Chapter 2 Mechanisms of passive mode-locking

2-1 Basic principles of mode-locking

Oscillation in an inhomogeneously broadened laser can take place at a number of frequencies, which are separated by (assuming a refractive index of n = 1)

$$\omega_q - \omega_{q+1} = \frac{\pi c}{l} \equiv \Omega \quad , \qquad (2\text{-}1.1)$$

where Ω is the free spectral range (or the mode spacing) in angular frequency and q is an integer. Now consider the total electric field on a reference plane resulting from such a multimode oscillation, say next to one of the mirrors, in the optical resonator. It can be taken, using complex notation [1], as

$$E(t) = \sum_m C_m e^{i[(\omega_0 + m\Omega)t + \varphi_m]} \quad , \qquad (2\text{-}1.2)$$

where C_m and φ_m are the amplitude and the phase of the m-th mode. The summation runs over all the oscillating modes and ω_0 is chosen arbitrarily as oscillation frequency of one of the modes (usually chosen as the one closest to the line center). One property of Eq. (2-1-2) is that $|E(t)|$ is periodic in time with a period of $\tau \equiv 2\pi/\Omega = 2l/c$, which is the round-trip transit time inside the resonator. Using Eq. (2-1-2), the field at t + τ can be written

$$E(t+\tau) = \sum_m C_m \exp\{i[(\omega_0 + m\Omega)(t + \frac{2\pi}{\Omega}) + \varphi_m]\}$$

$$= \sum_m C_m \exp\{i[(\omega_0 + m\Omega)t + \varphi_m]\} \exp i\, 2\pi \frac{\omega_0}{\Omega} + m$$

$$= E(t)\exp(i 2\pi \omega_0 t / \Omega) \quad . \qquad (2\text{-}1.3)$$

Notice that E(t+τ) is identical to E(t), except with a constant phase factor and the periodic property of E(t) depends on the fact that the modes are equally spaced and the phases ϕ_m

are fixed. In typical lasers the phases φ_m are likely to vary randomly with time. This causes the intensity of the laser output to fluctuate randomly and greatly reduces its usefulness for many applications where temporal coherence is important. It should be noted that this fluctuation takes place because of random interference between modes and not because of intensity fluctuations of individual modes.

Two ways in which the laser can be made coherent are: First, make it possible for the laser to oscillate at a single frequency only so that mode interference is eliminated. This can be achieved in a variety of ways, including shortening the resonator length *l*, thus increasing the mode spacing ($\Omega=\pi c/l$) to a point where only one mode has sufficient gain to oscillate. The second approach is to force the phases ϕ_m of all the modes to maintain their relative values (ideally zero, so that they all oscillate in phase). This is the so-called "mode locking" technique proposed and demonstrated in the early history of the laser. This mode locking causes the oscillation intensity to consist of a periodic pulse train with a period of $\tau \equiv 2\pi/\Omega = 2l/c$.

One of the most useful forms of mode locking results when the phases φ_m are made equal to zero. To simplify the analysis of this case, assume that there are N oscillating modes with equal amplitudes. Taking $C_m = 1/\sqrt{N}$ and $\varphi_m = 0$ in Eq. (2-1-2), we obtain

$$E(t) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{i(\omega_0 + m\Omega)t} = \frac{1}{\sqrt{N}} e^{i[\omega_0 + (N+1)\Omega/2]t} \frac{\sin(N\Omega t/2)}{\sin(\Omega t/2)}, \tag{2-1.4}$$

where the field is normalized to a constant energy (independent of N). The last equality is obtained by summing up the geometric series. The average laser power output is proportional to E(t)E*(t) and is given by

$$P(t) \propto \frac{1}{N} \frac{\sin^2(N\Omega t/2)}{\sin^2(\Omega t/2)}, \tag{2-1.5}$$

where the averaging is performed over a time that is long compared with the optical period $2\pi/\omega_0$ but short compared with the modulation period $2\pi/\Omega$.

Some of the analytic properties of P(t) are immediately apparent:

1. The power is emitted in a form of a train of pulses with a period $\tau=2l/c$, i.e., the round-trip transit time.

2. The peak power, $P(s\tau)$ (for s = 0, 1, 2, 3, . . .), is equal to N times the average power, where N is the number of modes locked together.

3. The peak field amplitude is equal to N times the amplitude of a single mode.

4. The pulse width of the main peaks, defined as the time from the peak to the first zero is $\tau_0=\tau/N$. This is approximately the FWHM of the main peaks of P(t) (for N >> 1). There are (N-2) sidelobes between the neighboring main peaks.

The number of oscillating modes can be estimated by $N \cong \Delta\omega/\Omega$, that is, the ratio of the transition line width $\Delta\omega$ (or gain bandwidth) to the frequency spacing between the modes. Using this relation, as well as $\tau = 2\pi/\Omega$ in $\tau_0 = \tau/N$, we obtain

$$\tau_0 \sim \frac{2\pi}{\Delta\omega} = \frac{1}{\Delta\nu} = \frac{\tau}{N}, \quad (2\text{-}1.6)$$

where $\Delta\upsilon$ is the gain bandwidth. Thus the temporal length of the mode-locked laser pulses is approximately the inverse of the gain line-width.

2-2 Passively mode-locking with nonlinear mirror

Saturable absorbers exhibit intensity-dependent light transmission. In mode-locked lasers this feature is used to perform amplitude discrimination and pulse shortening. Pulse shortening has been analyzed in the case of a fast absorber. Recently a novel nonlinear optical device utilizing second harmonic generation was proposed. It may exhibit intensity-dependent reflection or transmission with a fast time response in the sub-picosecond range. Successful mode locking of a Nd doped laser using this device was demonstrated. Since one of the important characteristics of a mode locker is pulse shortening in a single transit, pulse shortening is due to reflection by the nonlinear mirror as Fig. 2-1 shown below

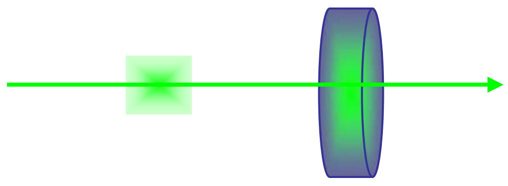

Fig. 2-1 The nonlinear mirror consists of a nonlinear crystal (NLC) for SHG and M-dichroic mirror

Fig. 2-1 illustrates the basic idea of the nonlinear mirror. An intense light beam at frequency ω generates second harmonic (SH) wave in a nonlinear crystal NLC. The total SH at 2ω and part of the fundamental F are reflected by a dichroic mirror M in the exact backward direction. In the second pass through the nonlinear crystal, partial reconversion of the second harmonic into the fundamental takes place. The higher the intensity of the incident beam the higher is the conversion into the second harmonic and the higher is the amplification of the reflected fundamental due to the presence of the second harmonic wave. Hence one should expect that the resultant reflectivity increases with the increase of the intensity of the incident beam.

