent external input $w(t)$:

$$\dot{x}(t) = -\mu x(t) + \alpha g(x(t)) + \beta g(x(t - \tau_1(t)) + w(t), \quad (3.17)$$

where $\mu > 0, \alpha > 0$ and $\beta \in \mathbb{R}$; $\tau_1(t)$ is a continuous function with $0 \leq \tau_1(t) \leq \tau \in \mathbb{R}$, for all $t \geq t_0$; $w(t)$ is a bounded continuous function defined for $t \geq t_0$; $g$ is an activation function of class $\mathcal{A}$ with $u < g(\xi) < v$, $g'(\xi) \le L := g'(0) = \max\{g'(\eta) : \eta \in \mathbb{R}\}$, for all $\xi \in \mathbb{R}$. Let $\rho = \max\{|u|, |v|\}$. We present the basic formations and propositions in Section 3.2.1. Some extensions form (3.17) to other scalar equations are given in Section 3.2.2. The proofs for the lemma, propositions and theorems in Section 3.2.1 and 3.2.2 are given in Section 3.2.3.

### 3.2.1 Formulations and properties

The main result (Theorem 3.2.4) in this subsection asserts that there exist three disjoint, bounded and closed intervals to which every solution of (3.17) converges, under certain parameter conditions. Adopting the arguments parallel to the ones in previous section (Section 3.1), we shall, iteratively, define sequences of upper and lower functions for the dynamics of (3.17) as time proceeds, to capture the asymptotical behavior of (3.17).

The first two conditions we impose on activation function $g$ and parameters are

Condition (A1c): $L > 2\mu/\alpha > 0$,
Condition (A2c): $L < \mu/|\beta|$.

Let us define $f(\xi) := -\mu\xi + \alpha g(\xi)$, where $g$ is the same as in (3.17). Then, $\tilde{f}'(\xi) = -\mu + \alpha g'(\xi)$, for any vertical shift $\tilde{f}$ of $f$. If condition (A1c) holds, there exist exactly two points $\bar{p}$, $\bar{q}$ with $\bar{p} < 0 < \bar{q}$ such that $\tilde{f}'(\bar{p}) = \tilde{f}'(\bar{q}) = 0$; $\tilde{f}'(\xi) > 0$ for $\xi \in (\bar{p}, \bar{q})$; and $\tilde{f}'(\xi) < 0$ for $\xi \in \mathbb{R} - [\bar{p}, \bar{q}]$. Restated, if $g'(0) > \mu/\alpha$, then $\bar{p}$ and $\bar{q}$ are the only two critical points of $\tilde{f}$, and $g'(\bar{p}) = g'(\bar{q}) = \mu/\alpha$, cf. Fig. 3.3. In addition, conditions (A1c), (A2c) imply $0 < (\mu - L|\beta|)/(\alpha + |\beta|) < \mu/\alpha$. Thus, there always exist two points $\tilde{p}$ and $\tilde{q}$, where $\tilde{p} < \bar{p} < \bar{q} < \tilde{q}$ such that

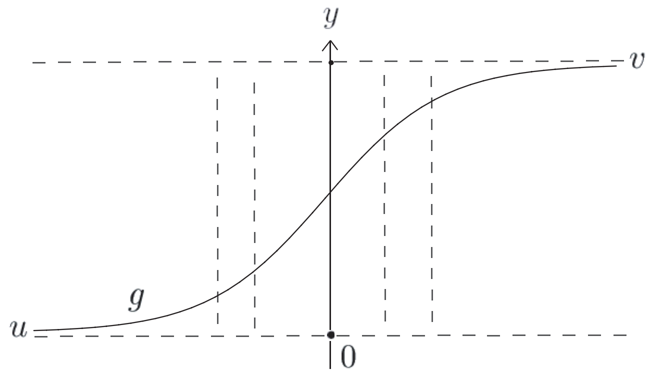
$$g'(\tilde{p}) = g'(\tilde{q}) = \frac{\mu - L|\beta|}{\alpha + |\beta|}.$$

We shall formulate the desired configuration and properties for equation (3.17) through the following quantities and functions. For $T \ge t_0$, let

$$\begin{aligned}
&w^{\min}(T) := \inf\{w(t) \mid t \ge T\},\ w^{\max}(T) := \sup\{w(t) \mid t \ge T\},\\
&\hat{f}^{(0)}(\xi, T) := -\mu\xi + \alpha g(\xi) + |\beta|\rho + w^{\max}(T),\\
&\check{f}^{(0)}(\xi, T) := -\mu\xi + \alpha g(\xi) - |\beta|\rho + w^{\min}(T).
\end{aligned}$$

For convenience of later uses, we denote

$$\hat{f}(\xi) := \hat{f}^{(0)}(\xi, t_0), \check{f}(\xi) := \check{f}^{(0)}(\xi, t_0). \tag{3.18}$$



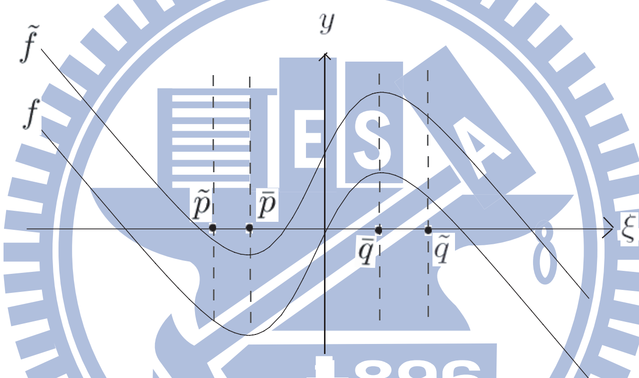




Figure 3.3: (a) The graph of activation function $g$ of class $\mathcal{A}$. (b) The configurations for functions $\tilde{f}$, $f$ with their critical points $\overline{p}, \overline{q}$, and the points $\tilde{p}, \tilde{q}$ at which $g$ has designated slopes.

Notably, $\hat{f}$ and $\check{f}$ are also vertical shifts of $f$. Let us introduce the third condition.

Condition (A3c): $\check{f}(\tilde{q}) > 0$, $\hat{f}(\tilde{p}) < 0$.

Under conditions (A1c), (A2c), (A3c), there exist three solutions $\hat{l}$, $\hat{m}$ and $\hat{r}$ (resp. $\check{l}$, $\check{m}$ and $\check{r}$) of $\hat{f}(\xi) = 0$ (resp. $\check{f}(\xi) = 0$). Moreover, $\check{l} < \hat{l} < \tilde{p} < \bar{p} < \hat{m} < \check{m} < \bar{q} < \tilde{q} < \check{r} < \hat{r}$. We further impose a slope condition on the middle part of the activation function. This condition actually covers (A1c).

Condition (A4c): $g^{'}(\xi) > 2\mu/\alpha$, for all $\xi \in [\hat{m}, \check{m}]$.

Let $\check{a}^{(0)}(T)$ (resp. $\check{b}^{(0)}(T), \check{c}^{(0)}(T)$) be the unique solution of $\check{f}^{(0)}(\cdot, T) = 0$ lying in interval $[\check{l}, \hat{l}]$ (resp. $[\hat{m}, \check{m}], [\check{r}, \hat{r}]$), and $\hat{a}^{(0)}(T)$ (resp. $\hat{b}^{(0)}(T), \hat{c}^{(0)}(T)$) be the unique solution of $\hat{f}^{(0)}(\cdot, T) = 0$ lying in $[\check{l}, \hat{l}]$ (resp. $[\hat{m}, \check{m}], [\check{r}, \hat{r}]$), cf. Fig. 3.4. The following functions can be defined iteratively for each fixed $T \geq t_0$: for $k \in \mathbb{N}$,

$$
\begin{aligned}
\hat{f}^{(k)}_{\mathrm{l}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{a}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\check{a}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases} \\
\check{f}^{(k)}_{\mathrm{l}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{a}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{a}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0, \end{cases} \\
\hat{f}^{(k)}_{\mathrm{m}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases} \\
\check{f}^{(k)}_{\mathrm{m}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0, \end{cases} \\
\hat{f}^{(k)}_{\mathrm{r}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{c}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases} \\
\check{f}^{(k)}_{\mathrm{r}}(\xi, T) &:= \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geq 0, \\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{c}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0. \end{cases}
\end{aligned}
$$

These functions are all vertical shifts of $f$ for each fixed $T$. Herein, $\check{a}^{(k)}(T)$ (resp. $\check{b}^{(k)}(T)$, $\check{c}^{(k)}(T)$) is the unique solution of $\check{f}^{(k)}_{\mathrm{l}}(\cdot, T) = 0$ (resp. $\check{f}^{(k)}_{\mathrm{m}}(\cdot, T) = 0$, $\check{f}^{(k)}_{\mathrm{r}}(\cdot, T) = 0$) lying in interval $[\check{l}, \hat{l}]$ (resp. $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$), and $\hat{a}^{(k)}(T)$ (resp. $\hat{b}^{(k)}(T)$, $\hat{c}^{(k)}(T)$) is the unique solution of $\hat{f}^{(k)}_{\mathrm{l}}(\cdot, T) = 0$ (resp. $\hat{f}^{(k)}_{\mathrm{m}}(\cdot, T) = 0$, $\hat{f}^{(k)}_{\mathrm{r}}(\xi, T) = 0$) lying in $[\check{l}, \hat{l}]$ (resp. $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$). We also define $w^{\min}(\infty) := \lim_{T\to\infty} w^{\min}(T)$, $w^{\max}(\infty) := \lim_{T\to\infty} w^{\max}(T)$.

The following lemma summarizes the properties for zeros of the above-defined sequences of single-variable functions.

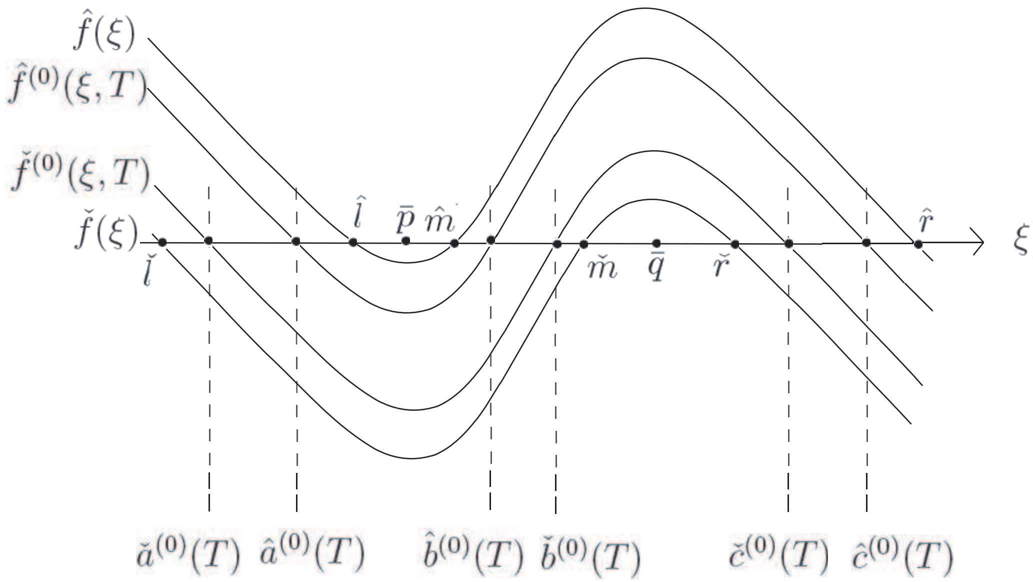



Figure 3.4: Configurations of functions $\hat{f}$, $\hat{f}^{(0)}$, $\check{f}^{(0)}$, and $\check{f}$, for fixed $T \geq t_0$.

**Lemma 3.2.1.** Assume that conditions (A2c)-(A4c) hold. Then, for each $T \geq t_0$, the sequences $\{\check{b}^{(k)}(T)\}_{k\geq 0}$, $\{\hat{b}^{(k)}(T)\}_{k\geq 0}$ $\{\check{a}^{(k)}(T)\}_{k\geq 0}$, $\{\hat{a}^{(k)}(T)\}_{k\geq 0}$, $\{\check{c}^{(k)}(T)\}_{k\geq 0}$, $\{\hat{c}^{(k)}(T)\}_{k\geq 0}$ can be defined iteratively. Moreover,
(i) for any fixed $k \in \mathbb{N} \cup \{0\}$, each of $\hat{b}^{(k)}(T)$, $\check{a}^{(k)}(T)$ and $\check{c}^{(k)}(T)$ is increasing, and each of $\check{b}^{(k)}(T)$, $\hat{a}^{(k)}(T)$, and $\hat{c}^{(k)}(T)$ is decreasing with respect to $T \geq t_0$;
(ii) for any $T \geq t_0$, there exist $\underline{b}(T), \overline{b}(T), \underline{a}(T), \overline{a}(T), \underline{c}(T), \overline{c}(T) \in \mathbb{R}$ such that $\hat{b}^{(k)}(T) \to \underline{b}(T)$, $\check{a}^{(k)}(T) \to \underline{a}(T)$, and $\check{c}^{(k)}(T) \to \underline{c}(T)$ increasingly, and $\check{b}^{(k)}(T) \to \overline{b}(T)$, $\hat{a}^{(k)}(T) \to \overline{a}(T)$, and $\hat{c}^{(k)}(T) \to \overline{c}(T)$ decreasingly, as $k \to \infty$;
(iii) there exist $\underline{b}, \overline{b}, \underline{a}, \overline{a}, \underline{c}, \overline{c} \in \mathbb{R}$, such that $\underline{b}(T) \to \underline{b}$, $\underline{a}(T) \to \underline{a}$, $\underline{c}(T) \to \underline{c}$ increasingly and $\overline{b}(T) \to \overline{b}$, $\overline{a}(T) \to \overline{a}$, $\overline{c}(T) \to \overline{c}$ decreasingly, as $T \to \infty$;
(iv) $\cap_{T\geq t_0}[\underline{b}(T), \overline{b}(T)] = [\underline{b}, \overline{b}]$, $\cap_{T\geq t_0}[\underline{a}(T), \overline{a}(T)] = [\underline{a}, \overline{a}]$, $\cap_{T\geq t_0}[\underline{c}(T), \overline{c}(T)] = [\underline{c}, \overline{c}]$;
(v) $0 \leq \overline{b}(T) - \underline{b}(T) \leq [w^{\max}(T) - w^{\min}(T)]/(\mu - |\beta|L)$, $0 \leq \overline{a}(T) - \underline{a}(T), \overline{c}(T) - \underline{c}(T) \leq [w^{\max}(T) - w^{\min}(T)]/(|\beta|L)$, for any $T \geq t_0$, moreover

$$0 \leq d_b := \overline{b} - \underline{b} \leq \frac{w^{\max}(\infty) - w^{\min}(\infty)}{\mu - |\beta|L},$$

$$0 \leq d_a := \overline{a} - \underline{a},\ d_c := \overline{c} - \underline{c} \leq \frac{w^{\max}(\infty) - w^{\min}(\infty)}{|\beta|L}.$$

In the following discussions, for an initial value $\phi \in C([-\tau, 0], \mathbb{R})$, we denote by $x(t) = x(t; t_0; \phi)$ the solution of (3.17) with $x(t_0 + \theta; t_0; \phi) = \phi(\theta)$, for $\theta \in [-\tau, 0]$.

**Definition 3.2.1.** A solution $x(t)$ of (3.17) is said to satisfy Property $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$, if, respectively,

for each $k \in \mathbb{N} \cup \{0\}$, $T \geq t_0, x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$, for all $t \geq T + k\tau$,

there exists $s \geq t_0$ such that $x(s) \in [\check{l}, \hat{l}]$,

there exists $s \geq t_0$ such that $x(s) \in [\check{r}, \hat{r}]$.

**Proposition 3.2.2.** Assume that conditions (A2c)-(A4c) hold.

(i) If $x(t)$ is a solution of (3.17) and for any fixed $T \geq t_0$, $k \in \mathbb{N}$, $x(t) \in [\hat{b}^{k-1}(T), \check{b}^{k-1}(T)]$ for all $t \geq T + (k-1)\tau$, then $x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$, for all $t \geq T + k\tau$;

(ii) If $x(t)$ is a solution of (3.17) and $x(s) > \check{b}^{(0)}(T)$ (resp. $x(s) < \hat{b}^{(0)}(T)$), for some $s \geq T \geq t_0$, then $x(t)$ satisfies Property $\mathcal{R}$ (resp. $\mathcal{L}$);

(iii) If the solution $x(t)$ of (3.17) satisfies Property $\mathcal{M}$, then $x(t) \to [\underline{b}(T), \overline{b}(T)]$ as $t \to \infty$, for any $T \geq t_0$; subsequently, $x(t) \to [\underline{b}, \overline{b}]$ as $t \to \infty$.

(iv) Each of $[\check{l}, \hat{l}]$ and $[\check{r}, \hat{r}]$ is a positively invariant interval for (3.17). Moreover, if $x(t)$ is a solution of (3.17), which satisfies Property $\mathcal{R}$ (resp. $\mathcal{L}$), then $x(t) \to [\underline{c}, \overline{c}]$ (resp. $[\underline{a}, \overline{a}]$), as $t \to \infty$.

**Proposition 3.2.3.** Assume that conditions (A2c)-(A4c) hold. Every solution $x(t)$ of (3.17) satisfies one of Properties $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$.

**Proof.** Let $x(t)$ be a solution of (3.17) which does not satisfy Property $\mathcal{M}$. Then there exist $k \in \mathbb{N} \cup \{0\}$, $T \geq t_0$, such that

$$x(t) \in \mathbb{R} - [\hat{b}^k(T), \check{b}^k(T)], \text{ for some } t \geq T + k\tau. \tag{3.19}$$

Set $\mathcal{K} := \{(k,T) : k \in \mathbb{N} \cup \{0\}, T \geq t_0, \text{ and (3.19) holds}\}$, $k_0 := \min\{k : \text{there exists } T \geq t_0 \text{ such that } (k,T) \in \mathcal{K}\}$. There are two possibilities: $k_0 \geq 1$ and $k_0 = 0$.

Case (i): If $k_0 \geq 1$, then for any $T \geq t_0$,

$$x(t) \in [\hat{b}^{k_0-1}(T), \check{b}^{k_0-1}(T)], \text{ for all } t \geq T + (k_0 - 1)\tau.$$

It follows from Proposition 3.2.2 (i) that $x(t) \in [\hat{b}^{k_0}(T), \check{b}^{k_0}(T)]$, for all $t \geq T + k_0\tau$, which is a contradiction to the definition of $k_0$.

Case (ii): If $k_0 = 0$, then there exist $T \geq t_0$ and $t \geq T$ such that $x(t) \in \mathbb{R} - [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$. $x(t)$ then satisfies Property $\mathcal{L}$ or $\mathcal{R}$, according to Proposition 3.2.2 (ii). $\square$

Combining Proposition 3.2.2(iii), (iv), and Proposition 3.2.3, we conclude the main result in this section.

**Theorem 3.2.4.** Assume that conditions (A2c)-(A4c) hold. Let $x(t)$ be a solution of (3.17). Then $x(t) \to [\underline{a}, \overline{a}]$, or $[\underline{b}, \overline{b}]$, or $[\underline{c}, \overline{c}]$, as $t \to \infty$; moreover,

$$0 \le d_b := \overline{b} - \underline{b} \le \frac{w^{\max}(\infty) - w^{\min}(\infty)}{\mu - |\beta| L},$$
$$0 \le d_a := \overline{a} - \underline{a},\ d_c := \overline{c} - \underline{c} \le \frac{w^{\max}(\infty) - w^{\min}(\infty)}{|\beta| L}.$$

### 3.2.2 Some extensions

For convergent dynamics of (2.8), we consider the following scalar equation with time-dependent external inputs $w(t)$ and $E(t)$:

$$\dot{x}(t) = -x(t) - \beta g(x(t-\tau)) + w(t) + E(t), \tag{3.20}$$

where $w(t)$ and $E(t)$ are bounded continuous functions defined for $t \ge t_0$, and $E(t) \to 0$ as $t \to \infty$; $g$ is an activation function of class $\mathcal{A}$ with $-1 < g(\xi) < 1$, $g'(\xi) \le L := g'(0) = \max\{g'(\eta) : \eta \in \mathbb{R}\}$, for all $\xi \in \mathbb{R}$.

Recall that Theorem 3.2.4 addresses convergence to three intervals for scalar equation (3.20). Such a delay independent result therein strongly relies on positiveness of $\alpha$, hence can not be applied to (3.20) directly. However, by adopting the idea of controlling the delay effect upon the motion of the equation in Section 3.1, we can still derive similar results for (3.20). Let us recompose some setting in Section 3.2.1 and introduce the conditions imposed in the following. We define

$$\hat{f}(\xi) := -\xi - \beta g(\xi) - \beta L\tau[4|\beta| + w^{\max}(t_0) - w^{\min}(t_0)] + w^{\max}(t_0),$$
$$\check{f}(\xi) := -\xi - \beta g(\xi) + \beta L\tau[4|\beta| + w^{\max}(t_0) - w^{\min}(t_0)] + w^{\min}(t_0).$$

We consider the condition:

Condition (A1a): $L > 1/(-\beta) > 0,\ \tau \le 1/[L(4|\beta| + w^{\max}(t_0) - w^{\min}(t_0))]$.

Let $\lambda$ be a fixed number in interval $(0, 1)$. Under condition (A1a), $\beta < 0$, there exist two points $\tilde{p}_\lambda$ and $\tilde{q}_\lambda$, where $\tilde{p}_\lambda < \bar{p} < \bar{q} < \tilde{q}_\lambda$ such that

$$g^{'}(\tilde{p}_\lambda) = g^{'}(\tilde{q}_\lambda) = (1-\lambda)/(-\beta). \tag{3.21}$$

We introduce the following conditions:

Condition (A2a)$^\lambda$: $\check{f}(\tilde{q}_\lambda) > 0$, $\hat{f}(\tilde{p}_\lambda) < 0$, Condition $(\mathrm{A3a})^{\lambda}$: $g'(\xi) > (1+\lambda)/(-\beta)$, $\xi \in [\hat{m}, \check{m}]$.

Under conditions (A1a) and $(\mathrm{A2a})^{\lambda}$, there exist three zeros $\hat{l}$, $\hat{m}$ and $\hat{r}$ (resp. $\check{l}$, $\check{m}$ and $\check{r}$) of $\hat{f}$ (resp. $\check{f}$). Condition $(\mathrm{A3a})^{\lambda}$ prefers larger slope for the middle part of the activation function $g$. Now, let us introduce the result of a trichotomy for the dynamics of (3.20).

**Theorem 3.2.5.** Assume that conditions (A1a), $(\mathrm{A2a})^{\lambda}$ and $(\mathrm{A3a})^{\lambda}$ hold for some $\lambda \in (0,1)$. There exist three disjoint compact intervals $[\underline{a}, \bar{a}]$, $[\underline{b}, \bar{b}]$, $[\underline{c}, \bar{c}]$ and every solution of (3.20) converges to one of these intervals. Moreover

$$0 \le d_a := \overline{a} - \underline{a}, d_b := \overline{b} - \underline{b}, d_c := \overline{c} - \underline{c} \le \frac{w^{\max}(\infty) - w^{\min}(\infty)}{(1 - 2|\beta|L\tau)\lambda}.$$

At last, for convergent dynamics of (2.6), we consider the following scalar equation with time-dependent external input $E(t)$:

$$\dot{x}(t) = -\mu x(t) + \Sigma_{i=1}^{2}\gamma_i g(x(t-\tau_i)) + E(t), \tag{3.22}$$

where $\mu > 0, \gamma_1, \gamma_2 \in \mathbb{R}$; $\tau_1, \tau_2 \ge 0$; where $w(t)$ and $E(t)$ are bounded continuous functions defined for $t \ge t_0$, and $E(t) \to 0$ as $t \to \infty$; $g$ is an activation function of class $\mathcal{A}$ with $-1 < g(\xi) < 1$, $g'(\xi) \le 1 = g'(0) = \max\{g'(\eta) : \eta \in \mathbb{R}\}$, for all $\xi \in \mathbb{R}$. We denote $\tau := \max\{\tau_1, \tau_2\}$. By similar arguments to Section 3.2, we can conclude that there exist three points and every solution of (3.22) converges to one of them under the following condition:

Condition (Ab): $\Sigma_{i=1}^{2}\gamma_i > \mu$, $\Sigma_{i=1}^{2}|\gamma_i|\tau_i < \min\{1/4, [\Sigma_{i=1}^{2}\gamma_i g(\bar{q}_\gamma) - \mu\bar{q}_\gamma]/(4\Sigma_{i=1}^{2}|\gamma_i|),$ $[\mu\bar{p}_\gamma - \Sigma_{i=1}^{2}\gamma_i g(\bar{p}_\gamma)]/(4\Sigma_{i=1}^{2}|\gamma_i|)\}$, where

$$g'(\bar{p}_\gamma) = g'(\bar{q}_\gamma) = \mu/\Sigma_{i=1}^{2}\gamma_i. \tag{3.23}$$

**Theorem 3.2.6.** Assume that condition (Ab) holds. Let $x(t)$ be a solution of (3.22). Then there exist some $a < 0$ and $c > 0$ such that $x(t) \to a$, or 0 or $c$ as $t \to \infty$.

Notably, by modifying the formulation for Theorem 3.2.6, we can also derive $\tau_1$-independent or $\tau_2$-independent results.

**Remark 3.2.1.** The condition $\Sigma_{i=1}^{2}\gamma_i > \mu$ in Theorem 3.2.6 plays the dominant role for the convergence to multiple equilibrium points of (3.22). Indeed, if $\Sigma_{i=1}^{2}\gamma_i$ is small (smaller than $\mu$ basically) instead, then (3.22) will admit convergence to the origin.

### 3.2.3 Proofs of lemma, propositions and theorems

We provide the proofs for Lemma 3.2.1, Proposition 3.2.2, 3.2.3, Theorems 3.2.4-3.2.6 in this subsection. We only prove the case of $\beta > 0$ for Lemma 3.2.1, Proposition 3.2.2, 3.2.3 and Theorems 3.2.4; the case of $\gamma_i > 0$, $i = 1, 2$, for Theorem 3.2.6, as the arguments for the other cases are similar.

**Proof of Lemma 3.2.1.** The labelling in the proof corresponds to the one in the statement of Lemma 3.2.1.

(i) Let us show that for any $T \geq t_0$, $\check{b}^{(k)}(T)$ and $\hat{b}^{(k)}(T)$ are well-defined for all $k \in \mathbb{N} \cup \{0\}$. Assume that $\hat{b}^{(j-1)}(T)$, $\check{b}^{(j-1)}(T)$ have been defined, for a fixed $T \geq t_0$. Notably,

$$\begin{aligned}
\hat{f}_{\mathrm{m}}^{(j)}(\xi, T) &= -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(j-1)}(T)) + w^{\max}(T) \\
&\leq -\mu\xi + \alpha g(\xi) + \beta\rho + w^{\max}(t_0) = \hat{f}(\xi), \\
\check{f}_{\mathrm{m}}^{(j)}(\xi, T) &= -\mu\xi + \alpha g(\xi) - \beta g(\hat{b}^{(j-1)}(T)) + w^{\min}(T) \\
&\geq -\mu\xi + \beta g(\xi) - \beta\rho + w^{\min}(t_0) = \check{f}(\xi).
\end{aligned}$$

It follows that $\check{f}(\xi) \leq \check{f}_{\mathrm{m}}^{(j)}(\xi, T) \leq \hat{f}_{\mathrm{m}}^{(j)}(\xi, T) \leq \hat{f}(\xi)$, for all $\xi \in \mathbb{R}$. In addition, $\bar{p}$ and $\overline{q}$ are two critical points of $\check{f}(\cdot)$, $\hat{f}(\cdot)$, $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T)$, and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T)$, and $g'(\bar{p}) = g'(\bar{q}) = \mu/\alpha$, due to condition (A1c). There exists exactly one solution for each of $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ in interval $(\hat{m}, \check{m})$. Accordingly, both $\check{b}^{(j)}(T)$ and $\hat{b}^{(j)}(T)$ are well defined. Moreover, it is straightforward to observe that $\hat{b}^{(j)}(T_1) \geq \hat{b}^{(j)}(T_2)$ and $\check{b}^{(j)}(T_1) \leq \check{b}^{(j)}(T_2)$, due to $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T_1) \leq \hat{f}_{\mathrm{m}}^{(j)}(\cdot, T_2)$ and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T_1) \geq \check{f}_{\mathrm{m}}^{(j)}(\cdot, T_2)$, for any $T_1 \geq T_2 \geq t_0$. Thus, for each $k \in \mathbb{N} \cup \{0\}$, $\hat{b}^{(k)}(T)$ increases and $\check{b}^{(k)}(T)$ decreases, with respect to $T$. The arguments for $\check{a}^{(k)}(T)$, $\hat{a}^{(k)}(T)$, $\check{c}^{(k)}(T)$, $\hat{c}^{(k)}(T)$ are similar.

(ii) Let us show that for each $T \geq t_0$,

$$\hat{b}^{(k+1)}(T) \geq \hat{b}^{(k)}(T); \; \check{b}^{(k+1)}(T) \leq \check{b}^{(k)}(T), \text{ for all } k \geq 0. \tag{3.24}$$

Assume that (3.24) holds for some $k = j - 1$. Notably, $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$ satisfy $\hat{f}_{\mathrm{m}}^{(j+1)}(\cdot, T) = 0$ and $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ respectively; i.e.,

$$\begin{aligned}
-\mu\hat{b}^{(j+1)}(T) + \alpha g(\hat{b}^{(j+1)}(T)) + \beta g(\check{b}^{(j)}(T)) + w^{\max}(T) &= 0, &(3.25)\\
-\mu\hat{b}^{(j)}(T) + \alpha g(\hat{b}^{(j)}(T)) + \beta g(\check{b}^{(j-1)}(T)) + w^{\max}(T) &= 0. &(3.26)
\end{aligned}$$

The difference of (3.25) and (3.26) is

$$\mu[\hat{b}^{(j+1)}(T)-\hat{b}^{(j)}(T)]-\alpha g'(\xi)[\hat{b}^{(j+1)}(T)-\hat{b}^{(j)}(T)] = \beta g'(\zeta)[\check{b}^{(j)}(T)-\check{b}^{(j-1)}(T)], \quad (3.27)$$

where $\xi$ (resp. $\zeta$) is a number between $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$ (resp. $\hat{b}^{(j)}(T)$ and $\hat{b}^{(j-1)}(T)$). (3.27) then yields

$$\hat{b}^{(j+1)}(T) - \hat{b}^{(j)}(T) = \frac{\beta g'(\zeta)[\check{b}^{(j)}(T) - \check{b}^{(j-1)}(T)]}{\mu - \alpha g'(\xi)} \geq 0,$$

due to that $g'(\xi) > \mu/\alpha$ for $\xi$ between $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$. Thus, the first part of (3.24) holds for $k = j$. The second part can be proved similarly. It follows that for any $T \geq t_0$, $\lim_{k\to\infty} \hat{b}^{(k)}(T) = \underline{b}(T) \in \mathbb{R}$, and $\lim_{k\to\infty} \check{b}^{(k)}(T) = \overline{b}(T) \in \mathbb{R}$ respectively, since both of $\check{b}^{(k)}(T)$ and $\hat{b}^{(k)}(T)$ are bounded monotone sequences. The situations for $\check{a}^{(k)}(T), \hat{a}^{(k)}(T)$, and $\check{c}^{(k)}(T)$, $\hat{c}^{(k)}(T)$ are similar.

(iii) For each $k \in \mathbb{N} \cup \{0\}$, it has been shown in (i) that $\hat{b}^{(k)}(T_2) \leq \hat{b}^{(k)}(T_1)$, if $T_1 > T_2 \geq t_0$. Thus, $\lim_{k\to\infty} \hat{b}^{(k)}(T_2) \leq \lim_{k\to\infty} \hat{b}^{(k)}(T_1)$, i.e. $\underline{b}(T_2) \leq \underline{b}(T_1)$. Therefore, $\underline{b}(T) \to \underline{b} \in \mathbb{R}$ increasingly as $T \to \infty$, since $\underline{b}(T)$ is bounded above for all $T \geq t_0$. Similarly, $\overline{b}(T) \to \overline{b} \in \mathbb{R}$ decreasingly as $T \to \infty$. Similar proofs apply to $\underline{a}(T) \to \underline{a}$, $\overline{a}(T) \to \overline{a}$, $\underline{c}(T) \to \underline{c}$, and $\overline{c}(T) \to \overline{c}$.

(iv) It is straightforward to see that $\cap_{T\geq t_0}[\underline{b}(T), \overline{b}(T)] = [\underline{b}, \overline{b}]$, $\cap_{T\geq t_0}[\underline{a}(T), \overline{a}(T)] = [\underline{a}, \overline{a}]$, $\cap_{T\geq t_0}[\underline{c}(T), \overline{c}(T)] = [\underline{c}, \overline{c}]$.

