國 立 交 通 大 學

統 計 學 研 究 所

碩 士 論 文

容忍區間及新方法探討貨品接受之研究

Tolerance Interval And a New Approach for Lot Production Inspection

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指導教授 ：陳鄰安 博士 (Dr. Lin-An Chen)

中 華 民 國 九 十 六 年 六 月

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A Thesis
Submitted to Institute of Statistics
College of Science
National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Master
in
Statistics
June 2007

Hsinchu, Taiwan, Republic of China

中華民國九十六年六月

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摘　要

長期以來容忍區間被用於檢驗每批貨品是否有固定比率的合格產品且其信心是否夠高。我們在這篇文章正式把信心寫成參數的函數。因此，任何預測信心的方法都可用來檢驗它的有效性。我們研究了容忍區間的檢力，其中我們發現當參數已知時此一方法太過樂觀當參數未知時其方法又太悲觀。主要原因是此一方法未使用一批貨品之量的大小。我們進一步發展了信心的點估計方法其表現似乎很好。

關鍵字：假設檢定；檢定力；品質管制；容忍區間。

Tolerance Interval And a New Approach for lot Production Inspection

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ABSTRACT

The tolerance interval has long been a technique for manufacturer to verify if there is confidence large enough that ensures a proportion of production lot conforming to specification limits. This paper formally formulate this "confidence" in terms of lot size and parameters involved in the underlying distribution of the characteristic. With this formulation, any technique for prediction of manufacturer's confidence may be evaluated for its efficiency and it also provides a wide room for this prediction through statistical inferences for the unknown confidence. We then study the power of the tolerance interval in detecting if there is a reasonably large manufacturer's confidence. We found that when the parameters involved in the distribution are known, the predicted manufacturer's confidence is too optimistic in a value much higher than the true value and when the parameters involved in the distribution are unknown, the predicted manufacturer's confidence is too conservative in a value much lower than the true one. The inefficiency partly comes from the fact that tolerance interval does not use the information of lot size in prediction which is an ancillary statistic in the considered statistical model. For statistical inference of this unknown confidence, we introduce a point estimation technique that its results shown a power comparison seems to be very promising for the manufacturer.

Key words: Hypothesis testing; power; quality control; tolerance interval.

誌　　謝

可以完成這篇論文，首先要感謝我的指導教授--陳鄰安老師，在這一年裡細心的指導，不論是課業或者是處事態度上都給於很好的建議，讓我獲益良多。更要感謝交大統計所的所有老師們，也因為您們才能讓我更了解統計哲學。特別感謝--洪慧念老師、蔡明田老師以及黃信誠老師三位口試委員，撥冗指導。

更要感謝在這兩年裡和我一起成長的同學們。永在，提供解決有關電腦問題方法。建威，提供解決撰擬程式上的害蟲建議。益銘及俊睿，讓我的眼界更加的遼闊。煜淳及育辰，陪我一起運動健身。以及曾經給於我協助、一起打拼及一起同樂的好同學們。

最後要感謝我的雙親、哥哥和姊姊。感謝爸爸媽媽辛苦的供我讀書及給於我一個盡情發揮的空間，讓我在就學上能一路順遂。感謝哥哥姊姊給於我支持及鼓勵。

在此，將以本篇論文獻給我親愛的家人、師長及同學們，並致上我最誠摯的謝意。

吳志文　謹誌于

國立交通大學統計學研究所

九十六年六月

Contents

Tolerance Interval and a New Approach for Lot Production Inspection

Abstract

The tolerance interval has long been a technique for manufacturer to verify if there is confidence large enough that ensures a proportion of production lot conforming to specification limits. This paper formally formulate this "confidence" in terms of lot size and parameters involved in the underlying distribution of the characteristic. With this formulation, any technique for prediction of manufacturer's confidence may be evaluated for its efficiency and it also provides a wide room for this prediction through statistical inferences for the unknown confidence. We then study the power of the tolerance interval in detecting if there is a reasonably large manufacturer's confidence. We found that when the parameters involved in the distribution are known, the predicted manufacturer's confidence is too optimistic in a value much higher than the true value and when the parameters involved in the distribution are unknown, the predicted manufacturer's confidence is too conservative in a value much lower than the true one. The inefficiency partly comes from the fact that tolerance interval does not use the information of lot size in prediction which is an ancillary statistic in the considered statistical model. For statistical inference of this unknown confidence, we introduce a point estimation technique that its results shown a power comparison seems to be very promising for the manufacturer.

Key words: Hypothesis testing; power; quality control; tolerance interval.

1. Introduction

In manufacturing industry, specification limits for one characteristic of an item, saying LSL and USL, define the boundaries of acceptable quality for a manufacturing item (component). An item is said to be non-defective if the measured characteristic is between the limits, otherwise it is defective. For a manufacturer of a mass-production item, the tolerance interval is designed for a quality assurance problem. For a production lot, the manufacturer knows that unless a proportion, saying γ, of this lot is acceptable in the sense that the corresponding items are non-defective, he will lose money in this production. In practice, the manufacturer further expects a process that, in the long run, he or she will have lots of production not losing money at least a percentage, saying q_0, of the time. If this is true, the manufacturer is guaranteed that, in the long run, he or she will have lots of production of proportion γ or more conforming to specifications at least $100q_0\%$ of the time. The tolerance intervals are widely used in quality control and related prediction problems to monitor manufacturing processes to ensure product compliance with specifications, etc. An important application of the tolerance interval is to examine if the manufacturer's expectation is accomplished.

We summarize the application of tolerance interval for predicting the manufacturer's confidence into three steps:

(a) The first step is to set up a hypothesis. Suppose that the characteristic variable has a probability density function $f(x, \theta)$. The common purpose to construct the tolerance interval is to test the following hypotheses:

$$H_0 : \int_{LSL}^{USL} f(x,\theta)dx \leq \gamma \text{ vs. } H_1 : \int_{LSL}^{USL} f(x,\theta)dx > \gamma, \tag{1.1}$$

(see Goodman and Madansky (1962) for reference). In fact, a level α test for this hypothesis tries to answer if there is at least a proportion γ of the population conforming to specification limits with a confidence level $1-\alpha$.