First, we review the basic theory of the second harmonic generation derived from the Maxwell equations [2]. We consider the interaction of three waves with ($\omega_3 = \omega_1 + \omega_2$) via the second-order optical nonlinearity. Let the field be written as

$$\mathbf{E_i(t) = E_i^{\omega_1}(t) + E_i^{\omega_2}(t) + E_i^{\omega_3}(t)} \qquad (i = x', y', z'). \qquad (2\text{-}2.1)$$

with $\mathbf{E_i^{\omega_1}(t) = \frac{1}{2}\left(E_{0i}^{\omega_1} e^{i(\omega_1 t - k_1 z)} + c.c.\right) = \frac{1}{2}\left(a_{1i} E_1 e^{i(\omega_1 t - k_1 z)} + c.c.\right)}$ $\qquad (i = x', y', z')$

$$\mathbf{E_i^{\omega_2}(t) = \frac{1}{2}\left(E_{0i}^{\omega_2} e^{i(\omega_2 t - k_2 z)} + c.c.\right) = \frac{1}{2}\left(a_{2i} E_2 e^{i(\omega_2 t - k_2 z)} + c.c.\right)} \qquad (i = x', y', z')$$

$$\mathbf{E_i^{\omega_3}(t) = \frac{1}{2}\left(E_{0i}^{\omega_3} e^{i(\omega_3 t - k_3 z)} + c.c.\right) = \frac{1}{2}\left(a_{3i} E_3 e^{i(\omega_3 t - k_3 z)} + c.c.\right)}, \qquad (i = x', y', z'),$$

where we assume that all of the three fields are the normal modes of propagation along the z-direction in the nonlinear medium, with unique wave-numbers k_1, k_2, and k_3 and amplitudes E_1, E_2, and E_3. In the above equations, a_1, a_2, and a_3 are unit vectors representing the polarization direction of the normal modes of propagation of these three fields in the nonlinear medium. In formulating the interaction we adopt z-axis as the common direction of propagation, and (x', y', z') as the Cartesian components in the principal coordinates of the nonlinear medium. These three fields are propagating in the same direction (z-axis), but may assume different polarization states. Generally speaking, the direction of propagation (z) may not be parallel to one of the principal axes. The collinear propagation in the nonlinear medium is to ensure the maximum physical overlap. In the absence of nonlinear dielectric response, these three fields are solutions of the wave equation, and are propagating independently in the medium.

By combining the Maxwell's equations and eliminating the magnetic field, we can get wave equation in a form which includes the electric field and polarization only:

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2}{\partial^2 t}(\varepsilon_0 \mathbf{E} + \mathbf{P}) = \mu_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial^2 t} + \mu_0 \frac{\partial^2}{\partial^2 t} \mathbf{P}_{NL}, \qquad (2\text{-}2.2)$$

where $\mathbf{P}_{NL}$ stands for the nonlinear polarization and can be written as

$$\mathbf{(P_{NL})_i = 2d_{ijk} E_j E_k} \qquad (i, j, k = x', y', z'). \qquad (2\text{-}2.3)$$

It is important to note that **E** in the above equation is the sum of the three fields given by Eq. (2-2.1). As a result of the nonlinear polarization, the three fields are coupled. To obtain the coupled equations for the field amplitudes, we start with the nonlinear polarization at ($\omega_1 = \omega_3 - \omega_2$).

$$\mathbf{[P_{NL}^{\omega_3 - \omega_2}(z,t)]_i = d_{ijk} a_{3j} a_{2k} E_3 E_2^* e^{i[(\omega_3 - \omega_2)t \;\; (k_3 - k_2)z]} + c.c.} \qquad (2\text{-}2.4)$$

$$[\mathbf{P}_{NL}^{\omega_3-\omega_1}(z,t)]_i = d_{ijk}a_{3j}a_{1k}E_3E_1^{*}e^{i[(\omega_3-\omega_1)t\ \ (k_3-k_1)z]} + c.c. \qquad (2\text{-}2.5)$$

$$[\mathbf{P}_{NL}^{\omega_1+\omega_2}(z,t)]_i = d_{ijk}a_{1j}a_{2k}E_1E_2e^{i[(\omega_1+\omega_2)t\ \ (k_1+k_2)z]} + c.c., \qquad (2\text{-}2.6)$$

where we observe the convention of summation over repeated indices (i, j, k = x', y', z'). These nonlinear polarizations can be viewed as distributed dipole sources which can radiate and generate waves at the oscillating frequencies. We now substitute these nonlinear polarizations into the wave equation. By carrying out the indicated differentiation and assuming the following slowly varying amplitude approximation,

$$\frac{d^2}{dz^2}E_s << k_s\frac{d}{dz}E_s, \qquad (s = 1, 2, 3), \qquad (2\text{-}2.7)$$

we obtain, after few steps of algebra

$$\frac{d}{dz}E_1 = -i\omega_1\sqrt{\frac{\mu_0}{\varepsilon_1}}dE_3E_2^{*}e^{-i(k_3-k_2-k_1)z},$$

$$\frac{d}{dz}E_2^{*} = +i\omega_2\sqrt{\frac{\mu_0}{\varepsilon_2}}dE_1E_3^{*}e^{+i(k_3-k_2-k_1)z}, \qquad (2\text{-}2.8)$$

$$\frac{d}{dz}E_3 = -i\omega_3\sqrt{\frac{\mu_0}{\varepsilon_3}}dE_1E_2e^{+i(k_3-k_2-k_1)z},$$

where d is the effective second-order nonlinear coefficient

$$d = \sum_{ijk} d_{ijk}a_{1i}a_{2j}a_{3k}\ . \qquad (2\text{-}2.9)$$

The coupled equations (2-2.8) constitute the main result of this section. In arriving at the coupled equations, we have employed the cyclic symmetry of the nonlinear coefficients, i.e., $d_{ijk} = d_{jik} = d_{ikj}$. We will apply them in the following sections to some specific cases. We note the coupled equations are in agreement with the conservation of energy. It can be shown that

$$\frac{d}{dz}\left(\sqrt{\varepsilon_1}\left|E_1\right|^2 + \sqrt{\varepsilon_2}\left|E_2\right|^2 + \sqrt{\varepsilon_3}\left|E_3\right|^2\right) = 0, \qquad (2\text{-}2.10)$$

provided ($\omega_3 = \omega_1 + \omega_2$).

If we define the following new field variables A_1, A_2, and A_3,

$$A_m = \sqrt{\frac{n_m}{\omega_m}} E_m, \qquad m = 1,2,3, \tag{2-2.11}$$

where n_m is the index of refraction associated with wave E_m, and ω_m is the corresponding frequency. The beam intensity can be written

$$I_m = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} n_m |E_m|^2 = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \omega_m |A_m|^2 . \tag{2-2.12}$$

Since a photon's energy is $\hbar\omega$, it follows that $|A_m|^2$ is proportional to the photon flux of the beam at frequency ω_m, the proportional constant being independent of frequency. The coupled equations Eq. (2-2.8) can now be written

$$\begin{aligned} \frac{d}{dz} A_1 &= -i\kappa A_3 A_2^* e^{-i\Delta kz}, \\ \frac{d}{dz} A_2^* &= +i\kappa A_1 A_3^* e^{+i\Delta kz}, \\ \frac{d}{dz} A_3 &= -i\kappa A_3 A_2 e^{+i\Delta kz}, \end{aligned} \tag{2-2.13}$$

where the momentum mismatch Δk (or wave-number mismatch) and the coupling constant are given by

$$\Delta k = k_3 - (k_1 + k_2), \tag{2-2.14}$$

$$\kappa = d\sqrt{\frac{\mu_0\omega_1\omega_2\omega_3}{\varepsilon_0 n_1 n_2 n_3}} = \sum_{ijk} d_{ijk} a_{1i} a_{2j} a_{3k} \sqrt{\frac{\mu_0\omega_1\omega_2\omega_3}{\varepsilon_0 n_1 n_2 n_3}}, \tag{2-2.15}$$

where the summations are over all the components of the polarization unit vectors. Using Eq. (2-2.15), the conservation of energy becomes

$$\frac{d}{dz}\left(\omega_1|A_1|^2 + \omega_2|A_2|^2 + \omega_3|A_3|^2\right) = 0 \tag{2-2.16}$$

(for second harmonic generation, the above indices 1 = 2). Since the nonlinear behavior of the mirror is expected to be more pronounced for high conversion into the second harmonic, the depletion of the fundamental is considerable and thus cannot be neglected. Therefore we use the exact solution for the amplitudes of the light waves (fundamental and second harmonic), as given by Armstrong et al. The treatment here considers plane waves and the following assumptions are made:

I. perfect phase matching, i.e. $\Delta k = k_3 - 2k_1 = 0$.

II. non-critical, i.e. 90^0 phase matching for which the "walk off" effect is avoided.

III. quasi-stationary operation.

The phase modulation, which may limit the conversion efficiency, is not included in this treatment.

