A Study for Application of Regression Tolerance Interval for Lot Productions Conforming To Specifications and A New Approach

Abstract

The tolerance interval has long been a technique for manufacturer to verify if there is a reasonably large value confidence that ensures a proportion of production lot conforming to specification limits. This paper formally formulate this interest of "confidence" in terms of lot size and parameters involved in the underlying distribution of a product's characteristic. With this formulation, any technique for prediction of manufacturer's confidence may be evaluated for its effieciency and it also provides a wide room for this prediction through statistical inferences for the unknown confidence. We then study the power of the tolerance interval in detecting if there is a reasonably large manufacturer's confidence for the production. We found that when the parameters involved in the distribution are known, the predicted manufacturer's confidence is too optimistic in a value much more higher than the true confidence and when the parameters involved in the distribution are unknown, the predicted manufacturer's confidence is too conservative in a value much lower than the true one. The inefficiency partly comes from the fact that tolerance interval does not use the information of lot size which is an ancillary statistic in the considered statistical model. For statistical inference of this unknown confidence, we introduce a point estimation technique that its results including a power comparison seems to be very promising for the manufacturer.

Key words: Confidence interval; hypothesis testing; linear regression; power; quality control; tolerance interval.

1. Introduction

Statistical theory of interval estimation mostly deals with the confidence interval to contain a parameter θ. In many applications, we require an interval to contain the future random variable which is a prediction problem. Among the alternatives, intervals in the form of tolerance intervals is widely used in quality control and related prediction problems to monitor manufacturing processes, detect changes in such processes, ensure product compliance with specifications, etc.

In manufacturing industry, specification limits for one charateristic of an item, saying LSL and USL, define the boundaries of acceptable quality for an manufacturing item (component). An item is said to be non-defective if the measured characteristic is between the limits, otherwise it is defective. For a manufacturer of a mass-production item, the tolerance interval is designed for a quality assurance problem. For a process of production, the manufacturer knows that unless a proportion of a lot is acceptable in the sense that the corresponding items are non-defective, he will lose money in this production lot. In practice, the manufacturer further expects to be guarateed that, in the long run, he or she will have lots of production not lossing money at least a percentage of the time. The approach of tolerance interval is a major technique to solve this problem and this technique has been applied to the linear regression problem.

Consider a linear regression model

$$y = x'\beta + \epsilon$$

where y and ϵ are, respectively, the response and error variables and x represents the explanatory vector. For a given vector x_0, it is assumed that there are specification limits $\{LSL_{x_0}, USL_{x_0}\}$ indicating that when the observation of the corresponding response variable y_0 falls in the limits then this observation is acceptable, otherwise, it is not acceptable. Henceful, the manufacturer is interesting to get information about the following probability

$$P_{\beta,\sigma}(y_0 \in (LSL_{x_0}, USL_{x_0})) = \int_{LSL_{x_0}}^{USL_{x_0}} f_{y|x_0}(y)dy \tag{1.1}$$

(see this point in Goodman and Madansky (1962)). For manufacturer's question, the technique of tolerance interval consider three steps in the followings to answer it:

(a) Consider the probability in a hypothesis testing problem with hypotheses:

$$H_0 : P_{\beta,\sigma}(y_0 \in (LSL_{x_0}, USL_{x_0})) < \gamma \text{ vs } H_1 : P_{\beta,\sigma}(y_0 \in (LSL_{x_0}, USL_{x_0})) \geq \gamma \tag{1.2}$$

(see Goodman and Madansky (1962) for reference). A level α test for this hypothesis tries to answer if there is at least a proportion γ of the population conforming to specification limits with a confidence level $1-\alpha$.

(b) Construct a γ-content tolerance interval at confidence $1-\alpha$, $(T_1, T_2) = (t_1(y, X), t_2(y, X))$, that satisfies

$$P[P(y_0 \in (T_1, T_2)|y, X) \geq \gamma] \geq 1 - \alpha. \tag{1.3}$$

The construction of tolerance interval of (1.3) has been received intensive attention in literature (see for examples, Lieberman and Miller (1963), Limam, Thomas (1988)), Carroll and Ruppert (1991) and Jordan (1995)).

(c). Let (t_1, t_2) be the observation of the tolerance interval. The test for hypothes H_0 vs H_1 is:

$$\textbf{We reject } H_0 \textbf{ if } (t_1, t_2) \subset (LSL, USL). \tag{1.4}$$

For discussing testing hypotheses of (1.2) by rule of (1.4), see Goodman and Madansky (1962) and Owen (1964). When H_0 is rejected, statistically we are $100(1-\alpha)\%$ sure that at least $100\gamma\%$ of the population is conforming to the specification limits. So, in a long run, we will have lots having at least proportion γ of the distribution conforming to the specification limits at least a proportion $1-\alpha$ of the time. With the interest of resolving the manufacturer's qusetion, it is generally to make an extending conclusion in the following:

When H_0 is rejected, the lot of product is acceptable because we have confidence of a reasonably large value that at least $100\gamma\%$ of the population is conforming to specification limits (1.5)

where Schilling (1982) further argued that the reasonably large value of the confidence is exactly $1-\alpha$. For other references of this hypothesis testing technique, see Bowker and Goode (1952), Owen (1964) and Papp (1992). Our interest is the question: When a sample data support to reject H_0, is the extending inference in (1.5) appropriate?

The manufacturer wants to see if there is proportion γ of acceptable products with a reasonable large confidence, saying q_0. Probabilitically it is known that guaranteering proportion γ of the population conforming to the specification limits is not guaranteering proportion γ of the products in the lot conforming to the specification limits. How can we believe that the prediction of reasonably large confidence $1-\alpha$ for tolerance interval implying a reasonably large confidence q_0 for the manufacturer is an appropriate technique? We concern this qusetion since the manufacturer wants to assure a good chance to have lots accepted when lots are produced at permissible levels of quality. On the other hand, it is also important for the consumer to see if there is irregular degradation of levels of process quality in submitted lots. Henceful, any technique used for this prediction should be able to protect the benifits of both the consumer and manufacturer.

