國 立 交 通 大 學

財 務 金 融 研 究 所

碩 士 論 文

考慮市場流動性不完全下之選擇權訂價與避險

Pricing and Hedging Options under Illiquid Markets

研 究 生：黃克鈞

指導教授： 鍾惠民 博士

中 華 民 國 九 十 六 年 七 月

考慮市場流動性不完全下之選擇權訂價與避險

Pricing and Hedging Options under Illiquid Markets

研 究 生：黃克鈞　　Student：Ke-Chun Huang

指導教授：鍾惠民　博士　　Advisor：Dr. Huimin Chung

國 立 交 通 大 學

財務金融研究所

碩 士 論 文

A Thesis
Submitted to Graduate Institute of Finance
College of Management
National Chiao Tung University
in partial Fulfillment of the Requirements
for the Degree of
Master
of
Science in Finance

July 2007

Hsinchu, Taiwan, Republic of China

中華民國九十六年七月

考慮市場流動性不完全下之選擇權訂價與避險

研究生：黃克鈞　　　　　　　　　　指導教授：鍾惠民 博士

國立交通大學財務金融研究所

2007 年 7 月

摘要

本篇論文建構在 Frey 與 Patie 2002 年的模型上，考慮流動性為股價所控制的函數。 Frey 在標準的 Black-Scholes 偏微分方程式中，加入流動性變數，用此求取選擇權價格。本篇論文的目的為，改善 Frey 偏微分方程式中的人造條件，使得此非線性偏微分方程更加穩定。因此，利用拔靴法求取波動度的上界，用其取代原本不合理的波動度上界。數值理論部份，改善了希臘字母(Greeks)在流動性不佳的情況下，不穩定的狀態。透過希臘字母的趨勢變化，可以幫助交易者更了解回饋效果在不同流動性市場下的變化。實證部份，我們挑選了 CBOE 交易量前 50 的股票選擇權當作標的物，且用上述的非線性偏微分方程去計算個別選擇權的價格。結果顯示在不完全流動的市場下，改善後的偏微分方程，可提供更精確的選擇權價格。

關鍵字: 流動性、選擇權定價、非線性偏微分方程、拔靴法、希臘字母、回饋效果、有限差分法

Pricing and Hedging Options under Illiquid Markets

Student: Ke-Chun Huang Advisors: Dr. Huiming Chung

Graduate Institute of Finance

National Chiao Tung University

July 2007

ABSTRACT

In this paper we build on Frey and Patie's literature (2002), where liquidity is a deterministic function of stock price. Frey implements an important factor, liquidity, into the standard Black-Scholes partial differential equation (PDE) to calculate the option price. The objective of our model is to improve an artificial pattern of Frey PDE to make the nonlinear PDE more reliable. Therefore, we choose bootstrap method to obtain the upper bound of volatility to replace the unreasonable setting. In numerical research, Greeks become smoother than before while using the bigger liquidity parameter. It helps traders to realize the variation of Feedback effect under different liquid markets. In empirical study, we choose the top 50 stock options of CBOE as underlying assets and use the PDE which contains liquidity parameter to solve each option price. The result shows that using the improved PDE offers more precise option prices in illiquid market.

Keyword: Liquidity; Option Pricing; Nonlinear PDE; Bootstrap Method; Greek Letters; Feedback Effect; Finite Difference Method.

誌謝

首先感謝鍾惠民教授在這兩年對學生的關懷與指導，讓我能夠順利的從交大財金所畢業。另外也對口試委員呂育道、王耀輝、謝文良、戴天時教授敬上最深的敬意，謝謝諸位教授珍貴的建議，才能使得此篇論文更加的完善。

在交大財金所短短的兩年中，交到了許多的好朋友，生活真的是多彩多姿。淑惠爽朗的個性、三花超帶衰的生活、胖兔不計較的心態、維峻完美的謀略、殺手慢步調的幽默、凱秩驚人的天份、阿師超齡的年紀、大爺愛裝娘的怪癖、小郭多樣化的生活…等。這些點點滴滴都會一直埋藏在我心中，因為有你們的存在，我的研究所生活才會如此精采。如果再讓我選擇一次，我一定會祈求上帝能夠跟這群優秀又幽默的同學一起生活，感謝你們帶給我的種種回憶，使我能夠快樂的學習，在最困苦的時候能不畏懼困難往前走。