(b) The second step is to construct the tolerance interval as a test statistic. Suppose that we have a random sample $X = (X_1, ..., X_n)'$ from the distribution of the characteristic variable. The pioneer article by Wilks (1941) introduced a γ-content tolerance interval with confidence $1-\alpha$ as an interval $(T_1, T_2) = (t_1(X), t_2(X))$ that satisfies

$$P_\theta\{P_\theta(X_0 \in (T_1, T_2)|X) \geq \gamma\} \geq 1-\alpha \text{ for } \theta \in \Theta \tag{1.2}$$

where Θ is the parameter space and X_0 represents the future observation with the same distribution. A vast literature on developing tolerance interval has been proposed (see for example Wilks (1941), Wald (1943), Paulson

(1943), Guttman (1970) and, for a recent review, Patel (1986)). Comparing tolerance intervals based on criterion of expected length is the most popularly used selection technique. For normal tolerance interval, Eisenhart etc. (1947) constructed the shortest one. With the appealing property of shortest length, it is now popularly implemented in manufacturing industry and introduced in engineering texts. This criterion has also been a guide line for developing regression tolerance interval (see Goodman and Madansky (1962), Liman and Thomas (1988) and Mee et. al. (1991)).

(c) The third step is to set up a rule for hypotheses in (1.1) based an observed tolerance interval. Let (t_1, t_2) be the observation of a γ-content tolerance interval at confidence $1-\alpha = q_0$. The test for hypotheses H_0 vs H_1 of (1.1) is:

$$\textbf{We reject } H_0 \textbf{ if } (t_1, t_2) \subset (LSL, USL). \tag{1.3}$$

When H_0 is rejected, statistically we are $100(1-\alpha)\%$ sure that at least $100\gamma\%$ of the population is conforming to the specification limits. So, in a long run, we will have lots having at least proportion γ of the distribution conforming to the specification limits at least a percentage $1-\alpha$ of the time. With the interest of resolving the manufacturer's question, it is also generally making an extending conclusion as the following:

When H_0 is rejected, the lot of product is acceptable because we have confidence of a reasonably large value that at least $100\gamma\%$ of the population is conforming to specification limits (1.4)

where Papp (1992) further interpreted that the reasonably large value is $1-\alpha$. For other references, see Bowker and Goode (1952), Owen (1964) and Schilling (1982). Our interest concerns the question: When a sample data support to reject H_0, is making the extending inference in (1.4) appropriate?

The manufacturer wants to see if there is proportion γ of acceptable products with a reasonable large confidence q_0. Probabilitically it is known that guaranteeing proportion γ of the population conforming to the specification limits is not guaranteeing proportion γ of the products in a lot or even the process conforming to the specification limits. Is it true that the prediction of reasonably large confidence $1-\alpha$ for tolerance interval indicates a reasonably large confidence q_0 for the manufacturer? We concern this question since manufacturers want to assure a good chance to have lots accepted when lots are produced at permissible levels of quality. On the other hand, it is also important for the consumer to see if there is irregular degradation of levels of process quality in submitted lots. Henceful, any technique used for this prediction should be able to protect the benefits of both the consumer and manufacturer.

With setting that the characteristic variable of a product obeys some fixed probability distribution, the confidence that there is proportion γ or more non-defective items, i.e., conforming to specification limits, is completely determined by the underlying distribution and the lot size, which is an unknown constant involving unknown parameters. The construction of tolerance interval for solving the manufacturer's problem aims to predict this unknown confidence. However, the tolerance interval in (1.2) is constructed primarily for the confidence that the interval contains a proportion γ of the distribution. There may exists a big discrepancy between the resulted proportion γ of the sample space and the proportion γ of product lot. The size of discrepancy may be determined from the underlying distribution and the lot size, however, it is desired to discover.

The aim of this paper is to study this discrepancy and investigate if there is an alternative technique that is promising for predicting the manufacturer's confidence. We first explicitly formulate (in Section 2) the manufacturer's unknown confidence in terms of distribution parameters and the lot size. With this formulation, it provides a room for statistical inferences for this true confidence. Next, we study (in Sections 2 and 4) the power of the test using the tolerance interval to detect if there is a reasonably large confidence q_0. We will see that the tolerance interval leads to predicted confidence too optimistic in way that it is higher much more than the true one when the distribution parameters are known. On the other hand, when the unknown parameters are unknown the predicted confidence is too conservative in the way that its predicted confidence is lower much smaller than the true one. This verifies that the use of tolerance interval to predict the manufacturer's confidence is not appropriate. Third, we will introduce a point estimation technique (in Section 5) for prediction of the true confidence which provides a new approach for answering the manufacturer's question. We then further investigate the power of detecting the manufacturer's confidence through this new technique which leads to very promising results. Fourth, one observation for the inefficiency of the tolerance interval in detecting the manufacturer's confidence is that this interval does not use the information of lot size which represents an ancillary statistic in the statistical model. This provides another evidence that ancillary statistic is important in several statistical inference problems.

2. Formulation of Confidence and an Evaluation of Tolerance Interval When Parameters Are Known

Let X_0 be a random variable, representing the characteristic of a product from a manufacturing process, having a distribution. There are production lots produced from this process. For simplicity, the lots are all assumed to have the same size k. We want to see if there is confidence q_0 that there is proportion γ of products in lots conforming to specification limits $\{LSL, USL\}$. To evaluate this, we have a random sample $X_1, ..., X_n$ from the same process to evaluate the manufacturer's confidence. We need first introduce appropriate criterions for evaluation of a technique for this purpose. We consider its ability in correctly identifying the true confidence for reducing the following two errors:

(a) The first error is that the inferenced confidence is much higher than the true confidence. In this situation, the resulted confidence is too optimistic for the manufacturer.

(b) The second error is that the inferenced confidence is much lower than the true confidence. In this situation, the resulted confidence is too conservative for the manufacturer.

