國 立 交 通 大 學

運輸科技與管理學系所

碩士論文

車流動力學之連續減速模式

The Successive Deceleration Model of Kinetic Traffic Flow

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中華民國九十六年七月

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摘要

欲發展良好的交通控制解決交通問題需要有充足的資訊，在相關資訊中最重要且不易估算的就是車流量的預測，車流模式的發展就是朝此方向之研究，關於介觀車流的相關研究國內相關研究仍未完整，尚有許多介觀車流議題需要被討論與研究，所以本研究欲藉由回顧較具重要性的介觀的車流動力學模式，將其重要貢獻與特性整理，並且建構一可較真確描述道路車流情形之介觀車流模式。

本研究引入具有物理意義之減速度考慮項目，放鬆交互影響項中瞬時速度變化與未考慮有限空間因素影響，藉由將這些特性納入並建構一新的介觀車流之動力學方程之中；並且利用動差函數方式將本研究所建構的介觀模式積分，可以獲得以巨觀參數─密度 $c(x,t)$、平均速度 $\bar{v}(x,t)$ 與變異數 $\theta(x,t)$ 為主的巨觀方程式；而後再以特徵速度、均衡解與數值模擬加以分析該模式的特性，並獲得因本研究引入等減速度可比以往研究的模式可多加解釋的特性，比較不同超車機率下不同的加速度值，在低的超車機率下，等減速度會比較有差別，在高的超車機率下，等減速度看不出差別，因為可很自由的超車情況下，使用到等減速度的車輛相形較少，所以對整體較看不出差別。由此可知本研究所建構的介觀車流動力學模式可較真確描地述更多車流情形。另外在撰寫論文時，將本研究的想法與行為意義清楚傳達，並將過程中的數學推導詳細記錄，希冀能對於介觀車流有興趣者有所幫助，以期許將來能有利於實際應用的發展。

關鍵詞：介觀車流；車流動力學；瞬時速度變化

The Successive Deceleration Model of Kinetic Traffic Flow

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Abstract

In order to develop traffic control to solve traffic problem, we need sufficient information. The related and the most important information is to estimate quantity of traffic flow. Developing traffic flow model is one of ways to research. However, the research about mesoscopic traffic flow, gas-kinetic traffic flow, is not yet complete so we want to review some important researches and arrange the focus. And we construct a gas-kinetic traffic flow model in order to describe real traffic more.

In this research, we introduce uniform deceleration with physical meaning to relax instant velocity-changing and not consider finite space in interaction term and construct a new model. Then we use momentum function to obtain three macroscopic models with density $c(x,t)$, average velocity $\bar{v}(x,t)$ and variance $\theta(x,t)$. We use characteristic velocity, equilibrium and numerical simulation to analysis the accent of our model, and we know our model could describe more phenomenon than others.

Compared with different uniform decelerations in different passing probability conditions, we could observe that different uniform decelerations make difference in smaller passing probability. Different uniform decelerations make no difference in bigger passing probability, because cars use few uniform decelerations in dilute traffic. Therefore, our research could describe more traffic phenomenon. Besides, we explain our thought and the meaning of our model clear, and record mathematic processes explicitly. We hope this research could help those who are interested in mesoscopic traffic flow and help the development of real application.

Keyword: Mesoscopic Traffic Flow, Gas-Kinetic Traffic Flow, Instant velocity-changing.

誌　謝

細數在交大所過的日子，一轉眼間就過了六年，時間過得真快，六年的時間中，曾受過大家的照顧、關心與幫助，真的很感謝。首先從身邊的朋友開始，可愛的室友：碗倫、大鳥和芝帆，感謝你們在大學四年的時間裡陪我度過了最美好的宿舍時光，回想每次去吃好湯、唱歌，還有半夜不睡覺一直在聊吃的越聊越餓的日子，真的很有趣，也感謝韻涵，總會熱情支助我們的唱歌休閒團、幫忙我搬家，還會帶吃的給我，真的很謝謝你；也感謝大學同學：郁婷、堉慈、憲梅、佳欣、文誠與敬哲，這一路上不論在學業或者生活上的幫助，還有研究室的夥伴：叮咚、banana、小白雞、至剛、育婷學姊、健綸學長、昱光學長、黃恆學長、宜珊、胖胖還有小捲，感謝大家這段時間的照顧，很懷念實驗室大家一起出外景的日子，防曬乳總是用的特別快，每次出門總是會忘了帶點東西，一起去吃飯、打排球、一起在實驗室苦生計畫等等，雖然日子很苦，但也總覺得大家的人情味是寒冬的溫情，帶來生活的溫馨與樂趣。

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Chapter 1 Introduction

1.1 Research Motivation

In recent decades, the auto industry grows vigorously and the proportion of possession of the car rises year by year because of the factors, such as the demand of economic development. Accounting the materials of The Directorate General of Budget, Accounting and Statistics (DGBAS) of Executive Yuan [1], it shows that the registered car has already been up to 6,380,000 till 2004, there are 0.73 cars in average each family, and the traffic accident piece is up to 137221. There are 72.83 accidents in every ten thousand cars by accounting the vehicle, and average death rate is 1.4 people in every ten thousand cars by accounting the vehicle. Both are higher than in Japan and Britain, etc. Therefore, to develop effective method to control traffic is the important topic in order to solve the traffic problem in our country.

However, it needs sufficient information to develop good traffic control. The most important and difficult to estimate one is the prediction of the flow of car in relevant information, and the development of traffic is the research in this direction. The models of mathematics could be used in the description of various kinds of the systematic physical phenomenon, and the traffic systems include drivers, vehicle and road states, etc. We could built and construct the studies of mathematical models to describe complicated driver's behaviors. Therefore, this research wants to build the way to construct a road traffic flow situation.

The general traffic theory could be divided into three kinds: microscopic, macroscopic and mesoscopic. According to the descriptions of scholars, such as Hoogendoorn and Bovy (2001) [2], etc., Microscopic traffic flow model is describing the relationship between driving behavior, and uses parameters to describe the individual's behavior in detail, for instance: car-following model. Macroscopic traffic flow model based on the relation of flow, velocity and density treats every car in traffic flow which is unable to identify alone with the view of the continuous fluid. The basic theories have average concepts. Mesoscopic traffic flow model is correlated with macroscopic traffic flow model and microscopic traffic flow model. Based on the relation of distance and density, mesoscopic traffic flow model builds on the distances between two cars in microscopic traffic flow model and density in macroscopic traffic flow model. Mesoscopic traffic flow model describes the individual behavior in the form of probability distribution, which basic theories are set out by the dynamics theory.

Among three model of traffic flow, it is difficult to be used in dynamic simulation for carrying the simulation out wastes time. That's because macroscopic behavior is unable to catch and microscopic model has a lot of parameters. Therefore, in order to describe the micro behavior, and offer information of macroscopic behavior, we need to use mesoscopic traffic flow model. Mesoscopic traffic flow model could be said that it is a bridge connecting microscopic traffic flow model and macroscopic traffic flow model.

According to the scholars, such as Hoogendoorn and Bovy (2001) [2], etc., they divide mesoscopic traffic flow model into three kinds: headway distribution model, cluster model and gas-kinetic continuum model. This research wants to use the gas-kinetic equation of mesoscopic traffic flow to develop the foundation of the traffic flow.

1.2 Research Purpose

The traffic flow theory could be divided into three big classes: macroscopic, mesoscopic and microscopic traffic. Our nation has a lot of relevant research about macroscopic and microscopic traffic flow model at present, but the relevant research about mesoscopic traffic is not complete so far. There are still a lot of topics about mesoscopic traffic that need to be discussed and studied. For this reason, by reviewing the more important mesoscopic gas-kinetic traffic flow model, this research puts its important contribution and characteristic in order to make it easier to understand for those who interest mesoscopic gas-kinetic traffic flow model. By showing this research, it may offer some materials to the follow-up topic persons.

The research will focus on the gas-kinetic traffic flow model of mesoscopic traffic flow with the introduction of the physical significance of considerable items. Its properties will be included in and build a new mesoscopic traffic Kinetic equation. It could be more accurate to describe the traffic flow cases, and ease the conditions of mesoscopic traffic equations which have not been relaxed or unreasonable originally. Through mesoscopic model integration, using the GMM function, this research could get Macro-parameters － density $c(x,t)$ 、 average velocity $\bar{v}(x,t)$ and variance $\theta(x,t)$ combining the formula. Then we use the deterministic and numerical simulation to analysis the characteristics of the model. We hope that the establishment of mesoscopic gas-kinetic traffic flow model could be more realistic, and help those whom interested in mesoscopic traffic flow. We even hope that it could be good to real application.

1.3 Research Scope

This research focuses on the gas-kinetic equation of mesoscopic traffic flow, which has a relation between microscopic traffic flow and macroscopic traffic. Its relation could be put in order as the following picture 1-3-1 by Hoogendorn and Bovy (2001) [2].

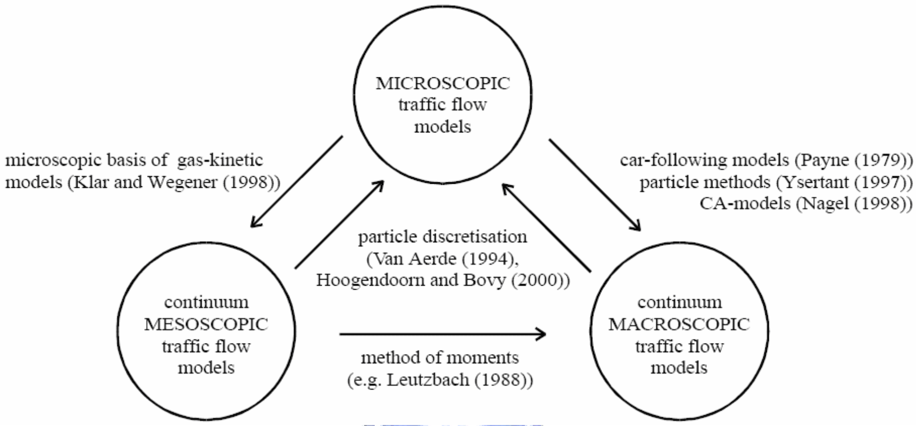

Fig. 1-3-1 The relation of microscopic, mesoscopic, and macroscopic traffic flow model

Data Source: Hoogendorn and Bovy (2001) [2]

This research range is fixed on the behavior factors needed considering in Microscopic. Considering these microcosmic factors, we establish mesoscopic traffic flow model to be macroscopic model via integration. It could be convenient to study its characteristics and be used in on-line dynamic simulation in Macroscopic traffic flow model in future.

Subdividing the research scope, the research scope of this research is as the following picture 1-3-2, and the research scope is fixed on the description of the non-interrupt traffic flow, for example: on freeway.

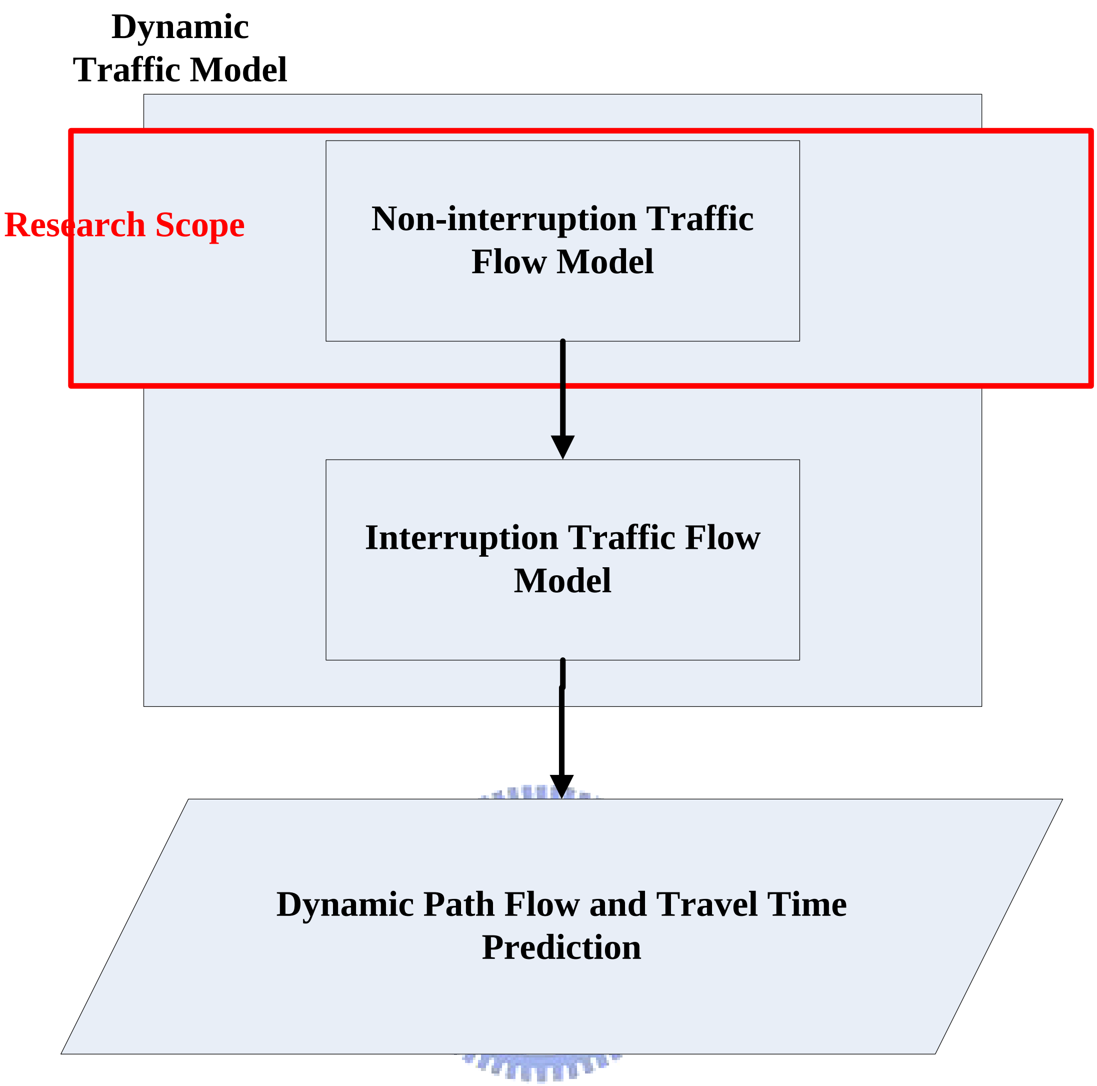

Fig. 1-3-2 Research Scope

Data Source: Cho and Lin (2004) [3]

1.4 Research Procedure

This research procedure is like Fig 1-4-1. Every step is illustrated in detail as following.

1. Describe and define the question

According to the development of mesoscopic traffic flow understands the development condition of gas-kinetic equation in mesoscopic traffic flow at present, submit the topic of the traffic gas-kinetic equation wanted to study, and do an intact description of the topic and define motivation and objection of this research.

2. Collect and review literature

Collect relevant literature probing into traffic gas-kinetic equation of the both in our country and abroad, and review the limiting conditions and contribution of these documents. From the velocity of vehicle assigned function to describe the traffic flow by Prigogine and Andrews (1960) [4] earliest, which set up the young type of gas-kinetic equation, to the follow-up research based on the Prigogine' s and Herman' s (1971) [5] models which are more expanded and improved, this research is by reviewing the literature pluses and minuses to discuss the restriction and suitability of these literature in order to help the traffic gas-kinetic equation of this research to be set up.

3. Construct the model

Build a conceptive mesoscopic gas-kinetic traffic flow model, and describe its every parameter and each physics meaning represented by consulting and collecting the literature and materials according to the question that we describe and define.

4. The deriving in Macroscopic traffic mode

Use the zeroth-order approximation of local equilibrium, and utilize moment function to integrate mesoscopic traffic flow model into Macroscopic traffic mode.

5. Analyze

Analyze the characteristic of the model, for example: characteristic velocity, equilibrium, and etc. Analyze the obtained result, and then analyze the rationality of its result. Last, observe the characteristics of the model by way of numerical simulation.

6. Conclusion and suggestion

Propose conclusion and suggestion to the course and result of this research.

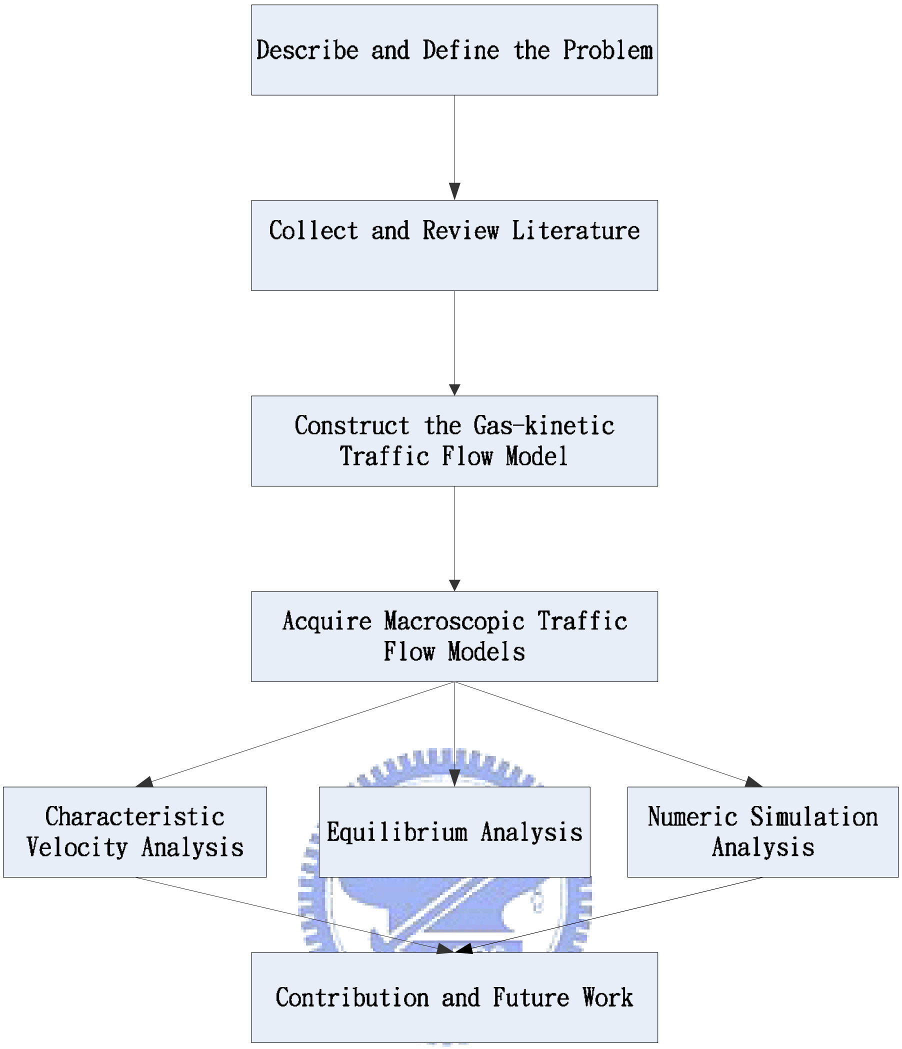

Fig. 1-4-1 The flowchart of This Research

Chapter 2 Literature Review

In this chapter, we go through the literature on the subject of kinetic traffic flow theory to understand the development of mesoscopic traffic flow at present. Then we set the topic to research. There are four sections. In the first section, we review one-lane and one-class traffic flow model. In the second section, we review multilane and multiclass traffic flow model. And we rearrange papers into figures in the third section. Finally, we conclude the results of the research topic.

2.1 One-Lane and One-Class

Gas-kinetic continuum model is proposed by Prigogine and Andrews (1960) [4] who describe traffic flow with velocity distribution function. Then Prigogine and Herman (1971) [5] arrange those papers about gas-kinetic continuum models into a book. Their model is assumed to be in an infinite freeway with low density, it does not considered interaction between drivers. There is a velocity distribution function $f(x,v,t)$ given certain time t and space x. Their kinetic traffic flow model could be representing as the following.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial t}\right)_{rel} + \left(\frac{\partial f}{\partial t}\right)_{\text{int}} \tag{2.1}$$

Where, the first term of the right-hand-side of the equation of (2.1) is "Relaxation Term". It results from the different between real and desired velocity, so it could be shown as the equation (2.2) with exponential law.

$$\left(\frac{\partial f}{\partial t}\right)_{rel} = -\frac{f - f_0}{T} \tag{2.2}$$

Where, $f_0(x,v,t)$ means desired velocity distribution function (Prigogine, 1961) [6], and it means the distribution of velocities that drivers want to drive. And the second term of right-hand-side is "Interaction Term".

$$(\frac{\partial f}{\partial t})_{\text{int}} = f(x,v,t)c(x,t)[\bar{v}(x,t)-v](1-p) \tag{2.3}$$

Where, p means passing probability, $c(x,t)$ means density, $\bar{v}(x,t)$ means average velocity. The interaction term shows that the faster (the latter) one car meets the other slower (the former) one without passing, the faster one needs to decelerate in order to avoid collision. Hence, the model of Prigogine and Herman (1971) [5] is the following.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{f - f_0}{T} + (1-p)c(\bar{v}-v)f \tag{2.4}$$

They use velocity distribution function $f(x,v,t)$ to show main functions of macroscopic traffic flow, local density and flow.