(v) It is obvious that $\overline{b}(T) - \underline{b}(T) \geq 0$, since $\check{b}^{(k)}(T) > \hat{b}^{(k)}(T)$ for any $k \in \mathbb{N} \cup \{0\}$, and any $T \geq t_0$. Next, we justify that $\overline{b}(T) - \underline{b}(T) \leq [w^{\max}(T) - w^{\max}(T)]/(\mu - |\beta|L)$, for any $T \geq t_0$. For such an assertion, we shall construct a mapping $\Gamma_T : H_T \to H_T$, for each $T \geq t_0$, where $H_T := [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \times [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \cap \{(y_1, y_2)|y_1 \leq y_2\} \subset \mathbb{R}^2$ and such a mapping is a contraction, mainly due to $g' > 2\mu/\alpha$, on $[\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$. The map $\Gamma_T$ thus admits an unique fixed point $(\underline{b}(T), \overline{b}(T))$. The difference of $\underline{b}(T)$ and $\overline{b}(T)$ can then be estimated to yield the assertion. Let us elaborate. For each $T \geq t_0$, we define the following functions:

$$\begin{aligned} h_T^{\max}(\xi, \gamma) &:= -\mu\xi + \alpha g(\xi) + \beta g(\gamma) + w^{\max}(T), \\ h_T^{\min}(\xi, \gamma) &:= -\mu\xi + \alpha g(\xi) + \beta g(\gamma) + w^{\min}(T). \end{aligned}$$

Notably, $\check{f}_{\rm m}^{(1)}(\xi, T) \leq h_T^{\min}(\xi, \gamma_1) \leq h_T^{\max}(\xi, \gamma_2) \leq \hat{f}_{\rm m}^{(1)}(\xi, T)$, if $\hat{b}^{(0)}(T) \leq \gamma_1 \leq \gamma_2 \leq \check{b}^{(0)}(T)$. For $(\xi, \gamma) \in H_T$, we define $\Gamma_T(\xi, \gamma) = (\xi_s, \gamma_s)$, where $\xi_s$ (resp. $\gamma_s$) is the unique point lying in $[\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$ satisfying $h_T^{\max}(\xi_s, \gamma) = 0$ (resp. $h_T^{\min}(\gamma_s, \xi) =$

0). Suppose $h_T^{\max}(\xi_s,\gamma)=0$, $h_T^{\max}(\xi_s',\gamma')=0$, then we derive

$$\mu(\xi_s'-\xi_s)-\alpha g^{'}(\eta)(\xi_s'-\xi_s)+\beta[g(\gamma)-g(\gamma')]=0,$$

where $\eta$ is between $\xi_s'$ and $\xi_s$. Subsequently, $|\xi_s'-\xi_s|\le|\beta|L|\gamma'-\gamma|/[\alpha(2\mu/\alpha)-\mu]=|\beta|L|\gamma'-\gamma|/\mu$, thanks to $g'(\eta)>2\mu/\alpha$, for $\eta\in[\hat{b}^{(0)}(T),\check{b}^{(0)}(T)]\subset[\hat{b}^{(0)}(t_0),\check{b}^{(0)}(t_0)]=[\hat{m},\check{m}]$. Similarly, we can prove that $|\gamma_s'-\gamma_s|\le|\beta|L|\xi-\xi'|/\mu$, if $h_T^{\min}(\gamma_s,\xi)=0$, $h_T^{\min}(\gamma_s',\xi')=0$. We thus establish

$$\|\Gamma_T(\xi,\gamma)-\Gamma_T(\xi',\gamma')\|_\infty=\|(\xi_s,\gamma_s)-(\xi_s',\gamma_s')\|_\infty\le\frac{|\beta|L}{\mu}\|(\xi,\gamma)-(\xi',\gamma')\|_\infty.$$

$\Gamma_T$ is thus a contracting mapping under our condition (A2c): $L<\mu/|\beta|$. Thus, there exists an unique fixed point of $\Gamma_T$ in $H_T$. Observe that $\Gamma_T^k(\hat{b}^{(0)}(T),\check{b}^{(0)}(T))=(\hat{b}^{(k)}(T),\check{b}^{(k)}(T))$, which converges to $(\underline{b}(T),\overline{b}(T))$ as $k\to\infty$. Thus $(\underline{b}(T),\overline{b}(T))\in H_T$ is the fixed point of $\Gamma_T$ and

$$\begin{aligned}
-\mu\underline{b}(T)+\alpha g(\underline{b}(T))+\beta g(\overline{b}(T))+w^{\max}(T) &= 0,\\
-\mu\overline{b}(T)+\alpha g(\overline{b}(T))+\beta g(\underline{b}(T))+w^{\min}(T) &= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{b}(T)-\underline{b}(T) &= [w^{\max}(T)-w^{\min}(T)]/[\alpha g'(\xi)-|\beta|g'(\xi)-\mu]\\
&\le [w^{\max}(T)-w^{\min}(T)]/[\mu-|\beta|g'(\xi)]\\
&\le [w^{\max}(T)-w^{\min}(T)]/[\mu-|\beta|L],
\end{aligned}$$

due to condition (A4c): $g'(\xi)\ge 2\mu/\alpha$ for $\xi\in[\underline{b}(T),\overline{b}(T)]\subset[\hat{b}^{(0)}(T),\check{b}^{(0)}(T)]\subset[\hat{m},\check{m}]$, condition (A2c): $\mu-|\beta|L>0$, and $g'(\xi)\le L$, for all $\xi$. Moreover, $\overline{b}-\underline{b}\le\overline{b}(T)-\underline{b}(T)\le[w^{\max}(T)-w^{\min}(T)]/[\mu-|\beta|L]$, for any $T\ge t_0$. We thus establish

$$0\le d_b:=\overline{b}-\underline{b}\le[w^{\max}(\infty)-w^{\min}(\infty)]/[\mu-|\beta|L].$$

The estimate for $\overline{c}(T)-\underline{c}(T)$ follows from

$$\mu[\overline{c}(T)-\underline{c}(T)]-\alpha g'(\xi)[\overline{c}(T)-\underline{c}(T)]-\beta g'(\xi)[\overline{c}(T)-\underline{c}(T)]+w^{\max}(T)-w^{\min}(T)=0,$$

for some $\xi\in[\check{c}^{(0)}(T),\hat{c}^{(0)}(T)]$, and

$$\overline{c}(T)-\underline{c}(T)\le\frac{w^{\max}(T)-w^{\min}(T)}{\mu-(\alpha+|\beta|)g'(\tilde{q})}=\frac{w^{\max}(T)-w^{\min}(T)}{|\beta|L}.$$

The estimate for $\overline{a}(T) - \underline{a}(T)$ is similar. The bounds for $\overline{a} - \underline{a}$, and $\overline{c} - \underline{c}$ can then be derived.

**Proof of Proposition 3.2.2.**

(i) Assume that $x(t) \in [\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)]$, for all $t \geq T + (k-1)\tau$. Then it is not difficult to derive that $\check{f}_{\rm m}^{(k)}(x(t), T) \leq \dot{x}(t) \leq \hat{f}_{\rm m}^{(k)}(x(t), T)$ for $t \geq T + k\tau$. Therefore, if the assertion does not hold, $x(t)$ eventually leaves $[\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)]$ after $t = T + k\tau$, and yields a contradiction, cf. Fig. 3.5. For a detailed proof, let us suppose the assertion does not hold, then there exists some $s \geq T + k\tau$ such that $x(s) \in [\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)] - [\hat{b}^{(k)}(T), \check{b}^{(k)}(T)]$. Suppose that $x(s) \in (\check{b}^{(k)}(T), \check{b}^{(k-1)}(T)] \neq \emptyset$ (the case $x(s) \in (\hat{b}^{(k-1)}(T), \hat{b}^{(k)}(T)] \neq \emptyset$ can be similarly discussed). Notably, $\check{f}_{\rm m}^{(k)}(\xi, T) := -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T) \geq \check{f}_{\rm m}^{(k)}(x(s), T) =: h_1 > 0$, for all $\xi \in [x(s), \check{b}^{(k-1)}(T)]$, in respecting the definition of $\check{f}_{\rm m}^{(k)}(\xi, T)$, cf. Fig. 3.5. In addition,

$$\begin{aligned} \dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s - \tau_1(s))) + w(s) \\ &\geq -\mu x(s) + \alpha g(x(s)) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T) \\ &= \check{f}_{\rm m}^{(k)}(x(s), T) = h_1 > 0, \end{aligned}$$

due to $s - \tau_1(s) \geq T + (k-1)\tau$. Therefore, $x(t)$ enters into $(x(s), \check{b}^{k-1}(T)]$ after $t = s$, and will never go back into $(-\infty, x(s)]$ again. Indeed, if there exists a time $s_1 > s$, such that $x(t) \in (x(s), \check{b}^{(k-1)}(T))$ for all $t \in (s, s_1)$, and $x(s_1) = x(s)$, then,

$$\begin{aligned} x(s_1) - x(s) &= \dot{x}(\tilde{s})(s_1 - s) \\ &= [-\mu x(\tilde{s}) + \alpha g(x(\tilde{s})) + \beta g(x(\tilde{s} - \tau_1(\tilde{s}))) + w(\tilde{s})](s_1 - s) \\ &\geq [-\mu x(\tilde{s}) + \alpha g(x(\tilde{s})) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T)](s_1 - s) \\ &= [\check{f}_{\rm m}^{(k)}(x(\tilde{s}), T)](s_1 - s) \geq h_1 \cdot (s_1 - s) > 0, \end{aligned}$$

for some $\tilde{s} \in (s, s_1)$, which is a contradiction. Thus, $x(t)$ stays in $[x(s), \check{b}^{k-1}(T)]$ for all $t \geq s$ with $\dot{x}(t) \geq h_1 > 0$. This is impossible and we conclude that $x(t) \in [\hat{b}^{(k)}(T), \check{b}^{(k)}(T)]$ for all $t \geq T + k\tau$.

(ii) We only prove the $\mathcal{R}$ case. This property holds mainly due to $\check{f}^{(0)}(x(t), T) \leq \dot{x}(t) \leq \hat{f}^{(0)}(x(t), T)$, for $t \geq T$. Therefore, if $x(s) \in (\check{b}^{(0)}(T), \infty)$ (resp. $(-\infty, \hat{b}^{(0)}(T))$) for some $s \geq T$, then $x(t)$ eventually enters into $[\check{r}, \hat{r}]$ (resp. $[\check{l}, \hat{l}]$), cf. Fig 3.4. Let us give detailed arguments. If $x(s) \in (\check{b}^{(0)}(T), \check{r})$, then $h_0 := \min\{\check{f}^{(0)}(x(s), s), \check{f}^{(0)}(\check{r}, s)\} > 0$, and $\check{f}^{(0)}(\xi, s) \geq h_0$, for $\xi \in [x(s), \check{r}]$, as observed from the graph of $\check{f}^{(0)}(\cdot, s)$ in

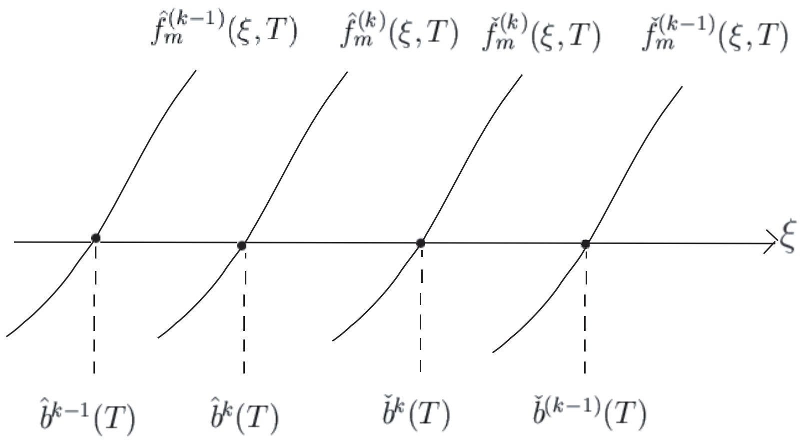



Figure 3.5: Configuration for the proof of Proposition 3.2.2 (i), for some $T \geq t_0$.

Fig. 3.6. In addition,

$$\begin{aligned}
\dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s-\tau_1(s))) + w(s), \\
&\geq -\mu x(s) + \alpha g(x(s)) - \beta\rho + w^{\min}(s), \\
&= \check{f}^{(0)}(x(s), s) \geq h_0.
\end{aligned}$$

Thus, $x(t)$ is increasing with a positive rate should it remain in $(\check{b}^{(0)}(T), \check{r})$. On the other hand, if $x(s) > \hat{r}$ (Fig. 3.6),

$$\begin{aligned}
\dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s-\tau_1(s))) + w(s) \\
&\leq -\mu x(s) + \alpha g(x(s)) + \beta\rho + w^{\max}(s) \\
&= \hat{f}^{(0)}(x(s), s) < 0.
\end{aligned}$$

Thus, $x(t)$ eventually enters into $[\check{r}, \hat{r}]$.

(iii) Let us show that $x(t) \to [\underline{b}(T), \overline{b}(T)]$, for any $T \geq t_0$. Assume otherwise that $x(t)$ does not converge to $[\underline{b}(T), \overline{b}(T)]$ as $t \to \infty$, for some $T \geq t_0$. Then, there exist $\varepsilon > 0$ and an increasing time sequence $\{t_n\}$ tending to $+\infty$, such that $x(t_n)$ does not belong to $[\underline{b}(T) - \varepsilon, \overline{b}(T) + \varepsilon]$ for all $n$. This contradicts to that for each $k \in \mathbb{N} \cup \{0\}$, $T > t_0$, $x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$ for all $t \geq T + k\tau$, by the assumption (Property $\mathcal{M}$), and that $\hat{b}^k(T)$ converges to $\underline{b}(T)$ increasingly, $\check{b}^k(T)$ converges to $\overline{b}(T)$ decreasingly, as $k \to \infty$. Moreover, since $\underline{b}(T)$ tends to $\underline{b}$ increasingly and $\overline{b}(T)$ tends to $\overline{b}$ decreasingly, as $T \to \infty$, we conclude that $x(t) \to [\underline{b}, \overline{b}]$, as $t \to \infty$.

(iv) First, both $[\check{l}, \hat{l}]$ and $[\check{r}, \hat{r}]$ are positively invariant sets for system (3.17) mainly because that $\check{f}(x(t)) \leq \dot{x}(t) \leq \hat{f}(x(t))$ for all $t \geq t_0$, cf. Fig. 3.4. More

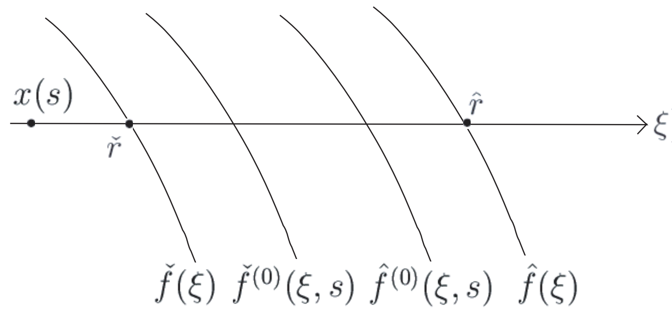



Figure 3.6: Configuration for the proof of Proposition 3.2.2 (ii), for some $s \geq T$.

precisely, assume that there exists $s \geq t_0$ such that $x(t) \in [\check{r}, \hat{r}]$ for $t_0 \leq t \leq s$ and $x(t_1) \notin [\check{r}, \hat{r}]$ for some $t_1 > s$. Let $s_1$ be the first time after time $s$ such that $x(s_1) = \check{r}$, and $x(t)$ leaves $[\check{r}, \hat{r}]$ after time $s_1$ and enters into $(-\infty, \check{r})$, without loss of generality. Then there exists $s_2 > s_1$ such that $\check{m} < x(t) < \check{r}$ for $t \in (s_1, s_2)$. A contradiction then arises as

$$\begin{aligned} x(s_2) - x(s_1) &= \dot{x}(s_3)(s_2 - s_1) \\ &= [-\mu x(s_3) + \alpha g(x(s_3)) + \beta g(x(s_3 - \tau_1(s_3))) + w(s_3)](s_2 - s_1) \\ &\geq \check{f}(x(s_3))(s_2 - s_1) > 0, \end{aligned}$$

for some $s_3 \in (s_1, s_2)$. Similar contradiction occurs if we consider $x(s_1) = \hat{r}$ and $x(t)$ enters into $(\hat{r}, \infty)$. The proof for positive invariance of $[\check{l}, \hat{l}]$ is similar.

Next, we assume that $x(t)$ satisfies Property $\mathcal{R}$, namely, there exists $s \geq t_0$ such that $x(s) \in [\check{r}, \hat{r}]$. We assert that for each $T \geq t_0$,

$$x(t) \to [\check{c}^{(k)}(T), \hat{c}^{(k)}(T)], \text{ as } t \to \infty, \text{ for all } k \geq 0. \tag{3.28}$$

We justify (3.28) by induction. Let $s_T := \max\{s, T\}$. It can be concluded that if $x(t_1) \in [\check{c}^{(0)}(T), \hat{c}^{(0)}(T)]$ for some $t_1 \geq s_T$, then $x(t) \in [\check{c}^{(0)}(T), \hat{c}^{(0)}(T)]$ for all $t \geq t_1$, by arguments similar to the previous ones for proving that $[\check{r}, \hat{r}]$ is positively invariant. If $x(t) \in [\check{r}, \check{c}^{(0)}(T))$, for all $t \geq s_T$, then

$$\begin{aligned} \dot{x}(t) &= -\mu x(t) + \alpha g(x(t)) + \beta g(x(t - \tau_1(t))) + w(t) \\ &\geq -\mu x(t) + \alpha g(x(t)) - \beta \rho + w^{\min}(T) \\ &= \check{f}^{(0)}(x(t), T) > 0, \end{aligned}$$

and yields a contradiction. Similarly, it can not hold that $x(t) \in (\hat{c}^{(k)}(T), \hat{r}]$, for all $t \geq s_T$. Hence, (3.28) holds for $k = 0$. Now, we assume that (3.28) holds for $k = j-1$, i.e., $x(t) \to [\check{c}^{(j-1)}(T), \hat{c}^{(j-1)}(T)]$, as $t \to \infty$. Let us illustrate that it also holds for $k = j$. Consider a point $x_U$ arbitrarily close to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$, and assume $x_U \leq \check{c}^{(j)}(T)$; there exists a function, say $f_U$, which is a vertical-shift of $\hat{f}_{\mathrm{r}}^{(0)}(\cdot, T)$ and $f_U$ has an unique zero at $x_U$, cf. Fig. 3.7. It can be derived that $\dot{x}(t) \geq f_U(x(t))$, as $t$ is large enough. Subsequently, it follows that $x(t)$ must become closer to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$ than to $x_U$, as $t \to \infty$. (3.28) thus holds for $k = j$. The arguments for $x_U \geq \hat{c}^{(j)}(T)$ and $\dot{x}(t) \leq f_U(x(t))$ are similar. Let us give detailed arguments. Assume that $x(t)$ does not converge to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$. Then, without loss of generality, there exist an $\varepsilon > 0$ and a time sequence $\{t_n\}$ with $t_n \geq s_T$ and $t_n \to \infty$, as $n \to \infty$, such that

$$x(t_n) \in [\check{r}, \check{c}^{(j)}(T) - \varepsilon); \tag{3.29}$$

moreover, $\check{c}^{(j)}(T) > \check{c}^{(j-1)}(T)$. Notably, $\check{c}^{(j)}(T)$ is the unique solution of the equation $-\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(j-1)}(T)) + w^{\min}(T) = 0$, which lies in $[\check{r}, \hat{r}]$. Thus, there exist $\delta_\varepsilon > 0$ and $x_U \in [\check{c}^{(j)}(T) - \frac{\varepsilon}{2}, \check{c}^{(j)}(T) + \frac{\varepsilon}{2}]$ such that $x_U$ is the unique solution of $f_U(\xi) := -\mu\xi + \alpha g(\xi) + \beta g(c_U) + w^{\min}(T) = 0$, where $c_U := \min\{\xi : \xi \in U\}$, $U := [\check{c}^{(j-1)}(T) - \delta_\varepsilon, \check{c}^{(j-1)}(T) + \delta_\varepsilon] \cap [\check{r}, \hat{r}]$, by continuity, cf. Fig. 3.7. On the other hand, there exists $\tilde{t}$ large enough such that $x(t) \geq c_U$, for all $t \geq \tilde{t}$, since $x(t)$ converges to $[\check{c}^{(j-1)}(T), \hat{c}^{(j-1)}(T)]$. It follows that

$$\begin{aligned} \dot{x}(t_N) &= -\mu x(t_N) + \alpha g(x(t_N)) + \beta g(x(t_N - \tau_1(t_N))) + w(t_N) \\ &\geq -\mu x(t_N) + \alpha g(x(t_N)) + \beta g(c_U) + w^{\min}(T) > 0, \end{aligned}$$

for some $t_N \geq \tilde{t} + \tau$, since $x(t_N) < \check{c}^{(j)}(T) - \varepsilon < x_U$. Moreover,

$$\begin{aligned} \dot{x}(t) &= -\mu x(t) + \alpha g(x(t)) + \beta g(x(t - \tau_1(t))) + w(t) \\ &\geq -\mu x(t) + \alpha g(x(t)) + \beta g(c_U) + w^{\min}(T) = f_U(x(t)) > 0, \end{aligned}$$

if $t \geq t_N$ and $x(t) \in (x(t_N), x_U)$. Therefore, $x(t)$ is increasing until it reaches $x_U$ and never goes back into $[\check{r}, \check{c}^{(j)}(T) - \varepsilon)$. This yields a contradiction to (3.29). We have therefore justified that (3.28) holds. Consequently, $x(t)$ converges to $[\underline{c}(T), \overline{c}(T)]$ for all $T \geq t_0$, and thus converges to $[\underline{c}, \overline{c}]$, as $t \to \infty$. The proof for $x(t)$ satisfying Property $\mathcal{L}$ and converging to $[\underline{a}, \overline{a}]$ is similar.

**Proof of Theorem 3.2.5.** We recompose the upper and lower functions in the

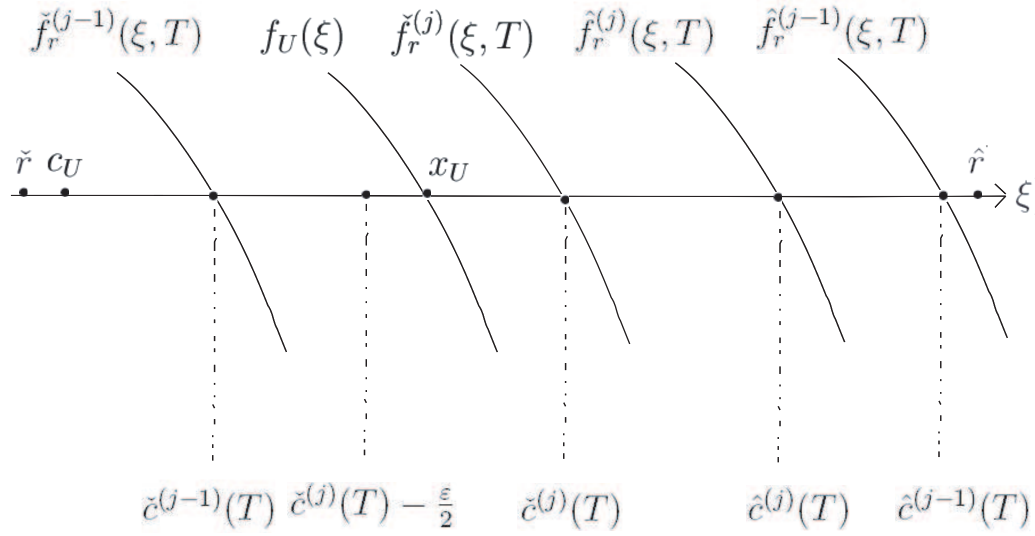



Figure 3.7: Configuration for the proof of Proposition 3.2.2 (iv), with fixed $T$.

formulation for Theorem 3.2.4 as follows:

$$\hat{f}^{(0)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}(\hat{A}^f) + w^{\max}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}(\check{A}^f) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}^{(0)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}(\check{A}^f) + w^{\min}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}(\hat{A}^f) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

$$\hat{f}_{\mathrm{l}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{l}}^{(k-1)}(\hat{a}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{l}}^{(k-1)}(\check{a}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{l}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{l}}^{(k-1)}(\check{a}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{l}}^{(k-1)}(\hat{a}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

$$\hat{f}_{\mathrm{m}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{m}}^{(k-1)}(\hat{b}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{m}}^{(k-1)}(\check{b}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{m}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{m}}^{(k-1)}(\check{b}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{m}}^{(k-1)}(\hat{b}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

$$\hat{f}_{\mathrm{r}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{r}}^{(k-1)}(\hat{c}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{r}}^{(k-1)}(\check{c}^{(k-1)}(T),T) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{r}}^{(k)}(\xi,T) := \begin{cases} -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\hat{f}_{\mathrm{r}}^{(k-1)}(\check{c}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta \geq 0; \\ -\mu\xi + (\alpha+\beta)g(\xi) - \beta L\tau\check{f}_{\mathrm{r}}^{(k-1)}(\hat{c}^{(k-1)}(T),T) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

where $\hat{A}^f$ (resp. $\check{A}^f$) is zero to $\hat{f}$ (resp. $\check{f}$). Then the proof follows from similar process as Theorem 3.2.4.

**Proof of Theorem 3.2.6.** We recompose the upper and lower functions in the formulation for Theorem 3.2.4 as follows:

$$
\begin{aligned}
&\hat{f}(\xi) := -\mu\xi + 2\Sigma_{i=1}^{2}|\gamma_i|,\ \check{f}(\xi) := -\mu\xi - 2\Sigma_{i=1}^{2}|\gamma_i|.\\
&\hat{f}^{(0)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) + (\Sigma_{i=1}^{2}|\gamma_i|\tau_i)(4\Sigma_{i=1}^{2}|\gamma_i|),\\
&\check{f}^{(0)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}|\gamma_i|\tau_i)(4\Sigma_{i=1}^{2}|\gamma_i|).\\
&\hat{f}_{\mathrm{l}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\check{f}_{\mathrm{l}}^{(k-1)}(\hat{a}^{(k-1)}),\\
&\check{f}_{\mathrm{l}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\hat{f}_{\mathrm{l}}^{(k-1)}(\check{a}^{(k-1)}),\\
&\hat{f}_{\mathrm{m}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\check{f}_{\mathrm{m}}^{(k-1)}(\hat{b}^{(k-1)}),\\
&\check{f}_{\mathrm{m}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\hat{f}_{\mathrm{m}}^{(k-1)}(\check{b}^{(k-1)}),\\
&\hat{f}_{\mathrm{r}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\check{f}_{\mathrm{r}}^{(k-1)}(\hat{c}^{(k-1)}),\\
&\check{f}_{\mathrm{r}}^{(k)}(\xi) := -\mu\xi + \Sigma_{i=1}^{2}\gamma_i g(\xi) - (\Sigma_{i=1}^{2}\gamma_i\tau_i)\hat{f}_{\mathrm{r}}^{(k-1)}(\check{c}^{(k-1)}).
\end{aligned}
$$

Then the proof follows from similar process as Theorem 3.2.4.

# Chapter 4

# Multistability for Hopfield-type Network

Most of the materials in this chapter has been published in [61]. In this chapter, we consider system (2.3):

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + J_i,$$

where $i = 1, 2, \cdots, n$; $g_i$ is the activation function of class $\mathcal{A}$. We establish the existence of $3^n$ equilibria for system (2.3) in Section 4.1. The main theorems of convergence of dynamics and stability of equilibria for system (2.3) are presented in Section 4.2 and 4.3 respectively. We demonstrate the theory by a numerical example in Section 4.4.

## 4.1 Existence of multiple equilibria

Let us introduce the following upper and lower bounds for each component of system (2.3):

$$\begin{aligned} \hat{F}_i(\xi) &:= -\mu_i \xi + \alpha_{ii} g_i(\xi) + \sum_{j \neq i} |\alpha_{ij}| \rho_j + \sum_{j=1}^{n} |\beta_{ij}| \rho_j + J_i, \\ \check{F}_i(\xi) &:= -\mu_i \xi + \alpha_{ii} g_i(\xi) - \sum_{j \neq i} |\alpha_{ij}| \rho_j - \sum_{j=1}^{n} |\beta_{ij}| \rho_j + J_i, \end{aligned}$$

where $\rho_i$ are the bounds for activation functions $g_i$, defined in (2.4).

Recall that $L_i$ is the largest slope of activation function $g_i$ at its inflection point, as defined in (2.4). We consider the following conditions which are the multi-dimensional versions of conditions (A1c),(A2c).

Condition (C1c): $L_i > 2\mu_i/\alpha_{ii} > 0$, for $i = 1, 2, \cdots, n$,

Condition (C2c): $\mu_i > \mu_i - L_i|\beta_{ii}| > \sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j\neq i} L_j|\beta_{ij}|$, for $i = 1, 2, \cdots, n$.

Notably, condition (C1c) implies $\alpha_{ii} > 0$, and the first inequality in condition (C2c) is equivalent to $\beta_{ii} \neq 0$, for all $i$. The discussions on critical points of $f$, $\hat{f}$ and $\check{f}$ and their vertical shifts in Section 3.2.1 are valid for $\hat{F}_i$, $\check{F}_i$, $i = 1, 2, \cdots, n$, as well as their vertical shifts. Accordingly, under condition (C1c), there exist critical points $\bar{p}_i$ and $\bar{q}_i$ of $\hat{F}_i$, $\check{F}_i$, which satisfy $g_i'(\bar{p}_i) = g_i'(\bar{q}_i) = \mu_i/\alpha_{ii}$. In addition, $\hat{F}_i$ and $\check{F}_i$ are strictly increasing in $(-\infty, \bar{p}_i)$, $(\bar{q}_i, \infty)$, and strictly decreasing in $(\bar{p}_i, \bar{q}_i)$, for $i = 1, 2, \cdots, n$. On the other hand,

$$0 < [\mu_i - (\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|)]/(\alpha_{ii} + |\beta_{ii}|) < \mu_i/(\alpha_{ii} + |\beta_{ii}|), \tag{4.1}$$

under conditions (C2c). Hence, there always exist exactly two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \bar{p}_i < \bar{q}_i < \tilde{q}_i$ such that

$$g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = [\mu_i - (\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|)]/(\alpha_{ii} + |\beta_{ii}|), \tag{4.2}$$

for $i = 1, 2, \cdots, n$. Next, we introduce

Condition (C3c): $\check{F}_i(\tilde{q}_i) > 0$ and $\hat{F}_i(\tilde{p}_i) < 0$, for all $i = 1, 2, \cdots, n$.

Under condition (C3c), there exist three solutions $\hat{l}_i^F$, $\hat{m}_i^F$, and $\hat{r}_i^F$ (resp. $\check{l}_i^F$, $\check{m}_i^F$ and $\check{r}_i^F$) to $\hat{F}_i(\cdot) = 0$ (resp. $\check{F}_i(\cdot) = 0$), for each $i = 1, 2, \cdots, n$. Moreover, $\check{l}_i^F < \hat{l}_i^F < \tilde{p}_i < \hat{m}_i^F < \check{m}_i^F < \tilde{q}_i < \check{r}_i^F < \hat{r}_i^F$. The following condition is the multi-dimensional version of condition (A4c).

Condition (C4c): $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ for all $\xi \in [\hat{m}_i^F, \check{m}_i^F]$, $i = 1, 2, \cdots, n$.

Let us introduce the following sets in $\mathbb{R}^n$

$$\begin{aligned}
\Omega_{\lambda_1\lambda_2\cdots\lambda_n} &= \Omega_1^{\lambda_1} \times \Omega_2^{\lambda_2} \times \cdots \times \Omega_n^{\lambda_n},\ \lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\},\ i = 1, 2, \cdots, n,\\
\tilde{\Omega}_{\lambda_1\lambda_2\cdots\lambda_n} &= \tilde{\Omega}_1^{\lambda_1} \times \tilde{\Omega}_2^{\lambda_2} \times \cdots \times \tilde{\Omega}_n^{\lambda_n},\ \lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\},\ i = 1, 2, \cdots, n,
\end{aligned}$$

which are defined through the following intervals

$$\begin{aligned}
&\Omega_i^{\mathrm{l}} = [\check{l}_i^F, \hat{l}_i^F],\ \Omega_i^{\mathrm{m}} = [\hat{m}_i^F, \check{m}_i^F],\ \Omega_i^{\mathrm{r}} = [\check{r}_i^F, \hat{r}_i^F],\\
&\tilde{\Omega}_i^{\mathrm{l}} = (-\infty, \hat{m}_i^F),\ \tilde{\Omega}_i^{\mathrm{m}} = \Omega_i^{\mathrm{m}},\ \tilde{\Omega}_i^{\mathrm{r}} = (\check{m}_i^F, \infty).
\end{aligned}$$

Herein, "l", "m", "r" represent respectively left, middle, and right. Through applying the contraction mapping principle, we derive the existence of $3^n$ equilibria for system (2.3).