We may consider that the characteristic variable of a product obeys some fixed probability distribution. With this probabilistic assumption, the confidence that there is proportion γ or more non-defective items, i.e., conforming to specification limits, is completely determined by the underlying distribution, which is an unknown constant when the distribution involves unknown parameters. The construction of tolerance interval for solving the manufacturer's problem aims to predict this unknown confidence. However, the tolerance interval in (1.3) is constructed primarily for the confidence that the interval contains a proportion γ of the distribution. There may have a big discrepancy between the proportion γ of the sample space conforming to the specification limits and the proportion γ of product lot conforming to the specification limits. The size of discrepancy may be determined from the underlying distribution, however, it is desired to discover.

Our aim in this paper is to achieve three purposes: First, we extend the idea of Chen, etc. (2006) developing regression type tolerance interval (in Sections 2 and 3). Second, we explicitly formulate the manufacturer's unknown confidence in terms of distribution parameters. With this formulation, it provides a room for statistical inferences for this true confidence. Third, we study the power of the ability that the test based on tolerance interval may conclude a reasonably large confidence q_0 when it does exist. We will see that the tolerance interval leads to predicted confidence too optimistic in way that it is higher much more than the true one. On the other hand, when the unknown parameters are unknown the version of tolerance interval is too conservative in way that its predicted confidence much lower than the true one. This verifies that the use of tolerance interval to predict the manufacturer's confidence is not appropriate. Fourth, we will introduce a point estimation technique for estimation of the true confidence which provides a new approach for answering the manufacturer's question. We then further investigate the power of detecting the manufacturer's confidence through this new technique which leads to very promising results.

2. Regression Tolerance Interval

Consider the linear regression model

$$y = X\beta + \epsilon \tag{2.1}$$

where y is a n-vector of observations for the response variable, X is a known $n \times p$ design matrix with $1'$s in the first column, β is unknown parameter n-vector and ϵ is a n-vector of independent and identically distributed disturbance variables.

Suppose that x_0 is a new design vector and y_0 is its corresponding response variable. The interest for developing the tolerance interval is stated in the following definition.

Definition 2.1. A γ-content regression tolerance interval at confidence $1-\alpha$ is a random interval $(T_1, T_2) = (t_1(y, X), t_2(y, X))$ for which

$$P[P(y_0 \in (T_1, T_2)|y, X) \geq \gamma] \geq 1 - \alpha. \tag{2.2}$$

In other words, a γ-content regression tolerance interval (T_1, T_2) contains at least $100\gamma\%$ of the population of the distribution of response variable

variable given x_0 with $(1-\alpha)$-confidence level. From (2.1), we wish to search T_1 and T_2 such that

$$P(C(y,X) \geq \gamma) \geq 1-\alpha. \tag{2.3}$$

with $C(y,X) = \int_{t_1(y,X)}^{t_2(y,X)} f_{y_0}(z)dz$ as the coverage of the interval $(t_1(y,X), t_2(y,X))$. The coverage $C(y,X)$ is, of course, a random variable. The distribution of the covearge $C(y,X)$ of this integral is, ingeneral, exceedingly complicated, and the question of how to chooses k exactly so as to meet the requirement (2.2) for preassigned γ and p is quite hard to answer (see this point in Guttman (1970)). The popular technique in obtaining a tolerance interval is through some approximation method (see Wald and Wolfowitz (1946)) and Ellison (1964)).

Let's define a regression type confidence interval of a population coverage interval where we will extend the result of Huang, etc. (2006) by showing that this confidence interval is a regression tolerance interval.

Definition 2.2. (a) An interval $(a(x_0,\beta,\sigma), b(x_0,\beta,\sigma))$, depending on parameters in general, is called a regression γ-content interval if it satisfies

$$P_{\beta,\sigma}(a(x_0,\beta,\sigma) \leq y_0 \leq b(x_0,\beta,\theta)) = \gamma.$$

(b) A random interval $(T_1,T_2) = (t_1(y,X), t_2(y,X))$ is called a $100(1-\alpha)\%$ C.I. of $(a(x_0,\beta,\sigma), b(x_0,\beta,\sigma))$ if it satisfies

$$1-\alpha = P(t_1(y,X) \leq a(x_0,\beta,\sigma) < b(x_0,\beta,\sigma) \leq t_2(y,X)).$$

Without using the classical approximation techniques, we prove that the confidence interval above is a tolerance interval for regression model.

Theorem 2.3. Suppose that $(a(x_0,\beta,\sigma), b(x_0,\beta,\sigma))$ is a $100\gamma\%$ coverage interval. If $(T_1,T_2) = (t_1(y,X), t_2(y,X))$ is a $100(1-\alpha)\%$ C.I. of $(a(x_0,\beta,\sigma), b(x_0,\beta,\sigma))$, then (T_1,T_2) is a γ-content tolerance interval at confidence coefficient $1-\alpha$ for future r.v. y_0.

Proof.

$$P[P\{t_1(y,X) \le y_0 \le t_2(y,X)|y,X\} \ge \gamma] = P\{F_{y_0}(t_2(y,X)) - F_{y_0}(t_1(y,X)) \ge \gamma\}$$
$$= P\{F_{y_0}(t_2(y,X)) - F_{y_0}(t_1(y,X)) \ge F_{y_0}(b(x_0,\beta,F)) - F_{y_0}(a(x_0,\beta,F))\}$$
$$\ge P\{F_{y_0}(t_1(y,X)) \le F_{y_0}(a(x_0,\beta,F)) < F_{y_0}(b(x_0,\beta,F)) \le F_{y_0}(t_2(y,X))\}]$$
$$\ge P\{t_1(y,X) \le a(x_0,\beta,F) < b(x_0,\beta,F) \le t_2(y,X)\}$$
$$= 1-\alpha,$$

as F_{y_0} is nondecreasing. □

3. Regression Tolerance Interval for Normal Distribution

In his original article, Wilks (1941) first pointed out that if the distribution of the variables of interest can be assumed to have a given functional form involving unknown parameters, methods having a greater efficiency than the nonparamter methods could be used for setting tolerance limits. In this section, we assume that the error variables are independent and identically distributed with normal distribution $N(0,\sigma^2)$.

In the simpler case that y is a normal random sample, Jilek and Likar consider two cases where one is known mean and unknown variance and the other one is known variance and unknown mean. We first extend their result to the regression case.