Local density function

$$c(x,t) = \int_0^\infty dvf(x,v,t) \tag{2.5}$$

Local flow function

$$q(x,t) = c(x,t)\bar{v}(x,t) = \int_0^\infty dvvf(x,v,t) \tag{2.6}$$

Dispersion of velocity

$$c\overline{(v-\bar{v})^2} = \int_0^\infty dv(v-\bar{v})^2 f(x,v,t) \tag{2.7}$$

Where, $f(x,v,t)$, $c(x,t)$, and $q(x,t)$ are continue functions. We could know that velocity distribution function $f(x,v,t)$ is a main role in mesoscopic traffic flow and it also relates microscopic and macroscopic traffic flow.

After Prigogine and Herman (1971) [5], there are many scholars improve their model. Paveri-Fontana (1975) [7] is one of them, and he considers Phase-Space Density (PSD) $g(x,v,w,t)$. Where, w means desired velocity. He replaces velocity distribution $f(x,v,t)$ and desired velocity distribution function $f_0(x,w,t)$ with PSD, showing as the followings.

$$f(x,v,t) = \int_0^\infty dwg(x,v,w,t) \tag{2.8}$$

$$f_0(x,w,t) = \int_0^\infty dvg(x,v,w,t) \tag{2.9}$$

In Pavery-Fontana's (1975) [7], he proposes a lot of shortcomings of the model of Prigogine and Herman (1971) [5] as the followings.

1. The constant relaxation time of relaxation term is unreasonable.
2. In interaction term
   - I When the faster car passes the slower one, the velocity of the faster car does not change.
   - II When the faster car passes the slower one, the velocity of the slower one does not change.
   - III Ignore the effect of car length.
   - IV The faster one car meets the other slower one without passing, the instant velocity-changing is unreasonable.
   - V They only consider the interaction of two car, not many cars.
   - VI The passing process, one car passes one car or one platoon, is unreasonable.
   - VII Assume two cars are vehicular chaos.

There are many scholars try to relax those above unreasonable assumptions.

2.2 Multilane and Multiclass

According to the section 2.1, we review the models of Prigogine and Herman (1971) [5] and Papery-Fontana (1975) [7]. Many researches depend on those above. Gas-kinetic traffic flow model advances to multilane and multiclass one. In this section, we first review the multilane model of Helbing (1997) [8-9]. He defines j as lane index, $g_j(x,v,w,t)$ as ML-PSD of j lane.

$$\partial_t g_j + v\partial_x g_j = (\partial_t g_j)_{rel} + (\partial_t g_j)_{\text{int}} + (\partial_t g_j)_{vc} + (\partial_t g_j)_{lc} + n_j^+ - n_j^- \quad (2.10)$$

Where, the first term of the right-hand-side $(\partial_t g_j)_{rel}$ is "Relaxation Term" and the second term $(\partial_t g_j)_{\text{int}}$ is "Interaction Term", and those meanings are the same as the previous section. The third term of the right-hand-side $(\partial_t g_j)_{vc}$ is "Velocity Diffusion Term", it describes the difference of velocities results to a perturbation of individual velocity. The fourth term of the right-hand-side $(\partial_t g_j)_{lc}$ is "Lane-Changing Term", it means the change of ML-PSD results from changing lane. The most right term $n_j^+ - n_j^-$ means the rate of car flow-in and flow-out.

Then, Hoogendoorn and Bovy (2000,001) [10-11] introduce attributes into models, they separate attribute $r = (b,k)$ into continue attribute b and discrete attribute k.

Continue attribute b:

1. The absolute value of desired velocity v^0 (not contain direction)

2. The angle of desired moving ω^0

3. The acceleration time T (it is also called relaxation time): It means that the time that a driver accelerate from their resent velocity or angle to their desired velocity or angle.

Discrete attribute k :

1. Lane j : Because of traffic law, control, and geometric design of road and so on, those reasons would affect the different behavior of driver on different lanes.

2. Driver u : Traffic status of drivers could be car, bus, van, bicycle, pedestrian and so forth. In the other hand, it could be described travel factors, entertainment trip, commercial trip, commutation trip and so on.

3. Driving Status s : There are two kinds of status in this research, free driver $s=1$ and driver in platoon $s=2$. This index could improve the interaction term of Pavery-Fontana (1975) [19], one car passes one car or one platoon.

4. Direction of Traffic Flow h : Many directions of traffic flows use the same space. For example, intersection.

5. Destination d : The different destination often affects driving behaviors. For example: In freeway, drivers would choose different gateways because of different destinations. It could leads to lane-changing.

Hoogendoorn and Bovy(2000,001) [10-11] introduce PSD $g(t,x,v,r)$ which mix continue and discrete attributes and they obtain gas-kinetic traffic flow models as the following.

$$\partial_t g_r + \overbrace{\nabla_x \cdot (g_r v)}^{(I)} + \overbrace{\nabla_v \cdot (g_r A_r)}^{(II)} + \overbrace{\nabla_{v^0} \cdot (g_r B_r)}^{(III)} = \overbrace{(\partial_t g_r)_{event}}^{(IV)} + \overbrace{(\partial_t g_r)_{cond}}^{(V)} \qquad (2.11)$$

Where, $\nabla_x = (\partial_{x1} \cdots \partial_{xn})$, $\nabla_v = (\partial_{v1} \cdots \partial_{vn})$ and $\nabla_{v^0} = (\partial_{v^0_1} \cdots \partial_{v^0_n})$, A_r means acceleration function, B_r means acceleration behavior.

I Convection: It is fundamental to balance flow-in and flow-out cars.

II Relaxation: It is also called " Acceleration Term", and it means the effect result from that current velocity accelerates to desired velocity.

III Adaptation of Continuous Attributes: This term describes effect results from continue attributes changing.

IV Event-based Non-continuum Processes: It means that effect results from continue and discrete attributes changing discretely by events. For instance, deceleration and lane-changing and so on.

V Condition-based Non-continuum Processes: It means that effect results from continue and discrete attributes changing discretely with certain conditions. For instance, spontaneous or postponed lane-changing, status changing (free or in platoon) and so forth.

2.3 Rearrange Papers

Since Prigogine and Herman (1971) [5] propose gas-kinetic traffic flow model, many scholars try to improve the limitations of their models step by step. Especially, Paveri-Fontana (1975) [7] introduce desired velocity as an independent variable into gas-kinetic traffic flow model and he points out the shortcomings of Prigogine and Herman (1971) [5]. The latter scholars' research base is on his paper. The following figure 2-3-1 shows that the more important contains of recent ten-year gas-kinetic traffic flow models.

Fig. 2-3-1 The More Important Contains of Recent Ten-Year Gas-Kinetic Traffic Flow Models

Scholars	Main Contributions
Takashi Nagatani (1997)[12]	He improves the gas-kinetic traffic flow model of Paveri-Fontana(1975)[7] according to the density of the former space, not the space where the car is to assume desired velocity. He represents velocity distribution function as discrete form. This research shows that there is a great difference between flow-in and flow-out the heavy traffic.
Helbing (1997) [9]	He constructs multilane gas-kinetic traffic and considers the behavior of lane-changing, the velocity diffusion owing to improper driving, and finite space related to car length.
C. Wanger (1997)[13]	He relaxes the instant velocity-changing with the average velocity between the former car and the latter one, and obtains macroscopic model to analysis numerical simulation.

Klar and Wegener (1999)[14]	He considers lane-changing, braking and acceleration, and sets the reaction micro threshold to construct gas-kinetic traffic flow model. Its velocity distribution function is considered the relation between the former and the latter car. He derives macroscopic models from the gas-kinetic traffic flow model.
Hoogendoorn and Bovy (2000, 2001) [10-11,15]	They consider continue and discrete attributes into constructing gas-kinetic traffic flow model which contains many factors as multilane, multiclass and so on. They also consider the situation of platoon and assume the car in platoon accelerates to the leader of platoon. They solve the interaction times which overestimate or underestimate, no relation with near cars, and the situation of dilute traffic, with considering platoon. Besides the traditional convection, relaxation and interaction terms, they also introduce the effects of adaptation of continuous attributes, event-based non-continuum and condition-based processes.
Cho and Lo (2002) [16]	They construct a self-consistent multiclass and multilane traffic flow model from traffic Boltzmann equation and traffic diffusion model ,and introduce two-dimension space and field. They propose that if the model that is only considered individual velocity to derive second-order momentum function has no physical meanings. It is also needed to consider equilibrium velocity.

The following figure 2-3-2 shows that several important researches rearrange on dimensions, scales, processes and operations. Where, dimensions (except time and space) contain velocity v, desired velocity w, horizontal position (lane) y and the others o. Scales contain continue c and discrete d. Processes contain deterministic d and stochastic s. Helbing (1997) [9], Hoogendoorn and Bovy (2001) [11] represent lanes with low index of velocity distribution function. Cho and Lo (2002) [16] consider two-dimensions (x, y) to vecor space $\boldsymbol{x}$. The other dimensions of Hoogendoorn and Bovy (2001) [11] are class and driving status (free traffic or in platoon) ,and the other dimension of Cho and Lo (2002) [16] is class.

Fig. 2-3-2 Rearrange the Characteristics of Important Papers

Gas-Kinetic Traffic Flow Models	**Dimensions**				**Scales**	**Processes**
	v	**w**	**y**	**o**		
Prigogine and Herman (1971) [5]	+				c	d
Paveri-Fontana (1975) [7]	+	+			c	d
Helbing (1997) [9]	+	+	+		c	d
C. Wanger (1997) [13]	+	+			c	d
Hoogendoorn and Bovy (2001) [11]	+	+	+	+	c d	d
Cho and Lo (2002) [16]	+		+	+	c	d

Dimensions (Except time and space): velocity v, desired velocity v_0, horizontal position (lane) y, and the others o.

Scales: continue c and discrete d.

Processes: deterministic d and stochastic s.

2.4 Summary

In this chapter, we summarize researches about mesoscopic traffic flow models and we could find that recent researches have improved to multilane and multiclass traffic flow models. However, there are many shortcomings that exist in the first model of Prigogine and Herman (1971)[5] do not improve, so the application is limited. These are needed to be improved.

Therefore, we want to try to relax the shortcomings of Prigogine and Herman (1971) [5]. Because the shortcoming that the faster one car meets the other slower one without passing, the instant velocity-changing is unreasonable is not completely relaxed, we try to relax in this researches. C. Wanger (1997) [13] has been tried to research on this topic, but he does not completely relax. The moving behavior of his model does not match the space changing. In order to relax the instant velocity-changing and to consider reasonable behavior, we also consider finite space in our model. This means that velocities change with the positions of cars. Hence, we would construct a gas-kinetic traffic flow model to describe more completely in next chapter.

Chapter 3 Construct Mesoscopic Traffic Flow Model

We could find that the model of Prigogine and Herman (1971) [5] still has some shortcoming needed to be improved with literature review. In this chapter, we want to relax assumptions of the interaction term of Prigogine and Herman (1971) [5], instant slowing-down and no consider finite space, to obtain more realistic traffic flow model.

3.1 The Question Description

Gas-kinetic traffic flow model has already been developed in decades, and a lot of them are based on Prigogine and Herman (1971) [5], and Paveri-Fontana (1975) [7]. Some research focuses on topic, multi-lane or multi-class, and the other focuses on relaxing the unreasonable assumptions of Prigogine and Herman(1971) [5], which are proposed by Paveri-Fontana (1975) [7] (As what we show in Chapter 2). This research does the latter. According to Chapter 2, literature review, we found few literatures about relaxing the instant slowing-down of interaction term. There has no more complete one. Therefore, we want to construct a model which could describe successive velocity-changing and finite space effect. We hope it could be more meaningful than previous ones.

We assume the same conditions as Prigogine and Herman (1971) [5]. Assume that there is an infinite long freeway, allowing passing. If the driver could not pass the former car, its interaction term (slowing-down behavior) could be described as the following equation (3.1)

$$\sum_j (1-p)[\, f_i \int_{v_i}^{\infty} dv_j\, f_j (v_j - v_i) - f_i \int_0^{v_i} dv_j\, f_j (v_i - v_j)] \tag{3.1}$$

In the above equation (3.1), car i means the car which we care, car j means other car which interact with car i (only consider two-cars interaction), and p means passing probability. So $(1-p)$ means the probability without passing. The equation (3.1) shows that the faster (the latter) one car meets the other slower (the former) one without passing, the faster one needs to decelerate in order to avoid collision. Hence, they assume that it decelerates its velocity to the velocity of the slower one. The first term of the equation (3.1) shows the condition the latter car (the faster), its velocity is v_j, meets the former car (the slower), its velocity is v_i. The second term shows the opposite condition, the latter car (the faster), its velocity is v_i, meets the former car (the slower), its velocity is v_j.

According to the section 2.1, we could know it has some shortcoming in the equation (3.1). Our research is main to relax instant slowing-down assumption of the interaction term and to relax no consideration of finite space. If the equation does not consider instant velocity-changing in slowing-down process, the headway between two cars should change with the velocity-changing of the later car. Therefore, we need to consider finite space in constructing model of continuous velocity-changing. We would construct a model that could describe the traffic situation of non-instant slowing-down and finite space consideration in next section.

3.2 Model Construction

In this section, the assumptions we used is the same as what Prigogine and Herman (1971) [5] did. Assume that there has an infinite long freeway, allowing passing, one car type, and each driver drives its different velocity and has the same desired velocity (cars of the same type have the same desired velocity). After passing, the car contains its origin velocity (the velocity before passing). If the faster (the latter) one car meets the other slower (the former) one without passing, the faster one needs to decelerate in order to avoid collision. We introduce the model of C. Wanger (1997)

[13] to improve the shortcoming, instant velocity-changing and no finite space consideration, of Prigogine and Herman (1971) [5] and to construct our model.

We use velocity distribution $g(x,v,w,t)$ defined by Paveri-Fontana (1975) [7]. Among it, w means desired velocity, and the relation between $g(x,v,w,t)$, $f(x,v,t)$ and $f_0(x,w,t)$ could be appeared like the following equation

$$f(x,v,t) = \int_0^{\infty} dwg(x,v,w,t) \tag{3.2}$$

$$f_0(x,w,t) = \int_0^{\infty} dvg(x,v,w,t) \tag{3.3}$$

According to the models of Paveri-Fontana (1975) [7] and C. Wanger (1997) [13], we modify the model of Prigogine and Herman (1971) [5]. Then our model is shown as below the equation (3.4).

$$\begin{aligned}
&\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} + \frac{\partial}{\partial v}\left(\frac{w-v}{T}g\right) \\
&\quad = \iint_{0 \le v_1 \le v_3} dv_1 dv_3 (1-p)(v_3 - v_1) f(x + d(v_1,v_3), v_1, t) g(x, v_3, w, t) \delta(v - \Phi(v_1,v_3)) \\
&\qquad - g(x,v,w,t) \iint_{0 \le v_1 \le v} dv_1 dv_2 (1-P)(v - v_1) f(x + d(v_1,v), v_1, t) \delta(v_2 - \Phi(v_1,v))
\end{aligned} \tag{3.4}$$

Where

$$d(v_1,v_3) = \tau v_1 + \frac{1}{c_{\max}} + \frac{1}{2}\frac{(v_3 - v_1)^2}{a} \tag{3.5}$$

$$d(v_1,v) = \tau v_1 + \frac{1}{c_{\max}} + \frac{1}{2}\frac{(v - v_1)^2}{a} \tag{3.6}$$

$$\Phi(v_3,v_1) = v_3 - a * \lambda \tag{3.7}$$

$$\Phi(v,v_1) = v - a * \lambda \tag{3.8}$$

Where the uniform deceleration $a \ge 0$, τ means reaction time, T means relaxation time, $c_{\max}$ means the heaviest density, λ means the time interval that the same each time interval of the latter car in deceleration process passes through. And $d(v_1, v_3)$ and $d(v_1, v)$ means the headway between the later and the former cars, p means passing probability, $(1-p)$ means probability without passing, $\Phi(v_1, v_3)$ and $\Phi(v_1, v)$ means the next velocity of the later car when it starts to decelerate.

Our mesoscopic traffic flow model constructs as the equation (3.4), where the left-hand side first and second terms, described the velocity-changing is caused by flow-in and flow-out, are called "convection term". The left-hand-side third term is called "relaxation term", it is also called "acceleration term". Because velocities of drivers are often different with their desired velocities, they would want to accelerate to their desired velocities. The relaxation term we follow the exponential law assuming by Prigogine and Herman (1971) [5] to show its effect in velocity distribution. The right-hand-side term of equation (3.4) is called "interaction term", and it means that the faster (the latter) one car meets the other slower (the former) one without passing, the faster one needs to decelerate in order to avoid collision. The first term of interaction term means the latter car v_3 meets the former one v_1, $v_1 < v_3$. According to the assumption, the latter car v_3 should decelerate to the former car without passing. $d(v_1, v_3)$ means the headway, so the headway should match the equation (3.5), the very two cars start to interact. By the way, the equation (3.7) shows the velocity through changing in the interaction process. It depends on the difference of velocities between two cars, the uniform deceleration and time interval.

In the same way, the second term of interaction term means the latter car v meets the former one v_1, $v_1 < v$, without passing, the faster one needs to decelerate in order to avoid collision. $d(v_1, v)$ means the headway, so the headway should match the equation (3.6), the very two cars start to interact. By the way, the equation (3.8) shows the velocity through changing in the interaction process. It depends on the difference of velocities between two cars, the uniform deceleration and time interval.

The second term of interaction term could also describe that successive deceleration affects $\frac{\partial g}{\partial t}$ to decrease.

The main characteristics of our model are shown as the following:

1. Headway: In this research, we express the headway between the very two car we care as the equations (3.5) and (3.6). In other researches likes [13, 17-19], they only consider the safety distance $\tau v + \frac{1}{c_{\max}}$ or $\tau v + \ell$, if they introduce the factor " finite space" into their model. There ℓ means car length. However, if velocity changes step by step, not instant changing, the above safety distance is not safe. The distance through decelerating process should be considered. If we take uniform deceleration formula to show the decelerating process, the headway should be shown as (reaction time*the velocity of the former car + the minimum headway + the distance required by deceleration). There "reaction time*the velocity of the former car + the minimum headway" is left to be safety distance for decelerating process. Hence, the headway should be represented the equation (3.5) and (3.6), and correspond to the safety distance in non-instant velocity-changing condition.

2. Velocity changing: We show the process of deceleration when the later (the faster) car meets the former (the slower) one as the equations of (3.7) and (3.8). We assume cars decelerate by uniform deceleration, so the velocity changing is related to the uniform deceleration. The velocity of the later car and the headway match the assumption of uniform deceleration. Therefore, the model could more reasonably describe drivers' behavior.

3. No adjust factor: In the past, scholars take adjust factor with considering finite space in their researches. Because of instant velocity-changing, they need to take

adjust factor to increase the frequency of interaction in considering finite space. However, our model is both considered finite space and successive velocity-changing, so it does not need to multiply an adjust factor.

3.3 Summary

We construct a standard and simple mesoscopic traffic flow model under the same assumptions as Prigogine and Herman (1971) [5], and Paveri-Fontana (1975) [7]. And we consider more, containing finite space and successive deceleration, in interaction term. If the headway satisfies the equations (3.5) or (3.6), the later (the faster) car would start to decelerate with constant uniform deceleration a. Hence, the velocity through deceleration process could be shown as the equations (3.7) and (3.8). The condition to occur interaction has been set in our model, and the deceleration process has also been regular. The situation of interaction would not happen at casual.

Chapter 4 Macroscopic model

The advantage of mesoscopic traffic flow model is that it could describe behaviors of microscopic traffic flow, and it could get macroscopic model by means of integration. The previous chapter constructs the mesoscopic traffic flow model by describing microscopic cars so this chapter uses microscopic traffic flow which chapter3 constructs to obtain macroscopic traffic flow model, flow conservation, equation of average velocity, variance, by the methods like integrations. It could be good to analyze the characteristic in the later chapter.