**Theorem 4.1.1.** There exist exactly $3^n$ equilibria for system (2.3) under conditions (C2c)-(C4c). Each region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ contains exactly one of these $3^n$ equilibria.

**Proof.** We will show that there exists exactly one equilibrium point in each $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$. Consider a fixed $\Omega = \Omega_{\lambda_1\lambda_2\cdots\lambda_n}$. Set $f_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi)$. For a given $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \Omega$, we define

$$h_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi) + \sum_{j=1,j\neq i}^{n} \alpha_{ij}g_j(y_j) + \sum_{j=1}^{n} \beta_{ij}g_j(y_j) + J_i,$$

for $\xi \in \mathbb{R}$, $i = 1, 2, \cdots, n$. Note that $\check{F}_i(\xi) \leq h_i(\xi) \leq \hat{F}_i(\xi)$, and all functions $\check{F}_i$, $h_i$, $\hat{F}_i$ are vertical-shifts of $f_i$. Thus, there exists an unique solution $y_i^*$ to equation $h_i(\cdot) = 0$, lying in $\Omega_i^{\lambda_i}$. We define a mapping $G_\Omega : \Omega \to \Omega$ by $G_\Omega(\mathbf{y}) = \mathbf{y}^*$, where $\mathbf{y}^* = (y_1^*, y_2^*, \cdots, y_n^*)$. Then $G_\Omega$ is continuous and we shall illustrate that it is a contraction map. Assume that $G_\Omega(\mathbf{y}) = \mathbf{y}^*$, $G_\Omega(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2, \cdots, n$

$$-\mu_i y_i^* + \alpha_{ii}g_i(y_i^*) + \sum_{j=1,j\neq i}^{n} \alpha_{ij}g_j(y_j) + \sum_{j=1}^{n} \beta_{ij}g_j(y_j) + J_i = 0,$$

$$-\mu_i x_i^* + \alpha_{ii}g_i(x_i^*) + \sum_{j=1,j\neq i}^{n} \alpha_{ij}g_j(x_j) + \sum_{j=1}^{n} \beta_{ij}g_j(x_j) + J_i = 0.$$

Then

$$(x_i^* - y_i^*)[\mu_i - \alpha_{ii}g_i'(\xi_i^*)] - \sum_{j=1,j\neq i}^{n} \alpha_{ij}g_j'(\eta_j^*)[x_j - y_j] - \sum_{j=1}^{n} \beta_{ij}g_j'(\eta_j^*)[x_j - y_j] = 0, \quad (4.3)$$

where $\xi_i^*$ is some number between $x_i^*$ and $y_i^*$; $\eta_j^*$ is some number between $x_j$ and $y_j$.

(i) If $\lambda_i =$"m", then $x_i^*, y_i^*, \xi_i^* \in [\hat{m}_i^F, \check{m}_i^F]$, and $g_i'(\xi_i^*) > 2\mu_i/\alpha_{ii}$, by condition (C4c). Hence

$$\begin{aligned}
|x_i^* - y_i^*| &= |\sum_{j=1,j\neq i}^{n} \alpha_{ij}g_j'(\eta_j^*)(x_j - y_j) + \sum_{j=1}^{n} \beta_{ij}g_j'(\eta_j^*)(x_j - y_j)|/|\alpha_{ii}g_i'(\xi_i^*) - \mu_i| \\
&\leq \{[\sum_{j=1,j\neq i}^{n} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|]/\mu_i\}\|\mathbf{x} - \mathbf{y}\|_\infty \\
&=: \tilde{\gamma}_i\|\mathbf{x} - \mathbf{y}\|_\infty,
\end{aligned}$$

and $0 < \tilde{\gamma}_i < 1$, owing to condition (C2c).

(ii) If $\lambda_i =$"r", then $x_i^*, y_i^* \in [\check{r}_i^F, \hat{r}_i^F]$, and $\xi_i^* > \tilde{q}_i$ (for $\tilde{q}_i < \check{r}_i^F$). Thus, $0 \le g_i^{'}(\xi_i^*) < [\mu_i - (\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^n L_j|\beta_{ij}|)]/[\alpha_{ii} + |\beta_{ii}|] < \mu_i/\alpha_{ii}$, as mentioned in (4.1). It follows that

$$\begin{aligned} |\alpha_{ii}g_i'(\xi_i^*) - \mu_i| &= \mu_i - \alpha_{ii}g_i'(\xi_i^*) \\ &> \mu_i - \alpha_{ii}[\mu_i - (\sum_{j=1,j\neq i}^n L_j|\alpha_{ij}| + \sum_{j=1}^n L_j|\beta_{ij}|)]/(\alpha_{ii} + |\beta_{ii}|) \\ &\ge \sum_{j=1,j\neq i}^n L_j|\alpha_{ij}| + \sum_{j=1}^n L_j|\beta_{ij}|. \end{aligned}$$

Subsequently, from (4.3)

$$\begin{aligned} |x_i^* - y_i^*| &\le \{[\sum_{j=1,j\neq i}^n L_j|\alpha_{ij}| + \sum_{j=1}^n L_j|\beta_{ij}|]/|\alpha_{ii}g_i'(\xi_i^*) - \mu_i|\}\|\mathbf{x} - \mathbf{y}\|_\infty \\ &=: \gamma_i\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

and $\gamma_i < 1$. The situation for $\lambda_i =$"l" is similar. Therefore, $G_\Omega$ is a contraction map and there exists an unique fixed point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$ of $G_\Omega$, lying in $\Omega$. Restated, for each $i = 1, 2, \cdots n$,

$$-\mu_i\overline{x}_i + \alpha_{ii}g_i(\overline{x}_i) + \sum_{j=1,j\neq i}^n \alpha_{ij}g_j(\overline{x}_j) + \sum_{j=1}^n \beta_{ij}g_j(\overline{x}_j) + J_i = 0. \tag{4.4}$$

Thus, $\overline{\mathbf{x}}$ is an unique equilibrium point of (2.3) lying in $\Omega$.

On the other hand, if $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$ is an equilibrium of (2.3), then (4.4) holds. Hence, $\overline{x}_i$ lies in one of $\Omega_i^{\mathrm{l}}, \Omega_i^{\mathrm{m}}, \Omega_i^{\mathrm{r}}$, for each $i$, and thus $\overline{\mathbf{x}}$ coincides with the unique equilibrium lying in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, $\lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$. System (2.3) therefore admits exactly $3^n$ equilibria. $\square$

## 4.2 Convergence

In the following discussions, we consider a fixed initial value $\phi \in C([-\tau, 0], \mathbb{R}^n)$, and the solution $\mathbf{x}(t) = \mathbf{x}(t; t_0; \phi) = (x_1(t; t_0; \phi), x_2(t; t_0; \phi), \cdots, x_n(t; t_0; \phi))$ to system (2.3), which is evolved from $\phi$ at $t = t_0$. For each $i = 1, 2, \cdots, n$, we write the $i$th component of system (2.3) in the following form:

$$\dot{\xi}(t) = -\mu_i\xi(t) + \alpha_{ii}g_i(\xi(t)) + \beta_{ii}g_i(\xi(t - \tau_{ii}(t))) + w_i(t), \tag{4.5}$$

where $w_i(t) = w_i(t;t_0;\phi) := \Sigma_{j=1,j\neq i}^{n}[\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i$ is regarded as a bounded function of $t$. The notations, Lemma 3.2.1, Propositions 3.2.2, 3.2.3, and Theorem 3.2.4 can all be adapted to (4.5). In particular, for $i = 1, 2, \cdots, n$, we define

$$\begin{aligned}\hat{f}_i(\xi) &= -\mu_i\xi + \alpha_{ii}g_i(\xi) + |\beta_{ii}|\rho_i + w_i^{\max}(t_0),\\ \check{f}_i(\xi) &= -\mu_i\xi + \alpha_{ii}g_i(\xi) - |\beta_{ii}|\rho_i + w_i^{\min}(t_0).\end{aligned}$$

Under conditions (C1c), (C2c), $\hat{f}_i, \check{f}_i$ admit similar properties as $\hat{f}, \check{f}$ in Section 3.2.1. In particular, there exist $\hat{l}_i^f$, $\hat{m}_i^f$, $\hat{r}_i^f$, $\check{l}_i^f$, $\check{m}_i^f$, $\check{r}_i^f$ which are the zeros of $\hat{f}_i, \check{f}_i$ respectively, and $\bar{p}_i$, $\bar{q}_i$ which are both the critical points of $\hat{f}_i$ and $\check{f}_i$. Notice that $\hat{F}_i, \check{F}_i$, and $\hat{f}_i, \check{f}_i$ share the same critical points $\bar{p}_i$, $\bar{q}_i$. According to our setting,

$$\check{F}_i(\xi) \leq \check{f}_i(\xi) \leq \hat{f}_i(\xi) \leq \hat{F}_i(\xi), \text{ for all } \xi \in \mathbb{R}.$$

Therefore, condition (C3c) implies that $\check{f}_i(\tilde{q}_i) \geq \check{F}_i(\tilde{q}_i) > 0$ and $\hat{f}_i(\tilde{p}_i) \leq \hat{F}_i(\tilde{p}_i) < 0$; in addition, $\bar{p}_i < \hat{m}_i^F < \hat{m}_i^f < \check{m}_i^f < \check{m}_i^F < \bar{q}_i$, $\check{l}_i^F < \check{l}_i^f < \hat{l}_i^f < \hat{l}_i^F < \tilde{p}_i$, and $\tilde{q}_i < \check{r}_i^F < \check{r}_i^f < \hat{r}_i^f < \hat{r}_i^F$, where $\tilde{p}_i$, $\tilde{q}_i$ are defined in (4.2), cf. Fig. 4.1. Moreover, we note that condition (C4c): $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ on $[\hat{m}_i^F, \check{m}_i^F]$ yields $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ on $[\hat{m}_i^f, \check{m}_i^f]$ since $[\hat{m}_i^f, \check{m}_i^f] \subset [\hat{m}_i^F, \check{m}_i^F]$.

According to Theorem 3.2.4, for each $i = 1, 2, \cdots, n$, there exist three disjoint, closed and bounded intervals $[\underline{a}_i, \overline{a}_i]$, $[\underline{b}_i, \overline{b}_i]$ and $[\underline{c}_i, \overline{c}_i]$ and the $i$th component $x_i(t)$ of the solution converges to one of them. Moreover, by Lemma 3.2.1, we can estimate the lengths of these intervals. Restated, $x_i(t) = x_i(t;t_0;\phi)$, the $i$-th component of solution starting from $\phi \in C([-\tau, 0], \mathbb{R}^n)$, converges to an interval $I_i$ of length $d_i$, and

$$d_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/\eta_i, \tag{4.6}$$

where $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\}$, $w_i^{\max}(\infty) = \lim_{T\to\infty} w_i^{\max}(T), w_i^{\min}(\infty) = \lim_{T\to\infty} w_i^{\min}(T), w_i^{\max}(T) := \sup\{w_i(t) \mid t \geq T\}$, and $w_i^{\min}(T) := \inf\{w_i(t) \mid t \geq T\}$. Notably, in (4.6), the magnitude of $d_i$ depends on the difference between $w_i^{\max}(\infty)$ and $w_i^{\min}(\infty)$ which are terms involving non-$i$ components of the solution and can not be measured without further elaboration. In the following, we employ an upper bound for $w_i^{\max}(\infty)$ and a lower bound for $w_i^{\min}(\infty)$, which are definite terms, and derive a rough estimate on $d_i$. From this estimate, we compute more precise upper (resp. lower) bounds for $w_i^{\max}(\infty)$ (resp. $w_i^{\min}(\infty)$) through an iterative process. This idea for estimating the magnitude of $d_i$ is illustrated and implemented in the following proposition.

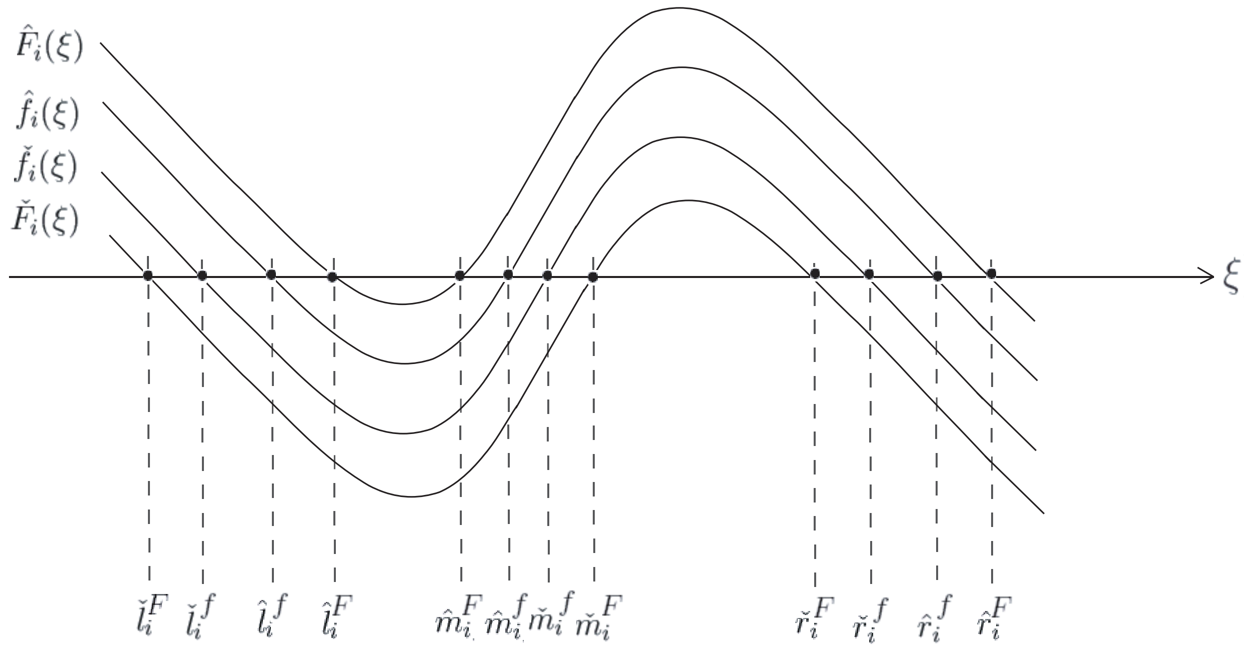



Figure 4.1: Configurations for functions $\hat{F}_i$, $\hat{f}_i$, $\check{f}_i$, $\check{F}_i$.

**Proposition 4.2.1.** Assume that conditions (C2c)-(C4c) hold. For each $i = 1, 2, \cdots, n$, there exists a sequence of intervals $\{I_i^{(k)}\}_{k=0}^{\infty}$ such that for each $k$, the $i$th component $x_i(t)$ of every solution $\mathbf{x}(t)$ to system (2.3) converges to $I_i^{(k)}$ as $t \to \infty$, and the length $d_i^{(k)}$ of $I_i^{(k)}$ satisfies

$$d_i^{(k)} \leq \{\sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k-1)}\}/\eta_i. \tag{4.7}$$

**Proof.** We prove the case of $\beta_{ii} > 0$. Let us define $d_i^{(0)} := 2\rho_i/L_i$, for $i = 1, 2, \cdots, n$. First, we illustrate that the assertion holds for $k = 1$ and $i = 1$. Set

$$\check{W}_1^{(1)}(\infty) := -\sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)\rho_j + J_1, \quad \hat{W}_1^{(1)}(\infty) := \sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)\rho_j + J_1.$$

Notable, $\check{W}_1^{(1)}(\infty) \leq w_1^{\min}(\infty) \leq w_1^{\max}(\infty) \leq \hat{W}_1^{(1)}(\infty)$. Recall $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\}$. We have shown that $x_1(t)$ converges to interval $I_1$ of length $d_1$, and

$$\begin{aligned} d_1 &\leq [w_1^{\max}(\infty) - w_1^{\min}(\infty)]/\eta_1 \\ &\leq [\hat{W}_1^{(1)}(\infty) - \check{W}_1^{(1)}(\infty)]/\eta_1 \\ &= [\sum_{j=2}^{n}|\alpha_{1j}| + \sum_{j=2}^{n}|\beta_{1j}|]L_j d_j^{(0)}/\eta_1. \end{aligned}$$

We may say that $x_1(t)$ converges to a closed and bounded interval $I_1^{(1)} \supset I_1$, whose length $d_1^{(1)}$ satisfies $d_1^{(1)} \leq [\Sigma_{j=2}^n |\alpha_{1j}| + \Sigma_{j=2}^n |\beta_{1j}|] L_j d_j^{(0)} / \eta_1$. Assume that the assertion holds for $k = 1$, $i = 1, 2, \cdots, \ell - 1$, $1 < \ell \leq n$ and $x_i(t)$ converges to a closed and bounded interval $I_i^{(1)} \supset I_i$ of length $d_i^{(1)} \leq \{\Sigma_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(1)} + \Sigma_{j=i+1}^n (|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(0)}\} / \eta_i$. Let us justify that the assertion also holds for $k = 1$ and $i = \ell$ as follows. Set

$$\begin{aligned}
\check{W}_\ell^{(1)}(\infty) &:= \sum_{j=1}^{\ell-1} \min_{\xi,\eta \in I_j^{(1)}} \{\alpha_{\ell j} g_j(\xi) + \beta_{\ell j} g_j(\eta)\} - \sum_{j=\ell+1}^{n} (|\alpha_{\ell j}| + |\beta_{\ell j}|)\rho_j + J_\ell, \\
\hat{W}_\ell^{(1)}(\infty) &:= \sum_{j=1}^{\ell-1} \max_{\xi,\eta \in I_j^{(1)}} \{\alpha_{\ell j} g_j(\xi) + \beta_{\ell j} g_j(\eta)\} + \sum_{j=\ell+1}^{n} (|\alpha_{\ell j}| + |\beta_{\ell j}|)\rho_j + J_\ell.
\end{aligned}$$

It follows that $x_\ell(t)$ converges to an interval $I_\ell^{(1)}$ whose length $d_\ell^{(1)}$ satisfies

$$\begin{aligned}
d_\ell^{(1)} &\leq [w_\ell^{\max}(\infty) - w_\ell^{\min}(\infty)] / \eta_\ell \\
&\leq [\hat{W}_\ell^{(1)}(\infty) - \check{W}_\ell^{(1)}(\infty)] / \eta_\ell \\
&= \{\Sigma_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|) L_j d_j^{(1)} + \Sigma_{j=\ell+1}^n (|\alpha_{\ell j}| + |\beta_{\ell j}|) L_j d_j^{(0)}\} / \eta_\ell.
\end{aligned}$$

Next, assume that the assertion holds for some $(k-1)$ and all $i = 1, 2, \cdots, n$. Namely, $x_i(t)$ converges to a closed and bounded interval $I_i^{(k-1)}$, whose length satisfies $d_i^{(k-1)} \leq \{\Sigma_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k-1)} + \Sigma_{j=i+1}^n (|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k-2)}\} / \eta_i$. Now, let us verify that the assertion holds for $k$ and $i = 1$ as well. Set

$$\begin{aligned}
\check{W}_1^{(k)}(\infty) &:= \sum_{j=2}^{n} \min_{\xi,\eta \in I_j^{(k-1)}} \{\alpha_{1j} g_j(\xi) + \beta_{1j} g_j(\eta)\} + J_1, \\
\hat{W}_1^{(k)}(\infty) &:= \sum_{j=2}^{n} \max_{\xi,\eta \in I_j^{(k-1)}} \{\alpha_{1j} g_j(\xi) + \beta_{1j} g_j(\eta)\} + J_1.
\end{aligned}$$

Thus, $x_1(t)$ converges to an interval $I_1^{(k)}$ whose length $d_1^{(k)}$ satisfies

$$\begin{aligned}
d_1^{(k)} &\leq [w_1^{\max}(\infty) - w_1^{\min}(\infty)] / \eta_1 \\
&\leq [\hat{W}_1^{(k)}(\infty) - \check{W}_1^{(k)}(\infty)] / \eta_1 \\
&= [\Sigma_{j=2}^n |\alpha_{1j}| + \Sigma_{j=2}^n |\beta_{1j}|] L_j d_j^{(k-1)} / \eta_1.
\end{aligned}$$

By continuing the above process, we can prove that for each $i = 2, \cdots, n$, $x_i(t)$ converges to an interval $I_i^{(k)}$ whose length is $d_i^{(k)} \leq \{\Sigma_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k)} + \Sigma_{j=i+1}^n (|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k-1)}\} / \eta_i$. $\square$

To establish further dynamical properties for system (2.3), we need the following condition which is stronger than condition (C2c).

Condition (C2c)$^*$: $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\} > \sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j\neq i} L_j|\beta_{ij}|$, for $i = 1, 2, \cdots, n$.

So far, we have considered a single solution to system (2.3), which is evolved from a given $\phi$ at $t = t_0$. From our previous derivations, it can be shown that every component of the solution converges to a sequence of closed intervals whose lengths $d_i^{(k)}$, $i = 1, 2, \cdots, n$, can be controlled by iterative formula (4.7). Next, it will be examined that for each $i$, $d_i^{(k)}$ converges to zero, as $k \to \infty$, via the Guass-Seidal iteration approach. Thus, the intervals to which each component of the solution converges degenerate into a single point. Hence the solution converges to a singleton.

**Theorem 4.2.2.** Assume that conditions (C2c)$^*$, (C3c) and (C4c) hold. Then the solution $\mathbf{x}(t) := \mathbf{x}(t; t_0; \phi)$ of (2.3) evolved from any initial value $\phi \in C([-\tau, 0], \mathbb{R}^n)$ converges to one of the $3^n$ equilibria of the system.

**Proof.** By Proposition 4.2.1, for each $i = 1, 2, \cdots, n$, we can find an interval sequence $\{I_i^{(k)}\}_{k=0}^{\infty}$ so that $x_i(t)$ converges to $I_i^{(k)}$ whose length satisfies (4.7), for each $k$. Below, we shall show that for all $i = 1, 2, \cdots, n$, $d_i^{(k)}$ converges to zero as $k$ tends to infinity. Set $z_i^{(0)} := d_i^{(0)}$, and for $i = 1, 2, \cdots, n$,

$$
\begin{aligned}
z_i^{(k)} &:= \{\Sigma_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j z_j^{(k)} + \Sigma_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j z_j^{(k-1)}\}/\eta_i, \ k \in \mathbb{N}, \\
\mathbf{z}^{(k)} &:= (z_1^{(k)}, z_2^{(k)}, \cdots, z_n^{(k)}), \ k \in \mathbb{N} \cup \{0\}.
\end{aligned}
$$

We observe that $\{z_i^{(k)} \mid i = 1, 2, \cdots, n\}$ are just the Gauss-Seidal iterations for solving the linear system

$$
\begin{aligned}
&(\mathbf{ML} + \mathbf{E})\mathbf{y} = \mathbf{0}, \qquad (4.8) \\
&\mathbf{M} := [m_{ij}]_{1\leq i,j\leq n}, m_{ii} = 0, m_{ij} = -|\alpha_{ij}| - |\beta_{ij}|, \text{ for } i \neq j, \\
&\mathbf{L} := \mathrm{diag}(L_1, L_2, \cdots, L_n), \mathbf{E} := \mathrm{diag}(\eta_1, \eta_2, \cdots, \eta_n).
\end{aligned}
$$

Notably, $\mathbf{ML} + \mathbf{E}$ is strictly diagonal-dominant [4, 74]; indeed, $\eta_i - \sum_{j\neq i}(|\alpha_{ij}| + |\beta_{ij}|)L_j > 0$, for all $i = 1, 2, \cdots, n$, by condition (C2c)$^*$. Accordingly, $\mathbf{z}^{(k)}$ converges to the unique solution of (4.8), which is zero, as $k \to \infty$.

Below, let us justify the following inequality:

$$0 \le d_i^{(k)} \le z_i^{(k)}, \text{ for } i = 1, 2, \cdots, n, k \in \mathbb{N} \cup \{0\}. \quad (4.9)$$

It is obvious that for $i = 1, 2, \cdots, n$, $0 \le d_i^{(k)}$, for $k \in \mathbb{N} \cup \{0\}$ and (4.9) holds for $k = 0$. In addition, (4.9) holds for $i = 1$ and $k = 1$ since $d_1^{(1)} \le \{\sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)L_j d_j^{(0)}\}/\eta_1 \le \{\sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)L_j z_j^{(0)}\}/\eta_1 = z_1^{(1)}$. We can continue to prove that (4.9) holds for $i = 2, 3 \cdots, n$ and $k = 1$. Assume that (4.9) holds for all $i = 1, 2, \cdots, n$ and $k = \ell$, for some $\ell \ge 1$, then (4.9) also holds for $i = 1$, $k = \ell + 1$ due to that $d_1^{(\ell+1)} \le \{\sum_{j=2}^{n}(|\alpha_{\ell j}|+|\beta_{\ell j}|)L_j d_j^{(\ell)}\}/\eta_1 \le \{\sum_{j=2}^{n}(|\alpha_{\ell j}|+|\beta_{\ell j}|)L_j z_j^{(\ell)}\}/\eta_1 = z_1^{(\ell+1)}$. Assume that (4.9) holds for $0 \le k_0 - 1$ and all $i = 1, 2, \cdots, n$, and $k = k_0$, $i = 1, \cdots, (\ell - 1)$, then

$$\begin{aligned} d_\ell^{(k_0)} &\le \{\Sigma_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(k_0)} + \Sigma_{j=\ell+1}^{n}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(k_0-1)}\}/\eta_\ell \\ &\le \{\Sigma_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j z_j^{(k_0)} + \Sigma_{j=\ell+1}^{n}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j z_j^{(k_0-1)}\}/\eta_\ell \\ &= z_\ell^{(k_0)}. \end{aligned}$$

Hence, for each $i = 1, 2, \cdots, n$, $d_i^{(k)}$ converges to zero as $k$ tends to infinity. Therefore, each $x_i(t)$ converges to a singe point and $\mathbf{x}(t)$ converges to a constant which is an equilibrium, as time tends to infinity. □

## 4.3 Stability of equilibria

Let us denote by $\overline{\mathbf{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ the equilibrium lying in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, $\lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$. The stability of all the $3^n$ equilibria of (2.3) can be concluded in the following theorem.

**Theorem 4.3.1.** Assume that conditions (C2c)*, (C3c) and (C4c) hold. Then, (i) every equilibrium $\overline{\mathbf{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ with $\lambda_i$ = "l", "r", for all $i = 1, 2, \cdots, n$, is asymptotically stable; (ii) the equilibrium $\overline{\mathbf{x}}_{\mathrm{m}\cdots\mathrm{m}}$ is unstable; (iii) every equilibrium $\overline{\mathbf{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ with $\lambda_i$ = "m" for some $i$ and $\lambda_j$ = "l", "r" for some $j$, is unstable.

**Proof.** It can be referred to Fig. 4.2 for the proof of Theorem 4.3.1.

(i) Consider an exterior region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, $\lambda_i$ = "l" or "r", $i = 1, 2, \cdots, n$. We show that the equilibrium $\overline{\mathbf{x}} := (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$ in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ is stable. Note that for each $i$, either $\overline{x}_i \in [\check{r}_i^F, \hat{r}_i^F]$ or $\overline{x}_i \in [\check{l}_i^F, \hat{l}_i^F]$. There exists $\varepsilon_i > 0$ such that $\check{r}_i^F - \varepsilon_i > \tilde{q}_i$ and $\check{l}_i^F + \varepsilon_i < \tilde{p}_i$, due to that $\check{r}_i^F > \tilde{q}_i$ and $\check{l}_i^F < \tilde{p}_i$. We shall illustrate that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\mathbf{x}_t - \overline{\mathbf{x}}\| \le \varepsilon$ for all $t \ge t_0$, for any

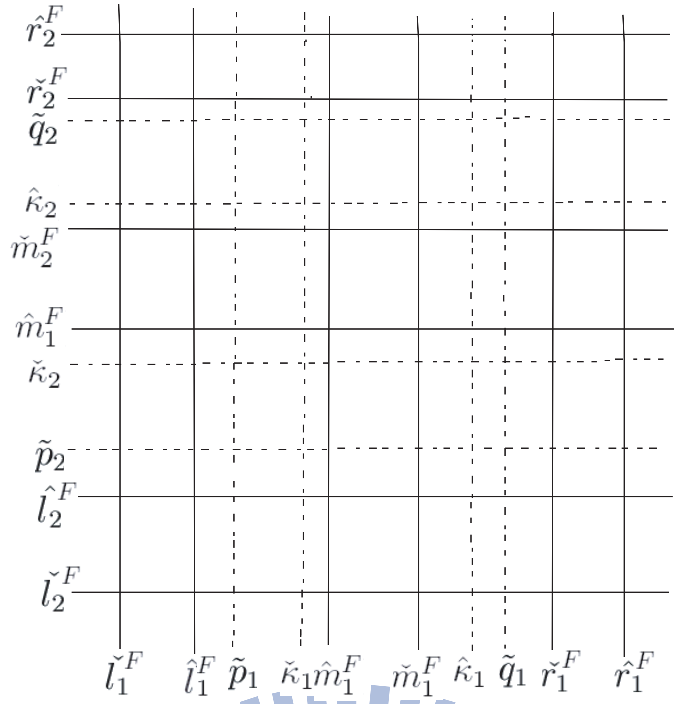



Figure 4.2: Configuration for the proof of Theorem 4.3.1.

$\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \overline{\mathbf{x}}\| \leq \delta$. For an $\varepsilon > 0$, we set $\delta := \min\{\varepsilon, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$. For an initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \overline{\mathbf{x}}\| \leq \delta$, the solution satisfies $x_i(s) > \tilde{q}_i$ if $\lambda_i =$ "r", and $x_i(s) < \tilde{p}_i$, if $\lambda_i =$ "l", for all $s \in [t_0 - \tau, t_0]$. It follows from similar argument as the proof of Proposition 3.2.2 (ii) that $x_i(t) > \tilde{q}_i$ for all $t \in [t_0 - \tau, \infty)$ or $x_i(t) < \tilde{p}_i$, for all $t \in [t_0 - \tau, \infty)$. We define $z_i(t) := x_i(t) - \overline{x}_i$, for $i = 1, 2, \cdots, n$. It follows from (2.3) that

$$\dot{z}_i(t) = -\mu_i z_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j'(\xi_j(t)) z_j(t) + \sum_{j=1}^{n} \beta_{ij} g_j'(\eta_{ij}(t)) z_j(t - \tau_{ij}(t))$$

where $\xi_j(t)$ is between $x_j(t)$ and $\overline{x}_j$, $\eta_{ij}(t)$ is between $x_j(t - \tau_{ij}(t))$ and $\overline{x}_j$, $i, j = 1, 2, \cdots, n$. It can be computed that

$$D_r|z_i(t)| \leq -\mu_i|z_i(t)| + \sum_{j=1}^{n} |\alpha_{ij}| g_j'(\xi_j(t)) |z_j(t)| + \sum_{j=1}^{n} |\beta_{ij}| g_j'(\eta_{ij}(t)) |z_j(t - \tau_{ij}(t))|,$$

for $t \geq t_0$, where $D_r$ denotes the right-hand derivative. Define $N(t) := \|\mathbf{z}_t\| = \max_{1 \leq i \leq n}\{\max_{s \in [t-\tau, t]} |z_i(s)|\}$. We shall show below that

$$D_r N(t) := \lim_{h \to 0^+} \frac{N(t+h) - N(t)}{h} \leq 0, \text{ for all } t \geq t_0. \tag{4.10}$$

For $t \geq t_0$, let $\tilde{I}(t) := \{i : |z_i(t)| \geq |z_j(t)|$, for all $j = 1, 2, \cdots, n\}$, and $i(t) := \min\{i \in \tilde{I}(t) : D_r|z_i(t)| \geq D_r|z_j(t)|$, for all $j \in \tilde{I}(t)\}$. Consider a fixed $t > t_0$, and denote $i(t)$ by $k$. If $N(t) = |z_k(t)| > |z_j(t-\tau)|$ for all $j = 1, 2, \cdots, n$, then either $N(t) > |z_j(s)|$ for all $j = 1, 2, \cdots, n$ and all $s \in [t-\tau, t)$ or $N(t) = |z_{i(s)}(s)|$ for some $s \in (t-\tau, t)$. For the former case, it can be derived that

$$
\begin{aligned}
D_r|z_k(t)| &\leq -\mu_k|z_k(t)| + \sum_{j=1}^{n} |\alpha_{kj}||g_j'(\xi_j(t))||z_j(t)| + \sum_{j=1}^{n} |\beta_{kj}||g_j'(\eta_{kj}(t))||z_j(t-\tau_{kj}(t))| \\
&\leq [-\mu_k + \alpha_{kk} g_k'(\xi_k(t)) + \sum_{j\neq k} |\alpha_{kj}| g_j'(\xi_j(t)) + \sum_{j=1}^{n} |\beta_{kj}| g_j'(\eta_{kj}(t))] N(t) \\
&< [-\mu_k + \alpha_{kk} g_k'(\gamma_k) + \sum_{j\neq k} |\alpha_{kj}| L_j + \sum_{j=1}^{n} |\beta_{kj}| L_j] N(t) \\
&\leq 0,
\end{aligned}
$$

for all $t \geq t_0$, where $\gamma_k = \tilde{p}_k$ or $\tilde{q}_k$, recalling that

$$
g_k'(\tilde{p_k}) = g_k'(\tilde{q_k}) = [\mu_k - (\sum_{j\neq i} L_j|\alpha_{kj}| + \sum_{j=1}^{n} L_j|\beta_{kj}|)]/(\alpha_{kk} + |\beta_{kk}|).
$$

Thus,

$$
\begin{aligned}
D_r N(t) &= \lim_{h\to 0^+} \frac{N(t+h) - N(t)}{h} \\
&= \lim_{h\to 0^+} \frac{|z_k(t+h)| - |z_k(t)|}{h} \\
&= D_r|z_k(t)| \leq 0.
\end{aligned}
$$

For the latter case,

$$
\begin{aligned}
D_r N(t) &= \lim_{h\to 0^+} \frac{N(t+h) - N(t)}{h} \\
&= \lim_{h\to 0^+} \frac{N(t) - N(t)}{h} = 0.
\end{aligned}
$$

For the other cases: $N(t) = |z_i(t-\tau)|$ for some $i \in \{1, 2, \cdots, n\}$; $N(t) = |z_i(s)|$ for some $i \in \{1, 2, \cdots, n\}$ and some $s \in (t-\tau, t)$ with $N(t) > |z_j(t-\tau)|$ and $N(t) > |z_j(t)|$ for all $j = 1, 2, \cdots, n$, (4.10) can also be justified. Hence, $N(t) = \|\mathbf{z}_t\| = \|\mathbf{x}_t - \overline{\mathbf{x}}\| \leq N(t_0) = \|\mathbf{z}_{t_0}\| = \|\mathbf{x}_{t_0} - \overline{\mathbf{x}}\| = \|\phi - \overline{\mathbf{x}}\|$ for all $t \geq t_0$. Therefore, $\overline{\mathbf{x}}$ is stable, hence asymptotically stable, in respecting Theorem 4.2.2.