We denote the real amplitude of the fundamental and the second harmonic with A_1 and A_2 and introduce a new variable,

$$\delta = C(A_1^2 + A_2^2)^{1/2} z, \quad (2\text{-}2.17)$$

derived by Stankov [3]. Here z is the distance, traveled by the waves in nonlinear crystal; C is a constant, which includes the second-order nonlinear susceptibility $\chi^{(2)}$ and the quantity in the brackets is the total power flux. Then the normalized amplitudes for the fundamental F and the second harmonic SH:

$$u = A_1 / (A_1^2 + A_2^2)^{1/2}, \quad (2\text{-}2.18)$$

$$v = A_2 / (A_1^2 + A_2^2)^{1/2}. \quad (2\text{-}2.19)$$

And substitute Eqs. (2-2.17), (2-2.18) and (2-2.19) into Eq (2-2.13) with the required conditions (I), (II) and (III) mentioned above, it gets

$$\frac{du}{d\delta} = -uv \bullet \sin\Theta, \quad (2\text{-}2.20)$$

$$\frac{dv}{d\delta} = u^2 \bullet \sin\Theta, \quad (2\text{-}2.21)$$

where $\Theta = 2\Phi_1 - \Phi_3$ is the phase difference between the two light waves with initial phase Φ_1 and Φ_2 for F and SH. Power conservation requires that:

$$u^2 + v^2 = 1. \quad (2\text{-}2.22)$$

The solution of these equations under certain initial conditions is given by:

$$v = \tanh(\delta + \delta_0), \quad (2\text{-}2.23)$$

$$u = \mathrm{sech}(\delta + \delta_0); \quad (2\text{-}2.24)$$

and we shall discuss this in more details, because it is crucial for the operation of the nonlinear mirror. First, if the amplitude of the SH is initial zero, $A_2(0) = 0$, the constant $\delta_0 = 0$ and the second harmonic intensity increase with propagation in nonlinear crystal. The situation is encountered when the incident light beam travels through the nonlinear crystal from the left to the right.

If $A_1(0) \neq 0$, $A_2(0) \neq 0$ and the initial phase difference at the entrance boundary of the NLC equals $\pi/2$, the second harmonic will be amplified first. In this case $\delta_0 > 0$ and the fundamental wave may be converted completely into the SH. If $\Theta = -\pi/2$ and $\delta_0 < 0$, then the fundamental will be amplified first. This situation takes place when the fundamental F and SH are reflected back by the mirror M and the phase shift Θ is properly adjusted. After the first pass through the nonlinear crystal the phase shift between F and SH is $+\pi/2$, a phase change of $-\pi$ may be obtained by tilting the nonlinear crystal or changing the distance between the nonlinear crystal and the dichroic mirror M (using dispersion in air).

The analysis of the nonlinear mirror is simplified by the fact that one may use relative intensities to evaluate the nonlinear reflection coefficient. A useful parameter is the power conversion efficiency into SH, which we denoted by η. Thus, we start with relative intensity of the fundamental $A^2(0) = 1$, which provides the conversion efficiency η. The normalized amplitudes of the SH and F after the first pass through the nonlinear crystal are:

$$\mathbf{v} = \sqrt{\boldsymbol{\eta}}, \tag{2-2.25}$$

$$\mathbf{u} = \sqrt{(\mathbf{1} - \boldsymbol{\eta})}. \tag{2-2.26}$$

The coordinate δ is then simplified given by:

$$\boldsymbol{\delta} = \mathbf{arctanh}\sqrt{\boldsymbol{\eta}}. \tag{2-2.27}$$

If the power reflection coefficient of dichroic mirror M for F and SH are R_ω and $R_{2\omega}$ respectively, the amplitudes of the reflected light waves at ω and 2ω are:

$$A_1' = u\sqrt{R_\omega}\ ,\qquad (2\text{-}2.28)$$

$$A_2' = v\sqrt{R_{2\omega}}\ .\qquad (2\text{-}2.29)$$

The total power flux, which enters the NLC after reflection by the mirror is ($A_1'^2 + A_2'^2$) and new coordinate δ, corresponding to this power flux is

$$\delta' = C(A_1'^2 + A_2'^2)^{1/2} \bullet z = \delta(A_1'^2 + A_2'^2)^{1/2}\ .\qquad (2\text{-}2.30)$$

Because $\delta = C\ A_1(0)\ z$ and $A_1(0) = 1$. One can find the new value of the normalized SH amplitude v" at the exit plane of the nonlinear crystal in the backward direction:

$$v'' = \tanh(\delta' - \delta_0)\ .\qquad (2\text{-}2.31)$$

The constant δ_0 is determined by the initial conditions for reflected waves at the entrance of the NLC in the backward direction

$$\delta = \operatorname{arctanh}[A_2' / (A_1'^2 + A_2'^2)^{1/2}]\ .\qquad (2\text{-}2.32)$$

Then the normalized intensity of the fundamental wave, emerging from the nonlinear mirror is:

$$u''^2 = 1 - v''^2\ .\qquad (2\text{-}2.33)$$

Here one should point out that the scale for the reflected waves is described by a factor $A_1'^2 + A_2'^2$. Therefore, a correction has to make for the intensity of the emerging waves. The intensity of the amplified fundamental represents the nonlinear coefficient R_{NL}, because the intensity of the incident fundamental is chosen to be equal to 1:

$$R_{NL} = u''^2 (A_1'^2 + A_2'^2)\ .\qquad (2\text{-}2.34)$$

Substituting the Eqs. (2-2.25)-(2-2.33) into (2-2.34), one obtains:

$$R_{NL} = [\eta R_{2\omega} + (1-\eta)R_\omega][1 - \tanh^2([\eta R_{2\omega} + (1-\eta)R_\omega]^{1/2}$$

$$\bullet \operatorname{arctanh}\sqrt{\eta} - \operatorname{arctanh}\{(\eta R_{2\omega})^{1/2} / [\eta R_{2\omega} + (1-\eta)R_\omega]^{1/2}\})]\ .\qquad (2\text{-}2.35)$$

An illustration of the variation of F and SH light wave powers is presents in Fig. 2-2.

The conversion efficiency into SH is 70%, the reflection coefficient of the mirror M is 0.99 for SH and 0.1 for F. After the first pass through the nonlinear crystal the fundamental intensity decreases to 30% of its initial value of which 10% are reflected back into nonlinear crystal. The injected intensity of 3% increase again (on left hand). This is the value of the nonlinear reflection coefficient R_{NL}.