4.1 Assumption

This research is assumed for the single car which is a simple situation. According to the assumption of Helbing (1995) [19, 20], non-equilibrium state could approach equilibrium quickly with the zeroth-order approximation of local equilibrium. According to the research [19, 21], we could know the velocity distribution of equilibrium approaching normal distribution. The velocity distribution could be expressed by the following

$$\begin{aligned} f(x,v,t) &\approx f_e(\bar{v}(x,t),\theta(x,t)) \\ &= \frac{c(x,t)}{\sqrt{2\pi\theta(x,t)}}\exp\{-[v-\bar{v}(x,t)]^2/[2\theta(x,t)]\} \end{aligned} \tag{4.1}$$

In the above equation (4.1), $f_e(\bar{v}(x,t),\theta(x,t))$ shows the velocity distribution in equilibrium, $\bar{v}(x,t)$ shows velocity of equilibrium, and $\theta(x,t)$ shows variance. Using equation (4.1) to get macroscopic model, the following are common skills

$$\int_{-\infty}^{\infty} dz z^{2k+1}\exp(\frac{-z^2}{4\theta}) = 0$$

$$\int_{-\infty}^{\infty} dz z^{2k}\exp(\frac{-z^2}{4\theta}) = \left[(-2)^{2k}+4^k\right]\theta^{k+1/2}\Gamma(\frac{1}{2}+k)$$

$$\int_{0}^{\infty} dz z^{2k+1}\exp(\frac{-z^2}{4\theta}) = 2\cdot 4\cdots 2k(2\theta)^{k+1}$$

$$\int_0^\infty dz z^{2k} \exp(\frac{-z^2}{4\theta}) = 1\cdot 3 \cdots (2k-1)\sqrt{\frac{\pi}{2}}(2\theta)^{k+1/2} \tag{4.2}$$

Besides, we use Tayler expansion to expand space so we would use $\frac{\partial f}{\partial x}$, and we set $z = v + w - 2\bar{v}(x,t)$, $y = v - w$, and then $\frac{\partial f}{\partial x}$ could be expressed as following

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\{\frac{c}{\sqrt{2\pi\theta}}\exp[\frac{-(z-y)^2}{4\cdot 2\theta}]\} \\
&= \frac{\partial c}{\partial x}\frac{1}{\sqrt{2\pi\theta}}\exp[\frac{-(z-y)^2}{8\theta}] + \frac{c}{\sqrt{2\pi}}(\frac{-1}{2})\theta^{-3/2}\frac{\partial\theta}{\partial x}\exp[\frac{-(z-y)^2}{8\theta}] \\
&\quad + \frac{c}{\sqrt{2\pi\theta}}\frac{(z-y)^2}{8}\theta^{-2}\frac{\partial\theta}{\partial x}\exp[\frac{-(z-y)^2}{8\theta}] \\
&\quad + \frac{c}{\sqrt{2\pi\theta}}\frac{-2(z-y)}{8}\theta^{-1}\frac{\partial z}{\partial x}\exp[\frac{-(z-y)^2}{8\theta}] \\
&= \left\{\frac{\partial c}{\partial x} - \frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x} + \frac{(z-y)^2}{8}c\theta^{-2}\frac{\partial\theta}{\partial x} - \frac{(z-y)}{4}\frac{\partial z}{\partial x}c\theta^{-1}\right\}\frac{1}{\sqrt{2\pi\theta}}\exp[\frac{-(z-y)^2}{8\theta}]
\end{aligned} \tag{4.3}$$

In order to use symbols conveniently, this chapter would use some numbers of variables to define as following

Average Velocity

$$\bar{v}(x,t) = \int dv v \frac{f(x,v,t)}{c(x,t)} \tag{4.4}$$

Velocity Variance

$$\theta(x,t) = \int dv (v - \bar{v}(x,t))^2 \frac{f(x,v,t)}{c(x,t)} \tag{4.5}$$

Traffic Pressure

$$\begin{aligned}
\mathrm{P}(x,t) &= \frac{1}{c(x,t)}\int dv(v-\bar{v}) f(x,v,t)\int dw(v-w) f(x,w,t) \\
&= \int dv(v-\bar{v})^2 f(x,v,t) \\
&= c(x,t)\theta(x,t)
\end{aligned} \tag{4.6}$$

Average Desired Velocity

$$\bar{v}_0(x,t) = \int dv \int dw w \frac{g(x,v,w,t)}{f(x,v,t)} \tag{4.7}$$

Flux of Velocity Variance

$$\begin{aligned} J(x,t) &= \frac{1}{c(x,t)}\int dv(v-\bar{v}(x,t))^2 f(x,v,t)\int dw(v-u)f(x,u,t) \\ &= \int dv(v-\bar{v}(x,t))^3 f(x,v,t) \end{aligned} \tag{4.8}$$

Covariance

$$\begin{aligned} C(x,t) &= \int dv\int dw(v-\bar{v}(x,t))(w-\bar{v}_0(x,t))\frac{g(x,v,w,t)}{c(x,t)} \\ &= \int dv(v-\bar{v}(x,t))(\tilde{v}_0(x,v,t)-\bar{v}_0(x,t))\frac{f(x,v,t)}{c(x,t)} \end{aligned} \tag{4.9}$$

where

$$\tilde{v}_0(x,v,t) = \int dww\frac{g(x,v,w,t)}{f(x,v,t)} \tag{4.10}$$

The zeroth-order approximation of local equilibrium is close to flux of velocity variance $J(x,t) \approx J_e(x,t) = 0$. Using the zeroth-order approximation of local equilibrium to get velocity variance $J(x,t)$, we could adapt $J(x,t)$ for the following.

$$\begin{aligned} J(x,t) &= \frac{1}{c(x,t)}\int dv(v-\bar{v}(x,t))^2 f(x,v,t)\int dw(v-u)f(x,u,t) \\ &= \frac{1}{c(x,t)}\int_{-\infty}^{\infty} d(\delta v)(\delta v)^2 f(x,\bar{v}+\delta v,t)\int_{-\infty}^{\infty} d(\delta u)(\delta v-\delta u)f(x,u+\delta u,t) \\ &= c(x,t)*2\int_{-\infty}^{\infty}dz\int_0^{\infty}dy(\frac{y+z}{2})^2 y\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\pi\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\pi\theta}) \\ &= c(x,t)\left[\frac{1}{2}\frac{1}{2}(4\theta)^2\frac{1}{\sqrt{4\pi\theta}}+0+\frac{1}{2}2\theta(4+4)\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta}\right] \\ &= c(x,t)\left[\frac{2\theta^{3/2}}{\sqrt{\pi}}+\frac{\theta^{3/2}}{\sqrt{\pi}}\right] \\ &= c(x,t)\frac{3\theta^{3/2}}{\sqrt{\pi}} \end{aligned} \tag{4.11}$$

We assume $v = \bar{v}(x,t)+\delta v$ and $w = \bar{v}_0(x,t)+\delta w$ to change the first column of RHS to the second column of RHS in equation (4.11). We assume $z = \delta v + \delta w$ and $y = \delta v - \delta w$, where Jacobin=1/2, from the second column to the third column, and use the zeroth-order approximation of local equilibrium. We could get flux of velocity variance $J(x,t)$ by the skills of equation (4.2).

Because of $J(x,t) \approx J_e(x,t) = 0$ and the equation (4.11), the high order could be ignored if θ^n, $n \geq 3/2$. In the following section, we could expect that macroscopic model we got is the Euler-like traffic equations.

4.2 Density Equation

The mesoscopic traffic flow model could get macroscopic model by the method of moment. We deal mesoscopic model with zeroth-order momentum in this section. That means we would deal mesoscopic model with integration and get macroscopic equation based on the macroscopic parameter, density $c(x,t)$.

Our mesoscopic mode would integrate velocity v and desired velocity w.

1. **Convection term integrates velocity v and desired velocity w.**

$$\begin{aligned}
&\int dv \int dw \frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} \\
&= \int dv \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} \\
&= \frac{\partial c}{\partial t} + \frac{\partial (c\bar{v})}{\partial x}
\end{aligned}$$

2. **Relaxation term integrates velocity v and desired velocity w.**

$$\int dv \int dw \frac{\partial}{\partial v}\left(\frac{w-v}{T} g\right) = 0$$

3. Interaction term integrates velocity v and desired velocity w.

$$\begin{aligned}
&\int dv \int dw[\iint_{0\le v_1\le v_3} dv_1 dv_3 (1-p)(v_3-v_1) f(x+d(v_1,v_3),v_1,t) g(x,v_3,w,t)\delta(v-\Phi(v_1,v_3)) \\
&\qquad - g(x,v,w,t) \iint_{0\le v_1\le v} dv_1 dv_2 (1-P)(v-v_1) f(x+d(v_1,v),v_1,t)\delta(v_2-\Phi(v_1,v))] \\
&= \int dv \iint_{0\le v_1\le v_3} dv_1 dv_3 (1-p)(v_3-v_1) f(x+d(v_1,v_3),v_1,t) f(x,v_3,t)\delta(v-\Phi(v_1,v_3)) \\
&\qquad - \int dv\, f(x,v,t) \iint_{0\le v_1\le v} dv_1 dv_2 (1-P)(v-v_1) f(x+d(v_1,v),v_1,t)\delta(v_2-\Phi(v_1,v)) \\
&= \int dv_2 \iint_{0\le v_1\le v_3} dv_1 dv_3 (1-p)(v_3-v_1) f(x+d(v_1,v_3),v_1,t) f(x,v_3,t)\delta(v_2-\Phi(v_1,v_3)) \\
&\qquad - \int dv_3\, f(x,v,t) \iint_{0\le v_1\le v_3} dv_1 dv_2 (1-P)(v_3-v_1) f(x+d(v_1,v_3),v_1,t)\delta(v_2-\Phi(v_1,v_3)) \\
&= 0
\end{aligned}$$

Change the first equal sign to the second sign by integrating the whole interaction item to separately integrating the passive and negative result of the interaction. Change the section equal sign to the third sign by changing variable v to v_2 of the passive result of the interaction, and changing variable v to v_3 of the negative result of the interaction. Because the passive and negative results of the interaction are the same, they could cancel and the answer is zero.

Use the result of integrations to velocity v and desired velocity w by convection term, relaxation term and interaction term to add, we could get macroscopic equation based on macroscopic parameter, density $c(x,t)$, as following equation (4.12)

$$\frac{\partial c}{\partial t} + \frac{\partial (c\bar{v})}{\partial x} = 0 \qquad (4.12)$$

4.3 Average velocity equation

This section will do the first-order momentum to mesoscopic model. That means that mesoscopic model times velocity and then integrates velocity. We could get macroscopic equation based on macroscopic parameter, average velocity $\bar{v}(x,t)$.

Our mesoscopic model times velocity v and integrates velocity v and desired velocity w.

1. **Convection term times velocity v and then integrates velocity v and desired velocity w.**

$$
\begin{aligned}
&\int dv \int dwv\left[\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x}\right] \\
&= \int dvv\frac{\partial f}{\partial t} + v^2\frac{\partial f}{\partial x} \\
&= \int dvv\frac{\partial f}{\partial t} + (v-\bar{v}+\bar{v})(v-\bar{v}+\bar{v})\frac{\partial f}{\partial x} \\
&= \int dvv\frac{\partial f}{\partial t} + \left[(v-\bar{v})^2 + 2(v-\bar{v})\bar{v} + \bar{v}^2\right]\frac{\partial f}{\partial x} \\
&= \frac{\partial(c\bar{v})}{\partial t} + \frac{\partial(c\theta)}{\partial x} + \frac{\partial(c\bar{v}^2)}{\partial x} \\
&= c\frac{\partial\bar{v}}{\partial t} + \bar{v}\frac{\partial c}{\partial t} + \frac{\partial(c\theta)}{\partial x} + c\bar{v}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial(c\bar{v})}{\partial x} \\
&= c\frac{\partial\bar{v}}{\partial t} + \frac{\partial(c\theta)}{\partial x} + c\bar{v}\frac{\partial\bar{v}}{\partial x}
\end{aligned}
$$

The second column is expanded to the form of the third column in order to make it become the integration of the forth column. We could get the above result by means of the definitions of equation (2.2), equation (2.3), equation (4.5), and equation (4.6).

2. **Relaxation term times velocity v and then integrates velocity v and desired velocity w.**

$$\int dv \int dw v \frac{\partial}{\partial v}\left(\frac{w-v}{T} g\right)$$
$$= -\int dv \frac{\tilde{v}_0 - v}{T} f$$
$$= -c \frac{(\bar{v}_0 - \bar{v})}{T}$$

3. **Interaction term times velocity v and then integrates velocity v and desired velocity w**

$$\int dv \int dw v [\iint_{0 \le v_1 \le v_3} dv_1 dv_3 (1-p)(v_3 - v_1) f(x + d(v_1, v_3), v_1, t) g(x, v_3, w, t) \delta(v - \Phi(v_1, v_3))$$
$$- g(x, v, w, t) \iint_{0 \le v_1 \le v} dv_1 dv_2 (1-p)(v - v_1) f(x + d(v_1, v), v_1, t) \delta(v_2 - \Phi(v_1, v))]$$
$$= \int dv \iint_{0 \le v_1 \le v_3} dv_1 dv_3 v (1-p)(v_3 - v_1) f(x + d(v_1, v_3), v_1, t) f(x, v_3, t) \delta(v - \Phi(v_1, v_3))$$
$$- \int dv v f(x, v, t) \iint_{0 \le v_1 \le v} dv_1 dv_2 (1-p)(v - v_1) f(x + d(v_1, v), v_1, t) \delta(v_2 - \Phi(v_1, v))$$

$$= \int dv_2 \iint_{0 \le v_1 \le v_3} dv_1 dv_3 v_2 (1-p)(v_3 - v_1) f(x + d(v_1, v_3), v_1, t) f(x, v_3, t) \delta(v_2 - \Phi(v_1, v_3))$$
$$- \int dv_3 f(x, v, t) \iint_{0 \le v_1 \le v_3} dv_1 dv_2 v_3 (1-p)(v_3 - v_1) f(x + d(v_1, v_3), v_1, t) \delta(v_2 - \Phi(v_1, v_3))$$
$$= \int dv_2 \iint_{0 \le v_1 \le v_3} dv_1 dv_3 (1-p)(v_2 - v_3)(v_3 - v_1) f(x + d(v_1, v_3), v_1, t) f(x, v_3, t) \delta(v_2 - \Phi(v_1, v_3))$$
$$= \iint_{0 \le v_1 \le v_3} dv_1 dv_3 (1-p)[(v_3 - a\lambda) - v_3](v_3 - v_1) f(x, v_3, t)[f(x, v_1, t) + d(v_1, v_3) \frac{\partial f(x, v_1, t)}{\partial x}]$$
$$= -(1-p) a\lambda \iint_{0 \le v_1 \le v_3} dv_1 dv_3 (v_3 - v_1) f(x, v_3, t)$$
$$* \{ f(x, v_1, t) + [\tau v_1 + \frac{1}{c_{\max}} + \frac{1}{2} \frac{(v_3 - v_1)^2}{a}] \frac{\partial f(x, v_1, t)}{\partial x} \}$$

Where $v_1 = \bar{v}(x,t) + \delta v_1, v_3 = \bar{v}(x,t) + \delta v_3$, and Jacobin=1

$$= -(1-p)a\lambda\int_{-\infty}^{\infty} d(\delta v_3) \int_{\delta v_1<\delta v_3} d(\delta v_1)(\delta v_3-\delta v_1) f(x,\bar{v}+\delta v_3,t)$$

$$* \{ f(x,\bar{v}+\delta v_1,t)+[\frac{1}{c_{\max}}+\tau(\bar{v}+\delta v_1)+\frac{1}{2}\frac{(\delta v_3-\delta v_1)^2}{a}]\frac{\partial f(x,\bar{v}+\delta v_1,t)}{\partial x}\}$$

Set $z=\delta v_3+\delta v_1$ and $y=\delta v_3-\delta v_1$, the $\delta v_3=\frac{z+y}{2}$, $\delta v_1=\frac{z-y}{2}$, and Jacobin=1/2

use the zeroth-order approximation of local equilibrium, equation (4.1) and equation(4.3)

$$= -(1-p)a\lambda\{c^2\int_{-\infty}^{\infty} dz\int_0^{\infty} dyy\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})$$

$$+c\int_{-\infty}^{\infty} dz\int_0^{\infty} dyy[\frac{1}{c_{\max}}+\tau(\bar{v}+\frac{z-y}{2})+\frac{y^2}{2a}]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})$$

$$*\left[\frac{\partial c}{\partial x}-\frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}+\frac{(z-y)^2}{8\theta}c\theta^{-2}\frac{\partial\theta}{\partial x}-\frac{(z-y)}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}\right]\}$$

Set $A=\frac{1}{c_{\max}}+\tau\bar{v}$, $B=\frac{\partial c}{\partial x}-\frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}$, $D=\frac{1}{8}c\theta^{-2}\frac{\partial\theta}{\partial x}$, $E=-\frac{1}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}$.

We take A, B, D, and E to the original equation and change its form as following.

$$= -(1-p)a\lambda\{c^2(\frac{1}{\sqrt{4\pi\theta}}2\theta)$$

$$+c\int_{-\infty}^{\infty} dz\int_0^{\infty} dyy[A+\tau\frac{(z-y)}{2}+\frac{y^2}{2a}]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})$$

$$\cdot[B+D(z-y)^2-E(z-y)]$$

Set

$$\Delta=\int_{-\infty}^{\infty} dz\int_0^{\infty} dyy[A+\tau\frac{(z-y)}{2}+\frac{y^2}{2a}]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})$$

$$*\left[B+D(z-y)^2-E(z-y)\right]\}$$

The calculation as following

$$\begin{aligned}\Delta = \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dy[ABy + AD(z^2y - 2y^2z + y^3) - AE(zy - y^2) + \frac{B}{2}\tau(zy - y^2) \\ + \frac{D}{2}\tau(z^3y + 3zy^3 - 3y^2z^2 - y^4) - \frac{E}{2}\tau(z^2y - 2y^2z + y^3) + \frac{By^3}{2a} \\ + \frac{D}{2a}(z^2y^3 - 2y^4z + y^5) - \frac{E}{2a}(y^3z - y^4)]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})\end{aligned}$$

We could cancel the above because of the first column of equation (4.2)

$$\begin{aligned}\Delta &= \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dy[ABy + AD(z^2y + y^3) - AE(-y^2) + \frac{B}{2}\tau(-y^2) + \frac{D}{2}\tau(-3y^2z^2 - y^4) \\ &\quad + \frac{E}{2}\tau(z^2y - 2y^2z + y^3) + \frac{By^3}{2a} + \frac{D}{2a}(z^2y^3 - 2y^4z + y^5) - \frac{E}{2a}(y^3z - y^4)] \\ &\quad * \frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta}) \\ &= AB\left(2\theta\frac{1}{\sqrt{4\pi\theta}}\right) + AD\left[8\theta^{3/2}\frac{\sqrt{\pi}}{2}2\theta\frac{1}{4\pi\theta} + 2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}\right] - AE\left(-\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}}\right) \\ &\quad + \frac{B}{2}\tau\left(-\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}}\right) + \frac{D}{2}\tau\left(-3\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}}\right) \\ &\quad - \frac{E}{2}\tau\left(2\theta\cdot 8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}\right) + \frac{B}{2a}\left(2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}\right) \\ &\quad + \frac{D}{2a}\left(-2(2\theta)^2 8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2\cdot 4(2\theta)^3\frac{1}{\sqrt{4\pi\theta}}\right) - \frac{E}{2a}\left(-1\cdot 3(2\theta)^{5/2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{4\pi\theta}}\right) \\ &= AB\frac{\theta^{1/2}}{\sqrt{\pi}} + \frac{6}{\sqrt{\pi}}AD\theta^{3/2} + AE\theta - \frac{B}{2}\tau\theta - 6\tau D\theta^2 - \frac{3}{\sqrt{\pi}}\tau E\theta^{3/2} + \frac{2}{a\sqrt{\pi}}B\theta^{3/2} + \frac{20}{a\sqrt{\pi}}D\theta^{5/2} + \frac{3}{a}D\theta^2\end{aligned}$$

Because we had set $B = \frac{\partial c}{\partial x} - \frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}$, $D = \frac{1}{8}c\theta^{-2}\frac{\partial\theta}{\partial x}$, $E = -\frac{1}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}$, we could decrease the numbers of symbols and join Δ as following

$$\begin{aligned}\Delta = (\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{-1/2} - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial\theta}{\partial x} \\ + (\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}\end{aligned}$$

Because of equation (4.11), we could ignore the high terms when θ^n , $n \ge 3/2$. The interaction term could be expressed as following.

Interaction term

$$= -(1-p)a\lambda c^2 \frac{1}{\sqrt{\pi}}\theta^{1/2} - (1-p)a\lambda c\{(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{-1/2} - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial \theta}{\partial x} + (\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}\}$$

Use the result of doing multiplication to v and integrations to velocity v and desired velocity w by convection term, relaxation term and interaction term to add, we could get as following.

$$\begin{aligned} & c\frac{\partial \bar{v}}{\partial t} + \frac{\partial P}{\partial x} + c\bar{v}\frac{\partial \bar{v}}{\partial x} + c\frac{(\bar{v}-\bar{v}_0)}{T} \\ & = -(1-p)a\lambda c^2 \frac{1}{\sqrt{\pi}}\theta^{1/2} - (1-p)a\lambda c\{(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{-1/2} \\ & \qquad - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial \theta}{\partial x} + (\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}\} \end{aligned}$$

For the above divides by density $c(x,t)$, we could get the macroscopic equation based on macroscopic parameter, average number $\bar{v}(x,t)$.

$$\begin{aligned} & \frac{\partial \bar{v}}{\partial t} + \frac{1}{c}\frac{\partial P}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial x} + \frac{(\bar{v}-\bar{v}_0)}{T} \\ & = -(1-p)a\lambda c \frac{1}{\sqrt{\pi}}\theta^{1/2} - (1-p)a\lambda\{(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{-1/2} \\ & \qquad - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial \theta}{\partial x} + (\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}\} \end{aligned} \tag{4.13}$$

In the derivation process, we assume $z = \delta v_3 + \delta v_1 = v_3 + v_1 - 2\bar{v}(x,t)$ so $\frac{\partial z}{\partial x} = -2\frac{\partial \bar{v}}{\partial x}$. Take it to equation (4.13), and we could get macroscopic equation (4.14) based on average number $\bar{v}(x,t)$.

$$
\begin{aligned}
&\frac{\partial \bar{v}}{\partial t}+\frac{1}{c}\frac{\partial P}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial x}-\frac{(\bar{v}_0-\bar{v})}{T}\\
&=-(1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2}-(1-p)a\lambda\{(\frac{A}{\sqrt{\pi}}\theta^{1/2}-\frac{\tau}{2}\theta)\frac{\partial c}{\partial x}+(\frac{A}{4\sqrt{\pi}}\theta^{-1/2}\\
&\qquad -\frac{\tau}{2}+\frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial \theta}{\partial x}+(-\frac{A}{2}+\frac{3\tau}{2\sqrt{\pi}}\theta^{1/2}-\frac{3}{2a}\theta)c\frac{\partial \bar{v}}{\partial x}\}
\end{aligned}
\tag{4.14}
$$

Where $A=\dfrac{1}{c_{\max}}+\tau\bar{v}$

4.4 Variance

This section will do the second-order momentum to mesoscopic model. That means that mesoscopic model times the square of velocity and then integrates velocity. We could get macroscopic equation based on macroscopic parameter, variance $\theta(x,t)$.