(ii) We shall show that $\overline{\mathbf{x}} := \bar{\mathbf{x}}_{\mathrm{mm}\cdots\mathrm{m}}$ is unstable. We choose an initial value which is close to the equilibrium $\overline{\mathbf{x}}$. Then the solution must move away from $\overline{\mathbf{x}} =$

$(\bar{x}_1, \cdots, \bar{x}_n)$. Such an assertion holds mainly because if the $i$th component $x_i(t)$ of solution remains close to $\bar{x}_i$ for all $i = 1, , 2, \cdots, n$, then the magnitude of $g_i'(x_i(t))$ will remain large and yield a contradiction. Notably, for $i = 1, 2, \cdots, n$, $g_i'(\xi) > 2\mu_i/\alpha_{ii}$, for all $\xi \in [\hat{m}_i^F, \check{m}_i^F]$, thus there exist $\hat{\kappa}_i$ and $\check{\kappa}_i$ such that $g'(\check{\kappa}_i) = g'(\hat{\kappa}_i) = 2\mu_i/\alpha_{ii}$, where $\check{\kappa}_i < \hat{m}_i^F < \check{m}_i^F < \hat{\kappa}_i$, for all $i = 1, 2, \cdots, n$. Set $\varepsilon_i := \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\}$, $\varepsilon := \min_{1\le i\le n}\{\varepsilon_i\}/2$. For any $\delta \in (0, \varepsilon)$, we choose the initial condition $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ with $\|\phi - \overline{\mathbf{x}}\| < \delta$, $\phi(s) \in \tilde{\Omega}_{\mathrm{mm}\cdots\mathrm{m}}$, for all $s \in [-\tau, 0]$, $\|\phi - \overline{\mathbf{x}}\| = |\phi_i(0) - \overline{x}_i|$ for some $i \in \{1, 2, \cdots, n\}$ and $\|\phi - \overline{\mathbf{x}}\| > |\phi_j(s) - \overline{x}_j|$, for all $j = 1, 2, \cdots, n$, $s \in [-\tau, 0)$. Now, let us show that there exist $j \in \{1, 2, \cdots, n\}$, and $t_1 > t_0$ such that $x_j(t_1) > \hat{\kappa}_j$ or $x_j(t_1) < \check{\kappa}_j$. Assume otherwise that

$$\check{\kappa}_i \le x_i(t) \le \hat{\kappa}_i, \text{ for all } t \ge t_0 - \tau, \ i = 1, 2, \cdots, n. \tag{4.11}$$

Notice that, under the assumption above, $g_i^{'}(x_i(t)) \ge 2\mu_i/\alpha_{ii}$ for all $t \ge t_0 - \tau$ and all $i = 1, 2, \cdots, n$. Let $z_i(t) = x_i(t) - \overline{x}_i$, and

$$B(t) := \max_{1\le i\le n}\{\max_{t_0-\tau\le s\le t} |z_i(s)|\}. \tag{4.12}$$

Then $B(t_0) = \max_{1\le i\le n}\{|z_i(t_0)|\} > 0$ and $B(t) > 0$ for all $t \ge t_0$. Let us show that

$$B(t) = \max_{1\le i\le n}\{|z_i(t)|\}, \text{ for all } t \ge t_0, \tag{4.13}$$

i.e., at least one component of $(|z_1(s)|, |z_2(s)|, \cdots, |z_n(s)|)$ will reach the value of $B(t)$ at time $t$. If otherwise, there is a $t > t_0$ so that $B(t) = |z_k(t_2)|$, for some $k \in \{1, 2, \cdots, n\}$ and some $t_2 \in [t_0, t)$, then either $B(t) = z_k(t_2)$ or $B(t) = -z_k(t_2)$. For the former case,

$$\begin{aligned}
\dot{z}_k(t_2) &= -\mu_k z_k(t_2) + \sum_{j=1}^{n} \alpha_{kj} g_j'(\xi_j(t_2)) z_j(t_2) + \sum_{j=1}^{n} \beta_{kj} g_j'(\eta_{kj}(t_2)) z_j(t_2 - \tau_{kj}(t_2)) \\
&\ge -\mu_k z_k(t_2) + \alpha_{kk}[2\mu_k/\alpha_{kk}] z_k(t_2) - \sum_{j\ne k} |\alpha_{kj}| g_j'(\xi_j(t_2)) |z_j(t_2)| \\
&\quad - \sum_{j=1}^{n} |\beta_{kj}| g_j'(\eta_{kj}(t_2)) |z_j(t_2 - \tau_{kj}(t_2))| \\
&\ge [\mu_k - \sum_{j\ne k} |\alpha_{kj}| L_j - \sum_{j=1}^{n} |\beta_{kj}| L_j] B(t) > 0,
\end{aligned}$$

owing to condition (C2c). For the latter case, we can also show that $\frac{d(-z_k)}{dt}(t_2) \ge [\mu_k - \sum_{j\ne k} |\alpha_{kj}| L_j - \sum_{j=1}^{n} |\beta_{kj}| L_j] B(t) > 0$. A contradiction to $B(t) = |z_k(t_2)|$ with

$t_2 \in [t_0, t)$ then arises. Thus, (4.13) holds. For any $t \geq t_0$, we define $k(t) := \min\{j : |z_j(t)| = B(t)\}$, then

$$
\begin{aligned}
D_r B(t) &\geq D_r |z_{k(t)}(t)| \\
&\geq [\mu_{k(t)} - \sum_{j \neq k(t)} |\alpha_{k(t)j}| L_j + \sum_{j=1}^{n} |\beta_{k(t)j}| L_j] B(t) \\
&\geq \min_{1 \leq i \leq n} \{\mu_i - \sum_{j \neq i} |\alpha_{ij}| L_j + \sum_{j=1}^{n} |\beta_{ij}| L_j\} B(t).
\end{aligned}
$$

It follows that $B(t)$ grows unboundedly as $t$ tends to infinity, which yields a contradiction to (4.11). We thus conclude that $\overline{\mathbf{x}}_{\mathrm{mm}\cdots\mathrm{m}}$ is unstable.

(iii) Consider a mixed region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, where $\mathcal{I} := \{i : \lambda_i = \text{"m"}\} \neq \emptyset$ and $\mathcal{E} := \{i : \lambda_i = \text{"l" or "r"}\} \neq \emptyset$. It will be shown that the equilibrium $\overline{\mathbf{x}} := (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$ in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ is unstable. We shall choose an initial value which is close to equilibrium $\overline{\mathbf{x}}$, then the evolved solution must move away from $\overline{\mathbf{x}}$. This is due to that if the $i$th component remains close to $\bar{x}_i$ for all $i \in \mathcal{I}$, then the magnitude of $g_i'(x_i(t))$ will remain large for all $i \in \mathcal{I}$. Moreover, it can be seen that the magnitude of $g_j'(x_j(t))$ keeps small for all $j \in \mathcal{E}$. In such a situation, there exists some $k \in \mathcal{I}$ such that $x_k(t)$ will move away from $\overline{x}_k$; subsequently a contradiction arises. To be more precise, let us define $\varepsilon_i := \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\}$, for $i \in \mathcal{I}$, and $\varepsilon_j := \overline{x}_j - \tilde{q}_j$ if $\lambda_j = \text{"r"}$, $\varepsilon_j := \tilde{p}_j - \overline{x}_j$ if $\lambda_j = \text{"l"}$, for $j \in \mathcal{E}$, and set $\varepsilon := \min_{1 \leq i \leq n}\{\varepsilon_i\}/2$. For $\delta \in (0, \varepsilon)$, we choose an initial condition $\phi$ satisfying: $\|\phi - \overline{\mathbf{x}}\| < \delta$, and $\phi_j(s) \neq \overline{x}_j$, for some $j \in \mathcal{I}$ and some $s \in [-\tau, 0]$, $\|\phi - \overline{\mathbf{x}}\| = |\phi_k(0) - \overline{x}_k|$, for some $k \in \mathcal{I}$ and $\|\phi - \overline{\mathbf{x}}\| > |\phi_i(s) - \overline{x}_i|$, for all $i \in \mathcal{E}$ and all $s \in [-\tau, 0]$. Below, let us claim that there exist $j \in \mathcal{I}$, and some $t > t_0$ such that $x_j(t) > \hat{\kappa}_j$ or $x_j(t) < \check{\kappa}_j$. Assume otherwise that $\check{\kappa}_i \leq x_i(t) \leq \hat{\kappa}_i$, for all $i \in \mathcal{I}$ and $t \geq t_0 - \tau$. Note that then $g_i'(x_i(t)) \geq 2\mu_i/\alpha_{ii}$, for all $t \geq t_0 - \tau$ and all $i \in \mathcal{I}$. Define $B(t)$ as (4.12) and $J(t) := \{j \in \mathcal{I} : |z_j(t)| \geq |z_i(t)|, \text{ for all } i \in \mathcal{I}\}$, $j(t) := \min\{\ell \in J(t) : D_r|z_\ell(t)| \geq D_r|z_j(t)|, \text{ for all } j \in J(t)\}$. There are two possibilities: $|z_{j(t)}(t)| \geq |z_i(t)|$, for all $t \geq t_0$, for all $i \in \mathcal{E}$, and $|z_k(t_3)| > |z_{j(t_3)}(t_3)|$, for some $t_3 > t_0$, and some $k \in \mathcal{E}$. For the first one, $B(t) := \max_{i \in \mathcal{I}}\{\max_{t_0 - \tau \leq s \leq t} |z_i(s)|\}$, for all $t \geq t_0$. Similar to previous discussion in (ii), we can also show that $B(t) = \max_{i \in \mathcal{I}}\{|z_i(t)|\}$. Subsequently, $B(t)$ will blow up and yield a contradiction. For the later situation, there exists $s_1 \in (t_0, t_3)$ such that $|z_{j(s)}(s)| \geq |z_j(s)|$ for all $j \in \mathcal{E}$ and all $s \in [t_0, s_1)$, and there exists $k \in \mathcal{E}$ such that $|z_k(s_1)| = |z_{j(s_1)}(s_1)|$, and $D_r|z_k(s_1)| \geq D_r|z_{j(s_1)}(s_1)|$. Thereafter, it can be shown that $B(s) := \max_{i \in \mathcal{I}}\{|z_i(s)|\}$, for all $s \in [t_0, s_1]$ as before.

Let us fix $s_1$ and denote $j(s_1)$ by $\ell$. There are four possible subcases: subcase (a): $B(s_1) = z_\ell(s_1) = z_k(s_1) > 0$; subcase (b): $B(s_1) = z_k(s_1) = -z_\ell(s_1) > 0$; subcase (c): $B(s_1) = -z_\ell(s_1) = -z_k(s_1) > 0$; subcase (d): $B(s_1) = z_\ell(s_1) = -z_k(s_1) > 0$. Let us consider subcase (a). Note that $x_\ell(t) \in [\check{\kappa}_\ell, \hat{\kappa}_\ell]$, for all $t \geq t_0 - \tau$, and either $x_k(t) > \tilde{q}_k$ or $x_k(t) < \tilde{p}_k$, for all $t \geq t_0 - \tau$. We compute that

$$\begin{aligned}
& D_r|z_\ell(s_1)| - D_r|z_k(s_1)| \\
\geq\ & (\mu_k - \mu_\ell)B(s_1) + [\alpha_{\ell\ell}g'_\ell(\xi_\ell(s_1)) - \alpha_{kk}g'_k(\xi_k(s_1))]B(s_1) - \sum_{j\neq\ell}|\alpha_{\ell j}|L_jB(s_1) \\
& -\sum_{j\neq k}|\alpha_{kj}|L_jB(s_1) - \sum_{j=1}^{n}[|\beta_{\ell j}|L_jB(s_1) + |\beta_{kj}|L_jB(s_1)] \\
\geq\ & \{(\mu_k - \mu_\ell) + \alpha_{\ell\ell}\frac{2\mu_\ell}{\alpha_{\ell\ell}} - \alpha_{kk}\frac{\mu_k - (\sum_{j\neq k}|\alpha_{kj}|L_j + \sum_{j=1}^{n}|\beta_{kj}|L_j)}{\alpha_{kk} + |\beta_{kk}|} \\
& -[\sum_{j\neq\ell}|\alpha_{\ell j}|L_j + \sum_{j\neq k}|\alpha_{kj}|L_j)] - \sum_{j=1}^{n}[|\beta_{\ell j}|L_j + |\beta_{kj}|L_j]\}B(s_1) \\
\geq\ & [\mu_\ell - \sum_{j\neq\ell}|\alpha_{\ell j}|L_j - \sum_{j=1}|\beta_{\ell j}|L_j]B(s_1) > 0,
\end{aligned}$$

which yields a contradiction. Other subcases can be similarly discussed. Hence, there exist $k \in \mathcal{I}$, and $t_3 > t_0$ such that $x_k(t_3) > \hat{\kappa}_i$ or $x_k(t_3) < \check{\kappa}_i$ and $|x_k(t_3) - \overline{x}_k| \geq \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\} > \varepsilon$. Therefore, there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon)$, there is an $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \overline{\mathbf{x}}\| < \delta$ and $\|\mathbf{x}_{t_3} - \overline{\mathbf{x}}\| > \varepsilon$, for some $t_3 > t_0$. Thereafter, $\overline{\mathbf{x}}$ is unstable. □

## 4.4 Numerical examples

We give a numerical example to illustrate the present theory.

**Example 4.5.1.** Consider the following two-dimensional system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$.

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -x_1(t) + 7g_1(x_1(t)) + 0.1g_2(x_2(t)) - 0.5g_1(x_1(t-1)) + 0.1g_2(x_2(t-1)) - 0.1 \\
\frac{dx_2(t)}{dt} &= -x_2(t) - 0.2g_1(x_1(t)) + 8g_2(x_2(t)) + 0.1g_1(x_1(t-1)) + 0.6g_2(x_2(t-1)).
\end{aligned}$$

Then $\hat{F}_1(\xi) = -\xi + 7g(\xi) + 0.6$, $\check{F}_1(\xi) = -\xi + 7g(\xi) - 0.8$, $\hat{F}_2(\xi) = -\xi + 8g(\xi) + 0.9$, $\check{F}_2(\xi) = -\xi + 8g(\xi) - 0.9$; $\tilde{p}_1 = -2.292431670$, $\tilde{q}_1 = 2.292431670$, $\tilde{p}_2 = -2.917401094$, $\tilde{q}_2 = 2.917401094$; $\check{m}_1^F = -0.10003918992$, $\check{m}_1^F = 0.1342679254$,

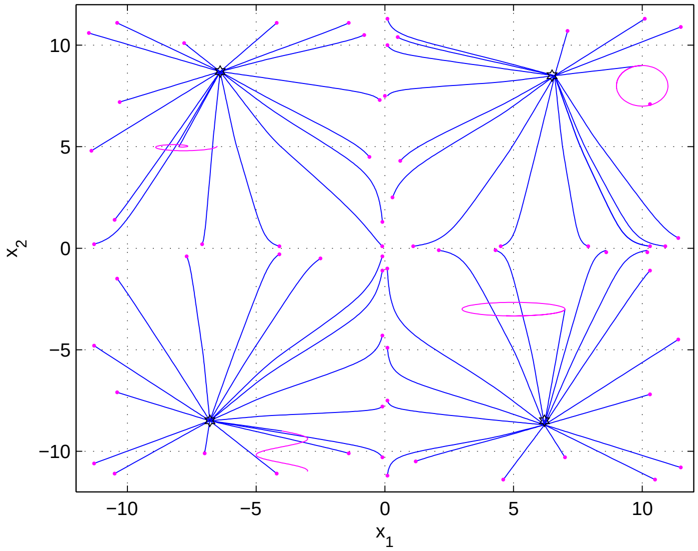



Figure 4.3: Numerical simulation for Example 4.5.1, with solutions evolved from initial functions at various locations.

$\hat{m}_2^F = -0.1293911878$, $\hat{m}_2^F = 0.1293911878$; $\check{\kappa}_1 = -1.238944365$, $\hat{\kappa}_1 = 1.238944365$, $\check{\kappa}_2 = -1.316957897$, $\hat{\kappa}_2 = 1.316957897$. Herein, $\check{\kappa}_1$ and $\hat{\kappa}_1$ are solutions of $g_1'(\cdot) = 2\mu_1/\alpha_{11} = 2/7$; $\check{\kappa}_2$ and $\hat{\kappa}_2$ are solutions of $g_2'(\cdot) = 2\mu_2/\alpha_{22} = 2/8$. It can be justified that conditions (C2c)$^*$, (C3c) and (C4c) hold as follows: condition (C2c)$^*$ holds since $\min\{\mu_1 - L_1|\beta_{11}|, L_1|\beta_{11}|\} = 0.5 > L_2|\alpha_{12}| + L_2|\beta_{12}| = 0.2$ and $\min\{\mu_2 - L_2|\beta_{22}|, L_2|\beta_{22}|\} = 0.4 > L_1|\alpha_{21}| + L_1|\beta_{21}| = 0.3$; condition (C3c) holds since $\check{F}_1(\tilde{q}_1) = 3.766139610 > 0$, $\hat{F}_1(\tilde{p}_1) = -3.966139610 < 0$, $\check{F}_2(\tilde{q}_2) = 4.135951278 > 0$ and $\hat{F}_2(\tilde{p}_2) = -4.135951278 < 0$; condition (C4c) holds since $[\hat{m}_1^F, \check{m}_1^F] \subset [\check{\kappa}_1, \hat{\kappa}_1]$ and $[\hat{m}_2^F, \hat{m}_2^F] \subset [\check{\kappa}_2, \check{\kappa}_2]$; subsequently $g_1'(\xi) > 2\mu_1/\alpha_{11}$ for $\xi \in [\hat{m}_1^F, \check{m}_1^F]$ and $g_2'(\xi) > 2\mu_2/\alpha_{22}$ for $\xi \in [\hat{m}_2^F, \check{m}_2^F]$. The Numerical simulation depicted in Fig. 4.3 demonstrates the convergence to four stable equilibria for solutions evolved from various initial conditions at different locations.

# Chapter 5

# Neural Network with Nearest-neighbor Coupling

In this chapter, we consider system (2.6):

$$\dot{x}_i(t) = -\mu x_i(t) + \alpha g_I(x_i(t-\tau_I)) + \beta[g_T(x_{i-1}(t-\tau_T)) + g_T(x_{i+1}(t-\tau_T))],$$

where $i \pmod N$; $g_I = g_T = g$ are the activation functions of class $\mathcal{A}$ with $-1 < g(\xi) < 1$, $g(0) = 0$, $g'(\xi) \leq g'(0) = 1$. We shall focus on the effect from scale of the network ($N$), self-decay ($\mu$), self-feedback strength ($\alpha$), coupling strength ($\beta$), delays ($\tau_I$, $\tau_T$), and the characteristic of $g$ upon upon synchrony, convergent dynamics and oscillation of (2.6). The presentation of this chapter is organized as follows. In section 5.1, we give a description of the dynamical scenarios extracted from the present investigations and compare with the existing results. The detailed arguments for establishing these scenarios are arranged in Sections 5.2 and 5.3. In Section 5.2, we focus on (2.6) of scale $N = 3$. Therein, global synchronization and convergence to three equilibria of (2.6) are investigated in Sections 5.2.1 and 5.2.2 respectively. Hopf bifurcation induced by the transmission delay at the trivial equilibrium is studied in Section 5.2.3. The investigations in Section 5.2 (including synchronization, convergence and delay-induced synchronous and asynchronous oscillations) can be carried over to system (2.3) of scale $N > 3$ by our approach. In particular, in Section 5.3, via a Guass-Seidal argument, we modify the approach in Section 5.2 to establish global synchronization for (2.3) of scale $N \geq 3$. We present some numerical illustrations in Section 5.4.

## 5.1 Description of dynamical scenarios

System (2.6) exhibits rich and a variety of dynamics corresponding to scale of the network, self-decay rate, self-feedback strength, coupling strength, delays, and characteristics of the activation function. We thus present a descriptive summary on the dynamical scenarios extracted from our investigation in this section. Some existing results and conjectures will also be reviewed.

First, we note that small scale may be advantageous for global synchronization of network (2.6) . In particular, network (2.6) of scale $N = 3$ has favorable structure than those of $N > 3$ in synchronization. Such a phenomenon is related to the coupling topology of (2.6) (each element of the network is coupled to the nearest ones). Theorems 5.2.1, 5.3.1 and Remark 5.3.1 depict such a result; in particular, Theorem 5.3.1 provides a criterion of synchronization for system (2.6), which favors small $N$. It is illustrated in Example 5.4.3 that (2.6) attains global synchronization under parameters satisfying the criterion of Theorem 5.3.1, as $N = 3$, but not for $N > 3$. However, it is generally nontrivial to find examples to show that a network of scale $N_1 > 3$ is apter to synchronize than another network of scale $N_2$ with $N_2 > N_1$. Nevertheless, there is a conjecture in [75]: when the scale of the network $N$ is odd, (2.6) can be synchronized if $|\alpha| + 2|\cos((N-1)\pi/N)||\beta| < 1$. This inequality is obviously stricter for larger $N$ and is thus consistent with our contention. Notably, while this conjecture provides a delay-independent criterion for synchronization of network (2.6) of odd scale, both delay-dependent and delay-independent criteria for synchronization of (2.6) of general scale are derived in our Theorem 5.3.1.

In the following, we focus on (2.6) with $N = 3$ to demonstrate our results and answer the conjectures mentioned in Section 2.2. The synchronization for (2.6) with $N > 3$ will be addressed in Section 5.3. Let us now summarize the synchrony, asynchrony, convergence, and oscillations for (2.6) respectively.

**Synchrony.**

The self-decay of (2.6) can promote synchronization of the network. Indeed, system (2.6) can be synchronized globally in spite of delays ($\tau_I$, $\tau_T$) if both self-feedback strength $\alpha$ and coupling strength $\beta$ are weak and dominated by self-decay rate $\mu$, cf. Fig. 5.1(d). For $\alpha, \beta$ in other regions, the magnitude of delays play an important role for synchronization of (2.6). If the self-feedback (resp. coupling) strength is dominated by the self-decay but the coupling (resp. self-feedback) strength is strong, then the excitatory coupling (resp. inhibitory self-feedback) is advantageous for synchronization of system (2.6) under small delay $\tau_T$ (resp. $\tau_I$), cf. Figs. 5.1(b), (c). Once self-feedback and coupling strength are both strong, system (2.6) can be synchronized if the self-feedback is inhibitory, the coupling is excitatory, and both delays $\tau_I$ and $\tau_T$ are small, cf. Fig. 5.1(a). Detailed arguments of these results are stated in Theorem 5.2.1 and Remark 5.2.1. Notably, Figs. 5.1(a)-(d) are depicted from the sufficient (parameter) conditions in Theorem 5.2.1. Since Fig. 5.1(d) describes delay-independent result, the precise way to read delay-dependent results in Figs. 5.1(b), (c), is to subtract the region in Fig. 5.1(d) from those regions in Figs. 5.1(b), (c).

As mentioned previously, system (2.6) can be synchronized if the self-feedback is inhibitory and strong with small $\tau_I$, or the coupling is excitatory and strong with small $\tau_T$. However, there is a qualitative difference between these two synchrony. The first synchrony is actually global convergence to the origin, whereas the second one is global convergence to multiple synchronous equilibria. The precise statement is given in Remark 5.2.3.

If $\tau_I = \tau_T$ is considered, then stronger results can be obtained. Namely, system (2.6) can be synchronized if $|\alpha-\beta| < \mu$ in spite of delay or $\beta-\alpha \geq \mu$ and $\tau_I$ is small. These parameter regions are depicted in Figs. 5.2(a), (b); notice that union of these regions are larger than union of the ones in Figs. 5.1(a)-(d). The detailed statements are summarized in Theorem 5.2.2. These results indicate that system (2.6) without delays ($\tau_I = \tau_T = 0$) can be synchronized if $\beta - \alpha > -\mu$. This can be interpreted as sufficiently strong inhibitory self-feedback or excitatory coupling can synchronize system (2.6) without delays. It is then natural to ask whether if sufficiently strong inhibitory self-feedback or excitatory coupling can also synchronize system (2.3) with delays. We shall see from Theorem 5.2.8 and Remark 5.2.4 that this depends on the delay size. Indeed, once the self-feedback strength $\alpha$ (resp. coupling strength $\beta$) is sufficiently stronger than coupling strength (resp. self-feedback strength), then synchrony for network $(2.6)_0$ with nonzero delays ($\tau_I$ and $\tau_T$ may be distinct) can be lost and nontrivial asynchronous oscillations are bifurcated from the origin at delay magnitude $\tau_I$ (resp. $\tau_T$) near bifurcation values (there are infinitely many such values). This highlights the difference between the effects from the self-feedback or coupling upon synchronization of the coupled network with delays and without delays.

Now, let us recall the conjecture in [8] ($\mu = 1$ therein): If $|\beta| < |1-\alpha|$ and $0 \leq \tau_I < \tau_S^{(1)}$ for some $\tau_S^{(1)}$, or $|\beta| < (|1-\alpha|)/2$ and $0 \leq \tau_I < \tau_S^{(2)}$ for

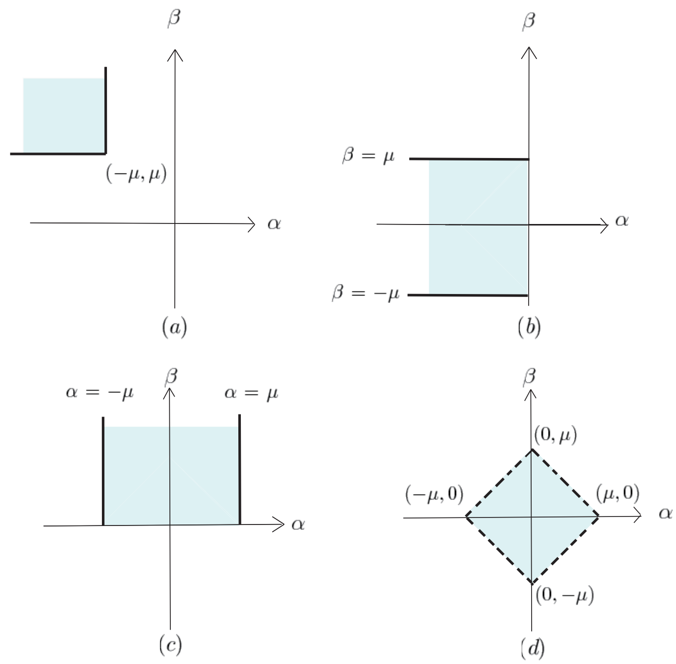



Figure 5.1: (a) The region of $(\alpha, \beta)$ that admits synchronization while $\tau_T$ and $\tau_T$ are small. (b) The region of $(\alpha, \beta)$ that admits synchronization in spite of $\tau_T$ while $\tau_I$ is small. (c) The region of $(\alpha, \beta)$ that admits synchronization in spite of $\tau_I$ while $\tau_T$ is small. (d) The region of $(\alpha, \beta)$ that admits synchronization in spite of $\tau_I$ and $\tau_T$.

some $\tau_S^{(2)}$, then (2.6) can be synchronized for all $\tau_T \geq 0$. Roughly speaking, these conditions require that $|\alpha|$ is relatively larger than $|\beta|$ and $\tau_I$ is small enough; i.e., the synchronization can be determined merely by the magnitude of coupling (not the sign of the coupling). This is incompatible with our result that "inhibitory" self-feedback strength ($\alpha < 0$, $|\alpha|$ large) is crucial for synchrony of system (2.6). In an example with parameters satisfying the condition of the conjecture and $\alpha > 0$ in Section 5.4, we illustrate that there exists a solution which converges to an asynchronous equilibrium. However, the conjecture may be positive under additional condition that $\alpha < \mu$ ($\mu = 1$ therein). We show in Theorem 5.2.1 that the conjecture holds for the parameter region in Fig. 5.1(b) with small $\tau_I$ (the smallness restriction on $\tau_I$ does not correspond to the one in [8]). As the self-feedback strength is strong but $\tau_I$ is not small, that $\tau_I$ can induce asynchrony provides an evidence to demonstrate that the magnitude of $\tau_I$ does matter for synchronization of system (2.6), for parameters in this range.

**Convergence and stability.**

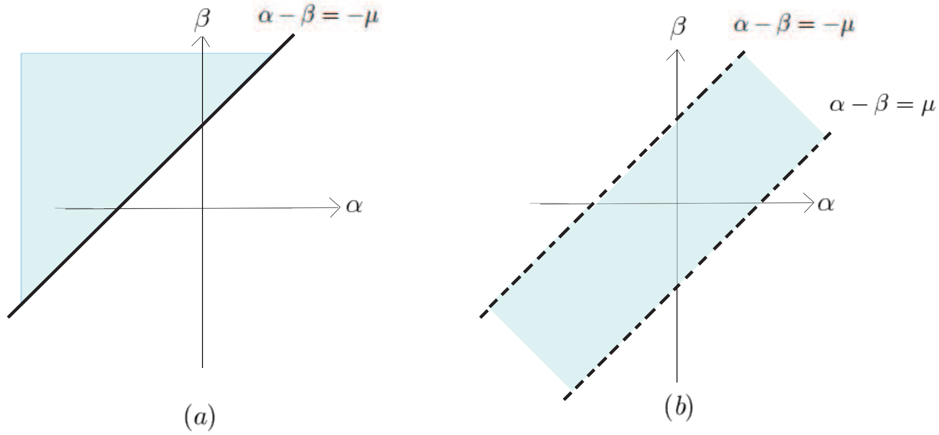



Figure 5.2: System (2.6) with $\tau_I = \tau_T$ attains synchronization if $(\alpha, \beta)$ lies in shaded region in (a) and $\tau_I = \tau_T$ is small, (b) in spite of delays.

System (2.6) of scale $N = 3$, with $\mu = 1$ and $\tau_I = \tau_T$ is considered [73]. Therein, the following results are concluded or addressed.

(i) the system achieves synchronization in spite of delay if $|\alpha - \beta| < 1$, cf. Fig. 5.2(b).

(ii) The system has three equilibria $(0, 0, 0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$ and $\mathbf{x}^- := (u^-, u^-, u^-)$ if $(\alpha, \beta) \in D_1 \cup D_2$, where

$$D_1 := \{(\alpha, \beta) : \alpha - \beta \leq -1 \text{ and } \alpha + 2\beta > 1\}, \tag{5.1}$$

$$D_2 := \{(\alpha, \beta) : |\alpha - \beta| < 1 \text{ and } \alpha + 2\beta > 1\}, \tag{5.2}$$

cf. Figs. 5.3(a), (b); moreover, the trivial equilibrium is unstable and the others are stable if $(\alpha, \beta) \in D_2$.

(iii) It was conjectured that the generic dynamics for system (2.6) is the convergence to $\mathbf{x}^{\pm}$, if $(\alpha, \beta) \in D_2$.

With 1 replaced by $\mu$ in (5.1), (5.2), we derive the following results including the unsolved in [73]:

(si) In addition to the above (i), the system achieves global synchronization if $\alpha - \beta \leq -\mu$, as the time lag is small, cf. Fig. 5.2(a).