Before consideration of any evaluation, let's formulate the framework of the statistical model we want to consider. We consider a manufacturing process that the products forms a sequence of independent and identically distributed random variables from a distribution with distribution function F_θ. We are interesting to see if a production lot of size k from this process is with proportion γ conforming to the specification limits $\{LSL, USL\}$. With this purpose, we have a random sample $X_1, ..., X_n$ drawn from this process for prediction. This random sample of size n is the only observable variables.

We first introduce a formulation of the true confidence. Let X be a random variable having distribution function F_θ with probability density function $f(x, \theta)$. Then the item (product) reliability representing the probability that an item is non-defective is

$$p_{item}(\theta) = \int_{LSL}^{USL} f(x,\theta)dx = F_\theta(USL) - F_\theta(LSL) \tag{2.1}$$

where F_θ is the distribution function. Suppose that the lot size is k (usually a large number). For this production lot, the number of acceptable products is with binomial distribution $b(k, p_{item}(\theta))$. Then the true confidence for having proportion γ of production lot conforming to specification limits is

$$q = \sum_{i=[k\gamma]}^{k} \binom{k}{i} p_{item}(\theta)^i (1 - p_{item}(\theta))^{k-i}. \tag{2.2}$$

Treating q as a function of $p_{item}(\theta)$, $1-q$ has the same properties as the operating characteristic (OC) curve. However, $1-q$ and OC curve are different since OC curve doesn't involve lot size k. Since confidence function q is increasing in item reliability $p_{item}(\theta)$, the interest of the manufacturer then is to see if the true item reliability makes q larger or equal to a prespecified value.

We assume that the manufacturer expects to have confidence q_0. Then, when a lot of production is with $q \geq q_0$ this lot attains the manufacturer's expectation. In this situation, in the long run, it will accept lots having at least proportion γ acceptable products at least a proportion q_0 of the time. The true confidence q often relies on the unknown parameters involving in the distribution of the characteristic such that it is also unknown to be predicted. The tolerance interval was designed to do this through the stated rule of the hypothesis testing. We will study the appropriateness of using the tolerance interval to detect the confidence with a simple situation.

Suppose that now X has a normal distribution $N(\mu, \sigma^2)$ where μ and σ are both known. From Wilks (1941), the natural tolerance interval as

$$(\mu - z_{\frac{1+\gamma}{2}}\sigma, \mu + z_{\frac{1+\gamma}{2}}\sigma) \tag{2.3}$$

is a γ-content tolerance interval with confidence 1. It is a 100% confidence tolerance interval. Suppose that we further assume that

$$(LSL, USL) = (\mu - z_{\frac{1+\gamma}{2}}\sigma, \mu + z_{\frac{1+\gamma}{2}}\sigma). \tag{2.4}$$

It is generally accepted the statement that when an item reliability $100\gamma\%$ is required, the production lot satisfying (2.4) should be accepted (see Schilling (1982)) and we may conclude that with 100% confidence that there is proportion γ of acceptable products in lots. Is this conclusion really true? With lot size k, the true confidence of this production lot with proportion γ or more acceptable products is

$$q = \sum_{i=[k\gamma]}^{k} \binom{k}{i} \gamma^i (1-\gamma)^{k-i}.$$

We assume that the manufacturer requires item proportion γ of acceptable products which, in this case of (2.4), is identical to the item reliability. We list the corresponding values of true confidence for $\gamma = 0.9, 0.95$ and several values of k.

Table 1. True confidence q for γ-content acceptable products

γ	$k = 100$	$1,000$	$10,000$	$100,000$
0.5	0.5397	0.5126	0.5039	0.5012
0.6	0.5432	0.5137	0.5043	0.5013
0.7	0.5491	0.5155	0.5049	0.5015
0.8	0.5594	0.5189	0.5059	0.5018
0.85	0.5683	0.5217	0.5068	0.5021
0.9	0.5831	0.5265	0.5084	0.5026
0.95	0.6159	0.5375	0.5118	0.5037

We have several conclusions drawn from the results in the above table:
(a) Although in all cases of γ and lot size the confidences are greater than 0.5, but are all less than 0.65. In this situation that the tolerance interval coincides with the specification limits, the manufacturer, in the long run, may accept lots having at least proportion γ acceptable products guaranteeing only proportion q less than 70% of the time. This result is not a surprise. Knowing the coverage interval only helps in knowing the item reliability that, in fact, doesn't improves in enlarging the true confidence.
(b) The true confidence q relies on lot size k, specification limits and the true item reliability $p_{item}(\theta)$ no matter if the parameter θ is known or not.
(c) For a given item reliability $p_{item}(\theta) = \gamma$, the true confidence q is increasing in lot size. On the other hand, for a given lot size, the true confidence decreases when γ increases. Furthermore, when the lot size increases to infinity $+\infty$, the true confidence q approaches to 0.5.
(d) It seems that it is too optimistic by using the tolerance interval to interpret the confidence for that the manufacturer may accept lots having at least proportion γ acceptable products.

Example 1. Schilling (1982) considered a case that the characteristic obeys the normal distribution $N(10, 1)$ which indicated that $(8.04, 11.96) = (10 - 1.96, 10 + 1.96) = (8.04, 11.96)$ is a $\gamma = 0.95$-content tolerance interval with confidence 100%. The author then assume that the specification limits are $LSL = 8.0$ and $USL = 12.0$. Since the tolerance interval is fully contained in the limits, he or she then claim that the product lot has to be accepted. However, the probabilities that there are more than proportion 0.95 of products conforming to the limits is

$$q = \sum_{i=k\times 0.95}^{k} \binom{k}{i} \gamma^i (1-\gamma)^{k-i}.$$

which given $q = 0.7789, 0.9844, 1$ when $k = 1000, 10,000, 100,000$ respectively. Let's consider the case that the specification limits be $LSL = 8.02$ and $USL = 11.98$. We may see that the confidences for the same $k's$ are $0.6676, 0.8642, 1$. $\square$

Let the proportion of acceptable products requiring by the manufacturer be fixed with γ. It is interesting to see, when the specification limits are wider or shorter than the tolerance interval of (2.3), the performances of the true confidence. From (2.2), any production lot has confidence a positive value when its item reliability $p_{item}(\theta)$ is in interval $(0, 1)$. That is, no matter how small the item reliability is, this process has chance to produce proportion γ acceptable products in a lot and, no matter how large ($<$ 1) the item reliability is, this process has chance to produce very small proportion acceptable products in a lot. Then, it is interesting to investigate the sensitivity of the confidence as a function of the item reliability. We set some values of $p_{item}(\theta)$ and proportion γ and list the corresponding confidences q in Table 2.