Our mesoscopic model (3.4) times the square of velocity v^2 and then integrates velocity v and desired velocity w.

1. **Convection term times the square of the velocity v^2 and then integrates velocity v and desired velocity w.**

$$
\begin{aligned}
&\int dv\int dw v^2(\frac{\partial g}{\partial t}+v\frac{\partial g}{\partial x})\\
&=\int dv\frac{\partial(fv^2)}{\partial t}+\frac{\partial(fv^3)}{\partial x}\\
&=\int dv\{\frac{\partial[f(v-\bar{v})^2]}{\partial t}+2\frac{\partial(fv\bar{v})}{\partial t}-\frac{\partial(f\bar{v}^2)}{\partial t}+\frac{\partial[f(v-\bar{v})^3]}{\partial x}-3\frac{\partial(fv\bar{v}^2)}{\partial x}\\
&\qquad +3\frac{\partial(fv^2\bar{v})}{\partial x}+\frac{\partial(f\bar{v}^3)}{\partial t}\}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\partial(c\theta)}{\partial t}+\frac{\partial(c\bar{v}^2)}{\partial t}+\frac{\partial(J)}{\partial x}+3\frac{\partial(c\bar{v}\theta)}{\partial x}+\frac{\partial(c\bar{v}^3)}{\partial x}\\
&=c\frac{\partial\theta}{\partial t}+\theta\frac{\partial c}{\partial t}+\bar{v}^2\frac{\partial c}{\partial t}+2c\bar{v}\frac{\partial\bar{v}}{\partial t}+\frac{\partial(cJ)}{\partial x}+2c\theta\frac{\partial\bar{v}}{\partial x}+2\bar{v}\frac{\partial(c\theta)}{\partial x}+c\bar{v}\frac{\partial\theta}{\partial x}+\theta\frac{\partial(c\bar{v})}{\partial x}\\
&\qquad +\bar{v}^2\frac{\partial(c\bar{v})}{\partial x}+2c\bar{v}^2\frac{\partial\bar{v}}{\partial x}\\
&=c\frac{\partial\theta}{\partial t}+2c\bar{v}\frac{\partial\bar{v}}{\partial t}+\frac{\partial(J)}{\partial x}+2c\theta\frac{\partial\bar{v}}{\partial x}+2\bar{v}\frac{\partial(c\theta)}{\partial x}+c\bar{v}\frac{\partial\theta}{\partial x}+2c\bar{v}^2\frac{\partial\bar{v}}{\partial x}
\end{aligned}
$$

$$= c\frac{\partial\theta}{\partial t} + \frac{\partial(J)}{\partial x} + 2c\theta\frac{\partial\bar{v}}{\partial x} + c\bar{v}\frac{\partial\theta}{\partial x} + 2c\bar{v}\{\frac{(\bar{v}_0 - \bar{v})}{T} - (1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2}$$

$$-(1-p)a\lambda[(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial\theta}{\partial x}$$

$$+(\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}]\}$$

$$= c\frac{\partial\theta}{\partial t} + 2c\theta\frac{\partial\bar{v}}{\partial x} + c\bar{v}\frac{\partial\theta}{\partial x} + 2c\bar{v}\{\frac{(\bar{v}_0 - \bar{v})}{T} - (1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2}$$

$$-(1-p)a\lambda[(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)\frac{\partial c}{\partial x} + (\frac{A}{4\sqrt{\pi}}\theta^{-1/2} - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial\theta}{\partial x}$$

$$+(\frac{A}{4} - \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{4a}\theta)c\frac{\partial z}{\partial x}]\}$$

The second column is expanded to the form of the third column in order to make it become the integration of the forth column. We could get the integration of the forth column by means of the definitions of equation (2.2), equation (2.3), equation (4.4), equation (4.5), and equation (4.8). We could change it to the result of the sixth column by equation (4.12), and the result of the seventh column by equation (4.13).

2. Relaxation term times the square of the velocity v^2 and then integrates velocity v and desired

$$\int dv \int dw v^2 \frac{\partial}{\partial v}\left(\frac{w-v}{T}g\right)$$

$$= -2\int dv \int dw v \frac{w-v}{T}g$$

$$= -2\int dv \int dw \frac{[(v-\bar{v})(w-\tilde{v}_0) - \bar{v}\tilde{v}_0 + w\bar{v} + v\tilde{v}_0] - [(v-\bar{v})^2 + 2v\bar{v} - \bar{v}^2]}{T} f$$

$$= \frac{-2c}{T}[C + \bar{v}\bar{v}_0 - \theta - \bar{v}^2]$$

3. Interaction term times the square of the velocity v^2 and then integrates velocity v and desired

$$\int dv \int dw v^2 [\iint_{0\le v_1 \le v_3} dv_1 dv_3 (1-p)(v_3-v_1) f(x+d(v_1,v_3),v_1,t) g(x,v_3,w,t)\delta(v-\Phi(v_1,v_3))$$

$$-g(x,v,w,t)\iint_{0\le v_1 \le v} dv_1 dv_2 (1-p)(v-v_1) f(x+d(v_1,v),v_1,t)\delta(v_2-\Phi(v_1,v))]$$

$$=\int dv v^2 [\iint_{0\le v_1 \le v_3} dv_1 dv_3 (1-p)(v_3-v_1) f(x+d(v_1,v_3),v_1,t) f(x,v_3,t)\delta(v-\Phi(v_1,v_3))$$

$$-\int dv f(x,v,t)\iint_{0\le v_1 \le v} dv_1 dv_2 (1-p)(v-v_1) f(x+d(v_1,v),v_1,t)\delta(v_2-\Phi(v_1,v))]$$

$$=\int dv_2 \iint_{0\le v_1 \le v_3} dv_1 dv_3 (1-p) v_2^2 (v_3-v_1) f(x+d(v_1,v_3),v_1,t) f(x,v_3,t)\delta(v_2-\Phi(v_1,v_3))$$

$$-\int dv_3 f(x,v,t)\iint_{0\le v_1 \le v_3} dv_1 dv_2 (1-p) v_3^2 (v_3-v_1) f(x+d(v_1,v_3),v_1,t)\delta(v_2-\Phi(v_1,v_3))$$

$$=\iint_{0\le v_1 \le v_3} dv_1 dv_3 (1-p)[(v_3-a\lambda)^2 - v_3^2](v_3-v_1) f(x,v_3,t)[f(x,v_1,t)+d(v_1,v_3)\frac{\partial f(x,v_1,t)}{\partial x}]$$

$$=-(1-p)a\lambda \iint_{0\le v_1 \le v_3} dv_1 dv_3 (v_3-v_1)(2v_3-a\lambda) f(x,v_3,t)$$

$$*\{f(x,v_1,t)+[\tau v_1+\frac{1}{c_{\max}}+\frac{1}{2}\frac{(v_3-v_1)^2}{a}]\frac{\partial f(x,v_1,t)}{\partial x}\}$$

Set $v_1=\bar{v}(x,t)+\delta v_1$, $v_3=\bar{v}(x,t)+\delta v_3$, and Jacobin=1

$$=-(1-p)a\lambda \int_{-\infty}^{\infty} d(\delta v_3)\int_{\delta v_1<\delta v_3} d(\delta v_1)(\delta v_3-\delta v_1)(2\delta v_3+2\bar{v}-a\lambda) f(x,\bar{v}+\delta v_3,t)$$

$$*\{f(x,\bar{v}+\delta v_1,t)+[\frac{1}{c_{\max}}+\tau(\bar{v}+\delta v_1)+\frac{1}{2}\frac{(\delta v_3-\delta v_1)^2}{a}]\frac{\partial f(x,\bar{v}+\delta v_1,t)}{\partial x}\}$$

Set $z=\delta v_3+\delta v_1$ and $y=\delta v_3-\delta v_1$, then $\delta v_3=\frac{z+y}{2}$, $\delta v_1=\frac{z-y}{2}$, and Jacobin=1/2 use the zeroth-order approximation of local equilibrium, equation(4.1)and equation (4.3)

$$
= -(1-p)a\lambda\{c^2\int_{-\infty}^{\infty}dz\int_0^{\infty}dy\,y(z+y+2\bar{v}-a\lambda)\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})
$$

$$
+c\int_{-\infty}^{\infty}dz\int_0^{\infty}dy\,y(z+y+2\bar{v}-a\lambda)[\frac{1}{c_{\max}}+\tau(\bar{v}+\frac{z-y}{2})+\frac{y^2}{2a}]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})
$$

$$
*\left[\frac{\partial c}{\partial x}-\frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}+\frac{(z-y)^2}{8\theta}c\theta^{-2}\frac{\partial\theta}{\partial x}-\frac{(z-y)}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}\right]\}
$$

Set $A=\frac{1}{c_{\max}}+\tau\bar{v}$, $B=\frac{\partial c}{\partial x}-\frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}$, $D=\frac{1}{8}c\theta^{-2}\frac{\partial\theta}{\partial x}$, $E=-\frac{1}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}$, and $F=2\bar{v}-a\lambda$. We take A, B, D, and E to the original equation and change its form as following.

$$
= -(1-p)a\lambda\{c^2\int_{-\infty}^{\infty}dz\int_0^{\infty}dy(yz+y^2+Fy)\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})
$$

$$
+c\int_{-\infty}^{\infty}dz\int_0^{\infty}dy\,y(yz+y^2+Fy)[A+\frac{\tau}{2}(z-y)+\frac{y^2}{2a}]\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})
$$

$$
*\left[B+D(z-y)^2-E(z-y)\right]\}
$$

$$
= -(1-p)a\lambda\{c^2[\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}}+F(2\theta)\frac{1}{\sqrt{4\pi\theta}}]
$$

$$
+c[AB(yz+y^2+Fy)+AD(-y^2z^2+y^4-y^3z+yz^3+Fyz^2-2Fy^2z+Fy^3)
$$

$$
-AE(-yz^2-y^3+Fyz-Fy^2)+\frac{\tau}{2}B(yz^2-y^3+Fyz-Fy^2)
$$

$$
+\frac{\tau}{2}D(-2y^2z^3+2y^4z-y^5+yz^4-Fy^4-3Fy^2z^2+3Fy^3z+Fyz^3)
$$

$$
-\frac{\tau}{2}E(-y^2z^2+y^4-y^3z+yz^3+Fyz^2-2Fy^2z+Fy^3)\}+\frac{B}{2a}(y^3z+y^4+Fy^3)
$$

$$
+\frac{D}{2a}(-y^4z^2+y^6-y^5z+y^3z^3+Fy^3z^2-2Fy^4z+Fy^5)
$$

$$
-\frac{E}{2a}(-y^5+y^3z+Fy^3z-Fy^4)]\}\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-z^2}{4\theta})\frac{1}{\sqrt{4\pi\theta}}\exp(\frac{-y^2}{4\theta})
$$

$$
\begin{aligned}
&= -(1-p)a\lambda c^2[\theta + \frac{F\theta^{1/2}}{\sqrt{\pi}}] \\
&\quad -(1-p)a\lambda c\{AB[\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}} + F(2\theta)\frac{1}{\sqrt{4\pi\theta}}] + AD[-\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}8\theta^{3/2}\frac{1}{4\pi\theta} \\
&\qquad +3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}} + F(2\theta)8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} + F2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad - AE[(2\theta)8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}} - F\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad +\frac{\tau}{2}B[(2\theta)8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}} - F\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad +\frac{\tau}{2}D[-F3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}} - 3F\cdot 3\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} \\
&\qquad\qquad + 2\theta\cdot 32\theta^{5/2}\frac{3}{2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2\cdot 4(2\theta)^3\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad\qquad -\frac{\tau}{2}E[-\sqrt{\frac{\pi}{2}}(2\theta)^{3/2}8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} + 3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}} \\
&\qquad\qquad + F(2\theta)8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} + F\cdot 2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad\qquad\quad +\frac{B}{2a}[3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}} + F2(2\theta)^2\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad +\frac{D}{2a}[-3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} + 1\cdot 3\cdot 5\sqrt{\frac{\pi}{2}}(2\theta)^{7/2}\frac{1}{\sqrt{4\pi\theta}} \\
&\qquad + F\cdot 2\cdot(2\theta)^2 8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} + F\cdot 2\cdot 4(2\theta)^3\frac{1}{\sqrt{4\pi\theta}}] \\
&\qquad -\frac{E}{2a}[2\cdot(2\theta)^2 8\theta^{3/2}\frac{\sqrt{\pi}}{2}\frac{1}{4\pi\theta} - 2\cdot 4(2\theta)^3\frac{1}{\sqrt{4\pi\theta}} - F\cdot 3\sqrt{\frac{\pi}{2}}(2\theta)^{5/2}\frac{1}{\sqrt{4\pi\theta}}]\}
\end{aligned}
$$

$$
\begin{aligned}
&= -(1-p)a\lambda c^2[\theta + \frac{F}{\sqrt{\pi}}\theta^{1/2}] \\
&\quad -(1-p)a\lambda c\{AB[\theta + \frac{F}{\sqrt{\pi}}\theta^{1/2}] + AD[-2\theta^2 + 6\theta^2 + \frac{2F}{\sqrt{\pi}}\theta^{3/2} + \frac{F}{\sqrt{\pi}}\theta^{3/2}] \\
&\qquad - AE[-F\theta + \frac{2}{\sqrt{\pi}}\theta^{3/2} - \frac{4}{\sqrt{\pi}}\theta^{3/2}] + \frac{\tau}{2}B[-F\theta + \frac{2}{\sqrt{\pi}}\theta^{3/2} - \frac{4}{\sqrt{\pi}}\theta^{3/2}] \\
&\qquad + \frac{\tau}{2}D[-6F\theta^2 - 6F\theta^2 + \frac{12}{\sqrt{\pi}}\theta^{5/2} - \frac{32}{\sqrt{\pi}}\theta^{5/2}] \\
&\qquad - \frac{\tau}{2}E[-2\theta^2 + 6\theta^2 + \frac{2}{\sqrt{\pi}}F\theta^{3/2} + \frac{4}{\sqrt{\pi}}\theta^{3/2}] + \frac{B}{2a}[\frac{4F}{\sqrt{\pi}}2\theta^{3/2} + 6\theta^2]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{D}{2a}[-12\theta^3+60\theta^3+\frac{8F}{\sqrt{\pi}}\theta^{5/2}+\frac{32F}{\sqrt{\pi}}\theta^{5/2}] \\
&-\frac{E}{2a}[-6F\theta^2+\frac{8}{\sqrt{\pi}}\theta^{5/2}-\frac{32}{\sqrt{\pi}}\theta^{5/2}]\}
\end{aligned}
$$

$$
\begin{aligned}
&=-(1-p)a\lambda c^2[\theta+\frac{F}{\sqrt{\pi}}\theta^{1/2}] \\
&\quad-(1-p)a\lambda c\{AB[\theta+\frac{F}{\sqrt{\pi}}\theta^{1/2}]+AD[4\theta^2+\frac{6F}{\sqrt{\pi}}\theta^{3/2}]-AE-\frac{2}{\sqrt{\pi}}\theta^{3/2}-F\theta] \\
&\quad+\frac{\tau}{2}B[-F\theta-\frac{2}{\sqrt{\pi}}\theta^{3/2}]+\frac{\tau}{2}D[-12F\theta^2-\frac{20}{\sqrt{\pi}}\theta^{5/2}]-\frac{\tau}{2}E[4\theta^2+\frac{6}{\sqrt{\pi}}F\theta^{3/2}] \\
&\quad+\frac{B}{2a}[\frac{4F}{\sqrt{\pi}}2\theta^{3/2}+6\theta^2]+\frac{D}{2a}[48\theta^3+\frac{40F}{\sqrt{\pi}}\theta^{5/2}]-\frac{E}{2a}[-6F\theta^2-\frac{24}{\sqrt{\pi}}\theta^{5/2}]\}
\end{aligned}
$$

Because we set $B=\frac{\partial c}{\partial x}-\frac{1}{2}c\theta^{-1}\frac{\partial\theta}{\partial x}$, $D=\frac{1}{8}c\theta^{-2}\frac{\partial\theta}{\partial x}$, $E=-\frac{1}{4\theta}\frac{\partial z}{\partial x}c\theta^{-1}$, we make the number of symbols simple and take them into above.

$$
\begin{aligned}
&=-(1-p)a\lambda c^2[\theta+\frac{F}{\sqrt{\pi}}\theta^{1/2}] \\
&\quad-(1-p)a\lambda c\{[+\frac{AF}{\sqrt{\pi}}\theta^{1/2}+(A-\frac{\tau}{2}F)\theta+(-\frac{\tau}{\sqrt{\pi}}+\frac{2F}{a\sqrt{\pi}})\theta^{3/2}+\frac{3}{a}\theta^2]\frac{\partial c}{\partial x} \\
&\quad+[\frac{AF}{4\sqrt{\pi}}\theta^{-1/2}-\frac{F}{2}+(-\frac{3\tau}{4\sqrt{\pi}}+\frac{3F}{2a\sqrt{\pi}})\theta^{1/2}+\frac{3}{2a}\theta]c\frac{\partial\theta}{\partial x} \\
&\quad+[(\frac{A}{2\sqrt{\pi}}-\frac{3\tau}{4\sqrt{\pi}}F)\theta^{1/2}+\frac{1}{4}AF+(-\frac{\tau}{2}+\frac{3F}{4a})\theta-\frac{3}{a\sqrt{\pi}}\theta^{3/2}]c\frac{\partial z}{\partial x}\}
\end{aligned}
$$

Because of equation (4.11), we could ignore the high terms when θ^n and $n\geq 3/2$ .The interaction term could be expressed as following.