(sii) $\mathbf{x}^{\pm}$ are stable in spite of delays as $(\alpha, \beta) \in D_1$ and $\alpha \geq 0$, $\beta \geq 0$;

(siii) The system achieves global convergence to the equilibria if $(\alpha, \beta) \in D_1 \cup D_2$ and the time lag is small, cf. Fig. 5.4(b).

These results are stated in Theorems 5.2.2, 5.2.3, and 5.2.5 respectively. Furthermore, parallel results to (si), (sii), (siii) can also be derived for system (2.3) with $\tau_I \neq \tau_T$, cf. Theorems 5.2.1, 5.2.3, and 5.2.4 respectively. There is a distinction between multistability of (2.6) induced from strong excitatory coupling and strong excitatory self-feedback. An extension of the investigations in [15, 16, 61] or Chapter 4 leads to the convergence to $3^n$ synchronous and asynchronous equilibria if the self-feedback strength is excitatory and sufficiently stronger than the coupling strength. Thus, we may say that "strong excitatory self-feedback"-induced multistability of (2.6) comprises coexistence of synchronous and asynchronous equilibria. On the other hand, "strong excitatory coupling"-induced multistability of (2.6) consists of multiple synchronous equilibria.

**Oscillation and asynchrony induced by delays.**

In the previous description, some criteria for synchrony and convergence of system (2.6) are dependent on delays (small size is favorable) and some are not. It is natural to ask how delays affect the dynamics of the system. We shall use the existence of standing wave as an evidence of asynchrony for system $(2.6)_0$. Roughly speaking, once the self-feedback (resp. coupling) strength is sufficiently strong, there exist synchronous and asynchronous nontrivial oscillations bifurcated from the origin as the corresponding delay $\tau_I$ (resp. $\tau_T$) is near certain values (there are infinitely many such values). Notably, the bifurcation scenario reveals that along the way of increasing $|\beta|$ (resp. $|\alpha|$), synchronous oscillations (resp. asynchronous oscillations) first appear at delay $\tau_T$ of magnitude (resp. $\tau_I$) near bifurcation values; after $|\beta|$ (resp. $|\alpha|$) passes certain value, both synchronous and asynchronous oscillations take place at delay $\tau_T$ (resp. $\tau_I$) of magnitude near bifurcation values. Detailed descriptions for these finding are stated in Theorem 5.2.8 and Remark 5.2.4.

Now, let us summarize some coarse-grained description on the collective dynamics for (2.6).

1. Small scale of the network, large self-decay, inhibitory self-feedback (resp. excitatory coupling) is advantageous for synchronization of (2.6), and the corresponding delay $\tau_I$ (resp. $\tau_T$) is required to be small if $|\alpha|$ (resp. $|\beta|$) is large.

2. Inhibitory self-feedback and excitatory coupling lead to distinct synchronous phases. Namely, strong inhibitory self-feedback promotes the convergence to the origin, while strong excitatory coupling advances the convergence to nontrivial synchronous equilibria, as delays are small.

3. Sufficiently strong inhibitory self-feedback or excitatory coupling can always synchronize (2.6) if the network is without delays, but may fail to synchronize if the

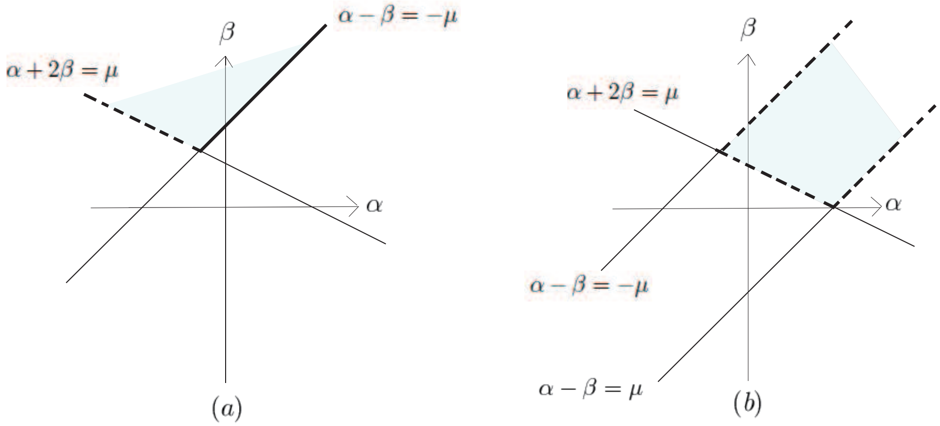



Figure 5.3: System (2.6) with parameters in regions (a) and (b), admits exactly three synchronous equilibria.

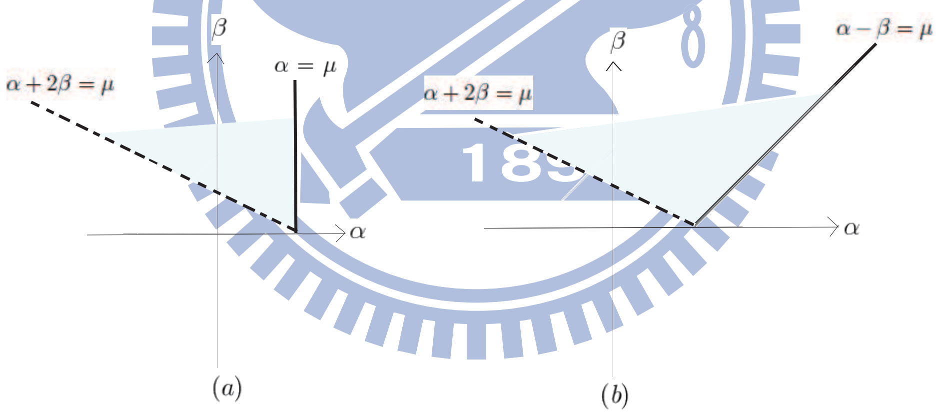



Figure 5.4: System (2.6) with parameters in shaded region in (a) and delays $\tau_I$ and $\tau_T$ are small, in (b) and $\tau_I = \tau_T$ is small, admits convergence to multiple equilibria.

network is with delays of substantial magnitude.

4. "Strong excitatory self-feedback"-induced multistability admits coexistence of synchronous and asynchronous equilibria, whereas "strong excitatory coupling"-induced multistability admits existence of synchronous equilibria.

5. The delay $\tau_I$ (resp. $\tau_T$) can lead to the emergence of synchronous or asynchronous nontrivial oscillation if the self-feedback strength (resp. coupling) is strong. The occurrence of synchronous and asynchronous oscillations are in order as the strength $|\beta|$ or $|\alpha|$ increases, cf. Remark 5.2.4.

6. The synchronization of network (2.6) may also depend on scale of the network. There exists a notable distinction in synchronization between systems (2.6) of scale $N = 3$ and $N > 3$.

## 5.2 Dynamics of the network with N=3

In this section, we focus on (2.6) of scale $N = 3$ to establish the synchronization and convergence to multiple synchronous equilibria of the network. Moreover, delayed Hopf bifurcation theory is employed to conclude the existence of nontrivial synchronous and asynchronous oscillations (standing waves) induced by transmission delay $\tau_T$. Notably, system (2.6) is a dissipative system, hence solution evolved from any initial condition $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^3)$ exists for all time $t \geq t_0$.

### 5.2.1 Global synchronization

We shall derive criteria for the global synchronization of (2.6); namely,

$$x_i(t) - x_{i+1}(t) \to 0, \text{ as } t \to \infty, \ i = 1, 2,$$

for every solution $(x_1(t), x_2(t), x_3(t))$ of (2.6). To this end, we consider the following difference system obtained from subtracting $x_{i+1}$-component from $x_i$-component in (2.6):

$$\dot{z}_i(t) = -\mu z_i(t) + \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))] - \beta[g(x_i(t-\tau_T)) - g(x_{i+1}(t-\tau_T))], \tag{5.3}$$

where $z_i(t) := x_i(t) - x_{i+1}(t)$, $i = 1, 2$. Obviously, each component of (5.3) satisfies (3.10) with $\gamma_1 = -\alpha$, $\gamma_2 = \beta$, $\tau_1 = \tau_I$, $\tau_2 = \tau_T$ and $w(t) = 0$. Moreover, $x_i(t)$ and $x_{i+1}(t)$ are eventually attracted by $[-(|\alpha| + 2|\beta|)/\mu, (|\alpha| + 2|\beta|)/\mu]$, as seen from the equation for $x_i$ and $x_{i+1}$ in (2.6). We denote

$$\hat{\alpha} := \begin{cases} \alpha, & \alpha \geq 0, \\ \alpha\tilde{L}, & \alpha < 0, \end{cases} \quad \check{\alpha} := \begin{cases} \alpha\tilde{L}, & \alpha \geq 0, \\ \alpha, & \alpha < 0, \end{cases} \tag{5.4}$$

$$\hat{\beta} := \begin{cases} \beta, & \beta \geq 0, \\ \beta\tilde{L}, & \beta < 0, \end{cases} \quad \check{\beta} := \begin{cases} \beta\tilde{L}, & \beta \geq 0, \\ \beta, & \beta < 0, \end{cases} \tag{5.5}$$

where

$$\tilde{L} := \min\{g'(\xi) : \xi \in [-(|\alpha| + 2|\beta|)/\mu, (|\alpha| + 2|\beta|)/\mu]\}. \tag{5.6}$$

Now, let us introduce four different conditions for synchronization of network (2.6).

Condition (S1b): $-\hat{\alpha} + \check{\beta} \geq 0$, $\tau_I|\alpha| + \tau_T|\beta| \leq \mu/(2\mu - \check{\alpha} + \hat{\beta})$;
restated,

$$\begin{cases} \beta \geq (1/\tilde{L})\alpha \text{ and } \tau_I|\alpha| + \tau_T|\beta| \leq \mu/(2\mu - \alpha\tilde{L} + \beta,) & \text{if } \alpha \geq 0, \beta \geq 0, \\ \tau_I|\alpha| + \tau_T|\beta| \leq \mu/(2\mu - \alpha + \beta), & \text{if } \alpha \leq 0, \beta \geq 0, \\ \beta \geq \tilde{L}\alpha \text{ and } \tau_I|\alpha| + \tau_T|\beta| \leq \mu/(2\mu - \alpha + \beta\tilde{L}), & \text{if } \alpha \leq 0, \beta \leq 0; \end{cases}$$

Condition (S2b): $\alpha = 0$, $|\beta| \leq \mu$; or $\alpha < 0$, $|\beta| \leq \mu$, $\tau_I \leq (\mu - |\beta|)/[\alpha(\alpha - 2\mu)]$;

Condition (S3b): $\beta = 0$, $|\alpha| \leq \mu$; or $\beta > 0$, $|\alpha| \leq \mu$, $\tau_T \leq (\mu - |\alpha|)/[\beta(\beta + 2\mu)]$;

Condition (S4b): $|\alpha| + |\beta| < \mu$.

It can be verified that each $i$th component of (5.3) satisfies condition (H1b) (resp. (H2b), (H3b)) under condition (S1b) (resp. (S2b), (S4b)), $i = 1, 2$. According to Theorem 3.1.9 (resp. Theorem 3.1.10, 3.1.11) with $w(t) = 0$, we conclude that $z_i(t) \to 0$, as $t \to \infty$, for every $z_i$ satisfying (5.3), $i = 1, 2$. Therefore, network (2.6) can be synchronized under one of conditions (S1b), (S2b) and (S4b). On the other hand, each $i$th component of (5.3) can also be regarded in the form as (3.10) with $\gamma_1 = \beta$, $\gamma_2 = -\alpha$, $\tau_1 = \tau_T$, $\tau_2 = \tau_I$ and $w(t) = 0$. Then every $i$th component system of (5.3) satisfies condition (H2b) under condition (S3b); hence $z_i(t) \to 0$, as $t \to \infty$, for every $z_i$ satisfying (5.3). Accordingly, (2.6) can be synchronized under condition (S3b). We thus conclude the following result.

**Theorem 5.2.1.** System (2.6) with $N = 3$ achieves global synchronization under one of conditions (S1b)-(S4b).

Notably, (S1b)-(S4b) are all sufficient conditions for synchronization of system (2.6). Observe that condition (S4b) is delay-independent; condition (S3b) is

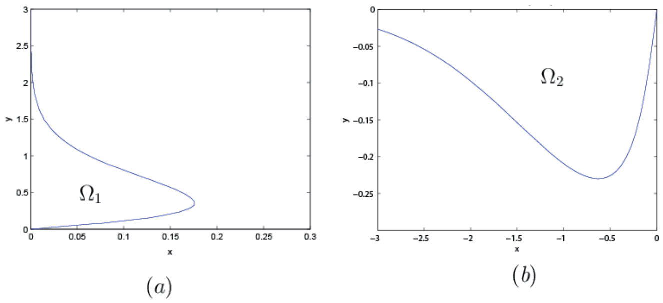



Figure 5.5: (a) $\Omega_1 = \{(\alpha, \beta) : \alpha \geq 0, \beta \geq 0 \text{ and } \beta \geq (1/\tilde{L})\alpha\}$; $\Omega_2 = \{(\alpha, \beta) : \alpha \leq 0, \beta \leq 0 \text{ and } \beta \geq \tilde{L}\alpha\}$, as $g(\xi) = \tanh(\xi)$ and $\mu = 1$.

$\tau_I$-independent or delay-independent on $\alpha$-axis with $|\alpha| < \mu$; condition (S2b) is $\tau_T$-independent or delay-independent on $\beta$-axis with $|\beta| < \mu$; (S1b) is $(\tau_I, \tau_T)$-dependent. In conditions (S1b)-(S3b), the inequalities involving delays $\tau_I, \tau_T$ all hold if $\tau_I$ and/or $\tau_T$ are small enough. The parameters $(\alpha, \beta)$ which satisfy the inequalities uninvolved with delays in conditions (S1b)-(S4b) are depicted in Figs. 5.1(a)-(d) respectively. In particular, let us interpret the region in Fig. 5.1(a) First, notice that the term $\tilde{L}$ in condition (S1b) actually depends on $\mu, \alpha, \beta$, cf. (5.6). The parameter $(\alpha, \beta)$ satisfying condition (S1b) may lie in the first, second or third quadrants of $(\alpha, \beta)$-plane. Indeed, condition (S1b) is always satisfied if $\beta$ is positive and $\alpha$ is negative, i.e., the second quadrant of $(\alpha, \beta)$-plane. However, in general, those parameters $(\alpha, \beta)$ satisfying condition (S1b) and lying in the first (resp. third) quadrant actually also lie in the parameter region depicted in Figs. 5.1(c) (resp. (b)), cf. Fig. 5.5. Note that Fig. 5.1(b) (resp. 5.1(c)) corresponds to condition (S2b) (resp. (S3b)) which provides $\tau_T$-independent (resp. $\tau_I$-independent) result. Therefore, precise reading of the parameter region for the $(\tau_I, \tau_T)$-dependent result under condition (S1b) is to subtract the parameter regions satisfying condition (S2b) or (S3b) from the second quadrant of $(\alpha, \beta)$-plane; as depicted in Fig. 5.1(a).

**Remark 5.2.1.** (i) These parameter regimes indicate that if the self-feedback (resp.coupling) strength is strong, then the self-feedback has to be inhibitory for synchronization of system (2.6); on the other hand, if the coupling strength is strong, then the coupling has to be excitatory for synchronization of system (2.6). (ii) Extracting from the results of Theorem 5.2.1, it can be observed roughly that large self-decay, inhibitory self-feedback and excitatory coupling are advantageous for (2.6) to be synchronized, while $\tau_I$ (resp. $\tau_T$) is required to be small if $|\alpha|$ (resp. $|\beta|$) is large. (iii) It will be shown that as $|\alpha|$ or $|\beta|$ gets larger, delay $\tau_I$ or $\tau_T$ can generate asynchrony, cf. Theorem 5.2.8 and Remark 5.2.4, and a numerical illustration in Example 5.4.2.

If we consider in particular $\tau_I = \tau_T$ for (2.6), then each $i$th component of (5.3) satisfies (3.10) with $\gamma_1 = -(\alpha - \beta)$, $\tau_1 = \tau_I$, $\gamma_2 = 0$ and $w(t) = 0$. We thus derive the following result.

**Theorem 5.2.2.** System (2.6) with $N = 3$ and $\tau_I = \tau_T$ attains global synchronization under one of the following conditions:
(i) $\alpha - \beta \le -\mu$ and $\tau_I = \tau_T \le \mu/[(\beta - \alpha)(2\mu - \alpha + \beta)]$;
(ii) $|\alpha - \beta| < \mu$.

**Remark 5.2.2.** (i) The parameter conditions in Theorem 5.2.2 are depicted in Figs. 5.2(a), (b) respectively. (ii) If we consider (2.6) without delays, i.e., $\tau_I = \tau_T = 0$, then by Theorem 5.2.2(i), sufficiently small $\alpha$ (resp. large $\beta$) yields synchronization of the system. It indicates that strong inhibitory self-feedback or excitatory coupling can synchronize (2.6) without delays.

### 5.2.2 Global convergence to multiple equilibria

In this subsection, we first investigate the stability of nontrivial synchronous equilibria $\mathbf{x}^{\pm}$ as $(\alpha, \beta) \in D_1$, then we derive criteria for global convergence to these synchronous equilibria for (2.6) with $(\alpha, \beta) \in D_1 \cup D_2$, where

$$D_1 := \{(\alpha, \beta) : \alpha - \beta \le -\mu \text{ and } \alpha + 2\beta > \mu\},$$
$$D_2 := \{(\alpha, \beta) : |\alpha - \beta| < \mu \text{ and } \alpha + 2\beta > \mu\}.$$

**Theorem 5.2.3.** System (2.6) with $N = 3$ has exactly three equilibria $(0, 0, 0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$ and $\mathbf{x}^- := (u^-, u^-, u^-)$ with $u^+ > 0$ and $u^- < 0$, if $(\alpha, \beta) \in D_1 \cup D_2$. If $(\alpha, \beta) \in D_1$ and $\alpha \ge 0$, $\beta \ge 0$, or $(\alpha, \beta) \in D_2$, then $\mathbf{x}^{\pm}$ is stable in spite of delays.

**Proof.** The existence of equilibria for $(\alpha, \beta) \in D_1 \cup D_2$ and the stability of $\mathbf{x}^{\pm}$ for $(\alpha, \beta) \in D_2$ can be established by similar arguments as in [73]. It remains to verify the stability of $\mathbf{x}^{\pm}$ for $(\alpha, \beta) \in D_1$, and $\alpha \ge 0$ and $\beta \ge 0$. The linearization of (2.6) about $\mathbf{x}^{\pm}$ is given by

$$\dot{v}_i(t) = -\mu v_i(t) + \alpha g'(u^+)v_i(t-\tau_I) + \beta g'(u^+)[v_{i-1}(t-\tau_T) + v_{i+1}(t-\tau_T)],\ i = 1,2,3. \tag{5.7}$$

Thus the characteristic equation for (5.7) is

$$\begin{aligned}
&\Delta_1(\lambda)\Delta_2^2(\lambda) = 0,\\
&\Delta_1(\lambda) := \mu + \lambda - \alpha g'(u^+)e^{-\lambda\tau_I} - 2\beta g'(u^+)e^{-\lambda\tau_T},\\
&\Delta_2(\lambda) := \mu + \lambda - \alpha g'(u^+)e^{-\lambda\tau_I} + \beta g'(u^+)e^{-\lambda\tau_T}.
\end{aligned}$$

We substitute $\lambda = \nu + iw$ with $w > 0$ into $\Delta_1(\lambda) = 0$, and collect the real and imaginary parts and obtain

$$\begin{aligned}
\nu + \mu &= g'(u^+)[\alpha e^{-\nu\tau_I}\cos(\tau_I w) + 2\beta e^{-\nu\tau_T}\cos(\tau_T w)],\\
w &= g'(u^+)[-\alpha e^{-\nu\tau_I}\sin(\tau_I w) - 2\beta e^{-\nu\tau_T}\sin(\tau_T w)].
\end{aligned}$$

Summing up the square of the equations gives

$$I_1(\nu) = I_2(\nu), \tag{5.8}$$

where $I_1(\nu) = (\nu+\mu)^2 + w^2$, $I_2(\nu) = [g'(u^+)]^2[\alpha^2 e^{-2\nu\tau_I} + 4\beta^2 e^{-2\nu\tau_T} + 4\alpha\beta e^{-\nu(\tau_I+\tau_T)}\cos((\tau_I - \tau_T)w)]$. Note that $u^+$ satisfies the stationary equation: $-\mu x + (\alpha + 2\beta)g(x) = 0$ which admits exactly three zeros $e_1$, $e_2$ and 0, where $e_1 < p^* < 0$, $0 < q^* < e_2$, and

$$g'(p^*) = g'(q^*) = \mu/(\alpha + 2\beta). \tag{5.9}$$

Obviously, $u^+ = e_2$; hence $g'(u^+) < \mu/(\alpha + 2\beta)$. If $\nu \geq 0$, then a contradiction to (5.8) occurs since $I_2(\nu) < [\mu/(\alpha + 2\beta)]^2[\alpha^2 + 4\beta^2 + 4\alpha\beta] = \mu^2 \leq I_2(\nu)$. Therefore, $\nu < 0$. If we substitute $\lambda = \nu + iw$ with $w > 0$ into $\Delta_2(\lambda) = 0$, it can also be verified that $\nu < 0$ by similar arguments. The proof is thus completed. □

**Theorem 5.2.4.** System (2.6) with $N = 3$ admits exactly three equilibria $(0,0,0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$, $\mathbf{x}^- := (u^-, u^-, u^-)$, and every solution of the system converges to one of these equilibria, under one of the following conditions:
(i) $\alpha \leq 0, \beta \geq 0$, $\alpha + 2\beta > \mu$, $|\alpha|\tau_I + |\beta|\tau_T < \mu/(2\mu - \alpha + \beta)$ and $|\alpha|\tau_I + 2|\beta|\tau_T < \tau^*$,
(ii) $\beta > 0$, $|\alpha| \leq \mu$, $\alpha + 2\beta > \mu$, $\tau_T \leq (\mu - |\alpha|)/[\beta(\beta + 2\mu)]$ and $|\alpha|\tau_I + 2|\beta|\tau_T < \tau^*$,
where

$$\tau^* := \min\{1/4, [(\alpha+2\beta)g(q^*) - \mu q^*]/[4(|\alpha|+2|\beta|)], [\mu p^* - (\alpha+2\beta)g(p^*)]/[4(|\alpha|+2|\beta|)]\}.$$

**Proof.** We only prove the first case, since the others can be treated similarly. We arrange (2.6) in the form

$$\dot{x}_i(t) = -\mu x_i(t) + \alpha f(x_i(t-\tau_I)) + 2\beta g(x_i(t-\tau_T)) + E_i(t), \tag{5.10}$$

where $E_i(t) = \beta[g(x_{i-1}(t-\tau_T)) + g(x_{i+1}(t-\tau_T)) - 2g(x_i(t-\tau_T))]$. Owing to $\alpha \leq 0$, $\beta \geq 0$ and $\alpha + 2\beta > \mu$, (2.6) has exactly three equilibria $(0,0,0)$, $\mathbf{x}^+$ and $\mathbf{x}^-$, by Theorem 5.2.3. Moreover, since $\alpha \leq 0$, $\beta \geq 0$ and $|\alpha|\tau_I + |\beta|\tau_T < \mu/(2\mu - \alpha + \beta)$, the system achieves global synchronization according to Theorem 5.2.1. Therefore, $E_i(t) \to 0$, as $t \to \infty$. Obviously, each component of (5.10) satisfies (3.22) with $\gamma_1 = \alpha$, $\gamma_2 = 2\beta$, $\tau_1 = \tau_I$, and $\tau_2 = \tau_T$. Note that $\bar{p}_\gamma = p^*$, $\bar{q}_\gamma = q^*$, as $\gamma_1 = \alpha$, $\gamma_2 = 2\beta$. Under the assumption of this theorem, every $i$th component of (5.10) satisfies condition (Ab). According to Theorem 3.2.6, every $i$th component $x_i(t)$ satisfying (5.10) converges to an element of $\{u^+, 0, u^-\}$. Hence the assertion is verified. □

If $\tau_I = \tau_T$ in (2.6) is considered in particular, we can modify the convergent criteria for (2.6) with multiple synchronous equilibria. We give the result without proof, as it is similar to the one for Theorem 5.2.4.

**Theorem 5.2.5.** System (2.6) with $\tau_I = \tau_T$ admits exactly three equilibria $(0,0,0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$ and $\mathbf{x}^- := (u^-, u^-, u^-)$ and all solutions of the system converge to one of these equilibria if (i) $(\alpha, \beta) \in D_1$ and $\tau_I = \tau_T < \{\tilde{\tau}, \mu/[(\beta-\alpha)(2\mu-\alpha+\beta)]\}$, or (ii) $(\alpha, \beta) \in D_2$ and $\tau_I = \tau_T < \tilde{\tau}$, where

$$\tilde{\tau} := \frac{\min\{1/4, [(\alpha+2\beta)g(p^*) - \mu p^*]/[4(\alpha+2\beta)], [\mu q^* - (\alpha+2\beta)g(q^*)]/[4(\alpha+2\beta)]\}}{\alpha+2\beta}.$$

The regions of $(\alpha, \beta)$ which admit convergence to multiple equilibria in Theorems 5.2.3, 5.2.4 are depicted in Fig. 5.4(a), (b) respectively.

**Remark 5.2.3.** (i) In Theorem 5.2.3 and 5.2.4, $\alpha + 2\beta > \mu$ and other conditions yield the existence of multiple equilibria for (2.6). Indeed, if $\alpha + 2\beta$ is sufficiently small instead, we can modify Theorem 5.2.3 and 5.2.4 to conclude that the system achieve global convergence to zero if delays $\tau_I$ and $\tau_T$ are small, cf. Remark 3.2.1. (ii) As mentioned in Remark 5.2.1(ii) and Remark 5.2.2(ii), inhibitory self-feedback or excitatory coupling is advantageous for (2.6) to be synchronized. However, there exist qualitative difference between the situations of strong inhibitory self-back and strong excitatory coupling. Roughly speaking, if the self-feedback is inhibitory (resp. coupling is excitatory) and sufficiently strong, then $\alpha + 2\beta$ is small (resp. large); consequently system (2.6) achieves global convergence to single equilibrium (resp. multiple equilibria) when delays are small.

### 5.2.3 Synchronous and asynchronous oscillations

In this subsection, we shall present the existence of nontrivial synchronous and asynchronous periodic solutions for (2.6) induced by transmission delay $\tau_T$. Similar discussions can be performed for bifurcation induced by self-feedback delay $\tau_I$. To focus on the effect of parameters $\alpha$ and $\beta$ upon the oscillations induced by delays, we set the parameter $\mu = 1$ in this subsection. Moreover, we shall seek for the existence of the standing wave solutions to serve as an evidence for asynchrony of (2.6). Due to the coupling topology of system (2.6),

$$\mathcal{S} := \{(y, y, y) : y \in C([-\tau_{\max}, 0]; \mathbb{R}\}$$

is positively invariant under the flow generated by system (2.6). On the other hand, $\mathcal{S}$ and $\mathcal{A}_\sigma$ are both positively invariant under the flow generated by system $(2.6)_0$, where

$$\mathcal{A}_\sigma := \{(y_1, y_2, y_3) : y_i = 0, y_j = -y_k \in C([-\tau_{\max}, 0]; \mathbb{R}), (i, j, k) = \sigma(1, 2, 3)\},$$

and $\sigma(1, 2, 3)$ is a permutation of index $(1, 2, 3)$. Hence, it allows us to consider system $(2.6)_+$ (resp. $(2.6)_-^\sigma$) which is a restriction of (2.6) on $\mathcal{S}$ (resp. $(2.6)_0$ on $\mathcal{A}_\sigma$) and consider only evolutions from points in $S$ (resp. $\mathcal{A}_\sigma$). First, let us focus on system $(2.6)_+$. Every component of $(2.6)_+$ satisfies

$$\dot{y}(t) = -y(t) + \alpha g(y(t - \tau_I)) + 2\beta g(y(t - \tau_T)). \tag{5.11}$$

The linearized system at the origin of (5.11) is

$$\dot{v}(t) = -v(t) + \alpha v(t - \tau_I) + 2\beta v(t - \tau_T). \tag{5.12}$$

Thus the characteristic equation for (5.12) is

$$\Delta_+(\lambda) := (1 + \lambda - \alpha e^{-\lambda\tau_I} - 2\beta e^{-\lambda\tau_T}) = 0. \tag{5.13}$$

We substitute $\lambda = iw$ with $w > 0$ into $\Delta_+(\lambda) = 0$ and collect the real and imaginary parts to yield

$$\begin{cases} 2\beta\cos(\tau_T w) = 1 - \alpha\cos(\tau_I w), \\ 2\beta\sin(\tau_T w) = -w - \alpha\sin(\tau_I w). \end{cases} \tag{5.14}$$

Summing up squares of these equations (5.14) gives

$$Q(w) = 4\beta^2, \tag{5.15}$$

where $Q(w) := w^2 + 2\alpha \sin(\tau_I w)w - 2\alpha \cos(\tau_I w) + \alpha^2 + 1$. Obviously, for all $w \geq 0$,

$$Q(w) \leq \tilde{Q}(w),$$

where $\tilde{Q}(w) := w^2+2|\alpha|w+(1+|\alpha|)^2$ is increasing for all $w \geq 0$. Direct computation gives $Q'(w) = [2 + 2\tau_I\alpha\cos(\tau_I w)]w + 2\alpha(1 + \tau_I)\sin(\tau_I w)$. Then,

$$Q'(w) \geq P(w), \text{ for all } w \geq 0, \tag{5.16}$$

where $P(w) = (2 - 2\tau_I|\alpha|)w - 2|\alpha|(1 + \tau_I)$. Obviously, if $\tau_I|\alpha| < 1$, $P(w) > 0$ for all $w \geq \tilde{w} := |\alpha|(1 + \tau_I)/(1 - \tau_I|\alpha|) \geq 0$. Therefore, $Q(w)$ is increasing on $[\tilde{w}, \infty)$ thanks to (5.16). Now, let us introduce the condition to guarantee the existence of purely imaginary roots of $\Delta_+(\lambda) = 0$.

Condition $(\text{B1b})_+$: $\tau_I|\alpha| < 1$ and $\tilde{Q}(\tilde{w}) < 4\beta^2$;

i.e.,

$$4\beta^2 > (1 + |\alpha|)^2 + |\alpha|^2[(\frac{1+\tau_I}{1-\tau_I|\alpha|})^2 + 2(\frac{1+\tau_I}{1-\tau_I|\alpha|})].$$

Notice that $Q(w) \leq \tilde{Q}(w)$ for all $w \geq 0$; $\tilde{Q}(w)$ is increasing for all $w \geq 0$; and $Q(w)$ is increasing on $[\tilde{w}, \infty)$. Subsequently, (5.15) admits exactly one positive zero, say $\omega^*_+$, under condition $(\text{B1b})_+$. We thus conclude that

**Lemma 5.2.6.** There exists exactly one pair of purely imaginary roots, say $\pm i\omega^*_+$, for characteristic equation (5.13) under condition $(\text{B1b})_+$. Herein, $\omega^*_+$ is the unique positive zero to (5.15).

On the other hand, every nontrivial component of $(2.6)^\sigma_-$ satisfies

$$\dot{y}(t) = -y(t) + \alpha g(y(t - \tau_I)) - \beta g(y(t - \tau_T)). \tag{5.17}$$

The linearized system at the origin of (5.17) is

$$\dot{v}(t) = -v(t) + \alpha v(t - \tau_I) - \beta v(t - \tau_T). \tag{5.18}$$

Then the characteristic equation for (5.18) is

$$\Delta_-(\lambda) := (1 + \lambda - \alpha e^{-\lambda\tau_I} + \beta e^{-\lambda\tau_T}) = 0. \tag{5.19}$$

We substitute $\lambda = iw$ with $w > 0$ into $\Delta_-(\lambda) = 0$ and collect the real and imaginary parts and obtain

$$\begin{cases} -\beta \cos(\tau_T w) = 1 - \alpha \cos(\tau_I w), \\ -\beta \sin(\tau_T w) = -w - \alpha \sin(\tau_I w). \end{cases} \tag{5.20}$$

Summing up the square of equations (5.20), we get

$$Q(w) = \beta^2. \tag{5.21}$$

Now, let us introduce the condition to guarantee the existence of purely imaginary roots for $\Delta_-(\lambda) = 0$.

Condition $(\text{B1b})_-$: $\tau_I|\alpha| < 1$ and $\tilde{Q}(\tilde{w}) < \beta^2$;

i.e.,

$$\beta^2 > (1 + |\alpha|)^2 + |\alpha|^2[(\frac{1+\tau_I}{1-\tau_I|\alpha|})^2 + 2(\frac{1+\tau_I}{1-\tau_I|\alpha|})].$$

**Lemma 5.2.7.** There exists exactly one pair of purely imaginary roots, say $\pm i\omega_-$, for characteristic equation (5.19) under condition $(\text{B1b})_-$. Herein, $\omega_-^*$ is the unique positive zero to (5.21).