最後要感謝我親愛的家人，跟你們生活在一起是全天下最幸福的事情，也許我門不是最有錢的家庭，但一定是最歡樂的家庭。希望我能夠越來越好，永遠永遠不會讓你們失望！

黃克鈞 謹誌

國立交通大學財務金融研究所

中華民國九十六年七月

Contents

1. Introduction

Since the Black-Scholes (BS) formula was innovated, this well-known formula has given traders a benchmark to trade option with appropriate price. Market participants initially take BS price as a standard and adopt hedging strategies to adjust their positions. However, investors gradually find that BS formula offers limited option price because of the restrictive BS assumptions. In order to release the assumptions to provide more reliable option prices, a number of new models have been developed for option valuation.

In this paper, we would like to release a specific assumption that markets are assumed to be completely elastic (perfect liquid market). Specifically, the hypothesis assumes the price do not change no matter how large amounts of an asset are traded. If all of the market participants are small investors, their trading strategies do not influence market prices and the market is perfectly liquid in this scenario. However, many markets are not perfectly liquid due to the presence of large investors. Large investors' trades have price impacts and thus they face illiquid market. According to Kyle(1985), large investors can use its market power to move prices in a certain direction. Based on Chan and Lakonishok (1995), large institutions with high proportion of trade tend to be associated with larger market impact. Therefore, it obviously violates the truth of reality and conveys an important message that markets are not perfectly liquid.

The absence of liquidity leads markets to face serious problems, such as financial crisis. A well-known event of illiquid problem is Long-Term Capital Management (LTCM) crisis. That is a hedge fund which the company manager longs less liquid instruments and shorts more liquid instruments at the same time. This trading strategy with high leverage initially brings significance profit to LTCM but the company's manager does not anticipate that their high risk manipulations would result in financial crisis. When the Russian debt crisis occurs, most of market liquidity suddenly deteriorates. Consequently, the LTCM not only loses all of their properties in a short period but also causes an inevitable financial disaster. Hence, there are numbers of researchers probing into the cause of market illiquidity.

In academic literature, there are three causes of illiquidity which are transaction costs, asymmetric information of asset and imperfect competition in asset market. In addition, another interesting theory causing market illiquidity is uncertainty. Uncertainty means that traders face a circumstance which they can not anticipate any changes and do not know how to management.

Moreover, many scholars try to explain the financial meaning of liquidity. Kyle (1985) explains market liquidity from three aspects which are tightness, depth and resilience of the market. Tightness measures the cost between buying and selling a position in a short period. Depth refers to the size of a transaction required to change prices. Resilience measures the speed of prices recovering fundamental after a sudden event occurs.

Schwartz (1988) explains that liquidity is the ability to trade immediately. Amihud and Mendelson (1989) divide the liquidity into two parts including price aspect and time aspect. In point of price aspect, if assets can be traded in specific time and minimize the concession from buyer and his counterparties, assets are more liquid. For time aspect, when buyer and his counterparties make acceptable concession, the less time used for transaction the more liquidity assets have. As I mentioned above, these ideas give us some concepts about market illiquidity.

During recent years, risk management has become a popular issue in the field of finance. Three types of risks are market risk, credit risk and liquidity risk. Bielecki and Rutkowski (2002) have successfully made an exceptional summary regarding market risk and credit risk. Meanwhile, for liquidity risk, Frey establishes an innovative nonlinear PDE to hedge portfolio perfectly. Frey and Patie (2002) continue this work and make an extension to describe the liquidity as the deterministic function of stock price. In addition, they offer some numerical results to show the illiquidity influence of hedging costs and Greeks' derivatives.

Furthermore, Esser and Moench (2003) generalize the liquidity model of Frey (2002) and incorporate the stochastic liquidity into stock process. In other words, they use their innovation to analyze the differences among large investors vary from market depth. Then, they make an observation of the variation of large investors' hedging strategies related to stock prices in illiquid markets as well. In addition, they verify that stochastic liquidity setting is more suitable for financial markets. Overall, their contribution and research provide a flexible model of hedging strategies under illiquid market.

From the above, we can easily say that there are more researchers investigating the modeling and hedging strategies while market illiquidity or large investors exist. In order to appreciate their contribution and dedication, we provide some further related references such as Kyle (1985), Jarrow (1994), Schonbucher and Willmott (2000) and Simona (2006).