Interaction term

$$
\begin{aligned}
&=-(1-p)a\lambda c^2[\theta+\frac{F}{\sqrt{\pi}}\theta^{1/2}] \\
&\quad-(1-p)a\lambda c\{[+\frac{AF}{\sqrt{\pi}}\theta^{1/2}+(A-\frac{\tau}{2}F)\theta]\frac{\partial c}{\partial x}+[\frac{AF}{4\sqrt{\pi}}\theta^{-1/2}-\frac{F}{2}+(-\frac{3\tau}{4\sqrt{\pi}}+\frac{3F}{2a\sqrt{\pi}})\theta^{1/2} \\
&\quad+\frac{3}{2a}\theta]c\frac{\partial\theta}{\partial x}+[(\frac{A}{2\sqrt{\pi}}-\frac{3\tau}{4\sqrt{\pi}}F)\theta^{1/2}+\frac{1}{4}AF+(-\frac{\tau}{2}+\frac{3F}{4a})\theta]c\frac{\partial z}{\partial x}\}
\end{aligned}
$$

Use the result of doing multiplication to v^2 and integrations to velocity v and desired velocity w by convection term, relaxation term and interaction term to add, we could get as following.

$$c\frac{\partial\theta}{\partial t}+2c\theta\frac{\partial\bar{v}}{\partial x}+c\bar{v}\frac{\partial\theta}{\partial x}+2c\bar{v}\{\frac{(\bar{v}_0-\bar{v})}{T}-(1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2}$$
$$-(1-p)a\lambda[(\frac{A}{\sqrt{\pi}}\theta^{1/2}-\frac{\tau}{2}\theta)\frac{\partial c}{\partial x}+(\frac{A}{4\sqrt{\pi}}\theta^{-1/2}-\frac{\tau}{2}+\frac{3}{2a\sqrt{\pi}}\theta^{1/2})c\frac{\partial\theta}{\partial x}$$
$$+(\frac{A}{4}-\frac{3\tau}{4\sqrt{\pi}}\theta^{1/2}+\frac{3}{4a}\theta)c\frac{\partial z}{\partial x}]\}-\frac{2c}{T}[C+\bar{v}\bar{v}_0-\theta-\bar{v}^2]$$
$$=-(1-p)a\lambda c^2[\theta+\frac{F}{\sqrt{\pi}}\theta^{1/2}]$$
$$-(1-p)a\lambda c\{[A\theta+\frac{AF}{\sqrt{\pi}}\theta^{1/2}-\frac{\tau}{2}F\theta]\frac{\partial c}{\partial x}+[\frac{AF}{4\sqrt{\pi}}\theta^{-1/2}$$
$$-\frac{F\tau}{2}+(-\frac{3\tau}{4\sqrt{\pi}}+\frac{3F}{2a\sqrt{\pi}})\theta^{1/2}+\frac{3}{2a}\theta]c\frac{\partial\theta}{\partial x}$$
$$+[(\frac{A}{2\sqrt{\pi}}-\frac{3\tau}{4\sqrt{\pi}}F)\theta^{1/2}+\frac{1}{4}AF+(-\frac{\tau}{2}+\frac{3F}{4a})\theta]c\frac{\partial z}{\partial x}$$

Divide the above by density $c(x,t)$, and take $F=2\bar{v}-a\lambda$ to the original equation. Layout time interval λ is a small value so $\lambda^2\approx 0$ and we could get macroscopic equation based on macroscopic parameter, variance $\theta(x,t)$

$$\frac{\partial\theta}{\partial t}+\frac{2P}{c}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\theta}{\partial x}-\frac{2c}{T}[C-\theta]$$
$$=-(1-p)a\lambda c\theta-(1-p)a\lambda\{A\theta\frac{\partial c}{\partial x}+[-\frac{3\tau}{4\sqrt{\pi}}\theta^{-1/2}+\frac{3}{2a}\theta)]c\frac{\partial\theta}{\partial x}+(\frac{1}{2\sqrt{\pi}}A\theta^{1/2}-\frac{\tau}{2}\theta)c\frac{\partial z}{\partial x}\} \tag{4.15}$$

We set $z=\delta v_3+\delta v_1=v_3+v_1-2\bar{v}(x,t)$ so $\frac{\partial z}{\partial x}=-2\frac{\partial\bar{v}}{\partial x}$, we could take it into equation (4.15), and get the macroscopic equation(4.16) based on macroscopic parameter, variance $\theta(x,t)$ as following.

$$\frac{\partial\theta}{\partial t}+\frac{2P}{c}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\theta}{\partial x}-\frac{2c}{T}[C-\theta]$$
$$=-(1-p)a\lambda c\theta-(1-p)a\lambda\{A\theta\frac{\partial c}{\partial x}+[-\frac{3\tau}{4\sqrt{\pi}}\theta^{-1/2}+\frac{3}{2a}\theta)]c\frac{\partial\theta}{\partial x}+(-\frac{1}{\sqrt{\pi}}A\theta^{1/2}+\tau\theta)c\frac{\partial\bar{v}}{\partial x}\} \tag{4.16}$$

Where $A=\frac{1}{c_{\max}}+\tau\bar{v}$

4.5 Summary

The traffic flow model that the chapter 3 constricted in this research which could describes non- instant velocity-changing and considers finite space could get three important macroscopic models. They are macroscopic equation (4.12) based on density function $c(x,t)$, macroscopic equation (4.14) based on average velocity $\bar{v}(x,t)$, and macroscopic equation (4.16) based on variance $\theta(x,t)$.

According to equation (4.12), the constructed model still has the characteristic of flow conservation. Comparing equation (4.14) and equation (4.16) with the preview studies, the result of the Interaction term is obvious different. The macroscopic model that the traditional mesoscopic model gets is usually the result of the multiplication of density, average velocity, and constant because of the result of the Interaction term. The result of the Interaction terms would change with gradient of density, average velocity or variance owing to considering the factors such as non- instant velocity-changing and space cooperated in this research which causes the result like equation (4.14), and equation (4.16). The others about model would make a deeper analysis in chapter 5.

Chapter 5 Characteristics Analysis of Macroscopic Models

We construct a mesoscopic traffic flow model that could describe non-instant velocity-changing and consider finite space in chapter 3. Then we get three macroscopic traffic flow models based on density $c(x,t)$, average velocity $\bar{v}(x,t)$, variance $\theta(x,t)$ in chapter 4, and they show as the equations (4.12), (4.14) and (4.16). Where, the equation (4.12) is flow conservation as result of other scholars, but the equations (4.14) and (4.16) are different from others. We would analysis these three macroscopic equations further in this chapter.

5.1 Characteristic Velocity Analysis

This section aims to calculate the characteristic velocity [17, 22-23] of the macroscopic model. First of all, rewrite the three macroscopic models from chapter 4 with symbols.

1. Density equation

$$\frac{\partial c}{\partial t}+\frac{\partial (c\bar{v})}{\partial x}=0 \tag{5.1}$$

It is the same as equation (4.12).

2. Average velocity equation

$$\begin{aligned}&\frac{\partial \bar{v}}{\partial t}+(\frac{\theta}{c}+A)\frac{\partial c}{\partial x}+(\bar{v}+B)\frac{\partial \bar{v}}{\partial x}+(1+C)\frac{\partial \theta}{\partial x}\\&=\frac{(\bar{v}_0-\bar{v})}{T}-(1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2}\end{aligned} \tag{5.2}$$

Where

$$A = (1-p)a\lambda(\frac{A}{\sqrt{\pi}}\theta^{1/2} - \frac{\tau}{2}\theta)$$
$$B = (1-p)a\lambda(-\frac{A}{2} + \frac{3\tau}{2\sqrt{\pi}}\theta^{1/2} - \frac{3}{2a}\theta)c$$
$$C = (1-p)a\lambda(\frac{A}{4\sqrt{\pi}}\theta^{-1/2} - \frac{\tau}{2} + \frac{3}{2a\sqrt{\pi}}\theta^{1/2})c$$

3. Variance equation

$$\frac{\partial\theta}{\partial t} + D\frac{\partial c}{\partial x} + (2\theta + E)\frac{\partial \bar{v}}{\partial x} + (\bar{v} + F)\frac{\partial\theta}{\partial x} = \frac{2}{T}[C-\theta] - (1-p)a\lambda c\theta \tag{5.3}$$

Where

$$D = (1-p)a\lambda A\theta$$
$$E = (1-p)a\lambda(-\frac{1}{\sqrt{\pi}}A\theta^{1/2} + \tau\theta)c$$
$$F = (1-p)a\lambda[-\frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{2a}\theta)]c$$

In order to get Characteristic Velocity, we need to rewrite the equation (5.1), equation(5.2) ,and equation (5.3) to Characteristic Form as equation (5.4).

$$\begin{bmatrix} c \\ \bar{v} \\ \theta \end{bmatrix}_t + \begin{bmatrix} \bar{v} & c & 0 \\ A + \frac{\theta}{c} & B + \bar{v} & 1 + C \\ D & E + 2\theta & F + \bar{v} \end{bmatrix}_x = 0 \tag{5.4}$$

Set

$$\boldsymbol{G} = \begin{bmatrix} \bar{v} & c & 0 \\ A + \frac{\theta}{c} & B + \bar{v} & 1 + C \\ D & E + 2\theta & F + \bar{v} \end{bmatrix}$$

The Eigenvalue ζ could be got from $\det[\zeta\boldsymbol{I} - \boldsymbol{G}] = 0$, where $\boldsymbol{I}$ is vector. Then we could get what ζ is.

$$\zeta \approx \bar{v} + \frac{1}{3}(B + F)$$
$$= \bar{v} + \frac{1}{3}(1-p)a\lambda c[(-\frac{A}{2} + \frac{3\tau}{2\sqrt{\pi}}\theta^{1/2} - \frac{3}{2a}\theta) + (-\frac{3\tau}{4\sqrt{\pi}}\theta^{1/2} + \frac{3}{2a}\theta)]$$
$$= \bar{v} + \frac{1}{3}(1-p)a\lambda c[-\frac{A}{2} + \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2}]$$

(5.5)

Because the other roots of $\det[\zeta \boldsymbol{I} - \boldsymbol{G}] = 0$ are imaginary roots, the Characteristic Velocity that this model gets is as the equation (5.6). (As appendix A)

$$\frac{dx}{dt} \approx \bar{v} + \frac{1}{3}(1-p)a\lambda c[-\frac{A}{2} + \frac{3\tau}{4\sqrt{\pi}}\theta^{1/2}] \qquad (5.6)$$

C. Wagner (1996) [17] shows that Paveri-Fontana (1975) [7] could get characteristic velocities as following on the condition of considering infinite space and instant velocity of deceleration.

$$\frac{dx}{dt} = \bar{v} \quad , \quad \frac{dx}{dt} = \bar{v} \pm \sqrt{3\theta} \qquad (5.7)$$

Because characteristic velocities also called local wave velocity that means characteristic velocity is density function, the following is the illustration of characteristic velocity that this research gets.

(1). According to the equation (5.6), we could know that the characteristic velocity which this research gets has relations to density. According to figure 5-1-1, that is a reasonable result.

(2). According to the equation (5.6), the characteristic velocity that this model constructs is positive. That means density wave passes down downward. According to basic macroscopic assumption, $q = c\bar{v}$ [24], as following figure 5-1-1, density velocity is positive when flow is smaller than the capacity, and density is negative on the other hand. Because we assume model on dilute traffic, the velocity of density wave should be positive value. This result is the same as result of the equation (5.6), and it is reasonable.

(3). If passing probability $p = 1$ that means which drivers can overtake cars at will, the characteristic velocity would be equal to the average velocity according equation (5.6). It means $\frac{dx}{dt} \approx \bar{v}$ and it is a reasonable result. Because drivers can overtake cars at will, cars are independent, and on dilute traffic, velocities would not interact. So velocity could achieve desired velocity, and that results that characteristic velocity and average velocity are the same.

(4). Because our model considers deceleration, when uniform deceleration $a = 0$ that shows drivers have no deceleration when the faster (the latter) car could not pass according to the equation (5.6). Hence, the two cars have the same velocities, and it is a reasonable result. This result is a special result that the researches of other scholars do not have.

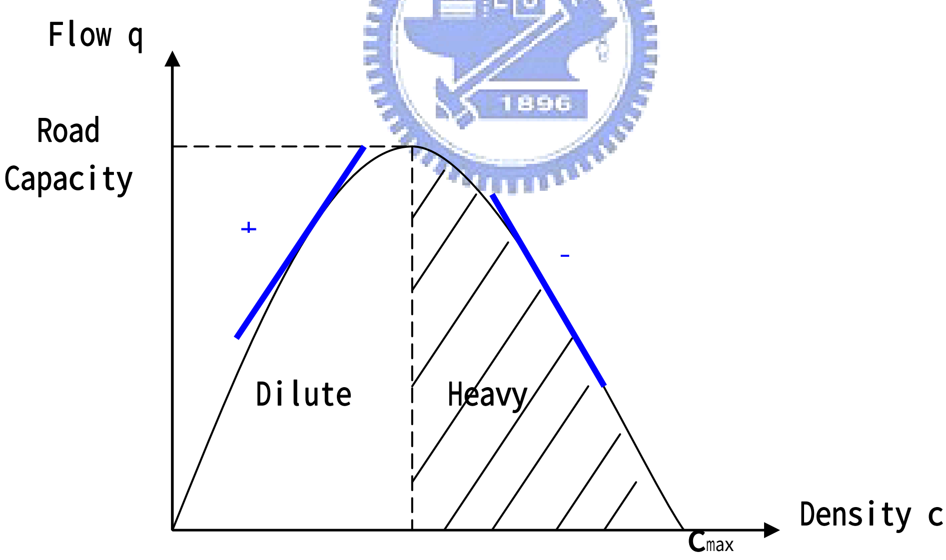

Fig. 5-1-1 Dilute and Heavy Traffic

Data Resource: Haberman, Richard (1977)[24]

5.2 Equilibrium Analysis

Macroscopic parameters density $c(x,t)$, average velocity $\bar{v}(x,t)$ and variance $\theta(x,t)$ would not change with time and space in equilibrium. It means that $\frac{\partial c}{\partial t}=0$, $\frac{\partial c}{\partial x}=0$, $\frac{\partial \bar{v}}{\partial t}=0$, $\frac{\partial \bar{v}}{\partial x}=0$, $\frac{\partial \theta}{\partial t}=0$, and $\frac{\partial \theta}{\partial x}=0$, so we could get the following equations.

1. The Relation Between Average Desired Velocity and Average Velocity

If it is in equilibrium, we could rewrite the equation (4.14) to the equation (5.8)

$$-\frac{(\bar{v}_0-\bar{v})}{T}=-(1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2} \tag{5.8}$$

Then we could get the relation between average desired velocity and average velocity as the equation (5.9).

$$\bar{v}_0-\bar{v}=T(1-p)a\lambda c\frac{1}{\sqrt{\pi}}\theta^{1/2} \tag{5.9}$$

That result of Paveri-Fontana (1975)[7] is the equation (5.10).

$$\bar{v}_0-\bar{v}=T(1-p)c\theta \tag{5.10}$$

The relation between average desired velocity and average velocity in equilibrium results from relaxation and interaction term. If it is not in equilibrium, the relation would depend on density gradient, average gradient and variance gradient. It is different from the equation (5.9). The relationship is explained as the following.

(1). From the equation (5.9), mean average desired velocity is faster than average velocity. It is a reasonable result.

(2). If the passing probability $p=1$ in equilibrium, the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$. It is reasonable result because cars could attain their desired velocities when they could pass at will. They do not affect each other in dilute density and independent conditions. Hence, the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$. If the passing probability

$p=0$ in equilibrium, it means cars could not pass each other. Therefore, their relaxation time T approximate infinity, this cause the result that the average desired velocity is different with the average velocity as we obtain.

(3). If the relaxation time $T=0$ in equilibrium, the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$. Because $T=0$ means the time that current velocity accelerates to the desired velocity is zero, it also means the current velocity is equal to the desired velocity. So this result is the same as the equation (5.9). The same as the above, if $T\to\infty$, it also means $p=0$. Hence, the average desired velocity is different with the average velocity.

(4). If the uniform deceleration $a=0$, it means there have no deceleration to avoid collision without passing. It also means that the velocity of the former is the same as the latter one, so it has no deceleration. Because every driver has his desired velocity, they would accelerate until he attains his desired velocity. It means that drivers have attained their desired velocity, so they have no acceleration. So the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$ in equilibrium. Besides, this is the special result that Paveri-Fontana (1975) [7] could not explain in equilibrium.

2. The Relation Between Covariance and Variance

If it is in equilibrium, we could rewrite the equation (4.16) to the equation (5.11)

$$-\frac{2}{T}[C-\theta]=-(1-p)a\lambda c\theta \qquad (5.11)$$

Then we could get the relation between covariance and variance as the equation (5.12).

$$C-\theta=\frac{T}{2}(1-p)a\lambda c\theta \qquad (5.12)$$

The following equation is result of Paveri-Fontana (1975)[7].

$$C-\theta=\frac{T}{2}(1-p)c\theta^{3/2} \tag{5.13}$$

The relation between covariance and variance in equilibrium results from relaxation and interaction term. If it is not in equilibrium, the relation would depend on density gradient, average gradient and variance gradient. It is different from the equation (5.12). The relationship is explained as the following.

(1). We could know covariance is larger than variance in equilibrium from equation (5.12). Covariance results from the difference between average velocity and velocity, average velocity and average desired velocity. Variance results from the difference between average velocity and velocity. So this is a reasonable result.

(2). When the passing probability $p=1$, it means that cars could pass in will. From the equation (5.12), we could know that covariance is equal to variance $C=\theta$ in equilibrium. It is reasonable result because cars could attain their desired velocities when they could pass at will. They do not affect each other in dilute density and independent conditions. Hence, the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$. According to the definitions of covariance and variance, so covariance is equal to variance in equilibrium. If the passing probability $p=0$ in equilibrium, it means cars could not pass each other. Therefore, their relaxation time T approximate infinity, this cause the result that covariance different with variance as we obtain.

(3). If the relaxation time $T=0$ in equilibrium, the average velocity is equal to the average desired velocity $\bar{v}_0=\bar{v}$. Because $T=0$ means the time that current velocity accelerates to the desired velocity is zero, it also means the current velocity is equal to the desired velocity. According to the definitions of covariance and variance, so covariance is equal to variance in equilibrium. The result is the same as the equation (5.12). The same as the above, if $T \to \infty$, it also means $p = 0$. Hence, covariance is different with variance.

(4). If the uniform deceleration $a = 0$, it means there have no deceleration to avoid collision without passing. It also means that the velocity of the former is the same as the latter one, so it has no deceleration. Because every driver has his desired velocity, they would accelerate until he attains his desired velocity. It means that drivers have attained their desired velocity, so they have no acceleration. So the average velocity is equal to the average desired velocity $\overline{v}_0 = \overline{v}$ in equilibrium. According to the definitions of covariance and variance, so covariance is equal to variance in equilibrium. Besides, this is the special result that the model of Paveri-Fontana (1975) [7] could not have.

5.3 Numerical Simulation Analysis

In this section, we would analysis the three macroscopic equations with numerical simulation. We would use "Upwind method to analysis". It is a simple first-order partial differential equation and it belongs to an explicit finite-difference method. According to Helbing (1999) [25], we set flow $Q = c\overline{v}$ and traffic pressure $P = c\theta$. Then the three macroscopic equations (4.12), (4.14) and (4.16) could be rewrite to be related to density c function as the following.

$$\frac{\partial c}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{5.14}$$

$$\begin{aligned}
&\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}[P + \frac{Q^2}{c} + (1-p)a\lambda\{\frac{5}{4c_\max\sqrt{\pi}}P^{1/2}c^{3/2} + \frac{11\tau}{4\sqrt{\pi}}QP^{1/2}c^{1/2} \\
&\qquad - \tau cP + \frac{3}{2a\sqrt{\pi}}P^{3/2}c^{1/2} - \frac{1}{2c_\max}cQ - \frac{\tau}{2}Q^2 - \frac{3}{2a}PQ\}] \\
&= \frac{cv_0 - Q}{T} - (1-p)a\lambda c^2
\end{aligned} \tag{5.15}$$

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial x}[\frac{3PQ}{c}+(1-p)a\lambda\{\frac{1}{c\_\max}Pc+2\tau QP-\frac{3\tau}{4\sqrt{\pi}}P^{3/2}c^{1/2}$$
$$+\frac{3}{2a}P^2-\frac{1}{c\_\max\sqrt{\pi}}P^{1/2}Qc^{1/2}-\frac{\tau}{\sqrt{\pi}}p^{1/2}Q^2c^{-1/2}\}] \tag{5.16}$$
$$=\frac{2[cC-P]}{T}-(1-p)a\lambda Pc$$

If we rewrite the equations (5.1), (5.2), and (5.3) as vector form as the following.

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f(u)}}{\partial x}=\mathbf{s(u)} \tag{5.17}$$

Where

$$\mathbf{u}=[c,Q,P] \tag{5.18}$$

$$\mathbf{f(u)}=[Q,P+\frac{Q^2}{c}+(1-p)a\lambda\{\frac{5}{4c\_\max\sqrt{\pi}}P^{1/2}c^{3/2}+\frac{11\tau}{4\sqrt{\pi}}QP^{1/2}c^{1/2}$$
$$-\pi cP+\frac{3}{2a\sqrt{\pi}}P^{3/2}c^{1/2}-\frac{1}{2c\_\max}cQ-\frac{\tau}{2}Q^2-\frac{3}{2a}PQ\},\frac{3PQ}{c}$$
$$+(1-p)a\lambda\{\frac{1}{c\_\max}Pc+2\tau QP-\frac{3\tau}{4\sqrt{\pi}}P^{3/2}c^{1/2}+\frac{3}{2a}P^2 \tag{5.19}$$
$$-\frac{1}{c\_\max\sqrt{\pi}}P^{1/2}Qc^{1/2}-\frac{\tau}{\sqrt{\pi}}p^{1/2}Q^2c^{-1/2}\}]$$

$$\mathbf{s(u)}=[0,\frac{cv_0-Q}{T}-(1-p)a\lambda c^2,\frac{2[cC-P]}{T}-(1-p)a\lambda Pc] \tag{5.20}$$

We use Dirichlet boundary condition, a period boundary condition, and it means that the simulation result would happen periodically. So we would simulate one cycle as present as the following.

$$\mathbf{u}(0,t)=\mathbf{u}(L,t) \tag{5.21}$$

Where, L means the end of the simulation road, and $x\in[L,L+\delta]$, δ means a small quantity of space. We assume the equation (5.9).

$$\mathbf{u}(x)=\mathbf{u}(L) \tag{5.22}$$

We separate time to 4000 equal grids and every time grid is 0.0001 hour. We separate space to 250 equal grids and every space grid is 0.004 km. Assume average car length is 0.005 km, desired velocity is 100km per hour ($v_0=100$ km/hr), relaxation time is 30 seconds ($T=30$ s), reaction time is 0.75 seconds ($\tau=0.75$ s), heavy density is 180 cars per km ($c\_max = 180$ cars/km), equilibrium density is 30 cars per hour ($c\_equi = 30$ cars/km), $\lambda = 0.0001$ hour is the same as the time grid, covariance $C = 144$ square km/ square hour, uniform deceleration $a = 60000$ km/square hour, passing probability $p = 0.99$. According to Helbing (1995) [26], he assumes the relation between equilibrium velocity and desired velocity as the equation (5.10) ,and the relation between covariance and variance as the equation (5.11).

$$v\_equi = v_0(\mathrm{pow}(1 + \exp((\mathrm{c\_equi/c\_max} - 0.25)/0.06), -1.0) - 3.72 * 10^{-6}) \tag{5.23}$$

$$\mathrm{var}iance\_equi = \mathrm{var}iance(\mathrm{pow}(1 + \exp((\mathrm{c\_equi/c\_max} - 0.25)/0.06), -1.0) - 3.72 * 10^{-6}) \tag{5.24}$$

<1>. Perturbation Simulation:

We add sine wave to the initial value of density, and see the result that perturbation causes. The following is the result of our research.