Theorem 3.1. Suppose that β is known. We let $S_1^2 = \frac{1}{n}\sum_{i=1}^n (y_i - x_i'\beta)^2$. Then the interval

$$(x_0'\beta - z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1, x_0'\beta + z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1)$$

is a γ-content tolerance interval at confidence coefficient $1-\alpha$ for future r.v. y_0.

Proof. A two sided regression γ-content interval for future r.v. X_0 is

$$(x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma, x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma).$$

Since $\frac{nS_1^2}{\sigma^2} \sim \chi^2(n)$, it is seen from the following derivation,

$$
\begin{aligned}
&1-\alpha = P(\frac{nS_1^2}{\sigma^2} \geq \chi^2_\alpha(n)) \\
&(\text{by the fact that } \frac{\sqrt{n}}{\sigma}S_1 \text{ and } \sqrt{\chi^2_\alpha(n))} > 0) \\
&= P(-\frac{\sqrt{n}}{\sigma}S_1 \leq -\sqrt{\chi^2_\alpha(n)} < \sqrt{\chi^2_\alpha(n)} \leq \frac{\sqrt{n}}{\sigma}S_1) \\
&= P(-z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1 \leq -z_{\frac{1+\gamma}{2}}\sigma < z_{\frac{1+\gamma}{2}}\sigma \leq z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1) \\
&= P(x_0'\beta - z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1 \leq x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma < x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma \leq x_0'\beta + z_{\frac{1+\gamma}{2}} \\
&\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1) \quad \square
\end{aligned}
$$

Table 1. γ-content tolerance interval at confidence coefficient $1-\alpha$ when β is known

γ-content interval	Tolerance interval
$(x_0'\beta - z_\gamma\sigma, \infty)$	$(x_0'\beta - z_\gamma\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1, \infty)$
$(-\infty, x_0'\beta + z_\gamma\sigma)$	$(-\infty, x_0'\beta + z_\gamma\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1)$
$(x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma, x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma)$	$(x_0'\beta - z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1, x_0'\beta + z_{\frac{1+\gamma}{2}}\sqrt{\frac{n}{\chi^2_\alpha(n)}}S_1)$

Normal distribution with known σ

Theorem 3.2. Consider the linear regression model where σ is known. Let $\hat{\beta} = (X'X)^{-1}X'y$, the least squares estimator of β. Then the interval

$$(x_0'\hat{\beta} - z_{\frac{1+\gamma}{2}}\sigma - z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0}, x_0'\hat{\beta} + z_{\frac{1+\gamma}{2}}\sigma + z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0})$$

is a γ-content tolerance interval at confidence coefficient $1-\alpha$.

Proof. Since $\hat{\beta}$ is the least squares estimator, we have $x_0'\hat{\beta} \sim N(x_0'\beta, \sigma^2 x_0'(X'X)^{-1}x_0)$.

It is seen from the following,

$$1-\alpha = P(-z_{1-\frac{\alpha}{2}} \leq \frac{x_0'\hat{\beta}-x_0'\beta}{\sigma\sqrt{x_0'(X'X)^{-1}x_0}} \leq z_{1-\frac{\alpha}{2}})$$

$$= P(-z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0} \leq x_0'\hat{\beta}-x_0'\beta \leq z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0})$$

$$= P(-z_{\frac{1+\gamma}{2}}\sigma - z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0} \leq x_0'\hat{\beta}-x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma < x_0'\hat{\beta}-x_0'\beta$$

$$+ z_{\frac{1+\gamma}{2}}\sigma \leq z_{\frac{1+\gamma}{2}}\sigma + z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0})$$

$$= P(x_0'\hat{\beta} - z_{\frac{1+\gamma}{2}}\sigma - z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0} \leq x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma < x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma \leq x_0'\hat{\beta}$$

$$+ z_{\frac{1+\gamma}{2}}\sigma + z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0}) \quad \square$$

Table 2. γ-content tolerance interval at confidence coefficient $1-\alpha$ when σ is known

γ-content interval	Tolerance interval
$(x_0'\beta - z_\gamma\sigma, \infty)$	$(x_0'\hat{\beta} - z_\gamma\sigma - z_{1-\alpha}\sigma\sqrt{x_0'(X'X)^{-1}x_0}, \infty)$
$(-\infty, x_0'\beta + z_\gamma\sigma)$	$(-\infty, x_0'\hat{\beta} + z_\gamma\sigma + z_{1-\alpha}\sigma\sqrt{x_0'(X'X)^{-1}x_0})$
$(x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma, x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma)$	$(x_0'\hat{\beta} - z_{\frac{1+\gamma}{2}}\sigma - z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0},$ $x_0'\hat{\beta} + z_{\frac{1+\gamma}{2}}\sigma + z_{1-\frac{\alpha}{2}}\sigma\sqrt{x_0'(X'X)^{-1}x_0})$

The tolerance interval for a normal distribution with mean and variance both unknown is first studied by Wilks (1941) and it has been re-treated by Wald and Wolfowitz (1946) and Odeh and Owen (1980). However, it has been all solved by approximation. An exact tolerance interval for this problem is solved by Huang, etc. (2006). We extend this result to the regression problem.

Theorem 3.3. Suppose that β and σ are all unknown. Then the interval

$$(x_0'\hat{\beta} - t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0},$$

$$x_0'\hat{\beta} - t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0})$$

is a two sided γ-content tolerance interval at confidence coefficient $1-\alpha$ and the following two intervals

$$(x_0'\hat{\beta} - t_{1-\alpha}(n-k, \frac{z_\gamma}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0}, \infty)$$

$$(-\infty, x_0'\hat{\beta} - t_{\alpha}(n-k, -\frac{z_\gamma}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0})$$

are one sided γ-content tolerance intervals at confidence coefficient $1-\alpha$.