To find the value of $\tau_T$ such that $\pm i\omega_\pm^*$ are the purely imaginary roots of $\Delta_\pm(\lambda) = 0$, we divide the second equation by the first of (5.14) or (5.20). Then

$$\tan(\tau_T w) = S(w)/C(w),$$
$$S(w) := -w - \alpha \sin(\tau_I w),$$
$$C(w) := 1 - \alpha \cos(\tau_I w).$$

Let us define

$$\eta_k^\pm := \frac{1}{\omega_\pm^*} \begin{cases} 3\pi/2 + 2(k-1)\pi, & \text{if } C(\omega_\pm^*) = 0,\ S(\omega_\pm^*) < 0, \\ \pi/2 + 2(k-1)\pi, & \text{if } C(\omega_\pm^*) = 0,\ S(\omega_\pm^*) > 0, \\ \tan^{-1}(S(\omega_\pm^*)/C(\omega_\pm^*)) + 2k\pi, & \text{if } C(\omega_\pm^*) > 0, \\ \tan^{-1}(S(\omega_\pm^*)/C(\omega_\pm^*)) + (2k-1)\pi, & \text{if } C(\omega_\pm^*) < 0. \end{cases} \tag{5.22}$$

Herein, $\eta_k^\pm$ is positive and $\Delta_\pm(\lambda) = 0$ has exactly one pair of purely imaginary roots $\pm\omega_\pm^*$ at the bifurcation value $\tau_T = \eta_k^\pm$. To apply the Hopf bifurcation theory, it suffices to verify the transversality condition:

Condition $(\text{B2b})_\pm$: $[\mathbf{R}(\omega_\pm^*, \eta_k^\pm)]^2 + [\mathbf{I}(\omega_\pm^*, \eta_k^\pm)]^2 \neq 0$, and $\Lambda(\omega_\pm^*) \neq 0$, where

$$\begin{aligned}
\mathbf{R}(\omega, \tau_T) &:= 1 + \tau_T + \alpha(\tau_I - \tau_T)\cos(\tau_I w), \\
\mathbf{I}(\omega, \tau_T) &:= \tau_T w - \alpha(\tau_I - \tau_T)\sin(\tau_I w), \\
\Lambda(\omega) &:= [1 + \alpha\tau_I\cos(\tau_I\omega)]\omega^2 + \alpha(1+\tau_I)\sin(\tau_I\omega)\omega.
\end{aligned}$$

**Theorem 5.2.8.** Assume that conditions $(\mathrm{B1b})_+$ and $(\mathrm{B2b})_+$ (resp. $(\mathrm{B1b})_-$ and $(\mathrm{B2b})_-$) hold for some fixed $k \in \mathbb{N}$. The Hopf bifurcation occurs at $\tau_T = \eta_k^+$ (resp. $\tau_T = \eta_k^-$), and a nontrivial synchronous periodic solution (resp. standing wave solution) is bifurcated from the zero solution of (2.6) (resp. $(2.6)_0$).

**Proof.** We only prove the first case. The others can be verified similarly. First, we derive that

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\Delta_+(\lambda)|_{\lambda=i\omega_+^*,\tau_T=\eta_k^+} \\
=\ & \{1 + \alpha\tau_I e^{-\lambda\tau_I} + 2\beta\tau_T e^{-\lambda\tau_T}\}|_{\lambda=i\omega_+^*,\tau_T=\eta_k^+} \\
=\ & \{1 + \alpha\tau_I e^{-\lambda\tau_I} + \tau_T(1 + \lambda - \alpha e^{-\lambda\tau_I})\}|_{\lambda=i\omega_+^*,\tau_I=\eta_k^+} \\
=\ & \mathbf{R}(\omega_+^*, \eta_k^+) + i\mathbf{I}(\omega_+^*, \eta_k^+).
\end{aligned}$$

Thus, $\frac{\partial}{\partial\lambda}\Delta_+(\lambda)|_{\lambda=i\omega_+^*,\tau_T=\eta_k^+} \neq 0$, under condition $(\mathrm{B2})_+$; hence there exists some $\delta > 0$ and a smooth function $\lambda : (\eta_k^+ - \delta, \eta_k^+ + \delta) \to \mathbb{C}$ such that $\Delta_+(\lambda(\tau_T)) = 0$ and $\lambda(\eta_k^+) = i\omega_+^*$. Differentiating $\Delta_+(\lambda(\tau_T)) = 0$ with respective to $\tau_T$ at $\tau_T = \eta_k^+$, we obtain

$$\lambda'(\eta_k^+) = \frac{-2\beta e^{-\lambda\tau_T}\lambda}{1 + \alpha\tau_I e^{-\lambda\tau_I} + 2\beta\tau_T e^{-\lambda\tau_T}}|_{\lambda=i\omega_+^*,\tau_T=\eta_k^+} = \frac{Q_1 + iQ_2}{W_1 + iW_2},$$

where $Q_1 = (\omega_+^*)^2 + \alpha\sin(\tau_I\omega_+^*)\omega_+^*$, $Q_2 = -\omega_+^* + \alpha\cos(\tau_I\omega_+^*)\omega_+^*$, $W_1 = 1 + \eta_k^+ + \alpha(\tau_I - \eta_k^+)\cos(\tau_I\omega_+^*)$ and $W_2 = \eta_k^+\omega_+^* - \alpha(\tau_I - \eta_k^+)\sin(\tau_I\omega_+^*)$. Therefore,

$$\begin{aligned}
&\mathbf{Re}\lambda'(\eta_k^+) \\
=\ & \{Q_1W_1 + Q_2W_2\}/(W_1^2 + W_2^2) \\
=\ & \{[1 + \alpha\tau_I\cos(\tau_I\omega_+^*)](\omega_+^*)^2 + \alpha(1+\tau_I)\sin(\tau_I\omega_+^*)\omega_+^*\}/(W_1^2 + W_2^2) \\
\neq\ & 0,
\end{aligned}$$

under condition $(\mathrm{B2})_+$. □

**Remark 5.2.4.** (i) In Theorem 5.2.8, condition $(\mathrm{B1b})_\pm$ plays the dominant ones, since condition $(\mathrm{B2b})_\pm$ is apter to be met. Basically, condition $(\mathrm{B1b})_\pm$ requires that $\tau_I$ is small and $|\beta|$ is relatively larger than $|\alpha|$. Notably, the restriction on magnitude of $\tau_I$ can be relaxed. Observe that the function $Q(w)$ in (5.15) and (5.21) is dominated by the leading term $w^2$, as $w$ is large. Therefore, if $|\beta|$ is sufficiently large, there exist exactly one positive zero for (5.15) and (5.21) given arbitrarily fixed $\tau_I$ and $\alpha$. Accordingly, large $|\beta|$ is advantageous for Hopf bifurcation to take place, hence synchronous or asynchronous oscillations induced by transmission delay $\tau_T$. (ii) Obviously, $(\mathrm{B1b})_+$ is weaker than $(\mathrm{B1b})_-$. We thus see that the $\tau_T$-induced synchronous oscillations appear ahead of the asynchronous oscillations along the way of increasing $|\beta|$. (iii) Similar formulations and arguments show that large $|\alpha|$ is advantageous to the occurrence of synchronous or asynchronous oscillations induced by transmission delay $\tau_I$. In contrast to (ii), the synchronous oscillations appear behind the asynchronous oscillations, along the way of increasing $|\alpha|$.

If system (2.6) of scale $N > 3$ is considered, the treatments in this section are still valid. Indeed, Hopf bifurcation for synchronous periodic solution of system (2.6) with general scale $N$ can be analyzed through the reduced system $(2.6)_+$. On the other hand, bifurcation analysis for anti-phase motion can be performed for system (2.6) of even scale through the reduced system $(2.6)_-^{\sigma}$.

## 5.3 Extension to $N \geq 3$

In this subsection, we shall discuss the synchronization for system (2.6) of general scale $N \geq 3$. The difference between the synchrony of system (2.6) of scale $N = 3$ and $N > 3$ will be addressed in Remark 5.3.1(ii). Notably, by arguments similar to the ones in Section 5.2, the convergence to multiple synchronous equilibria for (2.3) of scale $N \geq 3$ can also be established.

First, let us introduce the following conditions for global synchronization. These conditions can be regarded as the $N$-scale version of conditions (S1b)-(S4b) in Section 5.2, respectively.

Condition $(\mathrm{S1b})^*$: $-\hat{\alpha} + \check{\beta} \geq 0$, $\mu - \hat{\alpha} + \check{\beta} > (N-3)|\beta|$, $\tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)}$; more precisely,

$$\begin{cases} \beta \geq (1/\tilde{L})\alpha, \mu - \alpha + \beta\tilde{L} - (N-3)|\beta| > 0 \text{ and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)} & \text{if } \alpha \geq 0, \beta \geq 0, \\ \mu - \alpha\tilde{L} + \beta\tilde{L} - (N-3)|\beta| > 0 \text{ and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)}, & \text{if } \alpha \leq 0, \beta \geq 0, \\ \beta \geq \tilde{L}\alpha, \mu - \alpha\tilde{L} + \beta - (N-3)|\beta| > 0 \text{ and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)}, & \text{if } \alpha \leq 0, \beta \leq 0, \end{cases}$$

where

$$\tau_N^{(1)} := \min\{\frac{(|\alpha|+|\beta|)\mu}{(2\mu-\check{\alpha}+\hat{\beta})[|\alpha|+(N-2)|\beta|]}, \frac{\mu-\hat{\alpha}+\check{\beta}-(N-3)|\beta|}{2\mu-\check{\alpha}-\hat{\alpha}+\hat{\beta}+\check{\beta}}\}.$$

Condition (S2b)$^*$: $\alpha < 0$, $\mu - \alpha\tilde{L} > 2|\beta|$ and $\tau_I < \tau_N^{(2)}$, where

$$\tau_N^{(2)} := \min\{\frac{\mu}{(2\mu-\alpha)(|\alpha|+2|\beta|)}, \frac{\mu-\alpha\tilde{L}-2|\beta|}{\alpha(\alpha+\alpha\tilde{L}-2\mu)}\}.$$

Condition (S3b)$^*$: $\beta > 0$, $(|\alpha|+|\beta|)(\mu-|\alpha|) > (N-3)|\alpha\beta|$, $\mu+\beta\tilde{L}-|\alpha| > (N-3)|\beta|$ and $\tau_T < \tau_N^{(3)}$, where

$$\tau_N^{(3)} := \min\{\frac{(|\alpha|+|\beta|)(\mu-|\alpha|)-(N-3)|\alpha\beta|}{\beta(2\mu+\beta)[|\alpha|+(N-2)|\beta|]}, \frac{\mu+\beta\tilde{L}-|\alpha|-(N-3)|\beta|}{\beta(\beta+\beta\tilde{L}+2\mu)}\}.$$

Condition (S4b)$^*$: $|\alpha|+2|\beta| < \mu$.

**Theorem 5.3.1.** System (2.6) of scale $N \geq 3$ achieves global synchronization under one of conditions (S1b)$^*$-(S4b)$^*$.

**Proof.** The arguments for the $\tau_T$-dependent results under condition (S1b)$^*$ or (S3b)$^*$ are different from the ones for the $\tau_T$-independent results under condition (S2b)$^*$ or (S4b)$^*$. First, let us consider that case that condition (S1b)$^*$ holds. The difference system of (2.6) can be written as follows:

$$\begin{aligned}\dot{z}_i(t) &= -\mu z_i(t) + \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))] - \beta[g(x_i(t-\tau_T)) - g(x_{i+1}(t-\tau_T))] \\ &\quad + w_i(t),\ i = 1, \cdots, N, \end{aligned} \tag{5.23}$$

where $z_i(t) := x_i(t) - x_{i+1}(t)$, $w_i(t) = -\beta\Sigma_{j\in J_i}[g(x_j(t-\tau_T)) - g(x_{j+1}(t-\tau_T))]$, where $J_i := \{1, \cdots, N\} \setminus \{i, i-1, i+1 \pmod N\}$. Then each $i$th equation of (5.23) is of the form in (3.10) with $\gamma_1 = -\alpha$, $\gamma_2 = \beta$, $\tau_1 = \tau_I$, $\tau_2 = \tau_T$, and satisfies condition (H1b). According to Theorem 3.1.9, every $i$th component $z_i(t)$ of (5.23) converges to some interval $[-\rho_i, \rho_i] =: I_i$, as $t \to \infty$; moreover,

$$0 \leq \rho_i \leq |w_i|^{\max}(\infty)/\eta,$$

where

$$\eta := \mu - \hat{\alpha} + \check{\beta} - (\tau_I|\alpha| + \tau_T|\beta|)(2\mu - \check{\alpha} - \hat{\alpha} + \hat{\beta} + \check{\beta}),$$

and $\hat{\alpha}, \check{\alpha}, \hat{\beta}, \check{\beta}$ are as defined in (5.4), (5.5) for real numbers $\alpha, \beta$. We shall show that all $\rho_i$ are equal to zero; consequently, $z_i(t)$ converges to a singleton and the assertion thus follows. Now let us illustrate that for each $i$, there exists a sequence $\{\rho_i^{(k)}\}_{k=0}^{\infty}$ with $\rho_i^{(k)} \geq \rho_i$ for all $k$ and $z_i(t)$ converges to $[-\rho_i^{(k)}, \rho_i^{(k)}]$ as $t \to \infty$, for each $k$. We shall construct $\rho_i^{(k)}$ to satisfy

$$\rho_i^{(0)} := 2(N-3)|\beta|/\eta, \ i = 1, \cdots, N,$$

and for $k \geq 1$,

$$\begin{cases} \rho_1^{(k)} = \Sigma_{j=3}^{N-1} |\beta| \rho_j^{(k-1)}/\eta, \\ \vdots \\ \rho_i^{(k)} = (\Sigma_{j=1}^{i-2} |\beta| \rho_j^{(k)} + \Sigma_{j=i+2}^{N} |\beta| \rho_j^{(k-1)})/\eta, \\ \vdots \\ \rho_N^{(k)} = \Sigma_{j=2}^{N-2} |\beta| \rho_j^{(k)}/\eta. \end{cases}$$

The construction is similar to Proposition 4.2.1 and is sketched as follows. If such $\rho_i^{(k)}$, for $k = 1, \cdots, n-1, i = 1, \cdots, N$ and $k = n, i = 1, \cdots, \ell - 1 < N$ have been defined, then $|w_\ell(t)| = |-\beta \Sigma_{j \in J_\ell} [g(x_j(t-\tau_T)) - g(x_{j+1}(t-\tau_T))]| \leq \beta \Sigma_{j \in J_\ell} |z_j(t-\tau_T)|$. Hence,

$$0 \leq \rho_\ell \leq |w_\ell|^{\max}(\infty)/\eta \leq (\Sigma_{j=1}^{\ell-2} |\beta| \rho_j^{(n)} + \Sigma_{j=\ell+2}^{N} |\beta| \rho_j^{(n-1)})/\eta.$$

We observe that $\{\rho_i^{(k)} \mid i = 1, 2, \cdots, n\}$ are exactly the Gauss-Seidal iteration for solving the linear system

$$\mathbf{M}\mathbf{x} = \mathbf{0}, \tag{5.24}$$

where $\mathbf{M} := \eta I_N + \mathrm{circ}(0, 0, -|\beta|, \cdots, -|\beta|, 0)$. Herein,

$$\mathrm{circ}(\xi_0, \xi_1, \cdots, \xi_{n-1}) = \begin{pmatrix} \xi_0 & \xi_1 & \cdots & \xi_{n-1} \\ \xi_{n-1} & \xi_0 & \cdots & \xi_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1 & \xi_{n-1} & \cdots & \xi_0 \end{pmatrix},$$

where $\xi_j$ is a real number, $j = 0, 1, \cdots, n-1$. Notably, $\mathbf{M}$ is strictly diagonal dominant, cf. [4, 74] under condition (S1b)$^*$; hence $\eta - (N-3)|\beta| > 0$. Accordingly, $(\rho_1^{(k)}, \cdots, \rho_N^{(k)})$ converges to the unique solution of (5.24), which is zero, as $k \to \infty$. Thus for each $i$, sequence $\{\rho_i^{(k)}\}$ converge to zero as $k \to \infty$. Consequently, every component of the solution to (5.23), hence the solution itself, converges to zero.

For the case that condition (S3b)$^*$ holds, each component of (5.23) satisfies (3.10) with $\gamma_1 = \beta$, $\gamma_2 = -\alpha$, $\tau_1 = \tau_T$, $\tau_2 = \tau_I$ and satisfies condition (H2b). The assertion then follows from Theorem 3.1.10.

Now, let us justify the case of condition (S2b)$^*$. Note that the difference system derived from (2.6) can be regarded as another form different from (5.23) as follows:

$$\dot{z}_i(t) = -z_i(t) + \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))] + w_i(t), \; i = 1, \cdots, N, \quad (5.25)$$

where $w_i(t) = \beta[g(x_{i-1}(t-\tau_T)) - g(x_i(t-\tau_T)) + g(x_{i+1}(t-\tau_T)) - g(x_{i+2}(t-\tau_T))]$. Then (5.25) satisfies (3.10) with $\gamma_1 = -\alpha$, $\gamma_2 = 0$, $\tau_1 = \tau_I$ and satisfies condition (H2b) under condition (S2b)$^*$. According to Theorem 3.1.10, every $z_i$ of (5.25) converges to interval some $[-\tilde{\rho}_i, \tilde{\rho}_i]$; moreover,

$$0 \le \tilde{\rho}_i \le |w_i|^{\max}(\infty)/\tilde{\eta},$$

where

$$\tilde{\eta} := 1 - \alpha\tilde{L} + \tau_I\alpha(2 - \alpha - \alpha\tilde{L}).$$

The proof then follows process parallel to the first case of condition (S1b)$^*$, hence is omitted.

For the case of condition (S4b)$^*$, each $z_i$ of (5.25) can be regarded as in the form (3.10) with $\gamma_1 = -\alpha$, $\gamma_2 = 0$, $\tau_1 = \tau_T$ and satisfies condition (H3b). The assertion holds by Theorem 3.1.11. □

**Remark 5.3.1.** (i) If $N$ is large, then large $\mu$ and negative $\alpha$ of large magnitude are advantageous for conditions (S1b)$^*$-(S3b)$^*$. By Theorem 5.3.1, it turns out that system (2.6) can be synchronized if both magnitude of $\alpha$ and $\beta$ are small enough. Otherwise, strong inhibitory self-feedback play an important factor for synchronization of (2.6) if delays are small. (ii) If $N = 3$, the difference equation derived from (2.6) is nearly a decoupled system, cf. (5.3). If $N > 3$, the difference equation derived from (2.6) is a coupled system, cf. (5.23) or (5.25). Such a distinction between the structure of difference equations is the major reason for the disparity of synchrony for (2.6) of scale $N = 3$ and $N > 3$. In fact, Example 5.4.3 illustrates that under the same parameters, (2.6) can be synchronized globally as $N = 3$, but not as $N > 3$.

## 5.4 Numerical examples

We present four examples in this section. In Example 5.4.1, we illustrate the dynamics of synchronous oscillation. Example 5.4.2 demonstrates a transition from convergence of multiple synchronous equilibria to asynchronous oscillation as transmission delay $\tau_T$ increases. Example 5.4.3 shows that (2.6) of scale $N = 3$ with certain parameters is synchronous, but (2.6) of scale $N = 4$ with the same parameters is asynchronous. In Example 5.4.4, for some parameters satisfying the condition of the conjecture in [8] and $\alpha > 0$, we illustrate that there exists a solution which converges to an asynchronous equilibrium.

**Example 5.4.1.** Consider (2.6) with $\mu = 1$, $\alpha = -0.1$, $\beta = -0.9$, $\tau_I = 0.001$, $\tau_T = 8.2$, $N = 3$. The parameter $(\alpha, \beta) = (-0.1, -0.9)$ lies in Fig. 5.1(b), and condition (S2b) is met for $\tau_I = 0.001$; hence the system can be synchronized in spite of transmission delay $\tau_T$, according to Theorem 5.2.1. In addition, the parameters and delays satisfy condition of Theorem 5.2.8; therefore, there exist a nontrivial synchronous periodic solution induced by $\tau_T$ near $\tau_T = 8.178$. Fig. 5.6 illustrates that the solution of (2.6) tends to a synchronous periodic orbit as $t \to \infty$; in the panels, three different colors represent the evolutions of three components $x_1, x_2, x_3$.

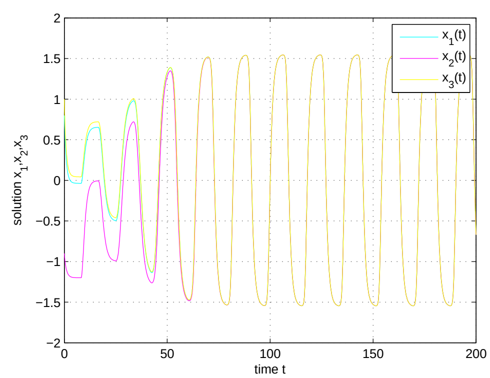



Figure 5.6: An orbit of (2.6) with $\mu = 1$, $\alpha = -0.1$, $\beta = -0.9$, $\tau_I = 0.001$, $\tau_T = 8.2$, which is evolved from $\phi(t) = (0.8, -0.9, 1)$. There exists a synchronous limit cycle.

**Example 5.4.2.** Consider (2.6) with $\mu = 1$, $N = 3$, $\alpha = 0.9$, $\beta = 2$, $\tau_I = 0.001$, $\tau_T = 0.001$ or $1.3$. If $\tau_T = 0.01$, the system satisfies condition (S3b), hence achieves global synchronization. Moreover, the system satisfies the assumptions of Theorem 5.2.3 and 5.2.4 (ii), hence the system achieves global convergence to three synchronous equilibria where the nontrivial ones are stable. Fig. 5.7(a) illustrates that the solutions plotted in blue converge to nontrivial stable equilibria; and the solution plotted in red converges to zero. Evolution for each component of the solution which converges to zero is illustrated in Fig. 5.7(b). If considering $\tau_T = 1.3$ instead, by Theorems 5.2.3 and 5.2.8, the nontrivial equilibria remain stable, but an asynchronous periodic solution is bifurcated from the origin. Fig. 5.7(c) illustrates that coexistence of asynchronous periodic oscillation plotted in red and two stable synchronous equilibria. Evolution of each component of this oscillation is illustrated in Fig. 5.7(d).

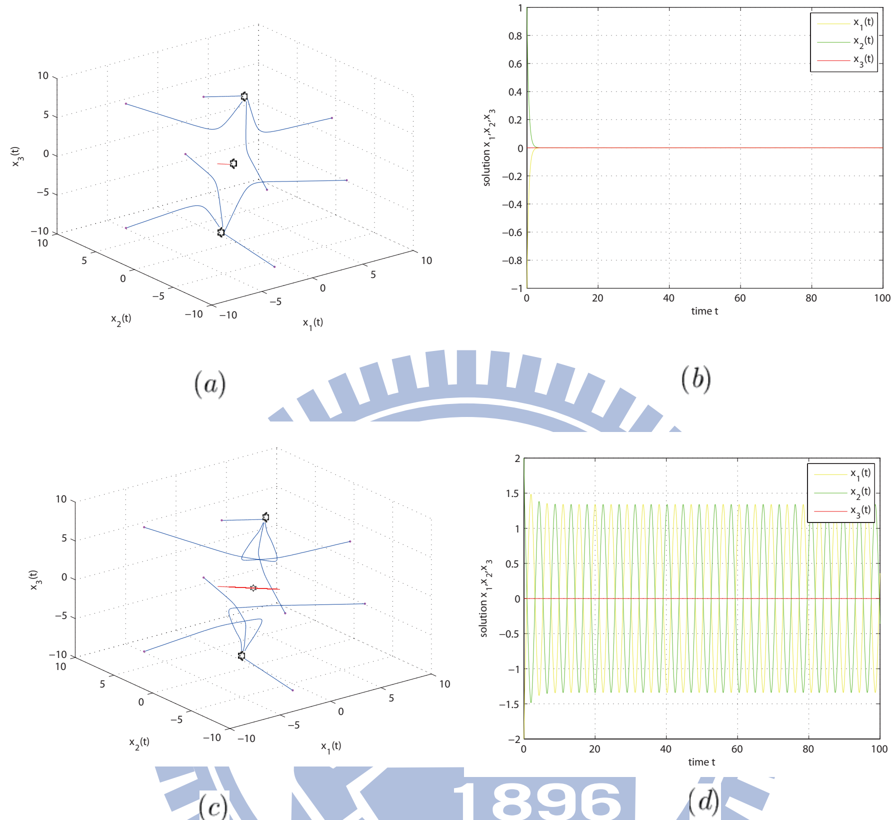



Figure 5.7: Consider $\mu = 1$, $\alpha = 0.9$, $\beta = -2$, $\tau_I = 0.001$. (a) Solutions of (2.6) with $\tau_T = 0.001$ evolved from various initial values converges to one of the three equilibria. (b) Evolution of three components of the solution plotted in red in (a). (c) Coexistence of asynchronous periodic oscillation plotted in red and two stable synchronous equilibria for (2.6) with $\tau_T = 1.3$. (d) The evolution of three components of the oscillation plotted in red in (c).

**Example 5.4.3.** Consider (2.6) with $\mu = 1$, $\alpha = 0$, $\beta = 1$, $\tau_I = 0.01$, $\tau_T = 10$. Such a system satisfies condition (S2b), hence can be synchronized as $N = 3$, according to Theorem 5.2.1. In addition, the synchronous phase contains at least two stable equilibria, according to Theorem 5.2.3, since this $(\alpha, \beta)$ lies in region $D_1$. However, Fig. 5.8 illustrates that as $N = 4$, there exists an asynchronous limit cycle.

**Example 5.4.4.** Consider system (2.6) with parameters $\mu = 1$, $\alpha = 6$, $\beta = -2$, and delays $\tau_I = 0$, $\tau_T = 5$. These parameters and delays satisfy the condition of the conjecture in [8], but there exists a solution which converges to an asynchronous equilibrium, cf. Fig. 5.9

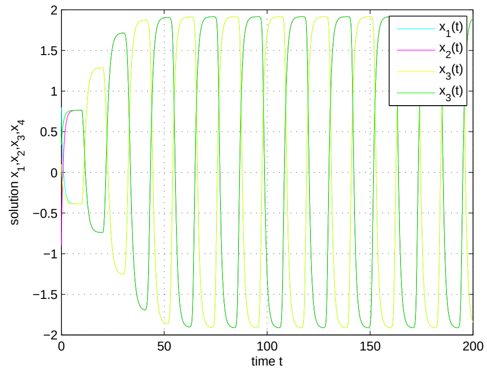



Figure 5.8: Asynchronous limit cycle of (2.6) with $\mu = 1$, $\alpha = 0$, $\beta = 1$, $\tau_I = 0.01$, $\tau_T = 10$ and $N = 4$. The orbit is evolved from $(0.8, -0.9, 0.1, 0.34)$.

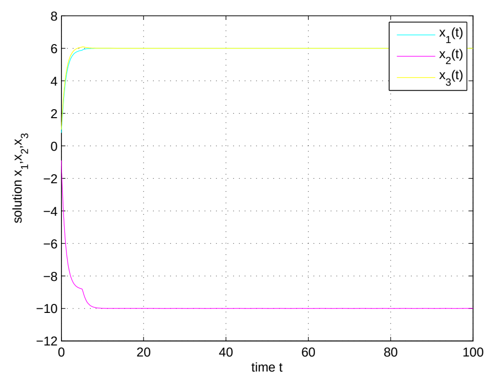



Figure 5.9: The orbit evolved from $(0.8, -0.9, 1)$ converges to an asynchronous equilibrium of system (2.6) with parameters $\mu = 1$, $\alpha = 6$, $\beta = -2$, and delays $\tau_I = 0$, $\tau_T = 5$.

# Chapter 6

# Synchronization and Oscillation for a System Comprising Two Subnetworks with Loop Structure

In this chapter, we consider system (2.8):

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \alpha_i g(x_{i-1}(t-\tau_I)), \ i = 1, 2\cdots, K-1 \text{ (mod K)}, \\ \dot{x}_K(t) = -\mu_K x_K(t) + \alpha_K g(x_{K-1}(t-\tau_I)) + cg(y_K(t-\tau_T)), \\ \dot{y}_i(t) = -\mu_i y_i(t) + \alpha_i g(y_{i-1}(t-\tau_I)), \ i = 1, 2\cdots, K-1 \text{ (mod K)}, \\ \dot{y}_K(t) = -\mu_K y_K(t) + \alpha_K g(y_{K-1}(t-\tau_I)) + cg(x_K(t-\tau_T)), \end{cases}$$

where $g$ is an activation function of class $\mathcal{A}$ with $g(0) = 0$. We set $\mu_i = 1$, $\alpha_i = 1$ for all $i$, $-1 < g(\xi) < 1$ and focus on the effect of dynamics for (2.8) from the gain of response function ($L$), the coupling strength between two loops ($c$), the internal delay ($\tau_I$) and the transmission delay ($\tau_T$). The presentation of this chapter is organized as follows. Global synchronization, anti-phase motion, convergence to trivial equilibrium are presented in Sections 6.1, 6.2, 6.3 respectively. Stability of nontrivial equilibria, their basins of attraction and convergence to multiple equilibria are summarized in Section 6.4. Hopf bifurcation induced by the internal delay at the trivial equilibrium is studied in Section 6.5. In Section 6.6, we summarize the dynamic scenarios corresponding to various coupling strength and delays. We present some numerical illustrations in Section 6.7.

## 6.1 Global synchronization for the coupled $K$-loops

In this section, we shall derive both transmission $\tau_T$-dependent and $\tau_T$-independent criteria for (2.8) to attain global synchronization; that is

$$x_i(t) - y_i(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \cdots, K$$

for solution $(x_1(t), \cdots, x_K(t), y_1(t), \cdots, y_K(t))$ of (2.8), starting from arbitrary initial condition. For this purpose, we shall consider the following difference system of (2.8):

$$\begin{cases} \dot{z}_i(t) = -z_i(t) + w_i(t), \; i = 1, \cdots, K-1, \\ \dot{z}_K(t) = -z_K(t) - c[g(x_K(t-\tau_T)) - g(y_K(t-\tau_T))] + w_K(t). \end{cases} \tag{6.1}$$

where $z_i(t) = x_i(t) - y_i(t)$, $i = 1, \cdots, K$; $w_i(t) = g(x_{i-1}(t-\tau_I)) - g(y_{i-1}(t-\tau_I))$, $i = 1, \cdots, K$ (mod $K$). Notice that both $x_K(t)$ and $y_K(t)$ are eventually attracted by $[-1-|c|, 1+|c|]$, as seen from the equation for $x_K$ and $y_K$ in (2.8) (with $\alpha_i, \rho$ set to one). We denote

$$\tilde{L}_c := \min\{g^{'}(\xi) : \xi \in [-1-|c|, 1+|c|]\}.$$

Obviously, every component of (6.1) is of the form (3.1). More precisely, the first $K-1$ components are in the form of (3.9) and the $K$th component is in the form of (3.1) with $\beta = c$. Now, we introduce the following $\tau_T$-dependent condition for global synchronization of (2.8):

Condition (S1a): $c > 0$, $\tau_T \leq 1/[L(2+cL)(1+c)]$, and $1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c) > L^K$.

The second inequality in condition (S1a) matches the second inequality in condition (H1a) with $|w_i|^{\max}(t_0) \leq 2$, for all $i$. The third inequality in condition (S1a) is needed for contraction of sequence of intervals in the following Theorem 6.1.2. Obviously, under condition (S1a), the $K$th component of (6.1) satisfies condition (H1a). By Theorem 3.1.4 and Corollary 3.1.8, there exist $K$ intervals $I_i := [-a_i, a_i]$, $i = 1, \cdots, K$, to which $z_i(t)$ converges respectively. Moreover,

$$\begin{cases} a_i \leq |w_i|^{\max}(\infty), \; i = 1, 2 \cdots, K-1, \\ a_K \leq |w_K|^{\max}(\infty)/[1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c)]. \end{cases} \tag{6.2}$$

By the similar arguments for Proposition 4.2.1 or Theorem 5.3.1, we can derive more precise estimates on $a_i$ through an iterative process. We thus give the following proposition without the proof.

**Proposition 6.1.1.** Assume that condition (S1a) holds. Then for any $i = 1, \cdots, K$, there exists a sequence of intervals $\{a_i^{(k)}\}_{k=1}^{\infty}$ such that for each $k$, the $i$th component $z_i(t)$ of every solution to (6.1) converges to $I_i^{(k)} := [-a_i^{(k)}, a_i^{(k)}]$ as $t \to \infty$, and $a_i^{(k)}$ satisfies

$$\begin{cases} 0 \le a_1^{(k)} = La_K^{(k-1)}, \\ 0 \le a_i^{(k)} = La_{i-1}^{(k)}, \ i = 2, \cdots, K-1, \\ 0 \le a_K^{(k)} = La_{K-1}^{(k)}/[1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c)], \end{cases} \tag{6.3}$$

where $a_K^{(0)} := 2/L$.