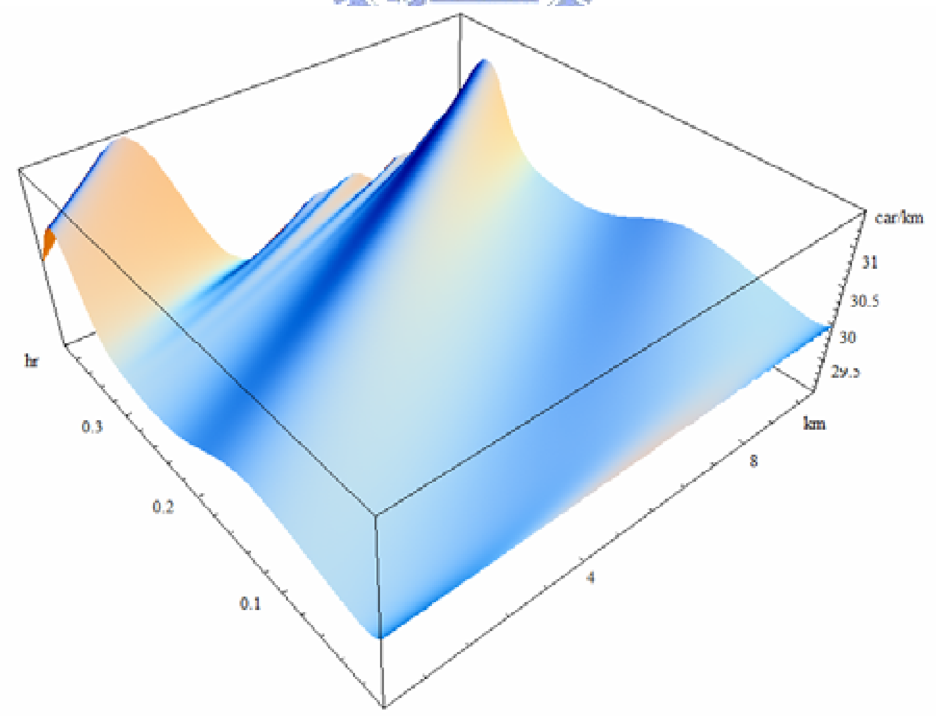

Fig. 5-3-1 Density Changing 1

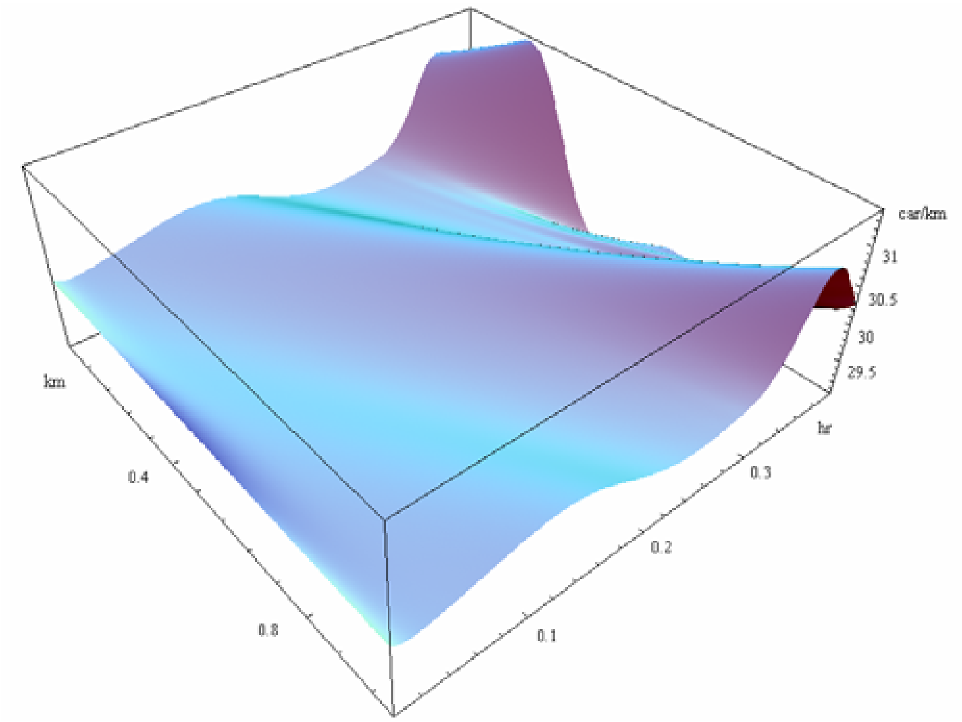

Fig. 5-3-2 Density Changing 2

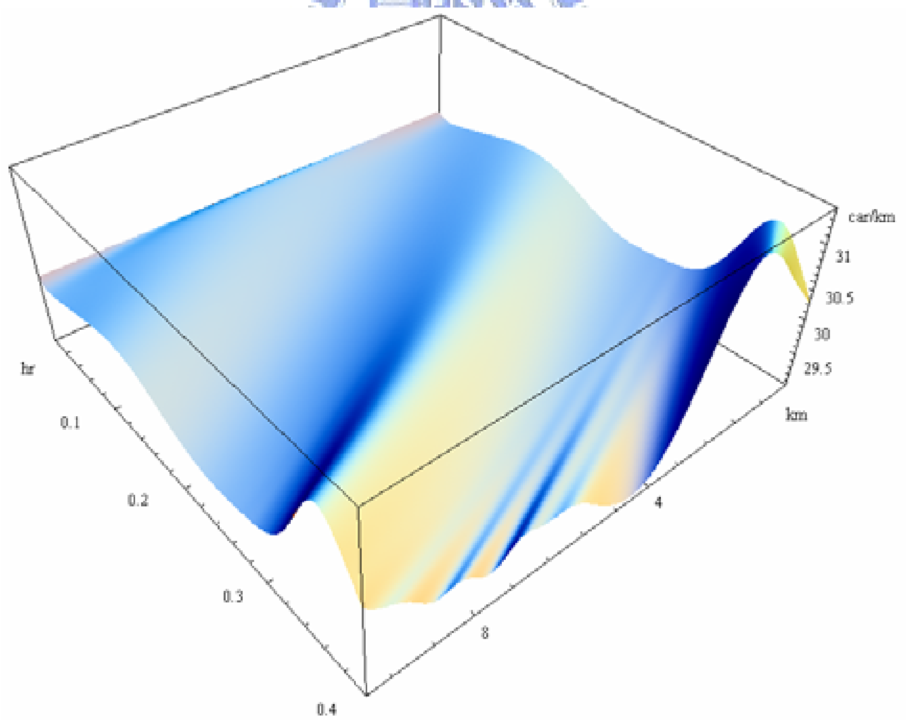

Fig. 5-3-3 Density Changing 3

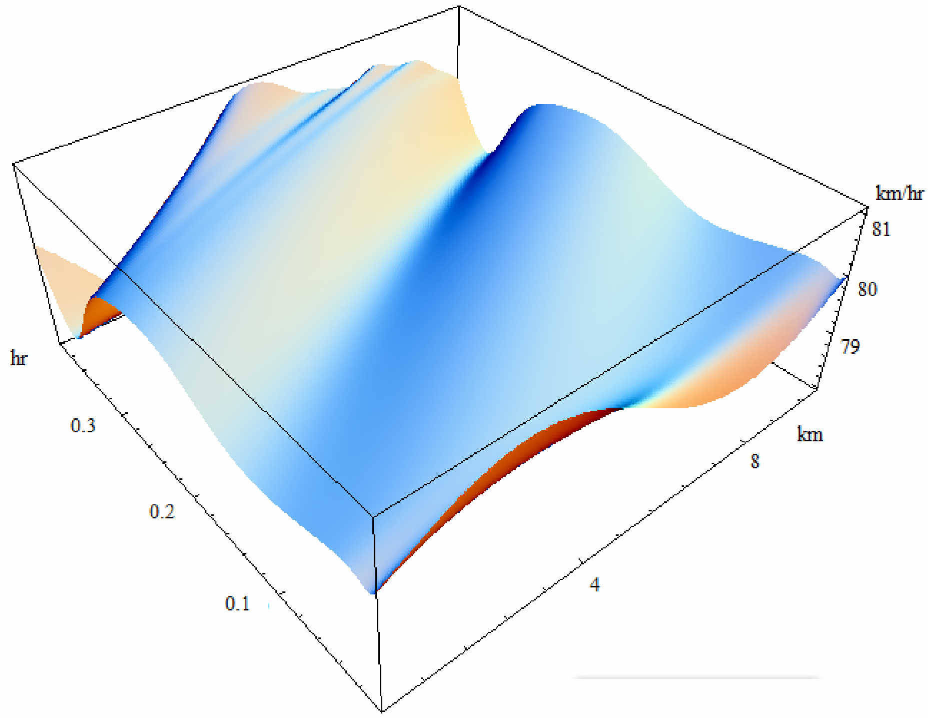

Fig. 5-3-4 Average Velocity Changing 1

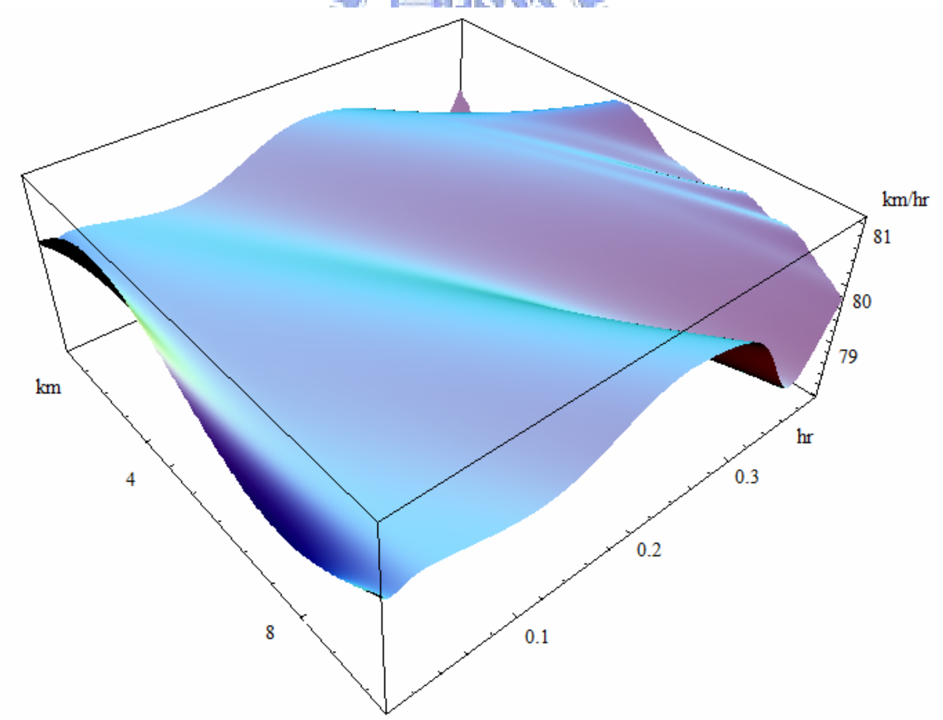

Fig. 5-3-5 Average Velocity Changing 2

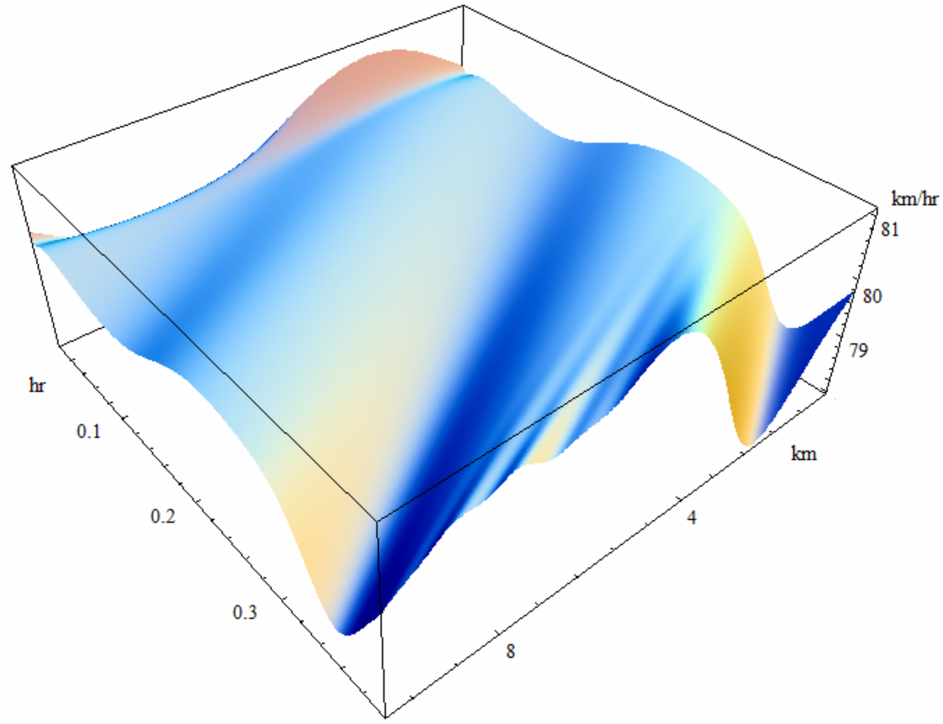

Fig. 5-3-6 Average Velocity Changing 3

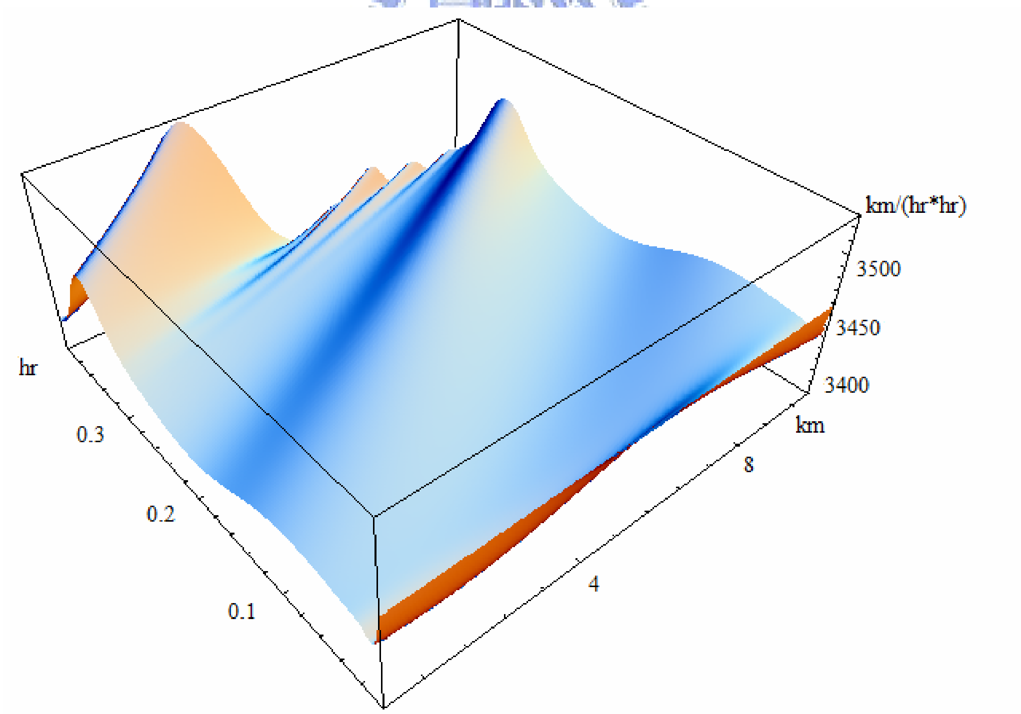

Fig. 5-3-7 Traffic Pressure Changing 1

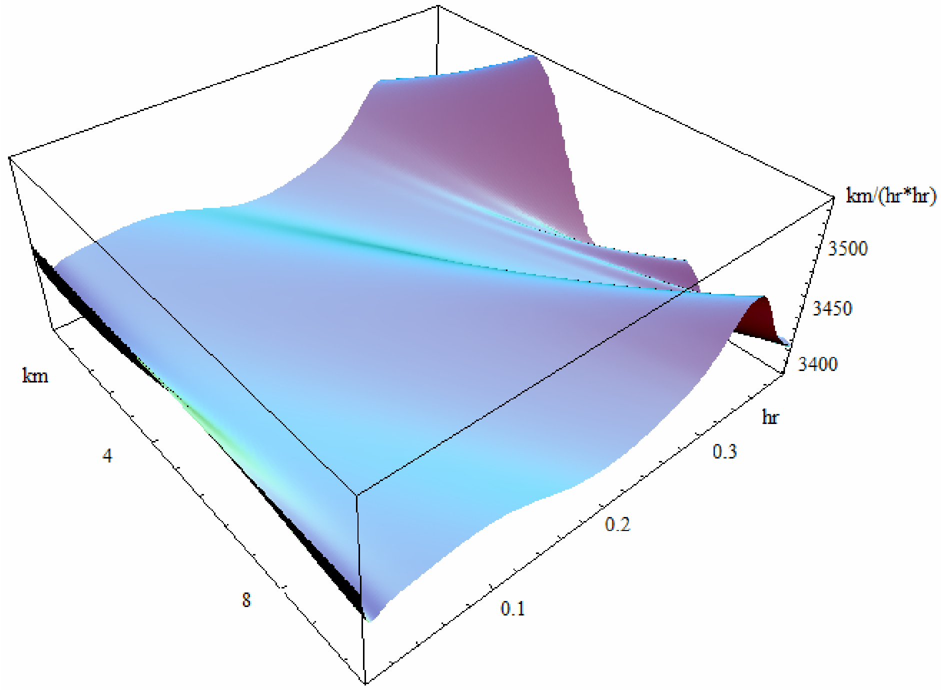

Fig. 5-3-8 Traffic Pressure Changing 2

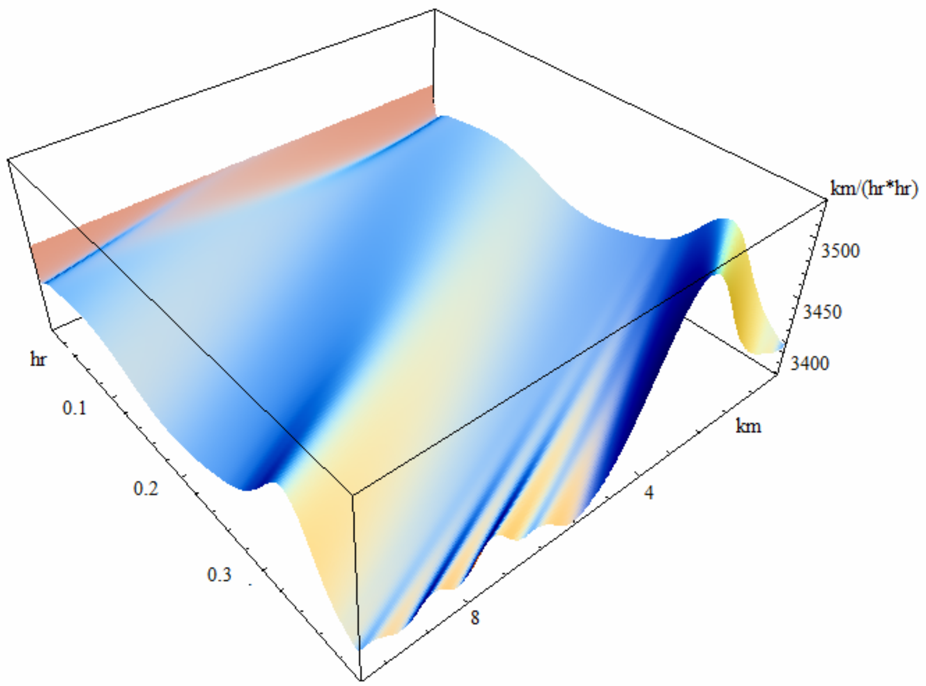

Fig. 5-3-9 Traffic Pressure Changing 3

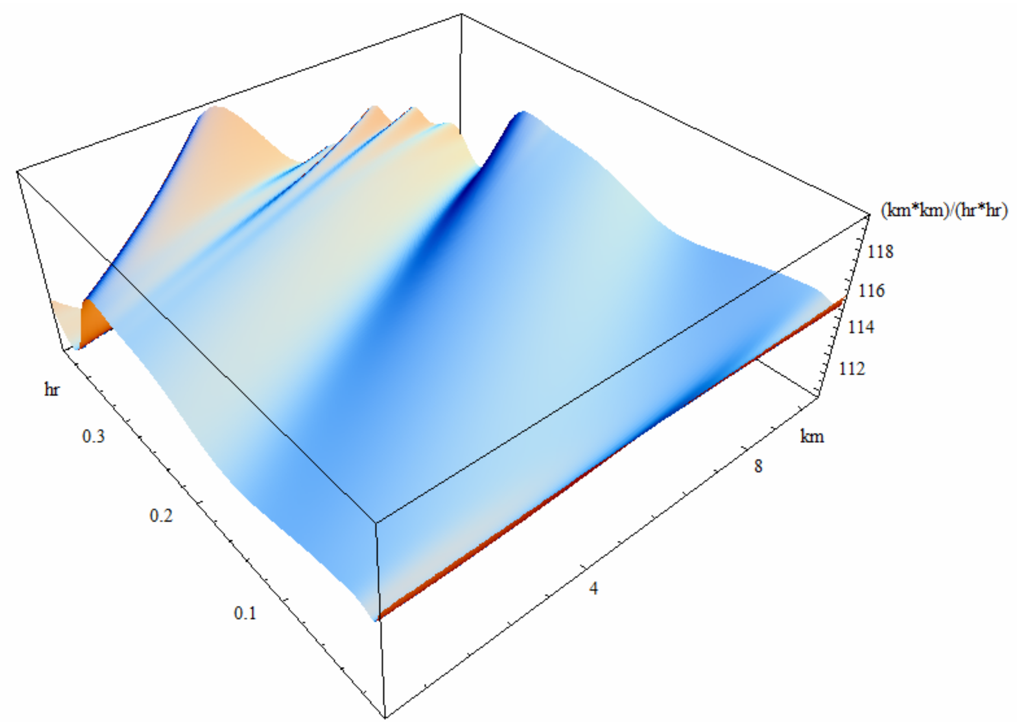

Fig. 5-3-10 Variance Changing 1

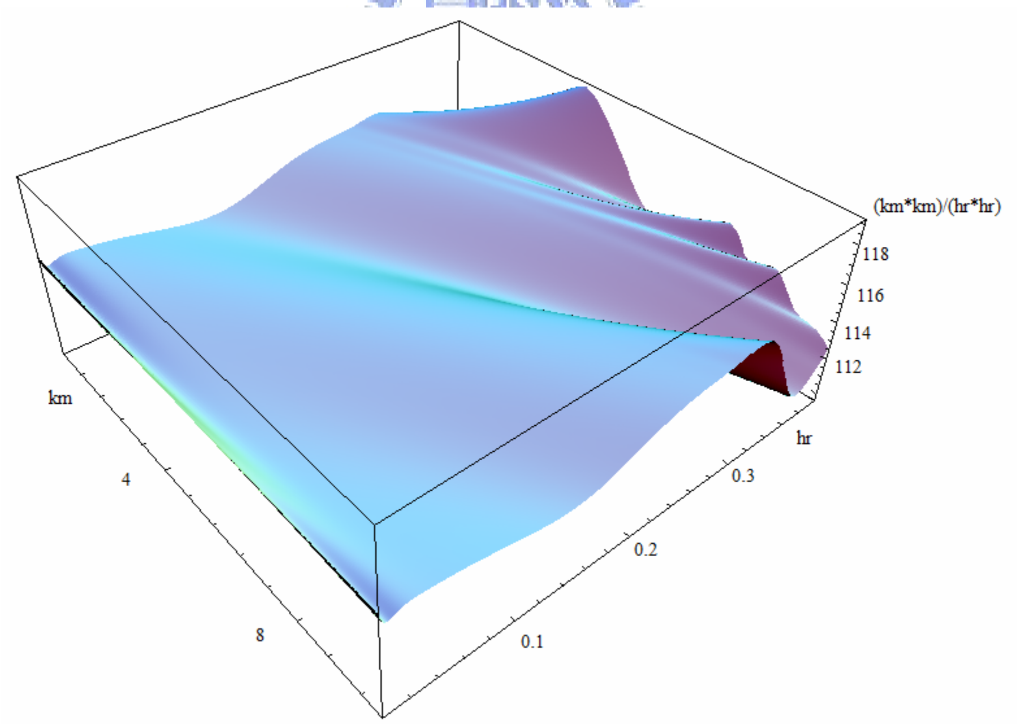

Fig. 5-3-11 Variance Changing 2

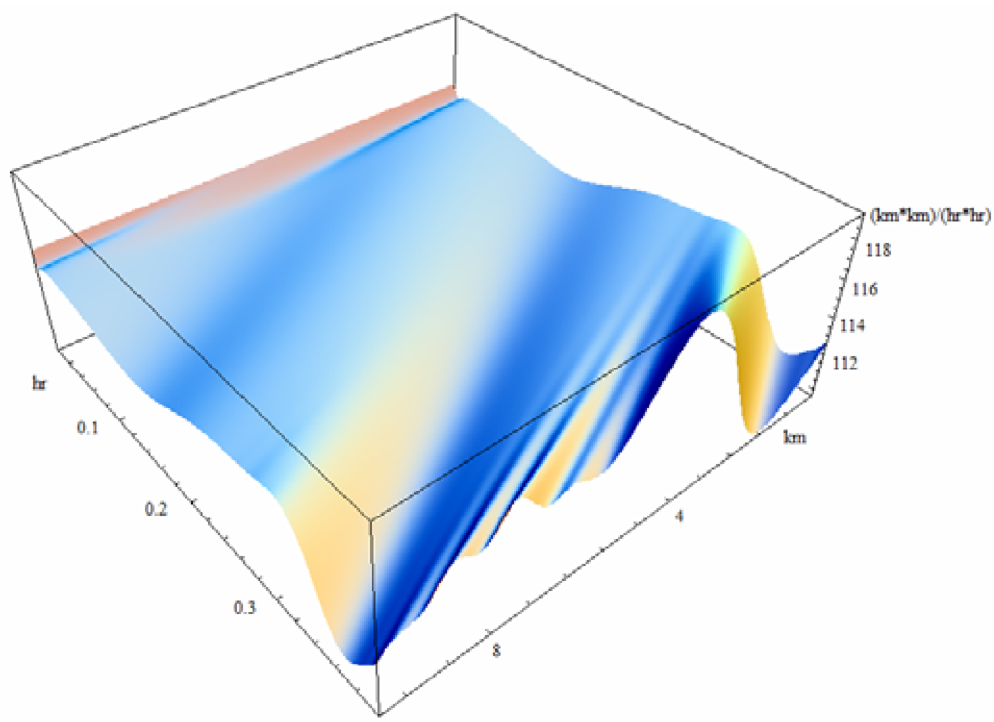

Fig. 5-3-12 Variance Changing 3

From the result of simulation, we could observe some result that match the real traffic, and we describe as the following.

(1). We could know the characteristic velocity is positive and it is the same result as the section 5.1. In dilute traffic, density wave could propagate downstream with time.

(2). When the traffic pressure becomes larger, the average velocity becomes slower. It is the same as general traffic theory. When drivers anticipate the density of downstream becomes heavier, traffic pressure becomes larger, drivers would decelerate and this causes the average velocity slower.

(3). When the density becomes heavier and average velocity becomes slower, variance would become smaller. Because cars could not pass, density would become heavier. The latter car decelerates to the velocity of the former car, so the average velocity becomes slower and variance becomes smaller.