Proof. Note that

$$\frac{x_0'\hat{\beta} - x_0'\beta + a\sigma}{\sigma\sqrt{x_0'(X'X)^{-1}x_0}} \sim N(\frac{a}{\sqrt{x_0'(X'X)^{-1}x_0}}, 1) \text{ for } a \in R.$$

This indicates that

$$\frac{x_0'\hat{\beta} - x_0'\beta + z_{(1+\gamma)/2}\sigma}{S\sqrt{x_0'(X'X)^{-1}x_0}} \sim t(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})$$

and

$$\frac{x_0'\hat{\beta} - x_0'\beta - z_{(1+\gamma)/2}\sigma}{S\sqrt{x_0'(X'X)^{-1}x_0}} \sim t(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}}).$$

Then

$$1-\alpha = P[x_0'\hat{\beta} - x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma < t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0}]$$

$$-P[x_0'\hat{\beta} - x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma < t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0}]$$

$$= P[x_0'\hat{\beta} - x_0'\beta < t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} - z_{\frac{1+\gamma}{2}}\sigma]$$

$$-P[x_0'\hat{\beta} - x_0'\beta < t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} + z_{\frac{1+\gamma}{2}}\sigma]$$

$$= P[t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} + z_{\frac{1+\gamma}{2}}\sigma < x_0'\hat{\beta} - x_0'\beta$$

$$< t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} - z_{\frac{1+\gamma}{2}}\sigma]$$

$$= P[t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} < x_0'\hat{\beta} - x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma$$

$$< x_0'\hat{\beta} - x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma < t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0})]$$

$$= P[x_0'\hat{\beta} - t_{1-\frac{\alpha}{2}}(n-k, \frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0} < x_0'\beta - z_{\frac{1+\gamma}{2}}\sigma$$

$$< x_0'\beta + z_{\frac{1+\gamma}{2}}\sigma < x_0'\hat{\beta} - t_{\frac{\alpha}{2}}(n-k, -\frac{z_{(1+\gamma)/2}}{\sqrt{x_0'(X'X)^{-1}x_0}})S\sqrt{x_0'(X'X)^{-1}x_0}].$$

This proves the theorem.

4. Formulation of Confidence and an Evaluation of Tolerance Interval When Parameters Are Known

Let y_0 be a random variable, representing the characteristic of a product, that corresponds with covariate x_0 from a manufacturing process. There are lots of products produced from this process. For simplicity, the lots are all assumed to have the same size k. We want to see if there is confidence q_0 that there is proportion γ of products in lots conforming to specification limits $\{LSL_x, USL_x\}$. To evaluate this, we have a sample $\begin{pmatrix} y_1 \\ X_1 \end{pmatrix}, ..., \begin{pmatrix} y_n \\ X_n \end{pmatrix}$ from the same process to evaluate the manufacturer's confidence. We need first introduce appropriate criterions for evaluation of a technique for this purpose. We consider its ability in correctly identifying the true confidence for reducing the following two errors:

(a) The first error is that the inferenced confidence is much higher than the true confidence. In this situation, the resulted confidence is too optimistic for the manufacturer.

(b) The second error is that the inferenced confidence is much lower than the true confidence. In this ituation, the resulted confidence is too conservative for the manufacturer.

Before consideration of any evaluation, let's formulate the framwok of the statistical model we want to consider. We have a linear regression model

$$y_i = x_i'\beta + \epsilon_i, i = 1, ..., n \tag{4.1}$$

where $\epsilon_i, i = 1, ..., n$ are independen and identically random variables with distribution function F_σ having mean zero and unknown variance σ^2. The vectors $\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}, ..., \begin{pmatrix} y_n \\ x_n \end{pmatrix}$ forms the only observable vectors. We are interesting to see if a production lot of size k from the same process is with proportion γ conforming to the specification limits $\{LSL_x, USL_x\}$.

The tolerance interval deals with the following problem: There is a collection of products forming a production lot, vectors $\begin{pmatrix} y^* \\ x^* \end{pmatrix}$, and each product is classified to be non-defective if y^* falls in specification limit interval

(LSL_{x^*}, USL_{x^*}) and to be defective if it lies outside the limit interval. However, for this production lot, the manufacturer knows that unless a proportion q of this lot is acceptable in sense that the corresponding products are non-defective, he will loss money.

For giving vector $\begin{pmatrix} y \\ x \end{pmatrix}$ following model (4.1) and its corresponding specification limits $\{LSL_x, USL_x\}$, the probability that variable y corresponding with covariates x is acceptable is

$$p_{spec}(x, \beta, \sigma) = \int_{LSL_x}^{USL_x} f_{y|x}(y)dy = F_\sigma(USL_x - x'\beta) - F_\sigma(LSL_x - x'\beta). \tag{4.2}$$

Consider production lot with size k is to be judged if there is proportion γ or more acceptable products. For simpler study, let's restrict the situation that the probability function $p_{spec}(x, \beta, \sigma)$ is independent of covariates x. The case that the specification limits of the form $LSL_x = x'\beta - l$ and $USL_x = x'\beta + l$ is one example. By letting $p_{spec}(\beta, \sigma) = p_{spec}(x, \beta, \sigma)$ for this case, the number of products representing by variables y are lying in their corresponding specification limits obeys the binomial distribution $b(k, p_{spec}(\beta, \sigma))$. Then, the confidence of $[k\gamma]$ or more acceptable products is

$$q = \sum_{i=[k\gamma]}^{k} \begin{pmatrix} k \\ i \end{pmatrix} p_{spec}(\beta, \sigma)^i (1 - p_{spec}(\beta, \sigma))^{k-i}. \tag{4.3}$$

which is the probability that there is proportion γ or more acceptable products. Treating q as a function of $p_{spec}(\beta, \sigma)$, $1 - q$ has the same properties as the operating characteristic (OC) curve. However, these two function are different since OC curve involves sample size n and $1 - q$ involves lot size k. The properties of OC curve indicates that confidence function q is increasing in item reliability $p_{spec}(\beta, \sigma)$. The interest of the manufacturer then is to see if the true item reliability makes q larger or equal to a prespecified value.

When the confidence of a production lot is with $q = q_0$ this lot attains the manufacturer's expectation to have γ-content acceptable products at confidence q_0. In this situation, the manufacturer achieves the desired result that, in the long run, it will accept lots having at least proportion γ

acceptable products at least a proportion q_0 of the time.