Table 2. Confidence q for γ-content acceptable products

$p_{item}(\theta)$	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.7	$4.986e-13$	0	0	0	0
0.75	$1.089e-4$	$8.770e-15$	0	0	0
0.8	0.5189	$2.644e-5$	0	0	0
0.85	0.9999	0.5217	$2.038e-06$	0	0
0.9	1	0.9999	0.5265	$5.995e-09$	0
0.95	1	1	1	0.5375	$2.797e-12$
0.99	1	1	1	1	0.5830

We have several conclusions drawn from Table 2:

1. When $p_{item}(\theta)$ is moderately below γ, the confidence q is approximately equal zero and when $p_{item}(\theta)$ is moderately larger than γ, it is approximately equal one.

2. When item reliability $p_{item}(\theta)$ is γ or lesser, the corresponding confidence q is about equal or smaller than 0.6.

3. For given a value of γ, the curve representing the confidence as a function of $p_{item}(\theta)$ is started from zero and rapidly climb up when $p_{item}(\theta)$ is increasing.

4. The interest is that when the true confidence q may be moderately large to ensure proportion γ or more acceptable products in a lot. From the table,

it is the case that item reliability $p_{item}(\theta)$ is larger than γ. This tells us how to improve the process.

3. Minimum Item Reliability for Achieving Proportion γ Acceptable Products with Confidence q_0

The use of tolerance interval to predict the manufacturer's confidence is shown to be too optimistic when the parameters involved in the underlying distribution are known. For case that parameters are unknown, can the classical tolerance intervals be also too optimistic or on the opposite way? To investigate this question, we need to prepare some further results on the relation between item reliability and manufacturer's confidence q.

Recall that the manufacturer expects to have proportion γ or more acceptable products with confidence q_0 or more. Let's define the minimum item reliability that guarantees proportion γ acceptable products with confidence q_0. That is, $p_{\gamma q_0}$ satisfies

$$\sum_{i=[k\gamma]}^{k} \binom{k}{i} (p_{\gamma q_0})^i (1-p_{\gamma q_0})^{k-i} = q_0. \tag{3.1}$$

Consider that the product's characteristic variable has a distribution function F_θ. Then, the manufacturer may expect to have proportion γ acceptable products with confidence q_0 only if the item reliability satisfies

$$p_{item}(\theta) = F_\theta(USL) - F_\theta(LSL) \geq p_{\gamma q_0}. \tag{3.2}$$

For a given pairs (γ, q_0), we list the item reliabilities to achieve exactly proportion γ of acceptable products with confidence q_0 in the following table.

Table 3. Minimum item reliability (k=1,000)

q	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.8	0.8099	0.8587	0.9071	0.9549	0.9918
0.85	0.8123	0.8608	0.9089	0.9562	0.9923
0.9	0.8152	0.8634	0.9111	0.9577	0.9929
0.95	0.8196	0.8673	0.9142	0.9599	0.9938
0.99	0.8277	0.8744	0.9200	0.9638	0.9952

Suppose that the characteristic variable of interest obeys a normal distribution $N(\mu, \sigma^2)$. Then item reliability, the probability that an item conforming to specifications, is

$$p_{item}(\mu, \sigma) = \int_{LSL}^{USL} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

and then the true confidence to have proportion γ acceptable products is

$$q = \sum_{i=[k\gamma]}^{k} \binom{k}{i} p_{item}(\mu, \sigma)^i (1 - p_{item}(\mu, \sigma))^{k-i}$$
$$= \sum_{i=[k\gamma]}^{k} \binom{k}{i} (\Phi(\frac{USL-\mu}{\sigma}) - \Phi(\frac{LSL-\mu}{\sigma}))^i (1 - (\Phi(\frac{USL-\mu}{\sigma}) - \Phi(\frac{LSL-\mu}{\sigma})))^{k-i}.$$

We denote the minimum item reliability as $p_{\gamma q_0}$. A production lot to have proportion γ acceptable products with confidence q_0 requires that

$$p_{item}(\mu, \sigma) = \Phi(\frac{USL-\mu}{\sigma}) - \Phi(\frac{LSL-\mu}{\sigma}) \geq p_{\gamma q_0} \qquad (3.3)$$

where $p_{\gamma q_0}$ may be found from Table 3.

One question very interesting is how short the specification limits that guarantees the minimum item reliability. Let the specification limits be the special form $\{LSL, USL\} = \{\mu - l\sigma, \mu + l\sigma\}$ and we denote $l_{\gamma q_0}$ as the l so that the item reliability produce item reliability exactly as

$$p_{\gamma q_0} = P(\mu - l_{\gamma q_0}\sigma \leq X \leq \mu + l_{\gamma q_0}\sigma).$$

We list $l_{\gamma q_0}$ in this design in the following table.

Table 4. Specification limits $(LSL, USL) = (\mu - l_{\gamma q_0}\sigma, \mu + l_{\gamma q_0}\sigma)$ to achieve item reliability exactly equal to $p_{\gamma q_0}$ (k=1,000)

q	$\gamma = 0.8$	0.85	0.9	0.95	0.99
0.8	1.3103	1.4710	1.6807	2.0043	2.6451
0.85	1.3172	1.4789	1.6898	2.0161	2.6672
0.9	1.3263	1.4889	1.7013	2.0309	2.6953
0.95	1.3397	1.5037	1.7184	2.0531	2.7380
0.99	1.3649	1.5317	1.7509	2.0954	2.8212

This table will be used in next section to study the power of the classical tolerance interval in detection of manufacturer's confidence.