<2>. Simulation of Different Passing Probability Effect:

In the same condition as the above, we could adjust different passing probability p to observe density, average velocity, traffic pressure and variance changing.

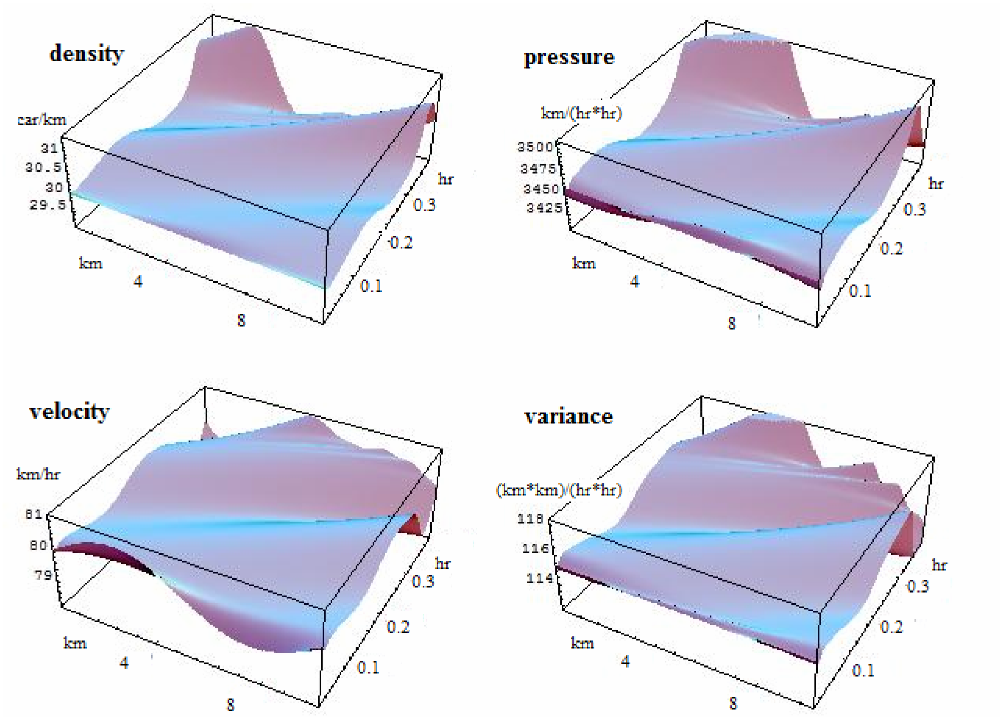

Fig. 5-3-13 Passing Probability =0.7

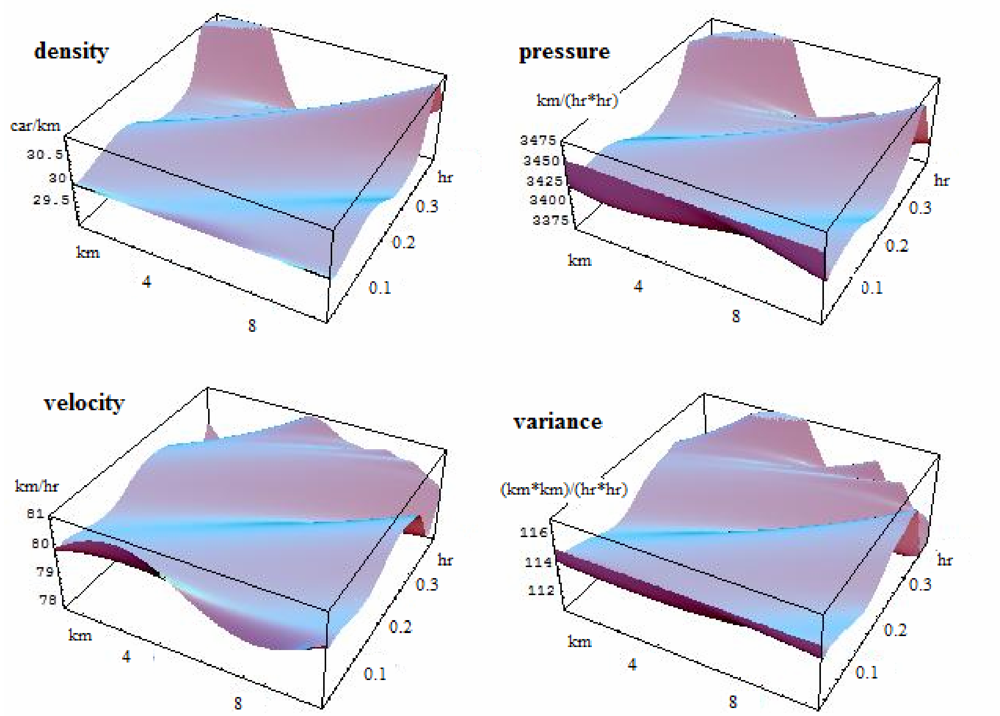

Fig. 5-3-14 Passing Probability =0.8

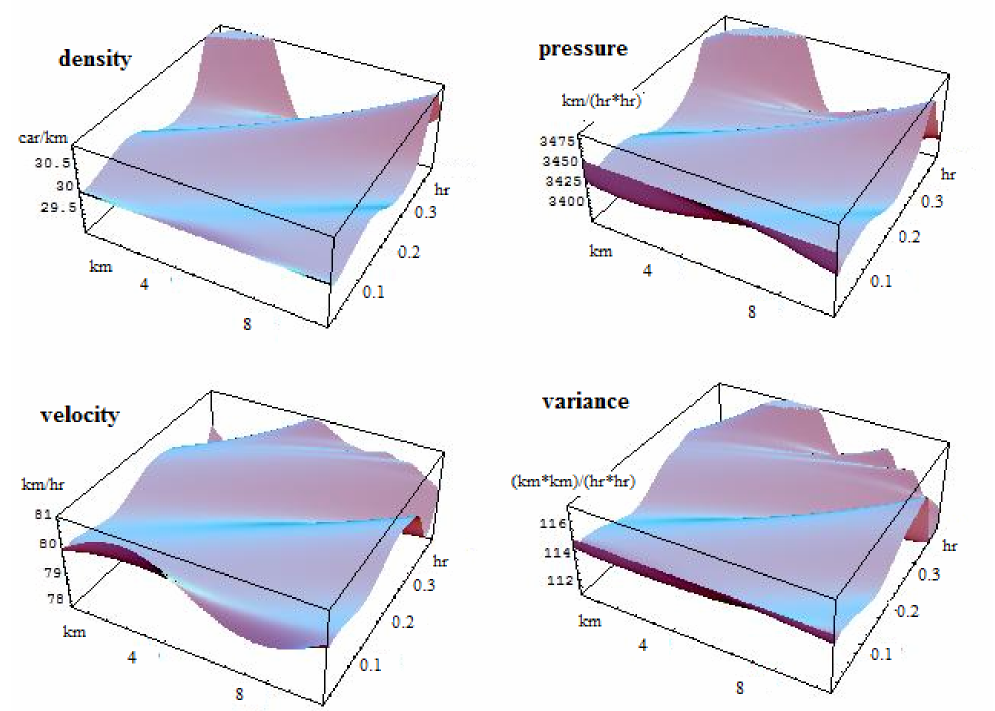

Fig. 5-3-15 Passing Probability =0.9

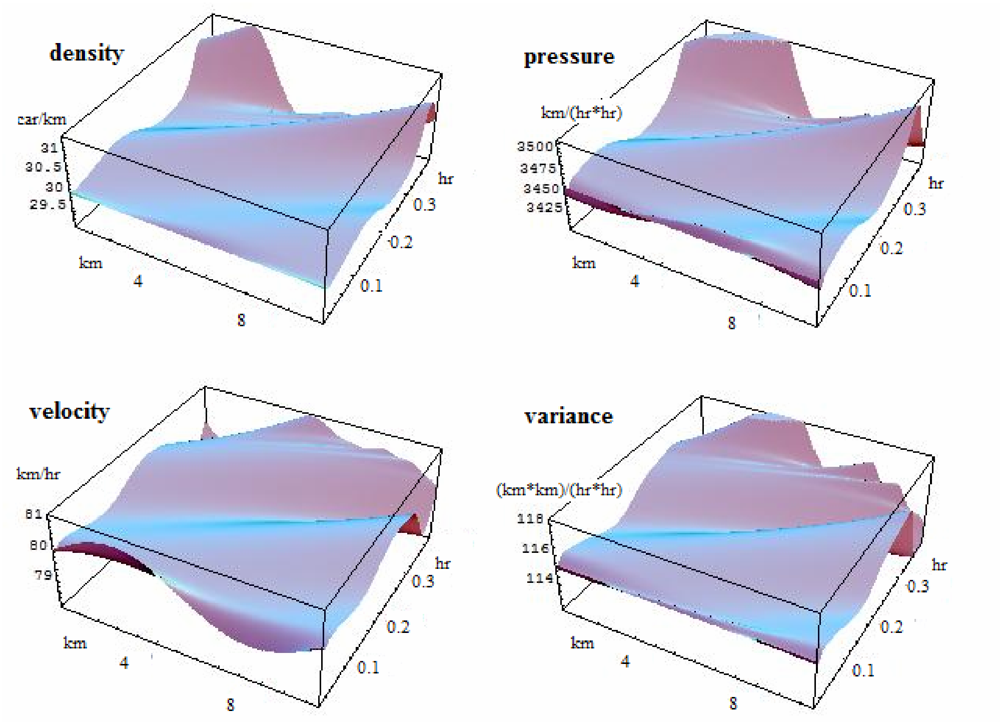

Fig. 5-3-16 Passing Probability =0.99

From figure 5-3-14 to figure 5-3-16, we could observe two characteristics.

1. When passing probability p becomes smaller, variance becomes larger. Because smaller passing probability p means cars could not pass each other easily, they could not attain their desired velocity easily. Therefore, variance becomes larger.

2. The whole changing trend does not change with passing probability p. Because we construct the model by assuming in dilute traffic, macroscopic model could approach local equilibrium quickly. Hence, the whole changing trend does not change with passing probability p.

<3>. Simulation of Different Uniform Deceleration Effect:

In the same conditions as the above, we compare density, average velocity, traffic pressure and variance with different uniform deceleration a and the same passing probability p. We use two kinds of passing probability p, $p = 0.99$ and $p = 0.7$, and three kinds of uniform deceleration a, $a = 60000$, $a = 600$ and $a = 6$. We show $a = 60000$ with blue, $a = 600$ with green and $a = 6$ with red.