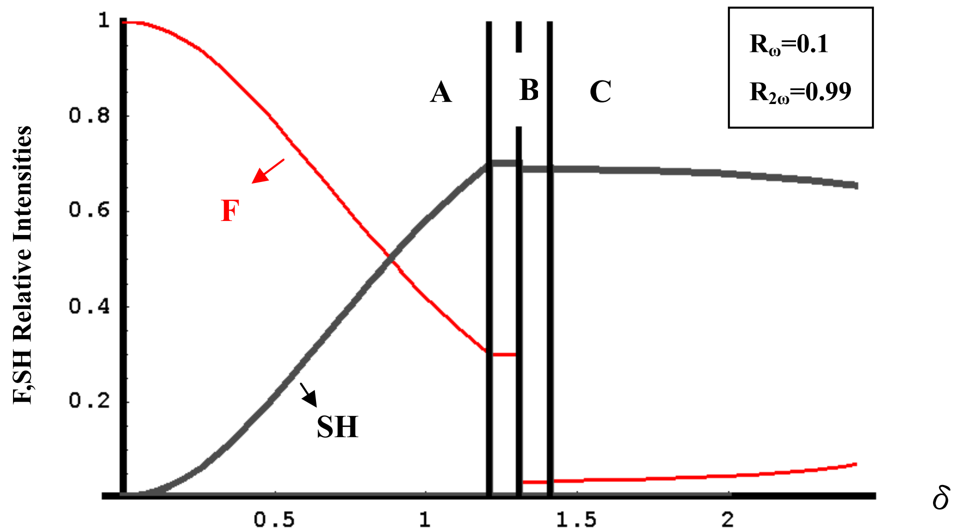

Fig. 2-2 Variation of the intensities of the fundamental (F) and the second harmonic (SH) as function of coordinate δ. The conversion efficiency into SH is 0.7, the power reflection coefficient for F and SH are 0.1 and 0.99 respectively. Note the different scales for forward (region A) and backward (region C) propagation. The region B corresponding to reflection by the mirror M.

2-3 Passive mode-locking with semiconductor saturable absorber mirror

The theory of CW mode-locking laser was first derived from the two papers of Haus's [4]. Much of passive mode locking treats in time domains, and especially, it was derived in frequency domain in [5]. Consider the laser cavity as Fig. 2.3

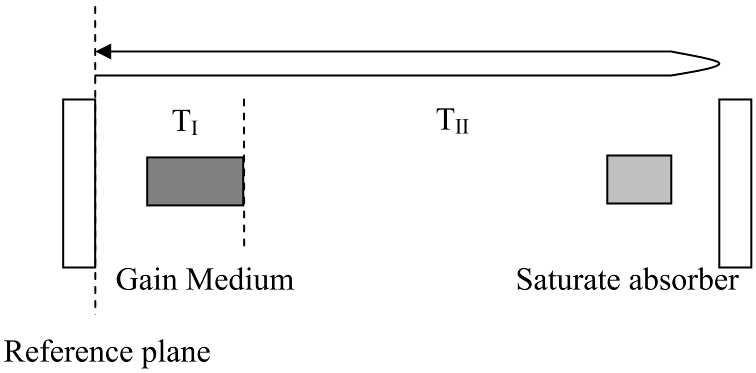

Fig. 2-3 Schematic of Laser with saturable absorber.

So we can interpret the electric field in the Fourier space of (2-3-1)

$$\mathbf{E_{nI}(\omega_k) = exp(\text{-}j\omega_k T_I)\, exp[G(\omega_k)]E_n(\omega_k)}, \tag{2-3.1}$$

where $\mathbf{G(\omega_k)}$ is frequency dependent gain of the laser medium and T_I is time delay in the gain medium. In time domain, it can transfer into

$$\mathbf{E_{nI}(t) = exp(G(\frac{d}{dt}))E_n(t - T_I)}, \tag{2-3.2}$$

where $\mathbf{G(\frac{d}{dt})}$ is interpreted in terms of the expansion of the exponential $\mathbf{exp[G(\omega_k)]}$ in $j\omega_k$, and replacement of the nth power of $j\omega_k$ by d^n/dt^n. After passage through the saturable absorber, the electric field is

$$\mathbf{E_{nI}(t) = exp[\text{-}L(t)]\, exp(G(\frac{d}{dt}))E_n(t - T_I - T_{II})}, \tag{2-3.3}$$

where L(t) is the power-dependent absorption coefficient of the saturable absorber. When the pulse returns to the reference plane, the delay time is equal to the cavity round-trip time, $T_R=2(T_I+T_{II})$, and two exponential have to reapplied. In order to take cavity loss into account, one multiples the result in addition by $\exp[-(\omega_0/2Q)T_R]$, which accounts for exponential decay of the pulse as determined by the Q of the axial modes. Here ω_0 is the center frequency of the pulse spectrum which has been assumed narrow.

The return pulse on the reference plane, now by definition the (n+1)-st pulse of the train, is given by

$$\mathbf{E_{n+1}(t) = exp(-\frac{\omega_0}{2Q}T_R)exp[G(\frac{d}{dt})]exp[-2L(t)]exp[G(\frac{d}{dt})]E_n(t - 2T_I - 2T_{II})}. \quad (2\text{-}3.4)$$

The first approximation we shall make is that the change on the pulse upon any one passage through the components of the systems is small, so that the exponential can be expanded to the first order. Furthermore, we assume that the gain has a Lorentzian line shape and can be expand to the second order in terms of the center frequency ω_0

$$\mathbf{G(\omega_k) = G(\omega_0)(1 + j\frac{\omega_k - \omega_0}{\omega_L})^{-1} \approx G(\omega_0)[1 - j\frac{\omega_k - \omega_0}{\omega_L} - (\frac{\omega_k - \omega_0}{\omega_L})^2]}. \qquad (2\text{-}3.5)$$

Because gain contains powers of ω-ω_0, it is convenient to write the electric field in terms of a slowly time-varying envelope $\upsilon(t)$ and $\exp(j\omega_0 t)$

$$\mathbf{E_n(t) = \upsilon_n(t)exp(j\omega_0 t)} \qquad (2\text{-}3.6)$$

with the Fourier transform

$$\mathbf{E_n(\omega_k) = \upsilon_n(\omega_k - \omega_0)}. \qquad (2\text{-}3.7)$$

Multiplication of $E_n(\omega_k)$ by $j(\omega_k$-$\omega_0)$ corresponding to d/dt-$j\omega_0$ in time domain which may be interpreted by a time derivative of $\upsilon_n(t)$ alone. Using Eqs. (2-3-2)-(2-3-5) and expanding the exponentials, one obtains

$$\mathbf{\upsilon_{n+1}(t) = [1 - \frac{\omega_0}{2Q}T_R - 2L(t) + 2G(\omega_0)(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt})]\upsilon_n(t - T_R)}. \qquad (2\text{-}3.8)$$

It convenient to simplify notation by introduction the following symbols:

$$\mathbf{\frac{2G(\omega_0)}{(\omega_0 / 2Q)T_R} \equiv g}, \qquad (2\text{-}3.9)$$

the gain normalized to the loss. Note that with no saturable absorber in the cavity, g = 1 corresponds to the threshold,

$$\mathbf{\frac{2L(t)}{(\omega_0 / 2Q)T_R} \equiv \frac{Q}{Q_A(t)}}, \qquad (2\text{-}3.10)$$

where Q_A may be interpreted as the amplitude- and hence time-, dependent Q as produced by the saturable absorber. We then have for Eq. (2-3-6)

$$\upsilon_{n+1}(t) = \upsilon_n(t - T_R) - \frac{\omega_0}{2Q} T_R [1 + \frac{Q}{Q_A(t)} - g(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt})]\upsilon_n(t - T_R). \quad (2\text{-}3.11)$$

The (n+1)-st pulse is a delayed version of the n-th pulse and has experienced the modifications expressed by the operator

$$-\frac{\omega_0}{2Q} T_R [1 + \frac{Q}{Q_A(t)} - g(1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt})]. \quad (2\text{-}3.12)$$

The first term is the effect of the linear cavity loss, the second term represents the modulation by the time-dependent inverse Q of the saturable absorber. The last term expresses the effect of the gain and dispersion of the laser medium.