4. Power Study for the Classical Tolerance Intervals

Suppose that we have $X_1, ..., X_n$ representing a random sample for the characteristic variable of interest and the specification limits for the characteristic are $\{LSL, USL\}$. Let (T_1, T_2) be a tolerance interval of Wilks (1941) constructed from the random sample. To test the hypotheses of (1.2), we consider the popularly rule (see Bowker and Goode (1952)) setting: rejecting H_0 if $(T_1, T_2) \subset (LSL, USL)$. Following the classical evaluation of hypothesis testing, we may analogously define a power function, in terms of specification limits, as

$$\pi(LSL, USL) = P_\theta((T_1, T_2) \subset (LSL, USL)), \tag{4.1}$$

which, from Papp(1992), provides the probability of concluding that there is proportion γ or more acceptable products in a lot with confidence $1-\alpha$ at the specified specifications. Basically (4.1) is to evaluate the rule that we conclude that there is proportion γ or more acceptable products with confidence q_0 when $(T_1, T_2) \subset (LSL, USL)$ occurs. Hence, we expects a tolerance interval to have lower power $\pi(LSL, USL)$ when $p_{item}(\theta) < p_{\gamma q_0}$ is true and large power when $p_{item}(\theta) \geq p_{\gamma q_0}$ with $q_0 = 1-\alpha$. We want to simulate the powers for the Eisenhart et al.'s tolerance interval for several combinations of specification limits. We also consider a Monte Carlo study with replication number m. By letting $(LSL, USL) = (-b, b)$, the simulated power of a tolerance interval (T_1, T_2) is defined as

$$\pi = \frac{1}{m}\sum_{j=1}^{m} I((t_1^j, t_2^j) \subset (-b, b)) \tag{4.2}$$

where (t_1^j, t_2^j) is the observation of (T_1, T_2) from the jth sample. The power of (4.2) simulates the chance of (4.1) that the tolerance interval (T_1, T_2) may conclude that the production lot includes a proportion γ of acceptable products with confidence $1-\alpha$.

To study (4.2), suppose that we have a random samaple $X_1, ..., X_n$ drawn from a distribution normal distribution $N(\mu, \sigma^2)$ where both μ and σ are unknown. The general form of a prediction interval for a future normal random variable is of the form

$$(\bar{X} - m^* S, \bar{X} + m^* S) \tag{4.3}$$

where the $100(1-\alpha)\%$ confidence interval (prediction interval) is the form with $m^* = t_{1-\frac{\alpha}{2}}(n-1)\sqrt{1+\frac{1}{n}}$ and where $t_{1-\frac{\alpha}{2}}(n-1)$ represents the $1-\frac{\alpha}{2}$th quantile of the central t-distribution with degrees of freedom. For the Wilks' tolerance interval, Eisenhart et al (1947) developed the shortest one which is now the most popular version of tolerance interval to deal with the manufacturer's problem when the characteristic variable does obey a normal distribution. We select values m^* from the table in Eisenhart et al. (1947). With replication $m = 100,000$, we generate random sample of size n from distribution $N(0,1)$. Let $\bar{x}_j$ and s_j^2 be the sample mean and sample variance for the jth sample. We compute this tolerance interval and study its powers of (4.2) with several sample sizes $n = 20, 30, 50$ and 100 and various values b. The simulated results are listed in Table 5.

Table 5. Powers of the minimum-width tolerance intervals ($(\gamma, 1-\alpha) = (0.9, 0.9)$)

Limits	$n = 20$	$n = 30$	$n = 50$	$n = 100$
(k)	(2.152)	(2.025)	(1.916)	(1.822)
$b = 1.4$ $q = 1.371e - 08$	0.0042	0.0024	0.0006	0.0000
$b = 1.5$ $q = 0.00071$	0.0107	0.0080	0.0041	0.0016
$b = 1.6$ $q = 0.17899$	0.0237	0.0217	0.0187	0.0151
$b = 1.645$ $q = 0.52786$	0.0325	0.0330	0.0328	0.0341
$b = 1.7013$ $q = 0.9$	0.0480	0.0542	0.0646	0.0863
$b = 1.8$ $q = 0.9995$	0.0834	0.1064	0.1484	0.2536
$b = 2.0$ $q = 1.0$	0.2129	0.3032	0.4714	0.7657
$b = 2.2$ $q = 1.0$	0.4043	0.5757	0.8067	0.9790
$b = 2.5$ $q = 1.0$	0.7113	0.8842	0.9860	0.9999
$b = 3.0$ $q = 1.0$	0.9649	0.9970	1.0000	1.0000
$b = 3.5$ $q = 1.0$	0.9986	0.9999	1.0000	1.0000

According to Papp(1992), when a γ-content tolerance interval with confidence $1-\alpha$ is contained in the specification limits, we may claim that there is percentage γ of acceptable products with confidence $1-\alpha$. We then have several conclusions drawn from the simulation results in above table:

(a) When the specification limit interval $(-b, b)$ is wider in sense that value b

is increasing the power for the tolerance interval to detect the manufacturing process to have confidence $1-\alpha$ is increasing.

(b) When $b < 1.7013$ the true confidence for a γ-content acceptable products is less than $1-\alpha = 0.9$. In this situation, the powers of this tolerance interval in all cases of sample size are all less than 0.05. That is, it is fewer than 0.05 to claim to have confidence 0.9.

(c) When $b \geq 1.7013$ the true confidence for a γ-content acceptable products is greater than or equal to $1-\alpha = 0.9$. In this situation, although the powers are increasing when b increases. However, they are still with very low percentages for cases such as $1.7013 \leq b \leq 2.0$ to claim that it is a process of $\gamma = 0.9$-content acceptable products with confidence $1-\alpha = 0.9$. This is a plan too conservative to both the consumer and manufacturer.