Suppose that now ϵ has a normal distribution $N(0, \sigma^2)$ and regression parameter β and error variance σ^2 are both known. From Wilks (1941), the following

$$(x'\beta - z_{\frac{1+\gamma}{2}}\sigma, x'\beta + z_{\frac{1+\gamma}{2}}\sigma) \tag{4.4}$$

is a γ-content tolerance interval with confidence 1. It is a 100% confidence tolerance interval. Suppose that we further let

$$(LSL_x, USL_x) = (x'\beta - z_{\frac{1+\gamma}{2}}\sigma, x'\beta + z_{\frac{1+\gamma}{2}}\sigma). \tag{4.5}$$

With this claim of 100% tolerance interval, it is interpreted that it will accept all the lots, without missing any one lot, having at least proportion γ acceptable products. Is this classical conclusion appropriate? Let number k be the size of a production lot (usually large). The true confidence of this production lot with proportion γ or more acceptable products is

$$q = \sum_{i=[k\gamma]}^{k} \binom{k}{i} \gamma^i (1-\gamma)^{k-i}.$$

We list the corresponding values of true confidence for $\gamma = 0.8, 0.85, 0.9, 0.95$ and several values of k.

Table 3. True confidence q for γ-content acceptable products

Lot size	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.9$	$\gamma = 0.95$
$k = 50$	0.6011	0.6045	0.6223	0.6580
$k = 100$	0.5838	0.6192	0.5528	0.5891
$k = 500$	0.5431	0.5299	0.5438	0.5322
$k = 1,000$	0.5147	0.5309	0.5223	0.5357
$k = 10,000$	0.5115	0.5067	0.5059	0.5108
$k = 100,000$	0.5013	0.5013	0.5013	0.5013

We have several conclusions drawn from the results in the above table:
(a) In all cases of γ and lot size the confidences are larger than, but close, to 0.5. In this situation that the tolerance interval coincides with the specification limits, the manufacturer, in the long run, may accept lots having at least proportion γ acceptable products guaranteeing only proportion q

close to 0.5 of the time. It is not a surprise. Knowing the coverage interval only helps in knowing the coverage probability that doesn't improves in the true confidence.

(b) The true confidence q relies on lot unit number k and the true value of the item reliability $p_{spec}(\beta, \sigma)$ no matter if the parameters β and σ are known or not.

(c) For a given coverage probability γ, the true confidence q approximates to 0.5 when the lot size increses to infinity.

(d) It seems that using the tolerance interval to interprete the confidence that the manufacturer may accept lots having at least proportion γ acceptable products is not appropriate.

From (4.3), any production lot has its true confidence with property: once an item reliability $p_{spec}(x, \beta, \sigma)$ is in interval $(0, 1)$, no matter how large and how small it is, its corresponding confidence should be between zero and one. However, when item reliability is close to one its corresponding confidence is also close to one and whe item reliability is close to zero its corresponding confidence is also close to zero. However, it is interesting to investigate the sensitivity of the confidence as a function of the item reliability. We set some values of $p_{spec}(x, \beta, \sigma)$ and percentages γ and list the corresponding confidences q in Table 4.

Table 4. Confidence for γ-content acceptable products

$p_{spec}(\beta, \sigma)$	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.75	0.0001	0	0	0	0
0.8	0.5189	0	0	0	0
0.85	0.9999	0.5217	0	0	0
0.9	1	0.9999	0.5265	0	0
0.95	1	1	1	0.5375	0
0.99	1	1	1	1	0.5830

We have several conclusions drawn from Table 4:

1. When $p_{spec}(x, \beta, \sigma)$ is moderately below γ, the confidence is approximately equal zero and when $p_{spec}(x, \beta, \sigma)$ is moderately larger than γ, it is approximately equal one.

2. When $p_{spec}(x, \beta, \sigma)$ is γ close to 0.5 the confidence q of γ-content acceptable products is only slightly more than 0.5.
3. For given a value of γ, the curve representing the confidence as a function of $p_{spec}(x, \beta, \sigma)$ is started from zero and rapidly climb up for domain of $p_{spec}(x, \beta, \sigma)$ is $(0, 1)$.
4. The interest is that when the true confidence q may be moderately large to ensure proportion γ or more acceptable products in a lot. From the table, it is the case that item reliability $p_{spec}(x, \beta, \sigma)$ is larger than γ. This tells us how to improve the process to achieve the aim when it is not.

5. Minimum Item Reliability for proportion γ Or More Acceptable Products With Confidence q

For the use of tolerance interval to predict the manufacturer's confidence is shown to be too optimistic when the parameters involved in the underlying distribution are known. For case that parameters are unknown, can the classical tolerance intervals be also too optimistic or on the opposite way? To investigate this question, we need to prepare some further results on the relation between item reliability and manufacturer's confidence q.

Recall that the manufacturer expects to have proportion γ or more acceptable products with confidence q_0 or more. Let's define the minimum item reliability that guarantees proportion γ acceptable products with confidence q as $p_{\gamma q}$ satisfying

$$\sum_{i=[k\gamma]}^{k} \binom{k}{i} (p_{\gamma q})^i (1 - p_{\gamma q})^{k-i} = q. \tag{5.1}$$

From (4.3), a process to have a γ-content acceptable products with confidence q_0 requires that

$$p_{spec}(\beta, \sigma) \geq p_{\gamma q_0}. \tag{5.2}$$

For given γ and q, we display the $p_{\gamma q}$ that solves equation (5.1) in the following table.