The fast absorber analysis contains a relation between the nth and (n+1)-st pulse passing through the laser discussed above. From Eq. (2-3-9), the modulation after one round-trip can be expressed as

$$\Delta\upsilon_n |_{loss} = -\frac{\omega_0 T_R}{2}[\frac{1}{Q} + \frac{1}{Q_A(t)}]\upsilon_n(t - T_R). \quad (2\text{-}3.13)$$

The laser medium modifies υ_n by

$$\Delta\upsilon_n |_{gain} = \frac{\omega_0 T_R}{2Q} g[1 + \frac{1}{\omega_L^2}\frac{d^2}{dt^2} - \frac{1}{\omega_L}\frac{d}{dt}]\upsilon_n(t - T_R). \quad (2\text{-}3.14)$$

This modification is of operator character which entails growth (the first term), spreading via “diffusion in time” (the second term), and a delay due to the change in dielectric susceptibility as caused by the laser medium (the third term).

We note first of all that a startup of single mode-locked pulses is characterized by the period T_R; relaxation oscillations usually have a much longer period. The laser line-width affects the process only for time durations of the order of the mode-locked pulse-width, which is much shorter than T_R. If one considers the limit where the change per pass is small enough, so that

$$\frac{\upsilon_{n+1}(t) - \upsilon_{n+1}(t - T_R)}{T_R} \sim \frac{d}{dt}\upsilon, \quad (2\text{-}3.15)$$

then one has the basic equation for υ

$$\frac{d\upsilon}{dt} = -\frac{\omega_0}{2Q}[1 + q - g]\upsilon, \quad (2\text{-}3.16)$$

where we have defined the normalized inverse Q of the saturable absorber

$$\frac{Q}{Q_A(t)} \equiv q(t). \tag{2-3.17}$$

In addition to the equation for the field envelope υ(t) in the cavity, one needs equations for the time-dependent gain g(t) and the time-dependent inverse Q of the absorber, i.e., q(t). We assume that the population difference n_A of the saturable absorber obeys the rate equation

$$\frac{d}{dt} n_A = -\frac{n_A - n_A{}^0}{T_A} - n_A \frac{P}{E_A} \tag{2-3.18}$$

and a similar equation for the population difference of the laser medium. Here $n_A{}^0$ is the equilibrium difference, T_A is the relaxation time, and E_A is the saturation energy. Now

$$\frac{n_A}{n_A{}^0} = \frac{Q_A{}^0}{Q_A}, \tag{2-3.19}$$

where $Q_A{}^0$ is the small-signal value of Q_A. Using Eq. (2-3-17) in Eq. (2-3-16) and the definition Eq. (2-3-15), we obtain the equation of motion for q,

$$\frac{d}{dt} q = -\frac{q - q_0}{T_A} - q \frac{P}{E_A}. \tag{2-3.20}$$

The equation of motion for the gain is correspondingly

$$\frac{d}{dt} g = -\frac{g - g_0}{T_L} - g \frac{P}{E_L}. \tag{2-3.21}$$

These equations assume that $1/Q_A$ and g follow the time variation of P in the same way as they adjust to its time average (i.e., the same saturation energy is used for both). This is only correct if the laser and the saturable absorber media are near the end mirrors. Modifications to take into account other positions will be considered later. Eq. (2-3-14) has a different interpretation when the period of the process under study is a submultiple of the cavity round-trip time T_R (self-starting of mode locking) from the one when the period of the process is much longer than T_R (relaxation oscillations).

2-3-1 Criterion of Q-switching

And we will focus later on "relaxation oscillation", which is when the energy inside the cavity grows at a rate usually much slower than the round-trip time. The laser gain gets progressively depleted to give away to loss and the energy decays. The pumping restores the gain while the field in the cavity is low, and the process repeats itself.

If one finds such oscillations, one may expect, at the very least, that the mode-locked pulse train is modulated by the relaxation oscillation. It is more likely that the existence of relaxation oscillations suppresses mode locking if the period of the relaxation oscillation is too short to allow for the buildup of mode-locked pulses within one cycle of the oscillation. The equation for the radiation inside the cavity is now a simple rate equation. It follows from Eq. (2-3-14) by multiplication by υ^* and addition of the complex conjugate as Haus dealt in [6]. One may normalize υ so that $|\upsilon|^2$ = P, the power traveling in one direction in the cavity. The differential equation for P is

$$\frac{d}{dt}P = -\frac{1}{T_c}[1+q-g]P, \qquad (2\text{-}3\text{-}1.1)$$

where $\frac{1}{T_c} = \frac{\omega_0}{Q}$.

The perturbation of Eq. (2-3-1.1) gives

$$\frac{d}{dt}\delta P + \frac{1}{T_c}[\delta q - \delta g]P_s = 0, \qquad (2\text{-}3\text{-}1.2)$$

where we have taken into account the fact that for the CW steady state the quantity $1 + q_s - g_s = 0$. If T_A is taken as short compared to the period of interest, a condition always met when relaxation oscillations tend to occur, then δq is an instantaneous function of $\delta P/P_A$

$$\delta q = -\frac{q_0}{(1+\frac{P_s}{P_A})^2}\frac{\delta P}{P_A}. \qquad (2\text{-}3\text{-}1.3)$$

Using Eqs. (2-3-1.3) in (2-3-1.2) and the differential equation Eq. (2-3-21) for δg, one has two coupled first-order differential equations for δP and δg. Assuming the time dependence exp(st), one obtains the determinantal equation

$$s^2 - s\{\frac{q_0 \frac{P_s}{P_A}}{(1+\frac{P_s}{P_A})^2 T_c} - (1+\frac{P_s}{P_L})\frac{1}{T_L}\} - \frac{1}{T_c T_L}[\frac{(1+\frac{P_s}{P_L})}{(1+\frac{P_s}{P_A})} q_0 \frac{P_s}{P_A} - \frac{g_0}{(1+\frac{P_s}{P_L})}\frac{P_s}{P_L}] = 0. \quad (2\text{-}3\text{-}1.4)$$

Instabilities are found (Re **s** > 0) when either the s independent term is negative, i.e.,

$$\frac{q_0}{(1+\frac{P_s}{P_L})}\frac{P_s}{P_L} > \frac{g_0}{(1+\frac{P_s}{P_L})^2}\frac{1}{P_L} \quad (2\text{-}3\text{-}1.5)$$

or the coefficient of s is negative

$$\frac{q_0 \frac{P_s}{P_A}}{(1+\frac{P_s}{P_A})^2} > (1+\frac{P_s}{P_L})\frac{T_c}{T_L}. \quad (2\text{-}3\text{-}1.6)$$

Condition (2-3-1.5) corresponds to exponential growth of perturbation. It has the simple interpretation

$$|\frac{d}{dP}q| < |\frac{d}{dP}g| \text{ at } P = P_S. \quad (2\text{-}3\text{-}1.7)$$

So it can give the stability criterion against Q-switching of cw-running laser.