(d) If, in the long run, there is a large proportion that the tolerance intervals rejected the null hypothesis H_0, the process must be in well control so that the specification limit interval (LSL, USL) is wide enough to have the tolerance intervals contained in the interval of specification limits.

(e) When there is larger chance that the tolerance interval will claim to have percentage γ of acceptable products with confidence $1-\alpha$. The actual confidence often is larger far from $1-\alpha$. For example, $b = 2.5$ is the situation that there is good chance to claim the manufacturer's expectation with confidence 0.9, however, the confidence is approximately near 1. This indicates that detection of manufacturer's question by the tolerance interval is too conservative.

Alternatively Huang, Chen and Welsh (2005) showed that

$$(\bar{X} - t_{1-\frac{\alpha}{2}}(n-1, \sqrt{n}z_{\frac{1+\gamma}{2}})\frac{S}{\sqrt{n}}, \bar{X} + t_{1-\frac{\alpha}{2}}(n-1, \sqrt{n}z_{\frac{1+\gamma}{2}})\frac{S}{\sqrt{n}}) \tag{4.3}$$

is also a γ content tolerance interval with confidence $1-\alpha$ where $t_\delta(\ell, m)$ is the δ-th quantile of noncentral t-distribution with degrees of freedom ℓ and noncentrality parameter m. The interest is that this tolerance interval is explicitly formulated. By letting $k_t = \frac{t_{1-\frac{\alpha}{2}}(n-1,\sqrt{n}z_{\frac{1+\gamma}{2}})}{\sqrt{n}}$, we list the simulation results of the powers of (4.1) for this tolerance interval in Table 5.

Table 6. Power of coverage-interval tolerance intervals $((\gamma, 1-\alpha) = (0.9, 0.9))$

Limits	$n = 20$	$n = 30$	$n = 50$	$n = 100$
(k_t)	(2.3960)	(2.2198)	(2.0649)	(1.9265)
$b = 1.4$	0.0009	0.0004	0.0001	0.0000
$b = 1.5$	0.0015	0.0009	0.0002	0.0000
$b = 1.6$	0.0065	0.0056	0.0041	0.0024
$b = 1.645$	0.0098	0.0088	0.0081	0.0063
$b = 1.7013$	0.0154	0.0160	0.0162	0.0206
$b = 1.8$	0.0291	0.0337	0.0477	0.0868
$b = 2.0$	0.0887	0.1288	0.2325	0.5043
$b = 2.2$	0.2023	0.3255	0.5669	0.9071
$b = 2.5$	0.4701	0.6979	0.9273	0.9991
$b = 3.0$	0.8675	0.9785	0.9997	1.0000
$b = 3.5$	0.9878	0.9996	1.0000	1.0000

Let's investigate the probability that H_0 will be accepted when there does exist a γ coverage interval contained in specification limits.

$$\frac{1}{m}\sum_{j=1}^{m} I(P_{X_0}\{(t_1, t_2)\} \geq \gamma \text{ and } (t_1, t_2) \subset (-b, b)).$$

Table 7. Power with γ content acceptable products for the minimum-width tolerance intervals $((\gamma, 1-\alpha) = (0.9, 0.9))$

Limits	$n = 20$	$n = 30$	$n = 50$	$n = 100$
(k)	2.152	2.025	1.916	1.822
$b = 1.645$	0.0000	0.0000	0.0000	0.0000
$b = 1.8$	0.0225	0.0373	0.0704	0.1589
$b = 2.0$	0.1270	0.2120	0.3750	0.6661
$b = 2.2$	0.3052	0.4748	0.7033	0.8797
$b = 2.5$	0.6084	0.7845	0.8842	0.9001
$b = 3.0$	0.8608	0.8965	0.8983	0.9002
$b = 3.5$	0.8952	0.8989	0.8983	0.9002
$b = 4.0$	0.8965	0.8989	0.8983	0.9002
$b = 4.5$	0.8966	0.8989	0.8983	0.9002

Table 8. Power with γ content acceptable products for the coverage-interval tolerance intervals $((\gamma, 1-\alpha) = (0.9, 0.9))$

Limits	$n = 20$	$n = 30$	$n = 50$	$n = 100$
(k_t)	2.3960	2.2198	2.0649	1.9265
$b = 1.645$	0.0000	0.0000	0.0000	0.0000
$b = 1.8$	0.0092	0.0149	0.0268	0.0659
$b = 2.0$	0.0598	0.1036	0.2052	0.4808
$b = 2.5$	0.4317	0.6626	0.8986	0.9766
$b = 3.0$	0.8279	0.9473	0.9725	0.9773
$b = 3.5$	0.9504	0.9683	0.9727	0.9773
$b = 4.0$	0.9630	0.9686	0.9727	0.9773
$b = 4.5$	0.9634	0.9686	0.9727	0.9773

5. Confidence Estimation and Its Power for a Process of γ-Content Acceptable Products

Let the interest of characteristic be a random variable having a distribution function F_θ and the specification limits are $\{LSL, USL\}$. We also have a random sample $X_1, ..., X_n$ drawn from this underlying distribution.

Let $\hat{\theta}$ be an estimator of the unknown parameter θ. Replacing F_θ by $F_{\hat{\theta}}$ and from (3.1), we have an estimator of the probability of a product to be acceptable as

$$\hat{p}_{item} = p_{item}(\hat{\theta}) = F_{\hat{\theta}}(USL) - F_{\hat{\theta}}(LSL).$$

For each (γ, q), there is item reliability $p_{\gamma q}$ such that when $p_{item} = p_{\gamma q}$ identity of (3.1) holds. First we want to simulate the efficiencies of estimating the item reliability $p_{item} = p_{\gamma q}$ in Table 3.