Table 5. Minimum specification coverage $p_{\gamma q}$ achieving for γ-content accptable products with confidence q (k=100,000)

q	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.8	0.8099	0.8587	0.9071	0.9549	0.9918
0.85	0.8123	0.8608	0.9089	0.9562	0.9923
0.9	0.8152	0.8634	0.9111	0.9577	0.9929
0.95	0.8196	0.8673	0.9142	0.9599	0.9938
0.99	0.8277	0.8744	0.9200	0.9638	0.9952

Suppose that the characteristic variable y of interest given covariates x obeys a normal distribution $N(x'\beta, \sigma^2)$ and the specification limits are $\{LSL_x, USL_x\} = \{x'\beta - \ell, x'\beta + \ell\}$. Then the probability that an observation of it to be acceptable is

$$p_{spec}(\sigma) = \int_{x'\beta-\ell}^{x'\beta+\ell} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x'\beta)^2}{2\sigma^2}} dy$$
$$= \Phi(\frac{\ell}{\sigma}) - \Phi(\frac{-\ell}{\sigma})$$

where Φ is the distribution function of the standard normal distribution. Then the confidence that there are γ percentage acceptable products is

$$p_\gamma(\sigma) = \sum_{i=[k\gamma]}^{k} \binom{k}{i} p_{spec}(\sigma)^i (1 - p_{spec}(\sigma))^{k-i}$$
$$= \sum_{i=[k\gamma]}^{k} \binom{k}{i} (\Phi(\frac{\ell}{\sigma}) - \Phi(\frac{-\ell}{\sigma}))^i (1 - (\Phi(\frac{\ell}{\sigma}) - \Phi(\frac{-\ell}{\sigma})))^{k-i}.$$

For a given σ and the specified specification limits $\{LSL_x, USL_x\} = \{x'\beta - \ell, x'\beta + \ell\}$, $p_\gamma(\sigma)$ is the true confidence for the production lot. Let $p_{\gamma q}$ be the minimum specification coverage defined in (5.1). A process to have a γ-content acceptable products with confidence $q(p_\gamma(\sigma) \geq q)$ requires that

$$p_{spec}(\sigma) \geq p_{\gamma q}. \tag{5.3}$$

One question very interesting is how small the difference of specification limits is that guarantees the desired process. Let's consider a simple situation that the error variable obeys the standard normal distribution $N(0, 1)$ and we consider the specification limits of the form $\{LSL_x, USL_x\} = \{x'\beta - l, x'\beta + l\}$ and denote $l_{\gamma q}$ as the minimum l that a tolerance interval achieves its corresponding minimum acceptability coverage, i.e., equality in (5.1) holds, as

$$p_{\gamma q} = P(-l_{\gamma q} \leq Z \leq l_{\gamma q}).$$

Table 6. Width $l_{\gamma q}$ for minimum specification coverage $p_{\gamma q}$ achieving for γ-content accptable products with confidence q (k=100,000)

q	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.8	1.3103	1.4710	1.6807	2.0043	2.6452
0.85	1.3174	1.4789	1.6898	2.0161	2.6672
0.9	1.3264	1.4889	1.7013	2.0309	2.6953
0.95	1.3397	1.5037	1.7184	2.0532	2.7380
0.99	1.3650	1.5318	1.7510	2.0956	2.8216

6. Power Study for the Regression Tolerance Intervals

We perform a simulation to study the role of a regression tolerance interval in estimating the manufacturer's confidence. With a simple linear regression model,

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$$

where error variables ϵ_i are iid standard normal and the predicted variable y_0 follows

$$y_0 = \beta_0 + \beta_1 x_0 + \epsilon_0.$$

The covariates x_i are independent normal random variables with mean i and variance 1. The sample size n, $n = 20, 30, 50, 100$ are considered and total of 100000 replications is performed. Let (t_{1j}, t_{2j}) be the tolerance interval for the jth sample. We first study if the regression tolerance interval of our approach does appropriate in playing its role of predicting the manufacturer's confidence. We first consider the power of the tolerance interval to be accepted as a process of γ-content acceptable products with confidence q as

$$\frac{1}{m}\sum_{j=1}^{m} I((t_{1j}, t_{2j}) \subset (LSL, USL)).$$

With $(LSL, USL) = (x'\beta - \ell, x'\beta + \ell)$, the following table list the results of the power.

Table 7. Simulated power as a γ-content acceptable products with confidence q (n=50)

	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.9$	$\gamma = 0.95$	$\gamma = 0.99$
$q = 0.9$					
$\ell = 1.5$	0.0061	0.0006	0	0	0
1.7013	0.0697	0.0141	0.0013	0	0
2.0	0.4216	0.1904	0.0447	0.0025	0
2.5	0.9332	0.8192	0.5580	0.1589	0.0018
3.0	0.9975	0.9895	0.9490	0.7106	0.0831
3.5	0.9999	0.9998	0.9980	0.9697	0.4868
4.0	1	1	0.9999	0.9986	0.8821
$q = 0.95$					
$\ell = 1.5$	0.0015	0.0002	0	0	0
1.7013	0.0234	0.0044	0.0003	0	0
2.0	0.2531	0.0901	0.0158	0.0006	0
2.5	0.8620	0.6889	0.3870	0.0786	0.0006
3.0	0.9930	0.9743	0.8908	0.5556	0.0378
3.5	0.9999	0.9993	0.9938	0.9306	0.3317
4.0	1	0.9999	0.9998	0.9957	0.7840

There are several conclusions may be drawn from the simulation results in above table:

(a) When the specification limit interval $(x'\beta - l, x'\beta + l)$ is wider in sense that value l is increasing the power for the tolerance interval to detect the manufacturing process to have confidence q is increasing.

(b) When $l < 1.7013$ the true confidence for a γ-content acceptable products is less than $q = 0.9$. In this situation, the powers of this tolerance interval in all cases of sample size are all less than 0.006. That is, it is fewer than 0.006 to claim to have confidence 0.9. This is a plan too conservative to both the consumer and manufacturer.

(c) In all cases of l, the true confidence for a γ-content acceptable products can't be greater than or equal to 0.9 in all situations of γ.

(d) If, in the long run, there is a large proportion that the tolerance intervals rejected the null hypothesis H_0, the process must be in well control so that the specification limit interval $(x'\beta - l, x'\beta + l)$ must be wide enough so that the tolerance intervals are contained in the limit interval.