By simplifying Eq. (2-3-1.6), we get

$$2P\frac{dq}{dP} < \frac{r}{T_L} \quad \text{with} \quad r = 1 + \frac{P}{P_L} \quad \text{and} \quad P_L = \frac{E_L}{\tau_L}, \quad (2\text{-}3\text{-}1.8)$$

where $\tau_L = T_L T_R$, r is the pump parameter that describes at how many times the threshold the laser operates, and P_L is the saturation power of the laser gain. The inequality (2-3-1.8) has a simple physical explanation. The right side of Eq. (2-3-1.8) is the relaxation time to equilibrium for the gain at a given pump power and constant laser power. The left side is the decay time of a power fluctuation of the laser at fixed gain. If the gain cannot react fast enough to fluctuations of the laser power, relaxation oscillations grow and result in passive Q-switching of the laser.

2-3-2 Criterion of Q-switching mode-locking

Reconsider the rate equations (2-3-20), (2-3-21) and (2-3-1.1) to understand the regime of Q-switched mode locking [7]. Figure 2.4 indicates that we can approximate the laser power as

$$\mathbf{P(T,t) = E_p(T)\sum_n f(t - nT_R)} \tag{2-3-2.1}$$

with $\int_{-T_R/2}^{T_R/2} \mathbf{f(t - nT_R)dt = 1}$, (2-3-2.2)

where $E_p\,(T = nT_R)$ is the pulse energy of the nth pulse, which only changes appreciably over many cavity round trips, and f(t) is the shape of the mode-locked pulses, which is not of interest for the time being. For simplicity we assume that the mode-locked pulses are much shorter than the recovery time of the absorber. In this case, the relaxation term for the absorber

$$\mathbf{q = \frac{q_0}{1 + P/P_A}} \quad \text{with} \quad \mathbf{P_A = \frac{E_A}{\tau_A}} \quad \mathbf{(\tau_A = T_A T_R)} \tag{2-3-2.3}$$

can be neglected for the duration of the mode-locked pulses.

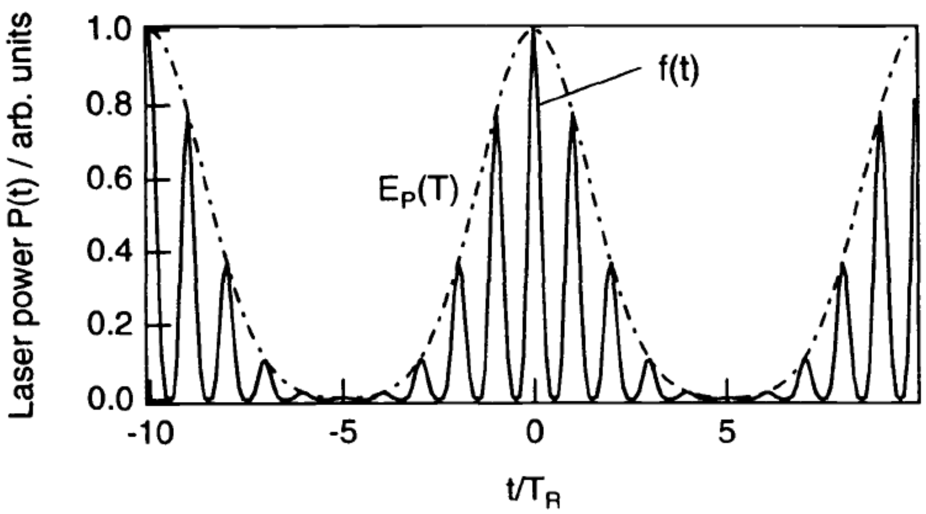

Fig. 2-4 Time dependence of the power when the laser operates in the Q-switched mode-locked regime

Since the absorber recovery time is assumed to be much shorter than the cavity round-trip time, the absorber is unsaturated before the arrival of a pulse. Thus, we obtain for the saturation of the absorber during one pulse

$$q(T,t) = q_0 \exp[-\frac{E_p(T)}{E_A}] \int_{-T_R/2}^{t} f(t')dt' . \qquad (2\text{-}3\text{-}2.4)$$

Therefore, the loss in pulse energy per round trip can be written as

$$q_p(T) = \int_{-T_R/2}^{T_R/2} f(t)q(T,t)dt = q_0 \frac{1 - \exp[\ E_p(T)/E_A]}{E_p(T)/E_A} . \qquad (2\text{-}3\text{-}2.5)$$

This equation shows that the saturable absorber essentially saturates with the pulse energy, and not with the average intensity of the laser as before in the case (2-3-2.3) of cw Q switching. Therefore, the absorber is much more strongly bleached at the same average power. After averaging Eqs. (2-3-21) and (2-3-1.1) over one round trip, we obtain the following two equations for the dynamics of the pulse energy and the gain on a coarse-grained time scale T:

$$T_R \frac{dE_p}{dT} = 2[g - l - q_p(E_p)]E_p , \qquad (2\text{-}3\text{-}2.6)$$

$$T_R \frac{dg}{dT} = -\frac{g - g_0}{T_L} - g\frac{E_p}{E_L} . \qquad (2\text{-}3\text{-}2.7)$$

The averaging is allowed because the saturation of the gain medium within one pulse is negligible due to the small interaction cross section of the solid state laser material. Comparing Eqs. (2-3-21), (2-3-1-1) and (2-3-2.3) with (2-3-2.5), (2-3-2.6) and (2-3-2.7), it becomes obvious that the stability criterion (2-3-1.8) applies also to Q-switched mode locking if we replace the formula (2-3-2.3) for cw saturation of the absorber by the formula (2-3-2.5) for pulsed saturation. Then we have

$$2E_p \left.\frac{dq_p}{dE_p}\right|_{cw\,mod} < \left.\frac{r}{T_L}\right|_{cw\,mod}$$

(2-3-2.8)

with $$2E_p \left.\frac{dq_p}{dE_p}\right|_{cw\,mod} = 2q_0 \frac{1 - \exp(-E_p/E_A)(E_p/E_A + 1)}{E_p/E_A} \qquad (2\text{-}3\text{-}2.9)$$

or again expressed in terms of the average power $P = E_p/T_R$, and $\chi_p = \frac{P_A}{P_L} T_A$ describes an effective stiffness of the absorber compared with the gain when the laser is cw mode locked at the same average power as the cw laser.

The theory presented so far yields guidelines for the design of an absorber that prevents Q-switching instabilities but still self-starts mode locking. A saturable absorber alone can shorten a pulse until it is roughly on the order of the absorber relaxation time if, of course, the gain band-width is sufficiently large. The pulse formation process is essentially given by the fast-saturable-absorber mode-locking model analyzed by Haus et al. This is the case for pico-second lasers where dispersion and self-phase-modulation are not the dominant pulse-shaping mechanisms.