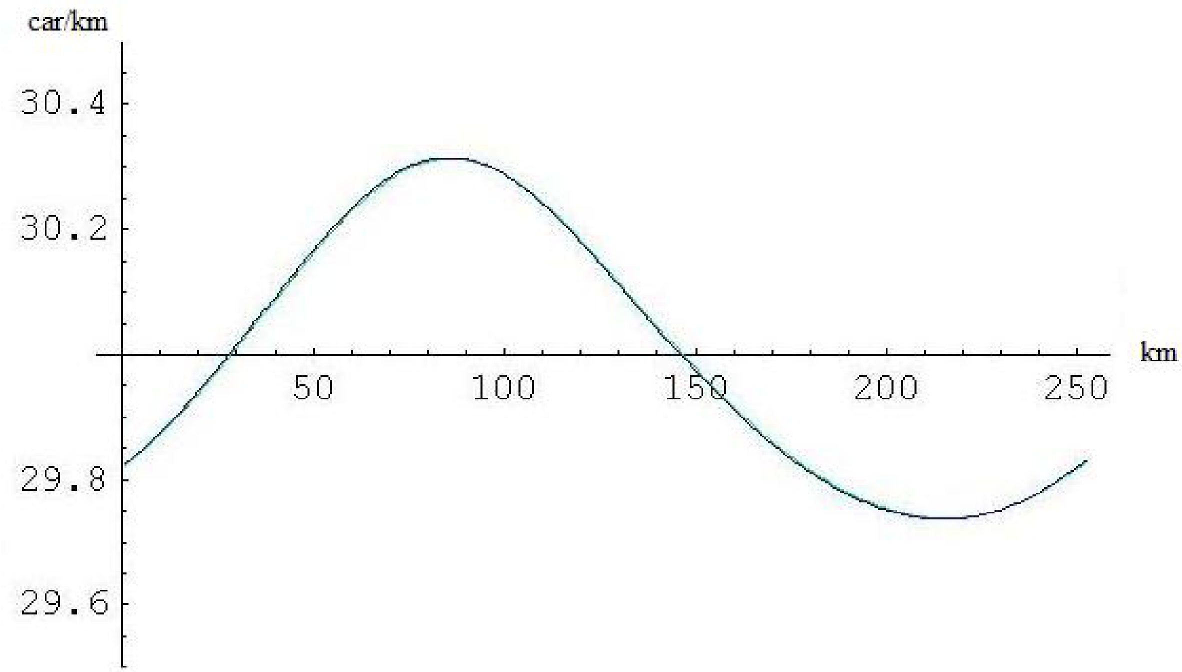

Fig. 5-3-17 Density in p=0.99 and Different a

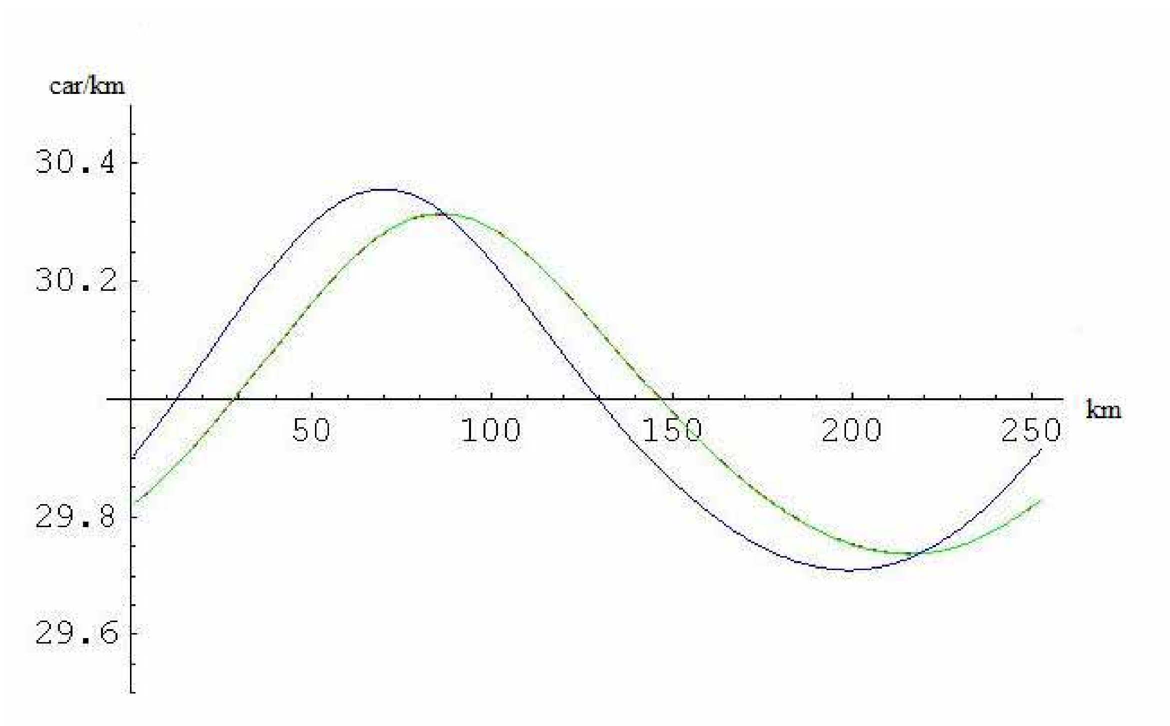

Fig. 5-3-18 Density in p=0.7 and Different a

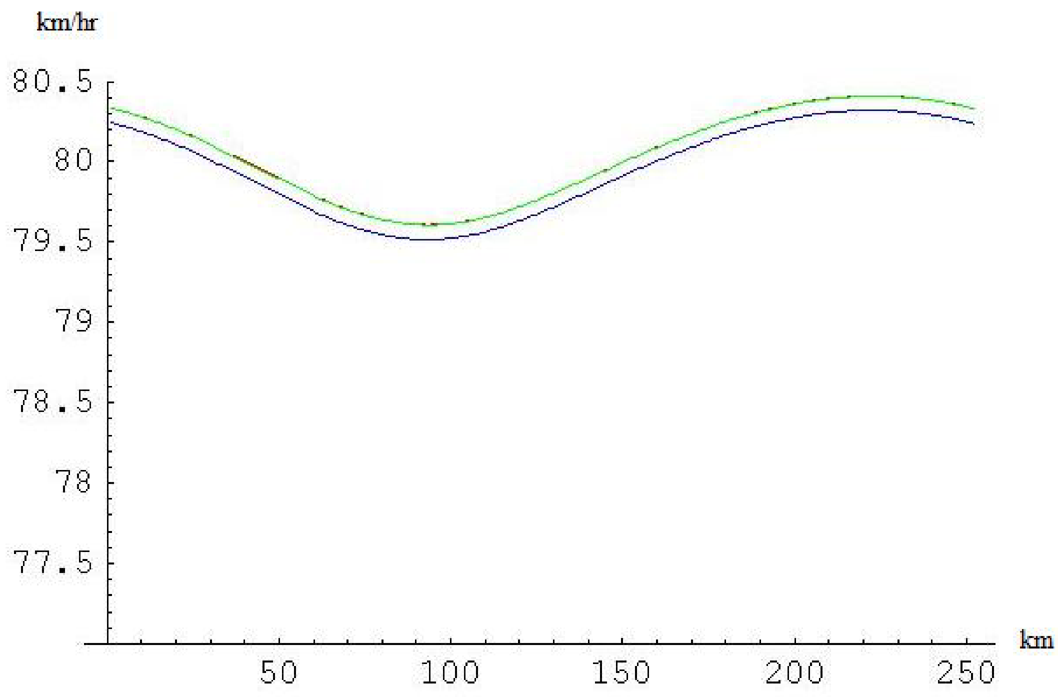

Fig. 5-3-19 Average Velocity in p=0.99 and Different a

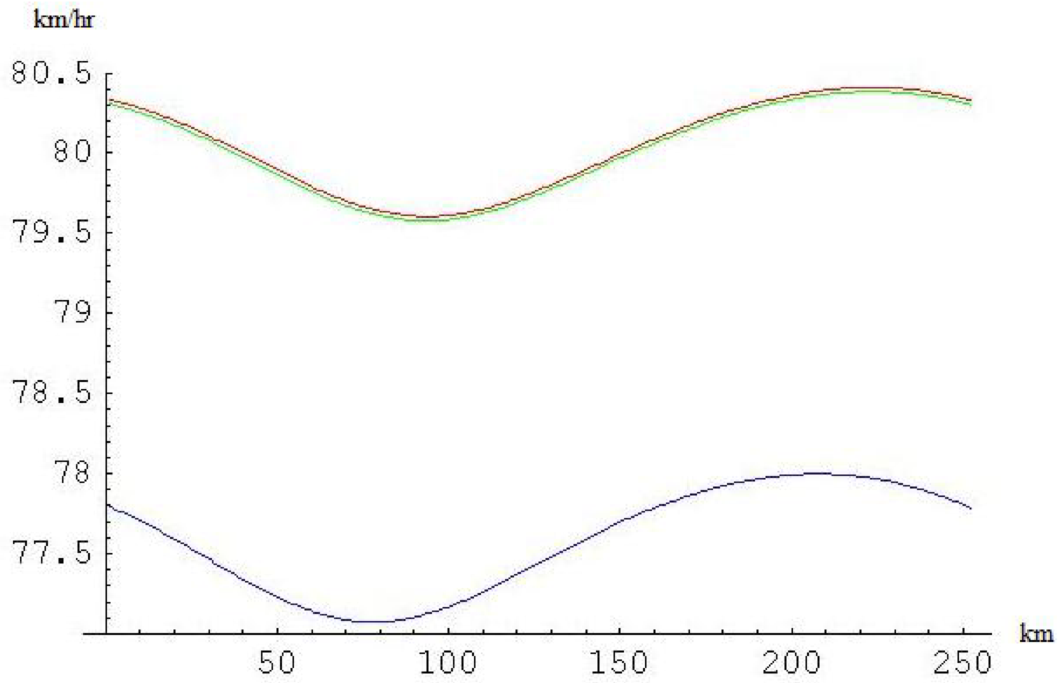

Fig. 5-3-20 Average Velocity in p=0.7 and Different a

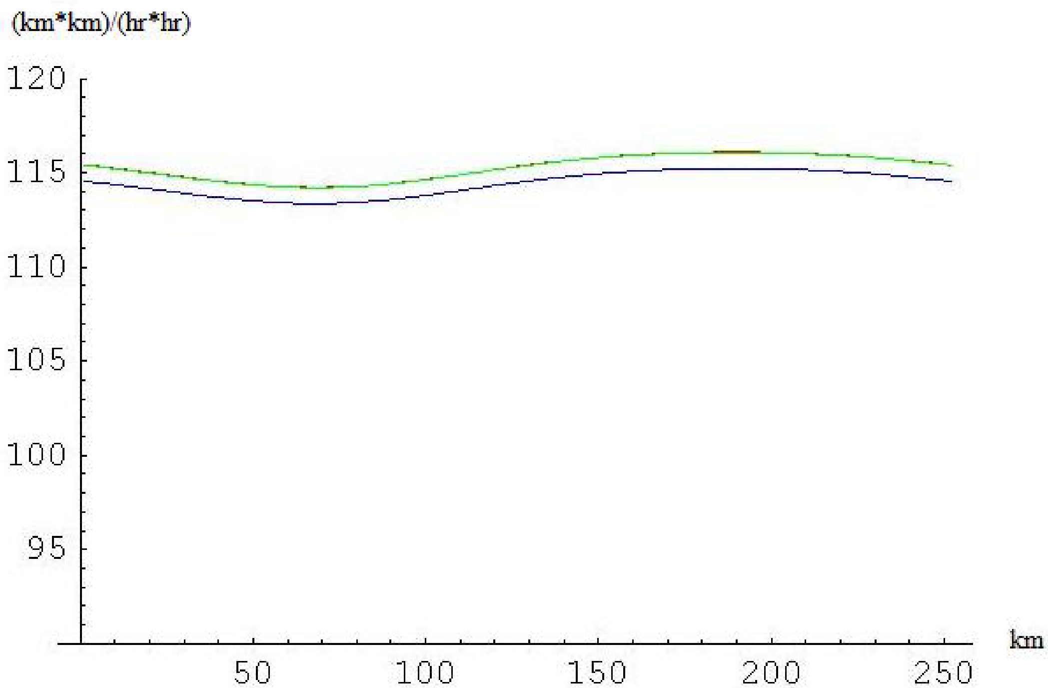

Fig. 5-3-21 Variance in p=0.99 and Different a

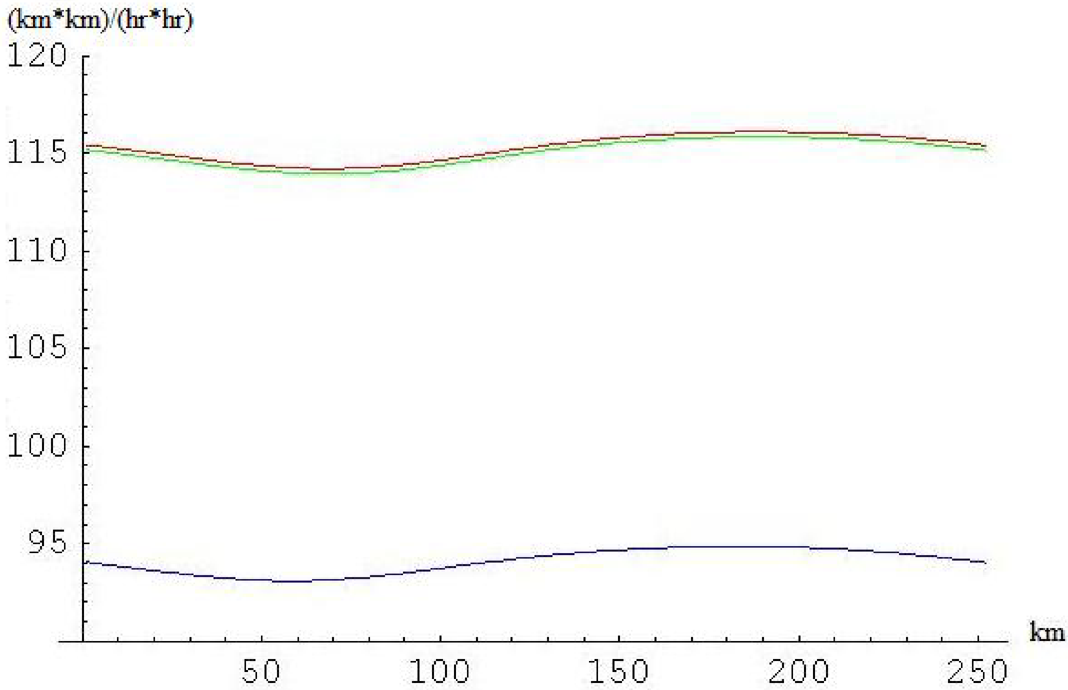

Fig. 5-3-22 Variance in p=0.7 and Different a

From figure 5-3-17 to figure 5-3-22, we could find that when passing probability becomes smaller, uniform deceleration $a = 60000$ affects density and average velocity more. By the way, we also could observe that when passing probability becomes smaller, different uniform deceleration a causes density and average velocity more different. In the view of mathematic, the value of uniform deceleration $a = 60000$ is larger, so it substitutes to affect interaction term more. Hence, the average velocity becomes slower and variance becomes smaller. In the view of traffic flow, the uniform deceleration becomes larger in dilute traffic, and it means that the letter car could approach to the velocity of the former car quickly. Compare to the traffic flow of smaller uniform deceleration, average velocity of the traffic flow of bigger uniform deceleration is slower and variance is smaller.

Besides, when passing probability is smaller, $a = 60000$ causes less effect to density than other two smaller uniform decelerations a. Because we assume approach local equilibrium quickly in dilute traffic, there have no heavy situation. However, different uniform decelerations would result in different velocities and variances, and they also affect the propagation of density wave. When the effect of interaction term becomes larger, it would vibrate the propagation of density wave more seriously.

Therefore, we could compare the effect of different uniform decelerations with different passing probability from figure 5-3-17 to figure 5-3-22, and we could observe that there has more difference in smaller passing probability. Because the number of cars use uniform deceleration is fewer with free passing, the whole has little difference. We could see the difference that cause by introducing uniform deceleration. This is the special point that other researches do not contain. Besides, it also shows that introducing uniform deceleration could describe traffic phenomenon more in heavy traffic.

5.4 Summary

From the first and second sections, we could know that our model could describe traffic flow more reasonable. The equations (5.6), (5.9), (5.12) and figure 5-1-1 show this research could describe more phenomenon than the model of C. Wagner and our model could explain the extreme condition. For example, because our model considers deceleration, when uniform deceleration $a = 0$ that shows drivers have no deceleration when the faster (the latter) car could not pass according to the equation (5.6). If the uniform deceleration $a = 0$, it means there have no deceleration to avoid collision without passing. It also means that the velocity of the former is the same as the latter one, so it has no deceleration. Because every driver has his desired velocity, they would accelerate until he attains his desired velocity. It means that drivers have attained their desired velocity, so they have no acceleration. So the average velocity is equal to the average desired velocity $\bar{v}_0 = \bar{v}$ in equilibrium. According to the definitions of covariance and variance, so covariance is equal to variance in equilibrium. These results are special results that the researches of other scholars do not have. Besides, we also discuss the extreme values of passing probability p and relaxation time T.

We analysis the three macroscopic equations that obtain from chapter 4 with numerical simulation “Upwind method” From figure 5-3-14 to figure 5-3-16, we could observe that when passing probability p becomes smaller, variance become larger. But the whole changing trend does not change with passing probability p. Besides, we could observe that there has more difference in smaller passing probability and see the difference that cause by introducing uniform deceleration from figure 5-3-17 to figure 5-3-22. This is the special point that other researches do not contain. Besides, it also shows that introducing uniform deceleration could describe traffic phenomenon more in heavy traffic.

Therefore, we could know that our model could describe more traffic phenomenon that other model could not explain. This means that it is meaningful to introduce uniform deceleration to relax instant velocity and consider finite space in gas-kinetic traffic flow model.

Chapter 6 Contribution and Future Works

This chapter aims to conclude the results of the preview chapters, put the result of this research in order, and summit suggestions in order to give reference to those interested in this topic. We hope this would be good to the development of mesoscopic traffic flow model in our nation.

6.1 Contribution

By means of the preview chapters, this section would conclude the emphasis of the preview chapters and sections, and illustrate the contribution of this research as following.

1. **Construct a mesoscopic traffic flow model with more physical meanings.**

   We introduce the uniform deceleration to relax instant velocity-changing and consider finite space, and construct a new mesoscopic traffic kinetic equation which could more close to the real traffic flow.

2. **Relax the original unreasonable assumptions of mesoscopic traffic flow model of Prigogine and Herman.**

   This research would relax infinite space and the assumption of instant velocity-change in interaction term of Prigogine and Herman's model.

3. **Describe more traffic phenomenon reasonably.**

   With integrating our mesoscopic model, we get macroscopic function based on macroscopic parameters, density $c(x,t)$, average velocity $\bar{v}(x,t)$ and variance $\theta(x,t)$.Then we analysis the model to obtain some characteristics about considering finite space and non-instant velocity-changing. For example, the results of different uniform decelerations are more different in lower passing probability. We know that our model could describe more traffic phenomenon that other model could not explain in chapter 5. This means that it is meaningful to introduce uniform deceleration to relax instant velocity and consider finite space in gas-kinetic traffic flow model.

4. **Describe the model and the meaning of characteristic in detail in this research.**

In this research, we want to explain the thoughts and meaning of behaviors and record the mathematical operation in detail. We hope would help those interested in mesoscopic traffic flow model and be good to the development of application.

6.2 Future Works

We simplify some complicated place in order to be convenient to express them in this research. Although that would not affect the correction of this research, that still limit us to illustrate traffic conditions. The followings are what we suggest to re-relax and research.

1. **Multi-lane and multi-class.**

Because this research emphasizes relaxing instant velocity-changing and infinite space, we simplify assumption of the lanes and class. The following researches could be improved toward the multi-lanes or multi-class. Nowadays, there several scholars discuss the complicated drivers' behaviors in mesoscopic traffic flow aspect.

2. **Consider the velocity distribution function of cars.**

We use normal function that assumes cars are independent in order to simplify in this research. In order to be close to the realistic traffic, we need to break down the assumption that every car is independent and we could adapt a velocity distribution function that could describe the interactions of cars. There are several famous scholars who try to research about velocity distribution function to describe more real traffic phenomenon at present.

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Appendix A

From section 5.1, we could know that eigenvalue ζ could be obtained by $\det[\zeta \boldsymbol{I} - \boldsymbol{G}] = 0$, where $\boldsymbol{I}$ is unit vector. Then we use Mathematica 5.1 to solve ζ, the real root is the following.

$$
\begin{aligned}
\xi \to\ & -(2^{1/3}(-B^2-3Ac-3E-3CE+BF-F^2-9\theta-6C\theta))/ \\
& (3 \\
& (2B^3+9ABc+27cD+27cCD+9BE+9BCE-3B^2F-18AcF+9EF+9CEF-3BF^2+2F^3+27B\theta+ \\
& 18BCq+18CFq+ \\
& \sqrt{(4(-B^2-3Ac-3E-3CE+BF-F^2-9\theta-6C\theta)^3+} \\
& \quad (2B^3+9ABc+27cD+27cCD+9BE+9BCE-3B^2F-18AcF+9EF+9CEF-3BF^2+2F^3+27B\theta+ \\
& \quad 18BC\theta+18CF\theta)^2))^{1/3})+ \\
& \frac{1}{3\,2^{1/3}} \\
& ((2B^3+9ABc+27cD+27cCD+9BE+9BCE-3B^2F-18AcF+9EF+9CEF-3BF^2+2F^3+27B\theta+18BC\theta+ \\
& 18CF\theta+ \\
& \sqrt{(4(-B^2-3Ac-3E-3CE+BF-F^2-9q-6Cq)^3+} \\
& \quad (2B^3+9ABc+27cD+27cCD+9BE+9BCE-3B^2F-18AcF+9eF+9CEF-3BF^2+2F^3+27B \\
& \quad \theta+18BC\theta+18CF\theta)^2))^{1/3})+\frac{1}{3}(B+F+3v)
\end{aligned}
$$

Because the definition of symbols A, B, C, D, E, and F of the equations (5.2) and (5.3), and time grid λ is a small value, we set $\lambda^2 \approx 0$. Then, we could simplify the above equation to the following.

$$
\begin{aligned}
\xi \to\ & -(2^{1/3}(-3E-9\theta))/(3(27B\theta+\sqrt{(4(-3E-9\theta-6C\theta)^3+(27B\theta)^2)})^{1/3})+ \\
& \frac{1}{3\,2^{1/3}}((27B\theta+\sqrt{(4(-3E-3CE+BF-F^2-9q-6Cq)^3+(27B\theta)^2)})^{1/3})+\frac{1}{3}(B+F+3v)
\end{aligned}
$$

We divide the above equation into three parts, the first row is the first term, the left-hand-side of the second row is the second term and the right-hand-side of the second row is the third term. Because the first and second term is more complex, we show the equation (5.5) with only the third term. But it does not mean that the first and the second terms have no meaning. It is a simple representation and its accent of characteristic velocity does not change owing to ignore the first and the second terms. Because these three terms all have $(1-p)a\lambda c$, we focus this point to discuss.

Appendix B - Symbols

1. Variables

x	one dimension position
y	parallel position (the second dimension)
t	time
v	one vehicle velocity
v_1	velocity
v_2	velocity
v_3	velocity
w	one vehicle desired velocity
r	total attribute
k	discrete attribute
b	continue attribute
o	other attribute
v^0	the absolute value of desired velocity (not contain direction)
ω^0	the angle of desired velocity
j	lane
u	driver (car class)

s	driving status
h	direction of traffic flow
d	destination

2. Constants

ℓ	car length
T	relaxation time
p	passing probability
a	uniform deceleration
τ	reaction time
$c\_\max$	the heaviest density

3. Functions

$c(x,t)$	density
$\bar{v}(x,t)$	average velocity
$\bar{v}_0(x,t)$	average desired velocity
$\theta(x,t)$	variance
$C(x,t)$	covariance
$q(x,t)$	flow

$P(x,t)$ traffic pressure

$J(x,t)$ flux of velocity variance

$J_e(x,t)$ flux of velocity variance in equilibrium

$\Phi(v_1,v_3)$ the velocity of the latter car during deceleration

$f(x,v,t)$ velocity distribution function of one vehicle

$f_0(x,w,t)$ desired velocity distribution function of one vehicle

$f_e(\bar{v}(x,t),\theta(x,t))$ velocity distribution function of one vehicle in equilibrium

$g(x,v,w,t)$ velocity distribution function of one vehicle (contain desired velocity)

4. Vectors

ζ eigenvalue

$\boldsymbol{I}$ unit vector