Let's study the following:

$$\frac{1}{m}\sum_{j=1}^{m} I(\text{there is } (t_1', t_2') \subset (t_{1j}, t_{2j}) \text{ such that } \int_{t_1'}^{t_2'} f_{y_0}(y)dy \geq \gamma \text{ and } (t_1', t_2') \subset (LSL, USL)).$$

Table 8. Simulated confidence for regression tolerance intervals ($\gamma = 0.9$)

True $1-\alpha$	$n = 20$	$n = 30$	$n = 50$	$n = 100$
$L = 0.0001$				
$1-\alpha = 0.8$	0.8369	0.8259	0.8169	0.8084
0.9	0.9377	0.9272	0.9197	0.9126
0.95	0.9784	0.9730	0.9666	0.9598
$L = 0.1$				
$1-\alpha = 0.8$	0.8803	0.8805	0.8901	0.9086
0.9	0.9595	0.9561	0.9567	0.9626
0.95	0.9869	0.9851	0.9842	0.9860
$L = 0.5$				
$1-\alpha = 0.8$	0.9422	0.9496	0.9615	0.9754
0.9	0.9830	0.9850	0.9891	0.9942
0.95	0.9957	0.9958	0.9973	0.9988
$L = 1.0$				
$1-\alpha = 0.8$	0.9527	0.9599	0.9682	0.9781
0.9	0.9871	0.9894	0.9923	0.9956
0.95	0.9968	0.9973	0.9980	0.9990

compute the following approximate confidence

$$\frac{1}{m}\sum_{j=1}^{m} I(CI \subset (t_{1j}, t_{2j})).$$

The approximate confidences of regression tolerance intervals are listed in Table 9.

Table 9. Simulated confidence for regression tolerance intervals

True $1-\alpha$	$n = 20$	$n = 30$	$n = 50$	$n = 100$
$\gamma = 0.9$				
0.8	0.8349	0.8234	0.8155	0.8087
0.9	0.9367	0.9286	0.9182	0.9136
0.95	0.9776	0.9716	0.9662	0.9595
$\gamma = 0.95$				
0.8	0.8409	0.8288	0.8170	0.8136
0.9	0.9367	0.9277	0.9187	0.9130
0.95	0.9775	0.9714	0.9666	0.9599

7. Confidence Estimation and Its Power for a Process of γ-Content Acceptable Products

Suppose that we have a sample linear regression model $y_i = x'\beta + \epsilon_i, i = 1, ..., n$ where $\epsilon_1, ..., \epsilon_n$ are independent and identically distributed random variables with normal distribution $N(0, \sigma^2)$. Again, we also let the specification limits be $\{LSL_x, USL_x\}$ The item reliability p_{spec} is

$$p_{spec}(x, \beta, \sigma) = \Phi(\frac{USL_x - x'\beta}{\sigma}) - \Phi(\frac{LSL_x - x'\beta}{\sigma}).$$

With a simple linear regression model, we let $\hat{\beta}_{ls}$ and s^2 be the least squares estimates of β and σ, respectively.

$$\begin{aligned} \hat{p}_{spec} &= p_{spec}(x, \hat{\beta}_{ls}, s) \\ &= \Phi(\frac{USL_x - x'\hat{\beta}_{ls}}{s}) - \Phi(\frac{LSL_x - x'\hat{\beta}_{ls}}{s}). \end{aligned}$$

We will perform a series of simulation. In this simulation, we consider the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, ..., n$ with specification limits $\{LSL_x, USL_x\} = \{\beta_0 + \beta_1 x - \ell, \beta_0 + \beta_1 x + \ell\}$. Consider that the error terms are randomly selected from the standard normal distribution $N(0, 1)$. Hence the estimate of the item reliability is

$$\hat{p}_{spec} = \Phi(\frac{\beta_0 + \beta_1 x + \ell - x'\hat{\beta}_{ls}}{s}) - \Phi(\frac{\beta_0 + \beta_1 x - \ell - x'\hat{\beta}_{ls}}{s})$$

where true regression parameters for data generation are $\beta_0 = \beta_1 = 1$. For given a pair (γ, q), there is pair $(p_{\gamma q}, \ell_{\gamma q})$ such that when the specification limits are $\{\beta_0 + \beta_1 x - \ell_{\gamma q}, \beta_0 + \beta_1 x + \ell_{\gamma q}\}$ the confidence of a proportion γ acceptable production lots is exactly equal to q. To have good estimation for confidence we should have good estimation technique of item reliability.

We choose $\ell_{\gamma q}$ corresponding with $\gamma = 0.9$ and confidence $q = 0.9$. With replication $m = 100,000$, let $\hat{p}^j_{spec}$ be the estimate of the item reliability at the jth replication. We define the empirical item reliability as

$$\bar{p}_{spec} = \frac{1}{m} \sum_{j=1}^{m} \hat{p}^j_{spec}.$$

Our aim is to see if $\hat{p}_{spec}$ is efficient to estimate the true item reliability $p_{\gamma q}$. The following table displays the simulation results of $\bar{p}_{spec}$ and the MSE's.

Table 10. Empirical item reliability $\bar{p}_{spec}$ associated with the true item reliability $p_{\gamma q}$ $(m = 100000, k = 1,000)$

	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.9$	$\gamma = 0.95$
$q = 0.8$	$p_{\gamma q} = 0.8099$	$p_{\gamma q} = 0.8587$	$p_{\gamma q} = 0.9071$	$p_{\gamma q} = 0.9549$
$n = 30$	0.8070	0.8548	0.9022	0.9495
$n = 50$	0.8082	0.8564	0.9042	0.9518
$n = 100$	0.8088	0.8575	0.9057	0.9533
$q = 0.9$	$p_{\gamma q} = 0.8152$	$p_{\gamma q} = 0.8634$	$p_{\gamma q} = 0.9111$	$p_{\gamma q} = 0.9577$
$n = 30$	0.8124	0.8597	0.9062	0.9524
$n = 50$	0.8135	0.8608	0.9081	0.9545
$n = 100$	0.8143	0.8624	0.9095	0.9561

The mean square error is

$$MSE = \frac{1}{m}\sum_{j=1}^{m}(\hat{p}^{j}_{spec} - p_{\gamma q})^2.$$

Table 11. MSE for empirical item reliability estimation $(m = 100000, k = 1,000)$

	$\gamma = 0.8$	0.85	0.9	0.95
$q = 0.8$				
$n = 30$	0.0032	0.0026	0.0017	0.0008
$n = 50$	0.0019	0.0015	0.0010	0.0004
$n = 100$	0.0009	0.0007	0.0005	0.0002
$q = 0.9$				
$n = 30$	0.0032	0.0025	0.0017	0.0007
$n = 50$	0.0019	0.0015	0.0010	0.0004
$n = 100$	0.0009	0.0007	0.0005	0.0002

We let the point estimate of the confidence $p_\gamma(\sigma)$ be its least squares estimate as

$$\hat{q} = \sum_{i=[k\gamma]}^{k}\binom{k}{i}(\Phi(\frac{USL_x - x'\hat{\beta}_{ls}}{s} - \Phi(\frac{LSL_x - x'\hat{\beta}_{ls}}{s})^i$$
$$(1 - [\Phi(\frac{USL_x - x'\hat{\beta}_{ls}}{s} - \Phi(\frac{LSL_x - x'\hat{\beta}_{ls}}{s}])^{k-i}.$$

The estimation of the unknown confidence uses the estimation of unknown parameters β and σ and lot size where the last one is an ancillary statistic.