Consider that we are dealing with the normal distribution $N(\mu, \sigma^2)$. For each (γ, q) there is $\ell_{\gamma q}$ such that $p_{\gamma q} = P(\mu - \ell_{\gamma q}\sigma \leq X \leq \mu + \ell_{\gamma q}\sigma)$ (see Table 4). When the item reliability is $p_{\gamma q}$ we may guarantee that the confidence of a lot with proportion γ products conforming in specification limits is q. Our aim is to see if the estimate $\hat{p}_{item}$ is efficient for estimating the true reliability $p_{item} = p_{\gamma q}$. Suppose that now this random sample is drawn from a normal distribution $N(\mu, \sigma^2)$ and choose the classical estimators $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. We then have

$$\hat{p}_{item} = \Phi(\frac{USL - \bar{X}}{S}) - \Phi(\frac{LSL - \bar{X}}{S}). \tag{5.1}$$

With the above process of parameter estimation, we perform this simulation with replication $m = 100,000$ and sample size $n = 30$. Let $\hat{p}_{\gamma}^{j}$ be the confidence estimate corresponding to the j-th, $j = 1, ..., m$, observation $x_{1j}, ..., x_{nj}$ drawing from the standard normal distribution $N(0,1)$ where we have then $(LSL, USL) = (-\ell_{\gamma q}, \ell_{\gamma q})$. Let $\hat{p}_{item}^{j}$ be the estimate of the item reliability at the jth replication. We then define the empirical average of ietem reliability estimation as

$$\bar{p}_{item} = \frac{1}{m}\sum_{j=1}^{m} \hat{p}_{item}^{j}.$$

In the following table, we display the simulated results of the averaging item reliability and true reliability $p_{\gamma q}$ in the vector form $\begin{pmatrix} \bar{p}_{item} \\ (p_{\gamma q}) \end{pmatrix}$.

Table 9. Empirical average for the estimation of item reliability $p_{\gamma q}$ when the underlying distribution is normal (k=1,000, m=100,000, n=30)

q	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.9$	$\gamma = 0.95$	$\gamma = 0.99$
0.8	0.8070 (0.8099)	0.8547 (0.8587)	0.9020 (0.9071)	0.9497 (0.9549)	0.9887 (0.9918)
0.85	0.8093 (0.8123)	0.8568 (0.8608)	0.9039 (0.9089)	0.9510 (0.9562)	0.9893 (0.9923)
0.9	0.8121 (0.8152)	0.8594 (0.8634)	0.9061 (0.9111)	0.9526 (0.9577)	0.9900 (0.9929)
0.95	0.8167 (0.8196)	0.8630 (0.8673)	0.9094 (0.9142)	0.9548 (0.9599)	0.9911 (0.9938)
0.99	0.8245 (0.8277)	0.8700 (0.8744)	0.9149 (0.9200)	0.9588 (0.9638)	0.9928 (0.9952)

In the next, we consider the mean square error

$$MSE = \frac{1}{m}\sum_{j=1}^{m}(\hat{p}_{item}^{j} - p_{\gamma q})^2.$$

The simulated results are displayed in the following table.

Table 10. Mean square error for the Estimation of item reliability p_{item} when the underlying distribution is normal (k=1,000, m=100,000, n=30)

q	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.9$	$\gamma = 0.95$	$\gamma = 0.99$
0.8	0.0031	0.0025	0.0017	0.0008	$1.08e-4$
0.85	0.0031	0.0025	0.0017	0.0007	$1.01e-4$
0.9	0.0031	0.0025	0.0016	0.0007	$9.09e-5$
0.95	0.0030	0.0024	0.0016	0.0007	$7.85e-5$
0.99	0.0029	0.0023	0.0015	0.0006	$5.68e-5$

From (3.2), we further have confidence estimator for a process of γ-content acceptable products as

$$\hat{q} = \sum_{i=[k\gamma]}^{k} \binom{k}{i} \hat{p}_{item}^{i}(1-\hat{p}_{item})^{k-i}. \tag{5.2}$$

The estimation of the unknown confidence uses the estimation of unknown parameters θ and lot size where the latter one is an ancillary statistic.

Suppose that now this random sample is drawn from a normal distribution $N(\mu, \sigma^2)$ and choose the classical estimators $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. We then have

$$\hat{p}_{item} = \Phi(\frac{USL-\bar{X}}{S}) - \Phi(\frac{LSL-\bar{X}}{S}).$$

With this, we further have

$$\hat{q} = \sum_{i=[k\gamma]}^{k} \binom{k}{i} (\Phi(\frac{USL-\bar{X}}{S}) - \Phi(\frac{LSL-\bar{X}}{S}))^{i}(1-(\Phi(\frac{USL-\bar{X}}{S}) - \Phi(\frac{LSL-\bar{X}}{S})))^{k-i}. \tag{5.3}$$

Let's study the efficiency of confidence estimation for having a proportion γ or more acceptable products. Let's first fix $\gamma = 0.9$. We generate observation of a random sample from the standard normal distribution and then compute its sample mean $\bar{x}$ and sample standard deviation $s = (\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2)^{1/2}$. With specification limits $(LSL, USL) = (-\ell, \ell)$ and $k = 1,000$, the confidence estimate for this sample is $\hat{p}_\gamma$ of (5.3) with $\bar{X}$ and S be replaced by $\bar{x}$ and s, respectively. We choose $(p_{\gamma q}, \ell)$ from Table 4 that guarantees an exact γ-content acceptable products with confidence q.

With the above process of parameter estimation, we perform this simulation with replication $m = 100,000$ and let $\hat{p}_{\gamma}^{j}$ be the confidence estimate corresponding to the j-th observation $x_{1j}, ..., x_{nj}$ for $j = 1, ..., m$. Now we are interesting in the following average confidence,

$$\bar{p}_{\gamma} = \frac{1}{m} \sum_{j=1}^{m} \hat{p}_{\gamma}^{j}.$$

The following table displays the simulated results of average confidence.