2-3-3 Criterion of contineous wave mode-locking

The nonlinear reflectivity $R(E_p)$ can be measured with the output of another cw mode-locked laser, which provides enough pulse energy to bleach the absorber. The pulse fluence in the absorber can be varied with an adjustable attenuator. A typical nonlinear reflectivity of a SESAM is shown in Fig. 2.5. The measured data are fitted with the function [8][9]

$$\mathbf{R}(\mathbf{E_P}) = \mathbf{R_{ns}} \frac{\ln\{1+\exp(-\Delta\mathbf{R})[\exp(\frac{\mathbf{E_P}}{\mathbf{E_{sat,A}}})-1]\}}{\frac{\mathbf{E_P}}{\mathbf{E_{sat,A}}}}, \qquad (2\text{-}3\text{-}3.1)$$

which was derived from a simple model for nonlinear pulse propagation in the absorber. This is equivalent to the equations used previously. The fit parameters are ΔR, $F_{sat,A}$, and R_{ns}, which is the reflectivity for high pulse energies and determines the nonsaturable loss $\Delta R_{ns}=1-R_{ns}$. ΔR is the maximum change in nonlinear reflectivity, which is also referred to as the maximum modulation depth of the SESAM device. For absorbers with ΔR smaller than approximately 10% (as used in all the experiments), one can simplify Eq. (2-3-3.1) to

$$R(E_P) = R_{ns}\{1 - \Delta R \frac{F_{sat,A} A_{eff,A}}{E_P}[1 - \exp(-\frac{E_P}{F_{sat,A} A_{eff,A}})]\}. \quad (2\text{-}3\text{-}3.2)$$

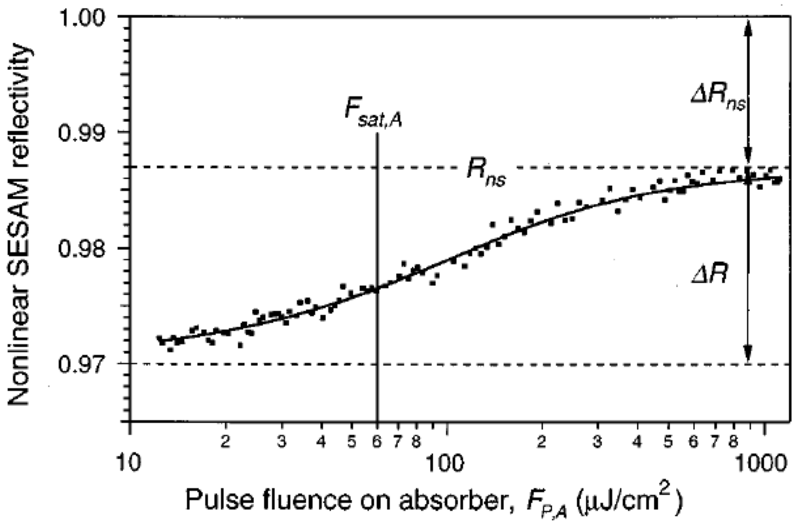

Fig. 2-5 The measured data are fitted with the function of Eq (2-3-3.1)

The nonlinear reflectivity $R(E_p)$ of the SESAM is related to the pulse energy loss per round trip $q_p(E_p)$. Note that it's not include any nonsaturable losses in q(t) Eq. (2-3.20) or

$$q_p(E_P) = q_0 \frac{F_{sat,A} A_{eff,A}}{E_P}[1 - \exp(-\frac{E_P}{F_{sat,A} A_{eff,A}})]. \quad (2\text{-}3\text{-}3.3)$$

Therefore the maximum modulation depth is given by $\Delta R = 1- \exp(-q_0) \sim q_0$ for $\Delta R << 1$. For passively mode-locked solid-state lasers we can assume that the modulation depth is small, i.e., $\Delta R << 1$. In addition, the nonsaturable losses should be as low as possible because they only degrade the laser performance. For stable mode locking we typically have to use a small output coupler transmission T_{out} of the order of a few percent, which results in the additional condition that $\Delta R_{ns} << T_{out} << 1$. Therefore we can make the approximation that $R_{ns} \sim 1$ and that

$$R(E_p) \sim \exp[-q_p(E_p)] \sim 1 - q_p(E_p). \quad (2\text{-}3\text{-}3.4)$$

The approximations made for Eq. (2-3-3.2) are then consistent with Eqs. (2-3-3.4) and (2-3-3.3) by means of $R_{ns} \sim 1$. The stability criterion against QML from relation (2-3-2.8) can be rewritten with the nonlinear reflectivity as [7]

$$\mathbf{E_p}\left|\frac{\mathbf{dR(E_p)}}{\mathbf{dE_p}}\right| < \frac{\mathbf{T_R}}{\boldsymbol{\tau}_\mathbf{L}}\mathbf{r} = \frac{\mathbf{T_R}}{\boldsymbol{\tau}_\mathbf{L}} + \frac{\mathbf{E_p}}{\mathbf{E_{sat,L}}}. \qquad (2\text{-}3\text{-}3.5)$$

The absorber parameters determine the left-hand side of relation Eq. (2-3-3.5), whereas, the right-hand side contains the laser material and laser cavity parameters.

To benefit from the full modulation depth of the saturable absorber in cw mode-locked lasers, the pulse energy must be high enough to bleach the absorber. To meet that condition, the pulse fluence in the SESAM should be approximately five times the absorber saturation fluence. With this approximation and the assumption that $R_{ns} = 1$, as well as with relations (2-3-3.4) and (2-3-3.3), it's obtained for the nonlinear reflectivity of the SESAM [10]

$$\mathbf{R(E_P) = 1 - \Delta R \frac{F_{sat,A} A_{eff,A}}{E_P}} \qquad (2\text{-}3\text{-}3.6)$$

or $$\mathbf{R(F_{P,A}) \sim 1 - \Delta R \frac{F_{sat,A}}{F_{P,A}}}, \qquad (2\text{-}3\text{-}3.7)$$

where $F_{P,A}=E_P/A_{eff,A}$ is the pulse fluence (i.e., pulse energy per unit area) incident upon the SESAM. At lower fluence the residual saturable absorption would contribute to the cavity loss and act against self-starting and efficient mode-locked operation. If the laser operates far above threshold (r >> 1), which is the case in most mode-locked lasers, we can neglect the first term on the right-hand side of relation (2-3-3.6), and the stability criterion against QML becomes independent of the upper-state lifetime of the considered laser material. The saturation energy is then the only relevant parameter of the gain medium. A laser material with a large stimulated-emission cross section σ_L is therefore desirable for stable cw mode locking. It also helps to choose a geometry of resonator with multiple passes through the gain medium to decrease the gain saturation fluence. Selecting inhomogeneously broadened gain materials with the same averaged σ_L would also reduce the gain saturation fluence because the class of laser ions with the highest cross sections dominates the gain saturation. Reducing the spontaneous lifetime, e.g., by lifetime quenching effects, does not affect the stability condition against QML.

With the approximations listed above, the stability condition (2-3-3.5) can be written in the following equivalent forms:

$$\mathbf{E_p^{\,2} > E_{sat,L}E_{sat,A}\Delta R}\,, \tag{2-3-3.8}$$

$$\mathbf{F_{p,A}^{\,2} > F_{sat,L}F_{sat,A}\Delta R\frac{A_{eff,L}}{A_{eff,A}}}\,, \tag{2-3-3.9}$$

$$\mathbf{P^2 > F_{sat,L}F_{sat,A}\Delta R A_{eff,L}A_{eff,A}\frac{1}{T_R^{\,2}}}\,. \tag{2-3-3.10}$$

With respect to the experimental verification of the theory, it is helpful to introduce the QML parameter $E_{sat,L}E_{sat,\,A}\Delta R$, because it contains all the parameters that determine the laser dynamics. We then define the critical intra-cavity pulse energy $E_{P,c}$ as the square root of the QML parameter:

$$\mathbf{E_{P,c} \equiv (E_{sat,L}E_{sat,A}\Delta R)^{1/2}}$$

$$\mathbf{= (F_{sat,L}A_{eff,L}F_{sat,A}A_{eff,A}\Delta R)^{1/2}}.$$

This is the minimum intra-cavity pulse energy, which is required for obtaining stable cw mode locking; i.e., for E_P. $E_{P,c}$ we obtain stable cw mode locking, and for E_P , $E_{P,c}$ we obtain QML. Note that, if we neglect the lifetime-dependent term in relation (2-3-3.6) and set the bracketed term in Eq. (2-3-3.2) as 1, both approximations lead to a slightly stricter stability criterion: A laser fulfilling the stability condition with these approximations will always fulfill the exact condition. For good stability of a mode locked laser against unwanted fluctuations of pulse energy, operation close to the stability limit [relations (2-3-3.8)–(2-3-3.10)] is not recommended.