We consider $\gamma = 0.9$ and confidence $q = 0.9$. By letting $\hat{q}^j$ as the jth estimate of the confidence, $j = 1, ..., m$. We simulate the empirical average of confidence as

$$\bar{q} = \frac{1}{m} \sum_{j=1}^{m} \hat{q}^j.$$

Table 12. Empirical average confidence p_γ $(m = 100000, k = 1,000, \gamma = 0.9, 1 - \alpha = 0.9)$

ℓ	$n = 30$	$n = 50$	$n = 100$
1.4	0.1254	0.0722	0.0185
1.5	0.2512	0.2124	0.1190
1.6	0.3960	0.4112	0.3665
1.7	0.5888	0.6148	0.6832
1.8	0.7432	0.8001	0.8990
2.0	0.9416	0.9681	0.9951
2.2	0.9891	0.9978	0.9999
2.5	0.9999	1	1

We have seen that the tolerance interval is not really appropriate in detecting the manufacturer's confidence for having a proportion γ of acceptable production lots for that it is too optimistic when the parameters involved in the distribution are known and it is too conservative when the parameters are unknown. It is then interesting to evaluate the manufacturer's confidence through the estimate of the unknown confidence. Consider a simulation that we randomly select errors from the standard normal distribution. We consider replication $m = 100,000$, $\gamma = 0.9$ and confidence $q_0 = 0.9$. By letting $\hat{q}^j$ as the jth estimate of the confidence, $j = 1, ..., m$. We define the power of this point estimator as

$$\pi = \frac{1}{m} \sum_{j=1}^{m} I(\hat{q}^j \geq q).$$

The following table list the simulation results of this study.

Table 13. Power for the Estimation of the confidence for a γ-content acceptable product when the underlying distribution is normal (k=100,000, m=100,000)

Spec L	$n = 30$	$n = 50$	$n = 100$
$l = 1.0$	0.0002	0	0
$l = 1.1$	0.0018	0.0003	0
$l = 1.2$	0.0084	0.0009	0
$l = 1.3$	0.0292	0.0065	0.0002
$l = 1.4$	0.0802	0.0340	0.0046
$l = 1.5$	0.1711	0.1101	0.0424
$l = 1.6$	0.3111	0.2675	0.1924
$l = 1.7$	0.4810	0.4840	0.4864
$l = 1.7013$	0.4854	0.4846	0.4944
$l = 1.8$	0.6566	0.7016	0.7834
$l = 1.9$	0.7958	0.8627	0.9417
$l = 2.0$	0.8950	0.9502	0.9912
$l = 2.2$	0.9811	0.9970	0.9999
$l = 2.5$	0.9994	0.9999	1
$l = 3.0$	1	1	1

We have several conclusions that may be drawn from the results in the above table:

(a) For every given sample size n, the effiicency is strictly increasing in the specification limit l. This fullfills our expectation.

(b) In the setting $\gamma = 0.9$ and $q_0 = 0.9$ by the manufacturer, $l = 1.7013$ guarantees to have proportion γ or more acceptable products with confidence exactly $q = 0.9$. In this situation, the effieciencies are about 0.48. There is probability 0.48 that we can detect that the lot of interest is acceptable. In Table 7, we see that the tolerance interval techniques can observe this fact with chance less than 0.1.

(c) When the specification limit l is a bit wider we have large chance to detect that the lot is acceptable. Comparing this table with Table 7, we see that this technique of confidence estimation is better than the tolerance interval technique simultanuously.

8. Concluding Remarks

We have several remarks illustrating our further concern and clarification of our study:

(a) We show that the use of tolerance interval to evaluate for the manufacturer the confidence for a proportion γ of production lot conforming to specifications is not appropriate.

(b) The inefficiency of the classical tolerance interval for detecting the manufacturer's confidence partly comes from the fact that it uses only the information contained in the sample. The lot size which is an ancillary statistic in this case is an important information that hasn't been considered in construction of tolerance interval. This is an example that an ancillary statistic provides important information for statistical inference.

(c) There are tolerance interval-like technique for deciding if we will accept a production lot (see Kirkpatrick (1970), Owen and Hua (1977), Weingarten (1982) and Mee (1984)). This technique does not employ the information of lot size and has not been popular in practical use. Hence it is not in our study.

For this confidence estimation technique, it may be argued that the lot size k may not be correctedly predicted or counted and we may have, especially when the size is huge, only an approximate number. Then, it is interesting to see how robust the estimation technique in it estiamted confidence when the lot size has not been correctly used. We design a simulation computing the average confidences with various large sizes to see its sensitiveness in terms of lot size.

Table 14. Simulated averaging confidences for some large lot sizes

l	$k = 100,000$	$k = 300,000$	$k = 500,000$	$k = 1000,000$
1.3	0.01449	0.01472	0.01478	0.01471
1.4	0.05897	0.06022	0.06055	0.06278
1.5	0.17367	0.1767	0.17818	0.17836
1.55	0.26367	0.26864	0.2698	0.26921
1.6	0.36821	0.37366	0.37848	0.37829
1.65	0.48829	0.49174	0.4926	0.49574
1.7	0.60048	0.60721	0.60875	0.61018
1.75	0.71045	0.71476	0.71825	0.72178
1.8	0.79859	0.80835	0.80823	0.80823
2.0	0.97638	0.97681	0.97801	0.97757
2.5	0.99999	1	1	1

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