So far, we have shown that every component of arbitrary solution to system (6.1) converges to a sequence of closed intervals whose lengths $2a_i^{(k)}$ can be controlled by iterative formula (6.3). Next, it will be examined that for each $i$, $a_i^{(k)}$ converges to zero as $k \to \infty$. Thus the interval to which each component of the solution converges degenerates into a single point. One can then conclude that system (6.1) achieves global convergence to zero; accordingly system (2.8) attains global synchronization. Such arguments are implemented in the following theorem.

**Theorem 6.1.2.** The coupled $K$-loops (2.8) attain global synchronization under condition (S1a).

**Proof.** Under condition (S1a), $a_1^{(k)}$ are decreasing with respective to $k$, since $a_1^{(j+1)} = La_K^{(j)} = L^2 a_{K-1}^{(j)}/[1+c\tilde{L}_c-\tau_T cL(2+cL+c\tilde{L}_c)] = L^K a_1^{(j)}/[1+c\tilde{L}_c-\tau_T cL(2+cL+c\tilde{L}_c)] < a_1^{(j)}$. By similar arguments, it can also be shown that $a_i^{(k)}$ are decreasing with respective to $k$ for $i = 2, \cdots, K$. Suppose $a_i^{(k)} \to \tilde{a}_i$, as $k \to \infty$, for $i = 1, \cdots, K$. By (6.3), it follows that

$$\begin{cases} 0 \le \tilde{a}_1 = L\tilde{a}_K, \\ 0 \le \tilde{a}_i = L\tilde{a}_{i-1}, \ i = 2, \cdots, K-1, \\ 0 \le \tilde{a}_K = L\tilde{a}_{K-1}/[1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c)]. \end{cases} \tag{6.4}$$

Subsequently, $0 \le \tilde{a}_1 = L^K \tilde{a}_1/[1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c)]$, and it yields that $\tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_K = 0$ under condition (S1a). This completes the proof.

Applying the same treatment as Theorem 6.1.2 and using Theorems 3.1.5, 3.1.6, we can derive criteria which are $\tau_T$-dependent as $c < 0$ and $\tau_T$-independent respectively, for synchronization of (2.8); namely

Condition (S2a): $-1/L < c < 0$, $\tau_T \le (1 + cL)/[L(2 + cL)(1 + |c|)]$ and $1 + cL + \tau_T cL(2 + cL + c\tilde{L}_c) > L^K$.

Condition (S3a): $0 < L < 1$ and $|c| \leq 1/L - 1$.

**Theorem 6.1.3.** The coupled $K$-loops (2.8) attains global synchronization under condition (S2a) or (S3a).

It is necessary that $L < 1$ for condition (S2a), but not for condition (S1a). Therefore, if $L > 1$, synchronization for (2.8) is apt to occur for excitatory coupling ($c > 0$). On the other hand, in Section 6.2, it will be seen that anti-phase motion for (2.8) is apt to take place under inhibitory coupling ($c < 0$), if $L > 1$. If $L < 1$, then the sign of $c$ is not that deterministic in these conditions.

Conditions (S1a) and (S2a) are dependent on $\tau_T$ and favor small $\tau_T$, but are independent of $\tau_I$. In Section 6.5, we shall show that, under condition (S1a) and some additional conditions, there exists nontrivial synchronous periodic solution induced by $\tau_I$. In addition, it will be shown that (2.8) actually achieves global convergence to the trivial equilibria under condition (S3a).

Notably, the third inequality in conditions (S1a), (S2a) become more difficult to hold if $L > 1$ and $K$ (the subnetwork size) is large. Thus, it is well possible that synchronization may be lost, when the sub-network size is too large.

## 6.2 Anti-phase motion for the coupled $K$-loops

In this section, we denote by $(2.8)_0$ system (2.8) with odd activation functions $g$ in class $\mathcal{A}$, ie., $g$ also satisfies $g(-\xi) = -g(\xi)$, for all $\xi \in \mathbb{R}$. We shall derive criteria for $(2.8)_0$ to admit global anti-phase motion; that is,

$$x_i(t) + y_i(t) \to 0, \text{ as } t \to \infty, \text{ for all } i = 1, \cdots, K,$$

for solution $(x_1(t), \cdots, x_K(t), y_1(t), \cdots, y_K(t))$ of (2.8), starting from any initial condition. We set $\tilde{y}_i(t) = -y_i(t)$, then from (2.8), we obtain

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + g(x_{i-1}(t - \tau_I)), \ i = 1, 2 \cdots, K-1 \ (\text{mod } K), \\ \dot{x}_K(t) = -x_K(t) + g(x_{K-1}(t - \tau_I)) - cg(\tilde{y}_K(t - \tau_T)), \\ \dot{\tilde{y}}_i(t) = -\tilde{y}_i(t) + g(\tilde{y}_{i-1}(t - \tau_I)), \ i = 1, 2 \cdots, K-1 \ (\text{mod } K), \\ \dot{\tilde{y}}_K(t) = -\tilde{y}_K(t) + g(\tilde{y}_{K-1}(t - \tau_I)) - cg(x_K(t - \tau_T)). \end{cases} \tag{6.5}$$

Showing that $(2.8)_0$ achieves global anti-phase motion amounts to justifying that (6.5) achieve global synchronization. By employing similar treatment in Section 6.1, we can conclude the following result.

**Theorem 6.2.1.** The coupled $K$-loops $(2.8)_0$ attains global anti-phase motion under condition (S3a) or (AP1) or (AP2), where

Condition (AP1): $c < 0$, $\tau_T \le 1/[L(2 - cL)(1 - c)]$ and $1 - c\tilde{L}_c + \tau_T cL(2 - cL - c\tilde{L}_c) > L^K$,

Condition (AP2): $1/L > c > 0$, $\tau_T \le (1 - cL)/[L(2 - cL)(1 + |c|)]$ and $1 - cL - \tau_T cL(2 - cL - c\tilde{L}_c) > L^K$.

## 6.3 Global convergence to trivial equilibrium

In this section, we shall derive both $\tau_T$-independent and $\tau_T$-dependent criteria for system (2.8) to admit global convergence to trivial equilibrium. For this purpose, we rewrite (2.8) as follows:

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + w_i(t), \ i = 1, 2 \cdots, K-1 \ (\text{mod } K), \\ \dot{x}_K(t) = -x_K(t) + cg(x_K(t - \tau_T)) + w_K(t) + v_K(t), \\ \dot{y}_i(t) = -y_i(t) + \tilde{w}_i(t), \ i = 1, 2 \cdots, K-1 \ (\text{mod } K), \\ \dot{y}_K(t) = -y_K(t) + cg(y_K(t - \tau_T)) + \tilde{w}_K(t) + \tilde{v}_K(t). \end{cases} \tag{6.6}$$

where $w_i(t) = g(x_{i-1}(t - \tau_I))$, $\tilde{w}_i(t) = g(y_{i-1}(t - \tau_I))$, $i = 1, 2 \cdots, K$; $v_K(t) = c[g(y_K(t - \tau_T)) - g(x_K(t - \tau_T))]$, and $\tilde{v}_K(t) = c[g(x_K(t - \tau_T)) - g(y_K(t - \tau_T))]$. We impose the following condition:

Condition (Ca):

$$\begin{cases} 0 < |c| < 1/L, \\ \tau_T \le \min\{1/[L(2 + |c|L)(1 + |c|)], (1 - |c|L)/[(1 + |c|)(2 - |c|L)L]\}, \\ \min\{1 + |c|\tilde{L}_c - \tau_T|c|L(2 + |c|L + |c|\tilde{L}_c), 1 - |c|L - \tau_T|c|L(2 - |c|L - |c|\tilde{L}_c)\} > L^K. \end{cases}$$

**Theorem 6.3.1.** Every solution of the coupled $K$-loops (2.8) converges to the trivial equilibrium as $t \to \infty$, under condition (Ca).

**Proof.** We merely prove the case of $c > 0$. The situation for $c < 0$ can be treated similarly. Notably, the latter two inequalities in condition (Ca) yield condition (S1a). By Theorem 6.1.2, (2.8) achieves global synchronization under condition (Ca), hence each of the first $K - 1$ components (resp. the $K$th component) of (6.6) is of the form (3.9) (resp. (3.8) with $\beta = -c$). Moreover, the $K$th component satisfies condition (H2a) under condition (Ca). Notice that both $x_K(t)$ and $y_K(t)$ are eventually attracted by $[-1 - |c|, 1 + |c|]$. By Corollary 3.1.7 and Corollary 3.1.8, there exist $K$ intervals $[-a_i, a_i]$, $i = 1, \cdots, K$, to which $x_i(t)$ converges respectively. Furthermore,

$$\begin{cases} a_i \leq |w_i|^{\max}(\infty), \ i = 1, 2 \cdots, K-1, \\ a_K \leq |w_K|^{\max}(\infty)/[1 - cL - \tau_T cL(2 - cL - c\tilde{L}_c)]. \end{cases} \tag{6.7}$$

Similar to Proposition 6.1.1, there exists a sequence of intervals $\{a_i^{(k)}\}_{k=1}^{\infty}$ such that for each $k$, the $i$th component $x_i(t)$ of every solution $x(t)$ to (6.6) converges to $I_i^{(k)} := [-a_i^{(k)}, a_i^{(k)}]$ as $t \to \infty$, and $a_i^{(k)}$ satisfies

$$\begin{cases} 0 \leq a_1^{(k)} = La_K^{(k-1)}, \\ 0 \leq a_i^{(k)} = La_{i-1}^{(k)}, \ i = 2, \cdots, K-1, \\ 0 \leq a_K^{(k)} = La_{K-1}^{(k)}/[1 - cL - \tau_T cL(2 - cL - c\tilde{L}_c)], \end{cases} \tag{6.8}$$

where $a_K^{(0)} := 2/L$. Due to the last inequality in condition (Ca), $\{a_i^{(k)}\}$ are decreasing with respective to $k$ for $i = 1, \cdots, K$, since $a_i^{(j+1)} = L^K a_i^{(j)}/[1 - cL - \tau_T cL(2 - cL - c\tilde{L}_c)] < a_i^{(j)}$. Suppose $\{a_i^{(k)}\}$ converges to $\tilde{a}_i$, for $i = 1, \cdots, K$. It follows from (6.8) that

$$\begin{cases} 0 \leq \tilde{a}_1 = L\tilde{a}_K, \\ 0 \leq \tilde{a}_i = L\tilde{a}_{i-1}, \ i = 2, \cdots, K-1, \\ 0 \leq \tilde{a}_K = L\tilde{a}_{K-1}/[1 - cL - \tau_T cL(2 - cL - c\tilde{L}_c)]. \end{cases}$$

Subsequently, $0 \leq \tilde{a}_1 = L^K \tilde{a}_1/[1 + c\tilde{L}_c - \tau_T cL(2 + cL + c\tilde{L}_c)]$. Hence, it yields that $\tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_K = 0$, thanks to the last inequality in condition (Ca). Therefore, every $x_i(t)$ converges to zero as $t \to \infty$. By similar arguments, we can show that every $y_i(t)$ converges to zero. This completes the proof.

By using the same techniques as Theorem 6.3.1, we can also derive the following theorem under $\tau_T$-independent condition (S3a).

**Theorem 6.3.2.** Every solution of the coupled $K$-loops (2.8) converges to the trivial equilibrium as $t \to \infty$, under condition (S3a).

This theorem indicates that synchronization under $\tau_T$-independent condition (S3a) is exactly convergence to trivial equilibrium. Several works in the literature obtained such kind of result by Lyapunov function technique. In [7], globally asymptotical stability (global convergence) of the origin for the coupled 3-loops without internal delay $\tau_I$ was addressed under condition independent of $\tau_T$.

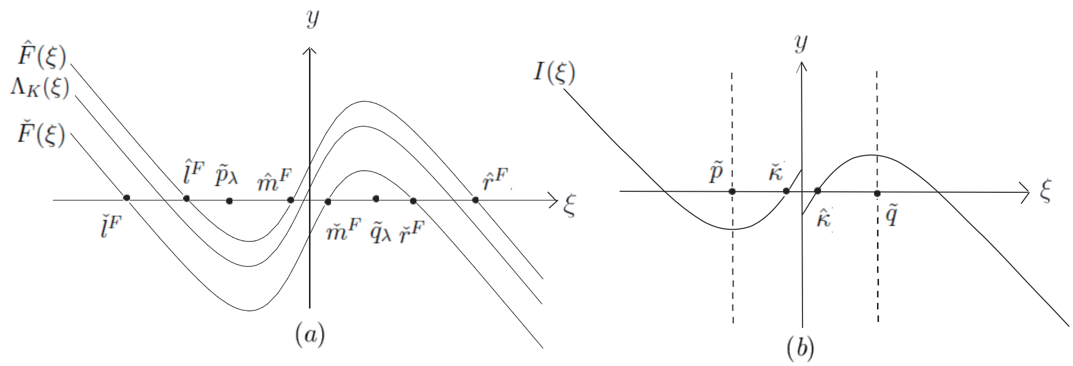



Figure 6.1: Configurations for (a) $\hat{F}, \check{F}, \Lambda_K$, (b) $I(\xi)$.

## 6.4 Global convergence to multiple equilibria

In this section, we shall establish the convergence to multiple synchronous equilibria for (2.8). In addition, the existence and stability of nontrivial equilibrium and basins of attraction for stable synchronous equilibria will be derived.

Let us consider the convergence of dynamics for (2.8). First, we define

$$\begin{aligned}\hat{F}(\xi) &= -\xi + cg(\xi) + 2\tau_T c(1+2c)L + 1,\\ \check{F}(\xi) &= -\xi + cg(\xi) - 2\tau_T c(1+2c)L - 1.\end{aligned}$$

For $0 < \lambda < 1$, we impose the following conditions:

Condition (C1a)$^\lambda$: $L > 1/c > 0$, $\tau_T < 1/[2L(1+2c)]$, $\lambda(1 - 2c\tau_T L) > L^K$.
Condition (C2a)$^\lambda$: $\check{F}(\tilde{q}_\lambda) > 0$, $\hat{F}(\tilde{p}_\lambda) < 0$.
Condition (C3a)$^\lambda$: $g'(\xi) > (1+\lambda)/c$, for all $\xi \in [\hat{m}^F, \check{m}^F]$.

Condition (C1a)$^\lambda$ is a multi-dimensional version of condition (A1a). Notably, $\tilde{p}_\lambda$ and $\tilde{q}_\lambda$ were defined in (3.21) where $-\beta$ is replaced by $c$. Under conditions (C1a)$^\lambda$ and (C2a)$^\lambda$, there exist exactly three zeros $\hat{l}^F$, $\hat{m}^F$and $\hat{r}^F$ (respectively $\check{l}^F$, $\check{m}^F$and $\check{r}^F$ ) of $\hat{F}(\xi) = 0$ (resp. $\check{F}(\xi) = 0$). Moreover, $\check{l}^F < \hat{l}^F < \tilde{p}_\lambda < \hat{m}^F < \check{m}^F < \tilde{q}_\lambda < \check{r}^F < \hat{r}^F$, cf. Fig. 6.1(a).

Let us introduce three sets in $\mathbb{R}^{2K}$ as follows:

$$\Omega_{\rm l} := \tilde{\Omega}_{\rm l} \times \tilde{\Omega}_{\rm l},\ \Omega_{\rm m} := \tilde{\Omega}_{\rm m} \times \tilde{\Omega}_{\rm m},\ \Omega_{\rm r} := \tilde{\Omega}_{\rm r} \times \tilde{\Omega}_{\rm r},$$

where $\tilde{\Omega}_{\rm l} = [-1,1]^{K-1} \times [\check{l}^F, \hat{l}^F]$, $\tilde{\Omega}_{\rm m} = [-1,1]^{K-1} \times [\hat{m}^F, \check{m}^F]$, $\tilde{\Omega}_{\rm r} = [-1,1]^{K-1} \times [\check{r}^F, \hat{r}^F]$. We can then derive the convergence to multiple synchronous equilibria for

(2.8). We say that an equilibrium $(\overline{x}_1, \cdots, \overline{x}_K, \overline{y}_1, \cdots, \overline{y}_K)$ of (2.8) is synchronous if $\overline{x}_i = \overline{y}_i$, for all $i = 1, \cdots, K$.

**Theorem 6.4.1.** Assume that conditions (S1a), (C1a)$^\lambda$-(C3a)$^\lambda$ hold for some fixed $\lambda \in (0, 1)$, then (2.8) achieves global convergence to synchronous equilibria; if in addition, $0 < L < 1$ and $\lambda \in (L, 1)$, then (2.8) admits exactly three synchronous equilibria. Each of regions $\Omega_{\rm l}$, $\Omega_{\rm m}$, $\Omega_{\rm r}$ contains one of these equilibria .

**Proof.** Recall that (2.8) can be rewritten into (6.6). Obviously, for $i = 1, \cdots, K-1$, the $i$th component of (6.6) is of the form as (3.9); hence, $x_i(t)$ converges to $[w_i^{\min}(\infty), w_i^{\max}(\infty)]$, as $t \to \infty$, as observed from the equation for $x_i$. Restated, for $i = 1, \cdots, K-1$, $x_i(t)$ converges to some compact interval $I_i$ whose length $d_i$ satisfies $d_i \le w_i^{\max}(\infty) - w_i^{\min}(\infty)$. The $K$th component of (6.6) has the form as (3.20) under condition (S1a) and satisfies conditions (A1a), (A2a)$^\lambda$ and (A3a)$^\lambda$, under condition (C1a)$^\lambda$, (C2a)$^\lambda$ and (C3a)$^\lambda$. By Theorem 3.2.5, $x_K(t)$ converges to some compact interval whose length $d_K$ satisfies $d_K \le [w_K^{\max}(\infty) - w_K^{\min}(\infty)]/[\lambda(1 - 2\tau_T cL)]$. By similar arguments as the ones for Theorem 4.2.2 or 6.3.1, we can show that each $x_i(t)$ actually converges to some singleton, for $i = 1, \cdots, K$. Similar arguments can apply to $y_i(t)$, for $i = 1, \cdots, K$.

Finding synchronous equilibrium for (2.8) amounts to solving

$$\begin{cases} -x_i + g(x_{i-1}) = 0, \ i = 1, \cdots, K-1, \ (\text{mod } K) \\ -x_K + g(x_{K-1}) + cg(x_K) = 0. \end{cases} \tag{6.9}$$

Note that under condition (S1a), all equilibria for (2.8) must be synchronous. Consider a fixed $\Omega \in \{\tilde{\Omega}_{\rm l}, \tilde{\Omega}_{\rm m}, \tilde{\Omega}_{\rm r}\}$. For a given $(\eta_1, \cdots, \eta_K) \in \Omega$, we define

$$\begin{cases} \Lambda_i(\xi) = -\xi + g(\eta_{i-1}), \ i = 1, \cdots, K-1, \ (\text{mod } K), \\ \Lambda_K(\xi) = -\xi + cg(\xi) + g(\eta_{K-1}). \end{cases}$$

Note that $|g(\cdot)| < 1$ and $\check{F}(\xi) \le \Lambda_K(\xi) \le \hat{F}(\xi)$. Thus, there exists a unique point $(\eta_1^*, \cdots, \eta_K^*) \in \Omega$ such that $\eta_i^*$ is the solution of equation $\Lambda_i(\cdot) = 0$ for all $i = 1, \cdots, K$, under conditions (C1a)$^\lambda$, (C2a)$^\lambda$, cf. Fig. 6.1 (a). Consequently, we can define a mapping $G_\Omega : \Omega \to \Omega$ by $G_\Omega(\eta_1, \cdots, \eta_K) = (\eta_1^*, \cdots, \eta_K^*) \in \Omega$. Thanks to conditions (C1a)$^\lambda$-(C3a)$^\lambda$ and $L < 1$, by using arguments similar to Theorem 4.1.1, we can show that $G_\Omega$ is a contraction mapping and there exists a unique fixed point $\overline{\mathbf{x}} = (\overline{x}_1, \cdots, \overline{x}_K)$ of $G_\Omega$, lying in $\Omega$. Restated, $\overline{\mathbf{x}}$ satisfies (6.9). Thus, $(\overline{x}_1, \cdots, \overline{x}_K, \overline{x}_1, \cdots, \overline{x}_K)$ is the unique equilibrium point of (2.8) lying in $\Omega \times \Omega$. □

**Remark 6.4.1.** (i) Let us observe what parameters satisfy conditions (S1a) and $(C1a)^{\lambda}$-$(C3a)^{\lambda}$. It can be seen that $L < 1$, sufficiently large $c$, and sufficiently small $\tau_T$ are apt to meet these conditions. (ii) The existence of equilibrium for (2.8) should have nothing to do with delay. In respecting the conditions (S1a) and $(C1a)^{\lambda}$-$(C3a)^{\lambda}$ involving delay, one can just take delay $\tau_T = 0$ in these inequalities, if existence of equilibrium is the only issue of concern. (iii) Notably, the third inequality in condition $(C1a)^{\lambda}$ becomes more difficult to hold if $L > 1$ and $K$ (the sub-network size) is large. (iv) Obviously, the equilibrium point lying in $\Omega_{\rm m}$ is the trivial and the others are nontrivial.

To discuss the stability of nontrivial equilibria in Theorem 6.4.1 and basins of attraction for the stable equilibria, some additional conditions are needed. First, we define

$$I(\xi) := \begin{cases} -\xi + 2cg(\xi) - 1 - c, & \text{if } \xi \geq 0, \\ -\xi + 2cg(\xi) + 1 + c, & \text{if } \xi < 0. \end{cases}$$

There exist two values $p_c < 0$ and $q_c > 0$ such that $g^{'}(p_c) = g^{'}(q_c) = 1/(2c)$ if $L > 1/(2c) > 0$. The first additional condition is

Condition (C4a): $L > 1/(2c) > 0$, $I(q_c) > 0$, $I(p_c) < 0$.

Under condition(C4a), there exit exactly two zeros of function $I$, say $\check{\kappa}$ and $\hat{\kappa}$, in intervals $(p_c, 0)$ and $(0, q_c)$ respectively, cf. Fig. 6.1(b). Next, we impose

Condition $(C5a)^{\lambda}$: $g^{'}(\check{\kappa}) \geq (1-\lambda)/c$, $g^{'}(\hat{\kappa}) \geq (1-\lambda)/c$.

Let us define the following two sets in $C([-\tau_{\rm max}, 0]; \mathbb{R}^6)$:

$$\Omega^+ := \{\phi : \phi \in C([-\tau_{\rm max}, 0]; \mathbb{R}^6),\ \phi_i(\theta) \geq \hat{\kappa},\ \theta \in [-\tau_T, 0], \text{for } i = K, 2K\},$$
$$\Omega^- := \{\phi : \phi \in C([-\tau_{\rm max}, 0]; \mathbb{R}^6),\ \phi_i(\theta) \leq \check{\kappa},\ \theta \in [-\tau_T, 0], \text{for } i = K, 2K\}.$$

**Theorem 6.4.2.** Under the conditions for the existence of nontrivial equilibria in Theorem 6.4.1, if in addition, (C4a) and $(C5a)^{\lambda}$ hold, then the nontrivial equilibria are asymptotically stable. Moreover, $\Omega^+$ (resp. $\Omega^-$) is contained in the basin of attraction for the equilibrium in $\Omega_{\rm r}$ (resp. $\Omega_{\rm l}$).

**Proof.** It can be justified that $\Omega^+$ and $\Omega^-$ are invariant under (2.8). We merely discuss the former case. We define the function $\tilde{I}(\xi) : (-\infty, \check{\kappa}] \cup [\hat{\kappa}, \infty) \to \mathbb{R}$ as follows:

$$\tilde{I}(\xi) := \begin{cases} -\xi + cg(\xi) + cg(\hat{\kappa}) - 1 - c, & \text{if } \xi \geq \hat{\kappa}, \\ -\xi + cg(\xi) + cg(\check{\kappa}) + 1 + c, & \text{if } \xi \leq \check{\kappa}. \end{cases}$$

We shall show that, for all $t \geq t_0$,

$$\dot{x}_K(t) > \tilde{I}(x_K(t)),\ \dot{y}_K(t) > \tilde{I}(y_K(t)). \tag{6.10}$$

If (6.10) holds, then both $x_K(t)$, $y_K(t)$ remain in $[\hat{\kappa}, \infty)$ for all $t \geq t_0$, due to $\tilde{I}(\hat{\kappa}) = I(\hat{\kappa}) = 0$ and $\tilde{I}(\check{\kappa}) = I(\check{\kappa}) = 0$. The invariance of $\Omega^+$ will then be justified. Let us now confirm (6.10). From (2.8), it can be seen that $x_K(t)$ satisfies, for $t \geq t_0$,

$$\dot{x}_K(t) = -x_K(t) + cg(x_K(t)) + g(x_{K-1}(t - \tau_I)) + c[g(y_K(t - \tau_T)) - g(x_K(t))].$$

The assertion holds for $t = t_0$. Indeed, $\dot{x}_K(t_0) > -x_K(t_0) + cg(x_K(t_0)) - 1 - c[1 - g(\hat{\kappa})] = \tilde{I}(x_K(t_0))$, since $y_K(t_0 - \tau_T) \geq \hat{\kappa}$ and $x_K(t_0) \geq \hat{\kappa}$ . By similar arguments, we can also verify that $\dot{y}_K(t_0) > \tilde{I}(y_K(t_0))$. Assume (6.10) holds for $t \in [t_0, \tilde{t})$ but does not at some $\tilde{t} > t_0$. One of the possibilities is that $\dot{x}_K(\tilde{t}) = \tilde{I}(x_K(\tilde{t}))$, and $\dot{x}_K(t) > \tilde{I}(x_K(t))$, $\dot{y}_K(t) > \tilde{I}(y_K(t))$ for all $t \in [t_0, \tilde{t})$. In this situation, $x_K(t)$ and $y_K(t)$ remain in $(\hat{\kappa}, \infty)$, for $t \in [t_0, \tilde{t})$. Then, $\dot{x}_K(\tilde{t}) > -x_K(\tilde{t}) + cg(x_K(\tilde{t})) - 1 - c[1 - g(\hat{\kappa})] = \tilde{I}(x_K(\tilde{t}))$ and yields a contradiction. The other possibilities can also be ruled out, by similar arguments. Therefore (6.10) is valid. The remaining arguments are similar to Theorem 4.3.1. We merely sketch the idea. Note that the two nontrivial equilibria lie in $\Omega^+$, $\Omega^-$ respectively. Thus, the solution starting near the nontrivial equilibrium remains in the invariant set $\Omega^+$ or $\Omega^-$. Hence, we have the slope control on the coupling terms, namely $cg'(x_i(t)) < 1$ for $i = K, 2K$, and $g'(x_i(t)) < 1$, for $i \neq K, 2K$, for all $t \geq t_0$, and so that the solution of (2.8) evolved near the nontrivial equilibrium is dominated by the degradation term $-x_i(t)$ in (2.8). Solutions starting near the equilibrium thus converge to the equilibrium.

According to Theorem 6.4.1 and the invariance, we conclude that $\Omega^+$ (resp. $\Omega^-$) is contained in the basin of attraction for the equilibrium lying in $\Omega_\mathrm{r}$ (resp. $\Omega_\mathrm{l}$). □

**Remark 6.4.2.** (i) Although conditions (C4a), $(\text{C5a})^\lambda$ seem complicated, it can be observed that they are apt to be satisfied for large $c$. (ii) In fact, it is not difficult to observe that the nontrivial equilibria are stable under the conditions on the existence of exactly three equilibria (Remark 6.4.1), and conditions (C4a), $(\text{C5a})^\lambda$. These conditions are independent of delays and are easy to satisfy if $0 < L < 1$ and $c$ is sufficiently large.

## 6.5 Bifurcation and oscillations

In Theorem 6.1.2, we have shown that if the transmission delay $\tau_T$ is small enough, (2.8) can attain global synchronization in spite of the magnitude of internal delay $\tau_I$. In this section, via bifurcation analysis, we shall show that there exist nontrivial synchronous periodic solutions for (2.8) induced by internal delay $\tau_I$. To simplify the presentation, we consider (2.8) with $K = 3$ in this section. Moreover, most of arguments in this section are similar to the ones in Section 5.2.3. Hence, we merely sketch the main process and adopt the same symbols for some settings used in Section 5.2.3.

First, let us consider a circle block matrix $\text{circ}(A_0, A_1, \cdots, A_{n-1})$ where $A_j$, $j = 0, 1, \cdots, n-1$, is a $k \times k$ matrix. Let $v_j = e^{\frac{2\pi j}{n} i}$, and define a function of matrices $G(x) = A_0 + xA_1 + \cdots + x^{n-1}A_{n-1}$. In [77], it is shown that

$$\det(\lambda I_{kn} - \text{circ}(A_0, A_1, \cdots, A_{n-1})) = \Pi_{j=1}^{n} \det(\lambda I_k - G(v_j)), \tag{6.11}$$

where $I_k$ is the $k \times k$ identity matrix. The linearization of (2.8) at the trivial equilibrium $(0, 0, 0, 0, 0, 0)$ is given by

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + Lu_3(t - \tau_I), \\ \dot{u}_2(t) = -u_2(t) + Lu_1(t - \tau_I), \\ \dot{u}_3(t) = -u_3(t) + Lu_2(t - \tau_I) + cLu_6(t - \tau_T), \\ \dot{u}_4(t) = -u_4(t) + Lu_6(t - \tau_I), \\ \dot{u}_5(t) = -u_5(t) + Lu_4(t - \tau_I), \\ \dot{u}_6(t) = -u_6(t) + Lu_5(t - \tau_I) + cLu_3(t - \tau_T). \end{cases} \tag{6.12}$$

For convenience, we set

$$d := cL.$$

Then the characteristic equation for (6.12) (cf. [34], [72]) is

$$\Delta(\lambda) := \det \begin{pmatrix} 1+\lambda & 0 & -Le^{-\lambda\tau_I} & 0 & 0 & 0 \\ -Le^{-\lambda\tau_I} & 1+\lambda & 0 & 0 & 0 & 0 \\ 0 & -Le^{-\lambda\tau_I} & 1+\lambda & 0 & 0 & -de^{-\lambda\tau_T} \\ 0 & 0 & 0 & 1+\lambda & 0 & -Le^{-\lambda\tau_I} \\ 0 & 0 & 0 & -Le^{-\lambda\tau_I} & 1+\lambda & 0 \\ 0 & 0 & -de^{-\lambda\tau_T} & 0 & -Le^{-\lambda\tau_I} & 1+\lambda \end{pmatrix} = 0.$$

Thanks to (6.11), the characteristic equation can be factored as

$$\Delta_+(\lambda)\Delta_-(\lambda) = 0,$$
$$\Delta_\pm(\lambda) := (1+\lambda)^2(1+\lambda \mp de^{-\lambda\tau_T}) - L^3 e^{-3\lambda\tau_I}.$$

We substitute $\lambda = iw$ with $w > 0$ into $\Delta_{\pm}(\lambda) = 0$ and collect the real and imaginary parts to yield

$$\begin{cases} L^3 \sin(3\tau_I w) = (w^3 - 3w) \pm (w^2 - 1)d \sin(\tau_T w) \pm 2d \cos(\tau_T w)w, \\ L^3 \cos(3\tau_I w) = (1 - 3w^2) \mp (1 - w^2)d \cos(\tau_T w) \mp 2d \sin(\tau_T w)w. \end{cases} \tag{6.13}$$

Summing up the square of equations (6.13) gives

$$Q_{\pm}(w) = L^6, \tag{6.14}$$

where $Q_{\pm}(w) := Q_1(w) \pm Q_2(w)$ and $Q_1(w) := w^6 + (3 + d^2)w^4 + (3 + 2d^2)w^2 + d^2 + 1$ and $Q_2(w) := d[2\sin(\tau_T w)w^5 - 2\cos(\tau_T w)w^4 + 4\sin(\tau_T w)w^3 - 4\cos(\tau_T w)w^2 + 2\sin(\tau_T w)w - 2\cos(\tau_T w)]$. Therefore the positive solution of (6.14) corresponds to the purely imaginary roots of $\Delta_{\pm}(w) = 0$.

Now let us introduce some settings and the condition imposed for purely imaginary roots of $\Delta_{\pm}(w) = 0$ as follows:

Define $P(w) := (6 - 2|d|\tau_T)w^5 - |d|(10 + 2\tau_T)w^4 + (12 + 4d^2 - |d||8 - 4\tau_T|)w^3 - |d|(12 + 4\tau_T)w^2 + (6 + 4d^2 - |d||8 - 2\tau_T|)w - |d|(2 + 2\tau_T)$. As $P$ is a polynomial, we set

$$\tilde{w} := \text{the largest zero of } P(w). \tag{6.15}$$

Condition (B1a)$_{\pm}$: $6 - 2|d|\tau_T > 0$, $\min\{Q_{\mp}(w) : w \in [0, \tilde{w}]\} > L^6$, and $\max\{Q_{\pm}(w) : w \in [0, \tilde{w}]\} < L^6$,

By similar arguments to Lemma 5.2.6, we can derive the following lemma.