2-4 Generalized model for passively Q-switched lasers with simultaneous mode-locking

In this Section, we will show the temporal change of photon density by the method derived by many pioneers such as John J. Degnan [11] [12], Michael Bass [13], YF Chen [14][15]. The case we discussed is the laser with an intra-cavity saturable absorber. The laser rod, saturable absorber and resonator mirrors are indicated. The following coupled rate equations describe the laser's operation:

$$\frac{d\Phi}{dt}=\frac{\Phi}{t_r}\{2\sigma nl-2\sigma_{gs}n_{gs}l_s-2\sigma_{es}n_{es}l_s-[\ln(\frac{1}{R})+L]\}, \qquad (2\text{-}4.1)$$

$$\frac{dn}{dt}=-\gamma c\sigma\Phi n, \qquad (2\text{-}4.2)$$

$$\frac{dn_{gs}}{dt}=-\frac{A}{A_s}c\sigma_{gs}\Phi n_{gs}. \qquad (2\text{-}4.3)$$

And in Q-switching mode-locking state, photon density shape can be expressed as:

$$\Phi(t)=\sum_{m=0}\Phi_m f(t-t_m), \qquad (2\text{-}4.4)$$

where $t_m = m\ t_r$, with t_r being the round trip time, m the number of round trip, Φ_m the relative amplitude of the mode locked pulses at m-th round trip, and f(t) assumed to be a sharp pulse centered at t = 0 which falls off rapidly in a time short compared to the resonator roundtrip transit time.

Consider the excited state absorption (ESA) effect in a four-level saturable absorber, the relative amplitude of the mode-locked pulses at time $t_m = m\ t_r$ after an additional roundtrip through the cavity is given by

$$\Phi_m=\Phi_{m-1}\exp\{2\sigma n(t_m)l-2\sigma_{gs}n_{gs}(t_m)l_s-2\sigma_{es}n_{es}(t_m)l_s-[\ln(\frac{1}{R})+L]\}, \qquad (2\text{-}4.5)$$

where σ is the stimulated emission cross section of the gain medium, $n(t_m)$ is population density of the gain medium at the m-th roundtrip, *l* is length of the gain medium, σ_{gs} is ground-state absorption (GSA) in the saturable absorber, σ_{es} is ESA cross section in the saturable absorber, $n_{gs}(t_m)$ is the absorber ground-state population density at the m-th roundtrip, $n_{es}(t_m)$ is absorber excited-state population density at the m-th roundtrip, R is reflectivity of the output mirror, and L is unsaturable intracavity roundtrip dissipative optical loss, respectively.

Introducing the variable $\boldsymbol{\beta = \sigma_{es} / \sigma_{gs}}$ and using the condition $n_{gs}(t_m) + n_{es}(t_m) = n_{so}$, Eq. (2-4.5) can be rewritten as

$$\boldsymbol{\Phi_m = \Phi_{m-1} \exp\{2\sigma n(t_m)l - [2(1-\beta)\sigma_{gs} n_{gs}(t_m) l_s} + \beta \ln(\frac{1}{T_0^{\,2}})] - \boldsymbol{[\ln(\frac{1}{R}) + L]\}}, \qquad (2\text{-}4.6)$$

where n_{so} is the total density of the absorber and $T_0 = \exp(-\sigma_{gs} n_{so} l_s)$ is the initial transmission of the saturable absorber. Note that the condition is an assumption introduced by Hercher [16] and adopted by Xiao and Bass [13] to simplify the analysis of passive saturable absorbers. This condition assumes:

1. the upper terminal level of the GSA relaxes infinitely fast (relative to the temporal duration of the optical pulse) to the lower level of the ESA.
2. the upper terminal level of the ESA behaves similarly. Namely, it is assumed that the saturable absorber atomic populations are totally contained in either the ground or excited states during the interaction with the optical pulse.

These approximations may not be valid for very short mode-locked pulse-widths. Since the Q-switched laser output pulses are much shorter than both the spontaneous lifetime and the pump period (time between output pulses), spontaneous relaxation and pumping can be safely neglected during the development of the output pulse. Therefore, the equation for the time rate of change of the population inversion density can be expressed as Eq. (2-4.2) where c is the speed of light and is the inversion reduction factor.

Dividing Eq. (2-4.2) by n, using (2-4.4) and normalized function

$$\boldsymbol{\int_{\infty}^{\infty} c\sigma f(t) dt = 1}, \qquad (2\text{-}4.7)$$

and integration over time from zero to t_m, $n(t_m)$ is given by

$$n(t_m) = n(0)\prod_{k=0}^{m-1} \exp(-\gamma\Phi_k), \tag{2-4.8}$$

where n(0) is the initial population inversion density in the gain medium. It can be determined from the condition that the round–trip gain is exactly equal to the round-trip losses just before the Q-switch opens, thus

$$n(0) = \frac{\ln(\frac{1}{T_0^{\ 2}}) + \ln(\frac{1}{R}) + L}{2\sigma l}. \tag{2-4.9}$$

The equation for the time rate of change of the absorber ground state population density is given by Eq. (2-4.3), where A/A_s is the ratio of the effective area in the gain medium and in the saturable absorber. Dividing Eq. (2-4.2) by (2-4.3) and integrated gives

$$n_{gs} = n_{so}[\frac{n(t_m)}{n(0)}]^{\alpha}, \tag{2-4.10}$$

where $\alpha = \frac{1}{\gamma}\frac{\sigma_{gs}}{\sigma}\frac{A}{A_s}$. (2-4.11)

The parameter α indicates how fast the saturable absorber is bleached. The larger the parameter, the faster the saturable absorber is bleached. Substituting Eqs. (2-4.8)-(2-4.10) into Eq. (2-4.6), the recurrence relation for Φ_m is given by:

$$\Phi_m = \Phi_{m-1} \exp\{[\prod_{k=0}^{m\ 1} \exp(-\gamma\Phi_k) - 1][\ln(\frac{1}{R}) + L]$$

$$+ [\prod_{k=0}^{m\ 1} \exp(-\gamma\Phi_k) - [\beta - (1-\beta)\prod_{k=0}^{m\ 1} \exp(-\gamma\Phi_k)^{\alpha}]]\ln(\frac{1}{T_0^{\ 2}})\}. \tag{2-4.12}$$

In terms of $\Phi(t)$, the instantaneous power coupled from the output mirror is given by Degnan:

$$P(t) = \frac{h\upsilon Al'}{t_r}\ln(\frac{1}{R})\Phi(t), \tag{2-4.13}$$

where hυ is the laser photon energy and Al’ is the cavity volume occupied by photons. Substituting Eq. (2-4.12) into Eq. (2-4.13), the output power can be expressed as

$$\mathbf{P(t)=\frac{h\upsilon Ac}{2}\ln(\frac{1}{R})\sum_{m=0}^{\infty}\varphi_m f(t-t_m)}. \tag{2-4.14}$$

In this simulation, we use hyperbolic secant as our pulse shape f(t) ~ $\text{sech}^2(t/\tau_p)$, where the parameter τ_p is related to the FWHM mode-locked pulse-width by τ (FWHM) = $1.76\tau_p$ [17]. Thus, we can simulate out the Q-switching mode-locking temporal results by (2-4.14) and we will present the results later in Chapter 6.

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