Table 11. Simulated averaging confidences (k=1,000, m=100,000)

Spec L	$n = 20$	30	50	100
$l = 1.4$	0.1664	0.1199	0.0682	0.0217
$l = 1.5$	0.2790	0.2459	0.1915	0.1209
$l = 1.6$	0.4219	0.4085	0.3916	0.3621
$l = 1.7013$	0.5697	0.5902	0.6245	0.6732
$l = 1.8$	0.7043	0.7504	0.8075	0.8881
$l = 2.0$	0.8932	0.9376	0.9758	0.9964
$l = 2.2$	0.9725	0.9910	0.9986	0.9999
$l = 3.0$	0.9999	1	1	1

6. Power Study and Robustness of Lot Size for Confidence estimation Technique

We have seen that the tolerance interval is not really appropriate in detecting the manufacturer's confidence for having a proportion γ of acceptable production lots for that it is too optimistic when the parameters involved in the distribution are known and it is too conservative when the parameters are unknown. It is then interesting to evaluate the manufacturer's confidence through the estimate of the unknown confidence. Consider a simulation that we randomly select observations from the standard normal distribution. We consider replication $m = 100,000$, $\gamma = 0.9$ and confidence $q_0 = 0.9$. By letting $\hat{q}^j$ as the jth estimate of the confidence, $j = 1, ..., m$. We define the power of this point estimator as

$$\pi = \frac{1}{m} \sum_{j=1}^{m} I(\hat{q}^j \geq q_0).$$

The following table list the simulation results of this study.

Table 12. Power for the Estimation of the confidence for having a proportion γ or more acceptable product when the underlying distribution is normal (k=1,000, m=100,000)

Spec L	$n = 20$	30	50	100
$l = 1.4$	0.1180	0.0744	0.0322	0.0049
1.5	0.2161	0.1682	0.1087	0.0429
1.6	0.3344	0.3061	0.2653	0.1927
1.7	0.4802	0.4838	0.4864	0.4850
1.7013	0.4890	0.4928	0.4959	0.5062
1.8	0.6208	0.6555	0.7060	0.7849
1.9	0.7460	0.7999	0.8643	0.9445
2.0	0.8455	0.8978	0.9518	0.9916
2.2	0.9548	0.9821	0.9967	0.9999
2.5	0.9966	0.9995	1.0	1.0
3.0	1.0	1.0	1.0	1.0

We have several conclusions that may be drawn from the results in the above table:

(a) For every given sample size n, the efficiency is strictly increasing in the specification limit l. This fulfills our expectation.

(b) In the setting $\gamma = 0.9$ and $q_0 = 0.9$ by the manufacturer, $l = 1.7013$ guarantees to have proportion γ or more acceptable products with confidence exactly $q = 0.9$. In this situation, the efficiencies are about 0.48. There is probability 0.48 that we can detect that the lot of interest is acceptable. In Table 5, we see that the tolerance interval techniques can observe this fact with chance less than 0.1.

(c) When the specification limit l is a bit wider we have large chance to detect that the lot is acceptable. Comparing this table with Table 5, we see that this technique of confidence estimation is better than the tolerance interval technique casewise.

For this confidence estimation technique, it may be argued that the lot size k may not be correctly predicted or counted and we may have, especially when the size is huge, only an approximate number. Then, it is interesting to see how robust the estimation technique in its estimated confidence when the lot size has not been correctly used. We design a simulation computing the average confidences with various large sizes to see its sensitiveness in terms of lot size.

Table 12. Simulated averaging confidences for some large lot sizes

Spec L	$n = 20$	30	50	100
$k = 100,000$				
$l = 1.5$	0.2645	0.2272	0.1715	0.0918
$l = 1.6474$	0.4756	0.4798	0.4775	0.4808
$l = 1.6482$	0.4783	0.4788	0.4817	0.4823
$l = 1.6507$	0.4787	0.4808	0.4842	0.4894
$l = 1.8$	0.6935	0.7389	0.8030	0.8892
$l = 2.0$	0.8905	0.9365	0.9775	0.9979
$l = 2.5$	0.9981	0.9998	1	1
$k = 300,000$				
$l = 1.5$	0.2669	0.2281	0.1739	0.0947
$l = 1.6474$	0.4787	0.4858	0.4874	0.4858
$l = 1.6482$	0.4807	0.4843	0.4863	0.4906
$l = 1.6507$	0.4850	0.4901	0.4943	0.4996
$l = 1.8$	0.6989	0.7440	0.8090	0.8948
$l = 2.0$	0.8910	0.9375	0.9780	0.9982
$l = 2.5$	0.9985	0.9999	1	1
$k = 500,000$				
$l = 1.5$	0.2723	0.2307	0.1745	0.0965
$l = 1.6474$	0.4802	0.4825	0.4872	0.4892
$l = 1.6482$	0.4827	0.4848	0.4898	0.4939
$l = 1.6507$	0.4865	0.4897	0.4963	0.5028
$l = 1.8$	0.6978	0.7420	0.8080	0.8963
$l = 2.0$	0.8921	0.9387	0.9783	0.9981
$l = 2.5$	0.9984	0.9998	1	1

7. Concluding Remarks

We have several remarks illustrating our further concern and clarification of our study:

(a) We show that the use of tolerance interval to evaluate for the manufacturer the confidence for a proportion γ of production lot conforming to specifications is not appropriate. This does not imply any in-appropriateness for other purposes.

(b) Concerning confidence q as a parameter in terms of unknown parameters θ, there are confidence interval and hypothesis testing that could be developed to infer the unknown q. Concerning the problem, if a process that may generally produce products with confidence q_0 a proportion γ of production lot conforming to specifications, we may want to test the hypothesis that if one other process may also produce the same quality.

(c) The inefficiency of the classical tolerance interval for detecting the manufacturer's confidence partly comes from the fact that it uses only the information contained in the random sample. The lot size which is an ancillary statistic in this case is an important information that hasn't been considered in construction of tolerance interval. This is an example that an ancillary statistic provides important information for statistical inference.

(d) There are tolerance interval-like technique for deciding if we will accept a production lot (see Kirkpatrick (1970), Owen and Hua (1977), Weingarten (1982) and Mee (1984)). This technique does not employ the information of lot size and has not been popular in practical use. Hence it is not in our study.

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