**Lemma 6.5.1.** Under condition (B1a)$_+$ (resp. (B1a)$_-$), there exist exactly one pair of purely imaginary roots, say $\pm i\omega_+^*$ (resp. $\pm i\omega_-^*$), for characteristic equation $\Delta(\lambda) = 0$. In particular, $\pm i\omega_+^*$ (resp. $\pm i\omega_-^*$) are the roots of $\Delta_+(\lambda) = 0$ (resp. $\Delta_-(\lambda) = 0$).

**Remark 6.5.1.** (i) Note that $Q_+(0) = (1 - d)^2$ and $Q_-(0) = (1 + d)^2$. Therefore

$$(1 - d)^2 < L^6 < (1 + d)^2 \text{ (resp. } (1 + d)^2 < L^6 < (1 - d)^2) \tag{6.16}$$

is a necessary condition for (B1a)$_+$ (resp. (B1a)$_-$) to hold. Moreover, $d = cL \neq 0$ is necessary, as seen from the inequality (6.16); in particular, $c = d/L > 0$ (resp. $c = d/L < 0$) is necessary for condition (B1a)$_+$ (resp. (B1a)$_-$). (ii) The following weaker condition

$$L^6 > (1 - d)^2 \text{ ( resp. } L^6 > (1 + d)^2) \tag{6.17}$$

can also provide the existence of zero to $Q_+(w) = 0$ (resp. $Q_-(w) = 0$), but the uniqueness of positive zero can not be guaranteed. However, the situation of multiple zeros can be ruled out with assistance of numerical computation. Basically, from (6.17), it can be observed that larger $L$ is advantageous to the occurrence of Hopf bifurcation, hence oscillation.

To find the value of $\tau_I$ such that $\pm i\omega_+^*$ (resp. $\pm i\omega_-^*$) are the purely imaginary roots of $\Delta_+(\lambda) = 0$ (resp. $\Delta_-(\lambda) = 0$), we divide the first equation of (6.13) by the second one and obtain

$$\begin{aligned}
&\tan(3\tau_I w) = S_\pm(w)/C_\pm(w),\\
&S_\pm(w) := (w^3 - 3w) \pm (w^2-1)d\sin(\tau_T w) \pm 2d\cos(\tau_T w)w,\\
&C_\pm(w) := (1-3w^2) \mp (1-w^2)d\cos(\tau_T w) \mp 2d\sin(\tau_T w)w.
\end{aligned}$$

Let us define $\eta_k^\pm$, $k \in \mathbb{N}$, as follows:

$$\eta_k^\pm := \frac{1}{3\omega_\pm^*}\begin{cases} 3\pi/2 + 2(k-1)\pi, & \text{if } C_\pm(\omega_\pm^*) = 0,\ S_\pm(\omega_\pm^*) < 0;\\ \pi/2 + 2(k-1)\pi, & \text{if } C_\pm(\omega_\pm^*) = 0,\ S_\pm(\omega_\pm^*) > 0;\\ \tan^{-1}(S_\pm(\omega_\pm^*)/C_\pm(\omega_\pm^*)) + 2k\pi, & \text{if } C_\pm(\omega_\pm^*) > 0;\\ \tan^{-1}(S_\pm(\omega_\pm^*)/C_\pm(\omega_\pm^*)) + (2k-1)\pi, & \text{if } C_\pm(\omega_\pm^*) < 0. \end{cases} \tag{6.18}$$

Herein, $\eta_k^+$ (resp. $\eta_k^-$) is positive and the critical value of bifurcation parameter with respect to $\tau_I$, at which $\Delta(\lambda) = 0$ has exactly one pair of purely imaginary roots $\pm i\omega_+^*$ (resp. $\pm i\omega_-^*$). To apply the Hopf bifurcation theory, it remains to verify the transversality condition:

Condition (B2a)$_\pm$: $[\mathbf{R}_\pm(\omega_\pm^*, \eta_k^\pm)]^2 + [\mathbf{I}_\pm(\omega_\pm^*, \eta_k^\pm)]^2 \neq 0$, and $\Lambda_\pm(\omega_\pm^*) \neq 0$,

where

$$\begin{aligned}
\mathbf{R}_\pm(\omega, \tau_I) \;&:=\; [-3 - 9\tau_I \pm 3d\tau_I\cos(\tau_T\omega) \mp d\tau_T\cos(\tau_T\omega)]\omega^2\\
&\quad \pm(2 + 6\tau_I + 2\tau_T)d\sin(\tau_T\omega)\omega + 3 + 3\tau_I \pm (-2 - 3\tau_I + \tau_T)d\cos(\tau_T\omega)\\
\mathbf{I}_\pm(\omega, \tau_I) \;&:=\; -3\tau_I\omega^3 \pm (\tau_T - 3\tau_I)d\sin(\tau_T\omega)\omega^2\\
&\quad +[6 + 9\tau_I \pm (-2 - 6\tau_I + 2\tau_T)d\cos(\tau_T\omega)]\omega \pm (2 + 3\tau_I - \tau_T)d\sin(\tau_T\omega)\\
\Lambda_\pm(\omega) \;&:=\; [9 \pm 3\tau_T d\cos(\tau_T\omega)]\omega^4 \pm (15 + 3\tau_T)d\sin(\tau_T\omega)\omega^3\\
&\quad +[9 + 6d^2 \pm (-12 + 3\tau_T)d\cos(\tau_T\omega)]\omega^2 + (3 + 3\tau_T)d\sin(\tau_T\omega)\omega.
\end{aligned}$$

**Proposition 6.5.2.** Assume that conditions (B1a)$_+$ and (B2a)$_+$ (respectively, (B1a)$_-$ and (B2a)$_-$) hold for some fixed $k \in \mathbb{N}$. The Hopf bifurcation occurs at $\tau_I = \eta_k^+$ (resp. $\eta_k^-$), and a periodic orbit is bifurcated from the zero solution of (2.8).

The proof of Proposition 6.5.2 is similar to the one of Theorem 5.2.8; hence omitted.

Let us recall Theorems 6.1.2 and 6.1.3 in which (2.8) attains synchronization under condition (S1a) or (S2a) or (S3a). As mentioned in Remark 6.5.1, $(1-d)^2 < L^6$ and $c > 0$ are necessary for condition $(B1a)_+$. It can be observed that $(1-d)^2 < L^6$ and $c > 0$ is compatible only with condition (S1a), but not (S2a) and (S3a). From this view point, positive coupling strength $c$ (with other conditions) leads to synchronous oscillation. On the other hand, similarly, we observe that negative coupling strength $c$ leads to anti-phase oscillation. Motivated by these observations, Theorems 6.1.2, 6.1.3, 6.2.1 and Proposition 6.5.2, we draw the following conclusion.

**Theorem 6.5.3.** Assume that conditions (S1a) (resp. (AP1)), $(B1a)_+$, and $(B2a)_+$ (resp. $(B1a)_-$, and $(B2a)_-$) hold for some fixed $k \in \mathbb{N}$. Then there exists a synchronous (respectively, anti-phase) periodic solution bifurcated from the trivial equilibrium, at internal delay $\tau_I = \eta_k^+$ (resp. $\tau_I = \eta_k^-$) for the coupled $K$-loops (2.8) (resp. $(2.8)_0$).

## 6.6 Description of dynamical scenarios

As mentioned earlier, coupled network system can exhibit a variety of interesting behaviors. We plan to depict the dynamical scenario for the coupled $K$-loops (2.8) under the influence of coupling strength, the gain of the activation function, the internal delay, and the transmission delay. Let us first mention some properties of the single $K$-loop (2.7). Notably, the dynamics of (2.7) without internal delay $\tau_I$ has been studied extensively in [7]. By similar approach as Lemma 2 in [31] and Lemma 4.1 in [72]), it can be shown that the trivial equilibrium of (2.7) is stable for all $\tau_I \geq 0$ if $L \leq 1$. On the other hand, there exist periodic solutions bifurcated from the zero solution, at suitable $\tau_I$ for (2.7) if $L > 1$. Therefore, the dynamical scenarios for cases $L \in (0, 1)$ and $L \in (1, \infty)$ are rather different. Below we shall discuss the two cases: $L \in (0, 1)$ and $L \in (1, \infty)$, for the coupled loops separately. Notice that the trivial equilibrium for the coupled loops (2.8) can become unstable as there exist periodic solutions bifurcated from the equilibrium, cf. Theorem 6.5.3. This already shows an effect from the coupling between these two loops.

Notably, what we have derived for global synchronization, global convergence to the origin are theories with sufficient conditions. When we say that (2.8) does not admit certain dynamics (such as synchronization), we may need computer simulations to support the arguments. In addition, caution must be used if saying that a system can not be synchronized merely through numerical simulation. It is not assured how long a simulation should be run to exclude the possibility of synchronization. On the other hand, anti-phase is an evidence of desynchrony that one can assure from analysis or numerical simulation. Our approach can establish anti-phase oscillations bifurcated from the zero solution at certain values of internal delay, and can also be extended to analyze bifurcation with respect to transmission delay.

**Effect of coupling strength.** Let us first consider the case that $L \in (0, 1)$, $c \geq 0$. It can be seen from Theorem 6.3.2 that if $c$, the coupling strength between two loops, is sufficiently small so that

$$c \leq 1/L - 1, \tag{6.19}$$

then the coupled loops (2.8) attain global convergence to the trivial equilibrium (hence global synchronization) in spite of delay magnitude of $\tau_I$ and $\tau_T$. In addition, if $c$ is larger so that (6.19) fails to hold, the coupled loops (2.8) can attain global synchronization if $\tau_T$ is small enough. Such an observation follows from that for arbitrarily large $c > 0$, condition (S1a) for Theorem 6.1.2 holds if $\tau_T$ is small enough. On the other hand, Theorem 6.5.3 concludes the birth of nontrivial synchronous periodic solutions, under certain criteria. It can be seen that the dominant condition $(\text{B1a})_+$ can not hold if $c = 0$ or is too small, cf. Remark 6.5.1. On the other hand, for some suitable magnitude of $c$ (not too large) such that $(\text{B1a})_+$ holds, the nontrivial synchronous periodic solutions of coupled loops can be induced by internal delays. Now, it is natural to ask what will occur if magnitude of $c$ is quite large. Theorem 6.4.1 has shown that system (2.8) with sufficiently large $c$ achieves global convergence to nontrivial synchronous equilibria (hence globally synchronized) in spite of internal lag ($\tau_I$) if transmission lag ($\tau_T$) is small enough. This result has justified the the numerical finding in [7].

Accordingly, we summarize a dynamical scenario for (2.8) as $c$ increases from zero to $\infty$:

global convergence to zero

$\rightarrow$ global synchronization ($\tau_T$ small)

$\rightarrow$ global synchronization with synchronous oscillation (induced by internal delays) ($\tau_T$ small)

$\rightarrow$ global convergence to nontrivial synchronous stable equilibria ($\tau_T$ small).

Notice that the effect of positive coupling strength on synchronization and negative coupling strength on anti-phase motion are counterparts to each other for the coupled loops. Indeed, if $L \in (0, 1)$, as $c$ varies from 0 to $-\infty$, $(2.8)_0$ goes through global convergence to zero $\rightarrow$ birth of anti-phase oscillation (induced by internal delays) $\rightarrow$ global convergence to nontrivial stable antisynchronous equilibria.

Roughly speaking, it is easier for the coupled loops (2.8) to attain global synchronization for the case $L \in (0, 1)$ than $L \in (1, \infty)$.

In the case that $L \in (1, \infty)$, the single loop (2.7) has at least three equilibria, and thus the decoupled system (2.8) with $c = 0$ can not be synchronized. In fact, we can observe that wether if the coupled loops (2.8) can attain synchronization strongly relies on the interaction between two coupled loops. Observe condition (S1a) in Theorem 6.1.2, it can be seen that coupling strength $c$ is necessary (can not be zero), and $\tau_T$ must be small enough. It has been illustrated in Example 6.7.2 that (2.8) fails to be synchronized if coupling strength is too small, and can be synchronized if coupling strength is suitably large. However, we found that large magnitude of $c$ does not always favor condition (S1a); it also depends on the nonlinearity of activation function $g$, and the magnitude of transmission delay. Similar observation was reported in [50] and [48]. In our numerical computation in Example 6.7.2, for some situation as transmission delay is too large, large coupling strength $c$ still can not synchronize the oscillators.

**Effect of delays.** As previous arguments, in some parameter regions of $L$ and $c$, it is necessary that transmission delay is small enough for (2.8) to be synchronized. Example 6.7.2 has illustrated that the coupled loops (2.8) can be synchronized as transmission lag $\tau_T$ is small enough, and (2.8) can not be synchronized if $\tau_T$ is too large. Notice that from our numerical evidence, if (2.8) becomes asynchronous induced by large transmission delay, stronger coupling strength $c$ does not promote the system to regain synchronization. In fact, the coupled loops can become asynchronous in the form of anti-phase oscillation. This can be confirmed by performing the bifurcation analysis as in Section 6.5, but with $\tau_T$ as bifurcation parameter. Then at a large coupling strength, there exists anti-phase periodic solution for certain $\tau_T$. Under our formulation, whether if the coupled loops (2.8) can be synchronized does not depend on internal delays, cf. Theorems 6.1.2, 6.1.3. But transmission delay plays a role in synchronization. It is then natural to ask how the internal delay affect the dynamics in (2.8). Following our result in Theorem 6.5.3, it can be seen that oscillation is generated by internal delay of certain magnitudes.

## 6.7 Numerical examples

We provide two numerical examples to illustrate the present theory.

**Example 6.7.1.** Consider the coupled 3-loops (2.8) with $g(\xi) = \tanh(0.999\xi)$, $\tau_I = 11.2$, $\tau_T = 0.001$, and $c = 400/999$. Then (2.8) satisfies condition (S1a). By Theorem 6.1.2, (2.8) attains global synchronization. It can be verified that condition $(\text{B1a})_+$ and $(\text{B2a})_+$, $k = 1, 2$, holds by direct computation. By Theorem 6.5.3, there exists a synchronous periodic solution bifurcated from the zero solution of (2.8). Fig. 6.2 illustrates that the solution of (2.8) tends to a synchronous periodic orbit as $t \to \infty$. Fig. 6.3 provides the the dynamics for each component of the solution in Fig. 6.2; in the panels, six different colors represent the evolutions of six components.

**Example 6.7.2.** Consider the coupled 3-loops (2.8) with $L = 1.02$. As $c = 0.5$ $\tau_I = 2$ and $\tau_T = 0.01$, it can be checked that (2.8) satisfies condition (S1a). In such a situation, Fig. 6.4 illustrates that the solution of (2.8), evolved from $(1.6 + 0.1\sin t, -1.6 + 0.1t, 1.6 + 0.1\sin t \cdot \cos t, -1.6 + 0.1\sin t, -1.6 + 0.1t, -1.6 + 0.1\sin t \cdot \cos t)$ is synchronized. If we consider (2.8) with smaller $c = 0.001$ instead, which does not satisfy condition (S1a), Fig. 6.5 illustrates that the solution of (2.8) converges to some asynchronous equilibrium point. If we consider (2.8) with the same parameters but with larger $\tau_T = 100$ instead, which does not satisfy condition (S1a), Fig. 6.6 illustrates that the solution of (2.8) appears to be anti-phase, hence not synchronized. If we increase the coupling strength to $c = 20$ and still hold $\tau_T = 100$, the system still exhibits anti-phase, hence not synchronized, as shown in Fig. 6.7.

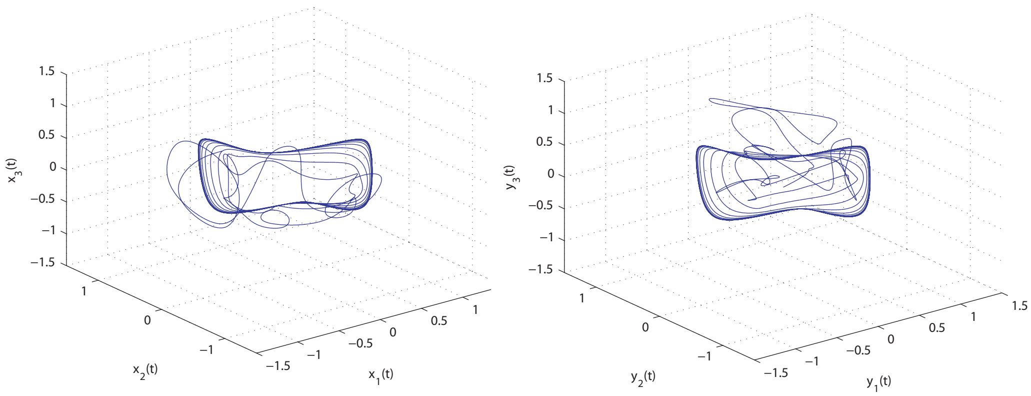



Figure 6.2: An orbit of (2.8) with $g(\xi) = \tanh(0.999\xi)$, $\tau_I = 11.2$, $\tau_T = 0.001$, and $c = 400/999$, which is evolved from $\phi(t) = (\sin t, \cos t, t, t\sin t, -t, \sin t \cdot \cos t)$. There exists a synchronous limit cycle.

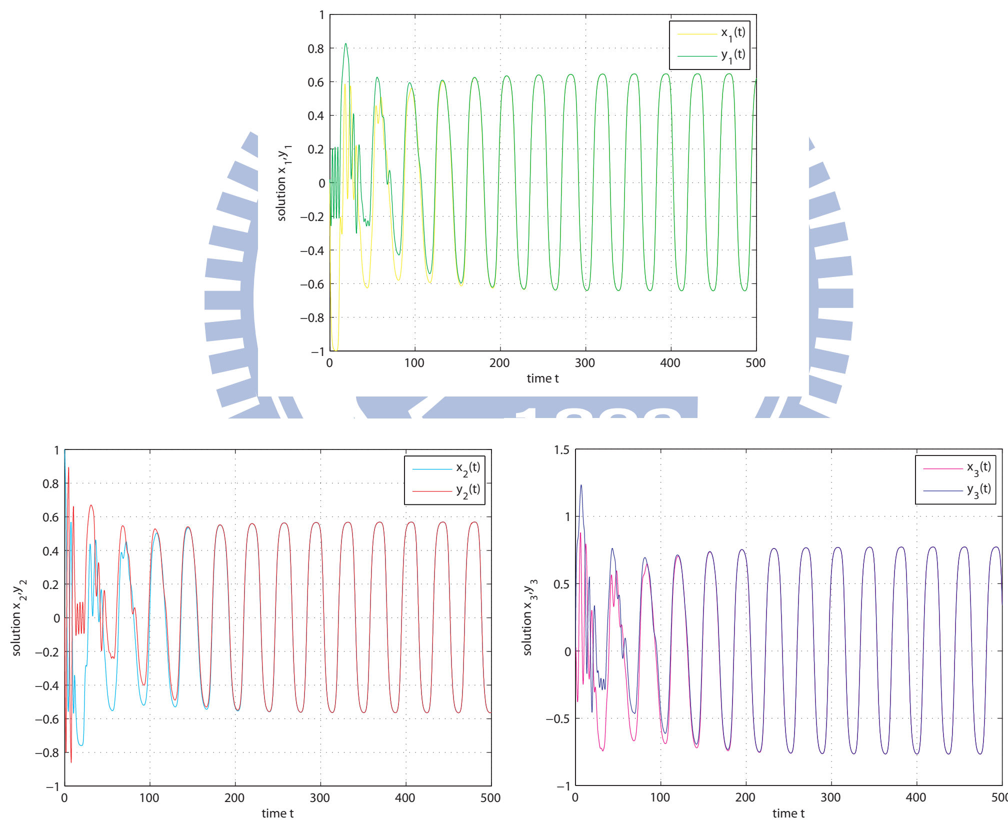



Figure 6.3: The dynamics for the corresponding component of solution in Fig. 6.2.

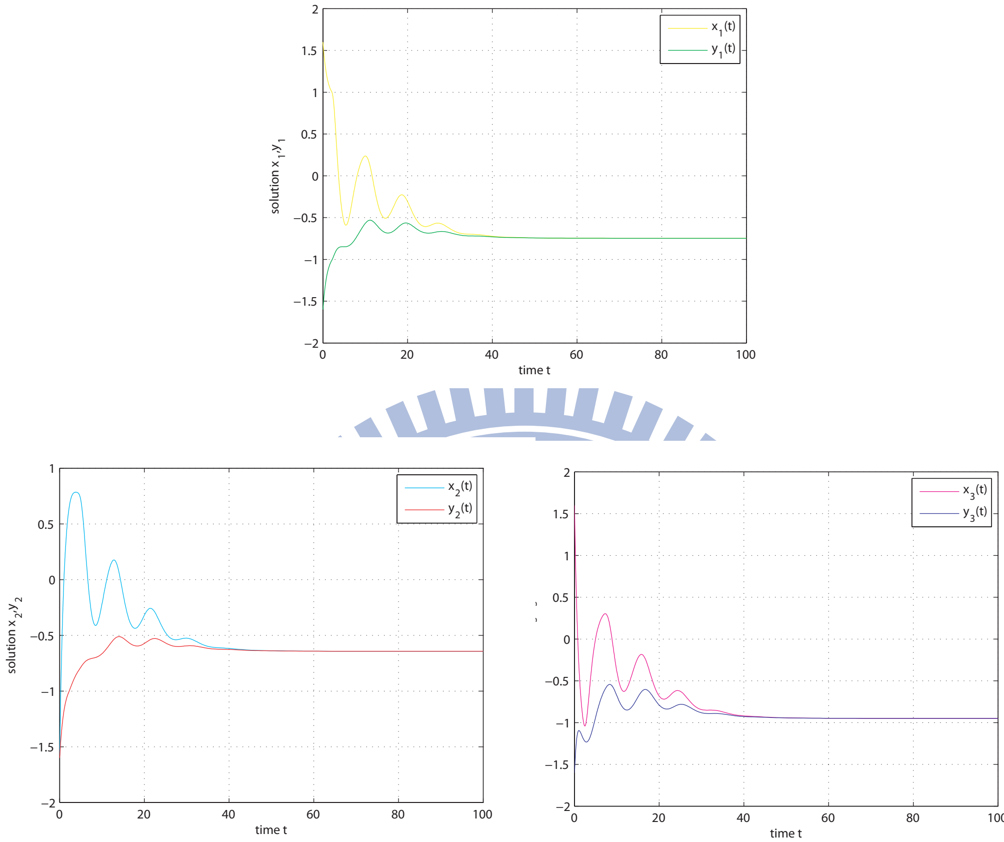



Figure 6.4: Evolutions of components $(x_i(t), y_i(t))$ for the solution of (2.8) with $g(\xi) = \tanh(1.02\xi)$, $\tau_I = 2$, $\tau_T = 0.01$, and $c = 0.5$, starting from $\phi(t) = (1.6 + 0.1 \sin t, -1.6 + 0.1t, 1.6 + 0.1 \sin t \cdot \cos t, -1.6 + 0.1 \sin t, -1.6 + 0.1t, -1.6 + 0.1 \sin t \cdot \cos t)$. It appears to be synchronized.

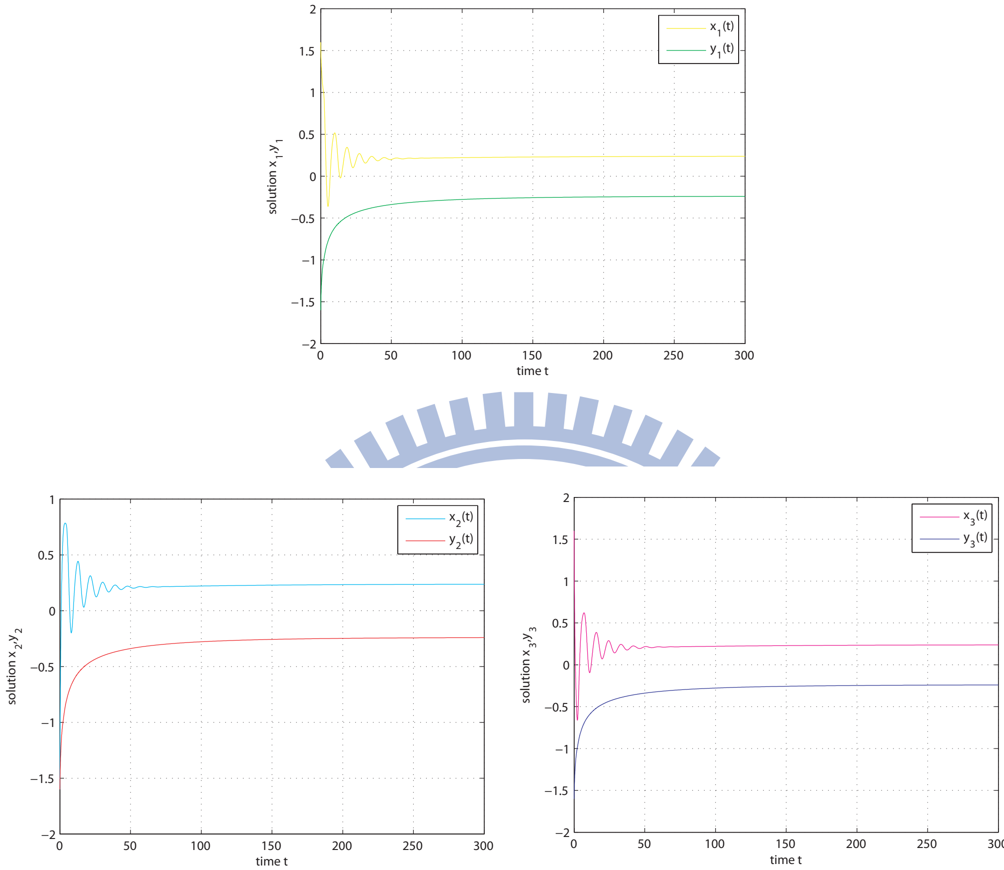



Figure 6.5: Evolutions of components $(x_i(t), y_i(t))$ for the solution of (2.8) with $g(\xi) = \tanh(1.02\xi)$, $\tau_I = 2$, $\tau_T = 0.01$, and $c = 0.001$, starting from $\phi(t) = (1.6 + 0.1\sin t, -1.6 + 0.1t, 1.6 + 0.1\sin t \cdot \cos t, -1.6 + 0.1\sin t, -1.6 + 0.1t, -1.6 + 0.1\sin t \cdot \cos t)$. The solution converges to asynchronous steady state.

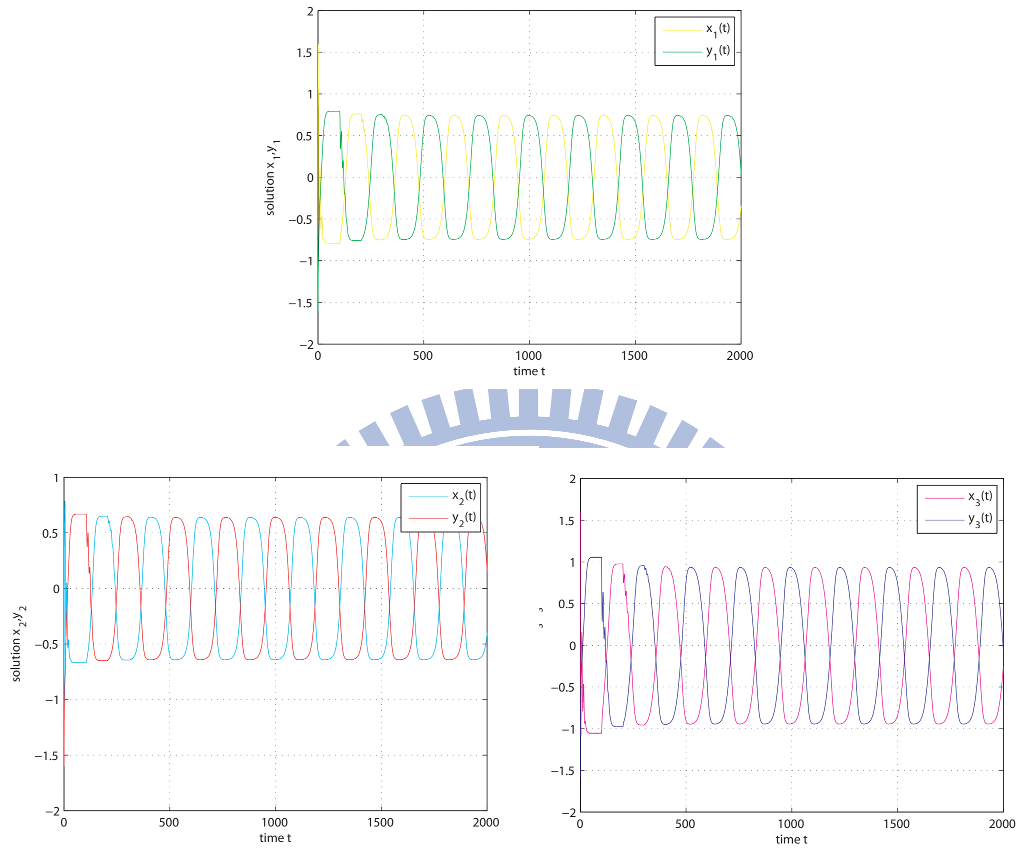



Figure 6.6: Evolutions of components $(x_i(t), y_i(t))$ for the solution of (2.8) with $g(\xi) = \tanh(1.02\xi)$, $\tau_I = 2$, $\tau_T = 100$, and $c = 0.5$, starting from $\phi(t) = (1.6 + 0.1\sin t, -1.6 + 0.1t, 1.6 + 0.1\sin t \cdot \cos t, -1.6 + 0.1\sin t, -1.6 + 0.1t, -1.6 + 0.1\sin t \cdot \cos t)$. The system tends to an anti-phase motion.

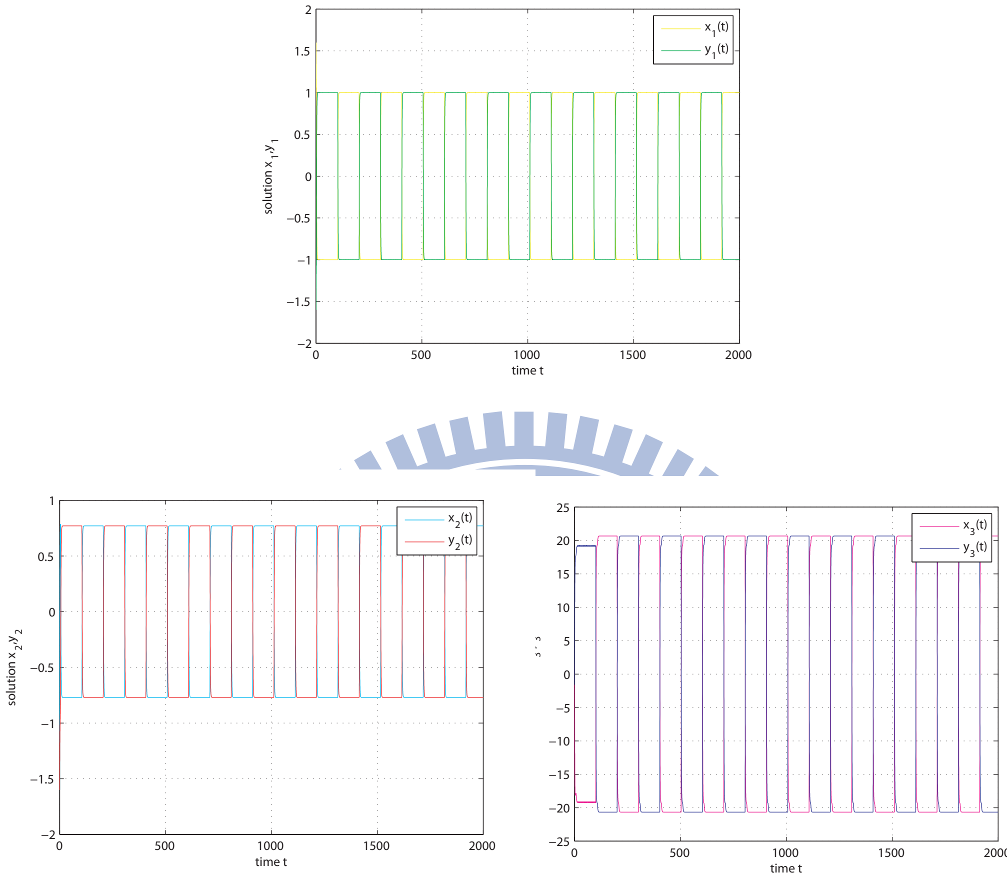



Figure 6.7: Evolutions of components $(x_i(t), y_i(t))$ for the solution of (2.8) with $g(\xi) = \tanh(1.02\xi)$, $\tau_I = 2$, $\tau_T = 100$, and $c = 20$, starting from $\phi(t) = (1.6 + 0.1\sin t, -1.6 + 0.1t, 1.6 + 0.1\sin t \cdot \cos t, -1.6 + 0.1\sin t, -1.6 + 0.1t, -1.6 + 0.1\sin t \cdot \cos t)$. The system retains an anti-phase motion, even with larger coupling strength $c$.

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