Thus, the core of our paper is to analyze the last type of risk, liquidity risk. We capture the notion of liquidity from Frey (2000) and our model builds on Frey and Patie (2002). We take liquidity as a deterministic function of price and implement it into nonlinear PDE. Using finite difference method (FDM), it helps to solve the problem of computing nonlinear PDE to obtain option prices. We choose least square method to estimate the liquidity. In order to make estimation more reliable, we use bootstrap method to calculate the upper bound of volatility of each asset to revise the artificial pattern. Moreover, Esser and Moench (2003) suggest the asymmetry relationship between the stock price and liquidity should be removed. All of these approaches help us to obtain more precise option prices and apply to dynamic hedging strategy to minimize the tracking error.

When it comes to trading strategy, we must mention feedback effect strategy. There are two types of feedback effect trading strategies. One is called positive feedback effect. For example, investors buy risky assets when the asset price is increasing and sell the risky asset when the asset price is declining. Similarly, there is a negative feedback effect. Its trading strategy is very similar to positive feedback. Large investors buy the risky asset while the asset price is declining and sell the risky asset while asset price is increasing. Those strategies are used to replicate convex payoff, long call, and concave payoff, short call, respectively. We would show these results in numerical section.

The remainder of the paper is organized as follows: in section 2, we introduce the basic assumption of Frey model (2002) and make a description of how to derive the nonlinear PDE. Afterwards, we modify the artificial pattern mentioned in Frey and Patie (2002) and explain why it is not appropriate. In section 3, numerical results would help us to realize the feedback effect. In section 4, estimate option price using data of CBOE top 50 companies is applied to our empirical study. Finally, we summarize the discoveries in this context and offer our conclusions in section 5.

2. The Deterministic Liquidity Model

Frey and Patie (2002) assume that financial markets trade between one riskless asset (bond) and one risky asset (stock or stock index). They consider bond markets as perfectly liquid markets. In other words, no matter how large shares of bonds are traded, it does not influence the bond prices. In this circumstance, large investors and small market participants have the same weight in bond markets. On the other hand, stock markets seem to be illiquid markets. For example, a significant trade would cause price impact. This phenomenon is in line with economy intuition that money markets are more liquid than the stock markets.

The cause of the illiquid market is the presence of large investors. Their trading strategies lead to price motion and bring feedback effect to underlying assets. For example, the price of an asset increases (declines) when the institution investors buy (sell) the asset. This assumption is supported by Holthausen and Leftwich (1987) that large transactions are related to price impact. Therefore, Frey and Patie argue that large investors' manipulations violate the assumption that market is perfectly liquid.

In order to release the above assumption, Frey incorporates the liquidity parameter, ρ, in dynamic process of asset price. Initially, liquidity is treated as a constant which means that the liquidity of the asset does not change through time. Frey and Patie treat liquidity as the deterministic function of time and stock price. Our model builds on this concept and revises an artificial pattern of Black-Scholes PDE.

In the following, we offer the basic assumptions and asset price dynamics of Frey and Patie (2002). First, we introduce the derivation of nonlinear PDE. Second, we provide some essential knowledge of trading strategy. Third, we use the introduction of the tracking error to investigate the performance between perfect and imperfect market liquidity hedging strategy. Finally, we take the advantage of numerical method to solve the nonlinear PDE and revise the problems of the artificial pattern which may produce irrational volatility.

2.1 Basic assumptions of the deterministic liquidity model

The core of the basic assumption explains the trading strategy which the large investors have followed. Instead of building on microeconomic equilibrium, Frey argues that the variation of asset price dynamics originates from large investor's stock holdings or trading strategy (α_t). Furthermore, Frey takes the influence of price impact as exogenous. Therefore, we illustrate some assumptions of asset trading strategies as follows:

(A1) The stockholdings $(\alpha_t)_t$ are left-continuous (i.e. $\alpha_t = \lim_{\substack{< \\ s\to t}} \alpha_s$).

(A2) The right-continuous process α^+ with $\alpha_t^+ = \lim_{\substack{< \\ s\to t}} \alpha_s$ is a semimartingale.

(A3) The downward-jumps of trading strategy are bounded: $\Delta\alpha_t^+ := \alpha_t^+ - \alpha_t > -1/\overline{\rho}$ for some $\overline{\rho} > 0$.

These three assumptions provide large investors' trading strategy as a smooth function of the stock price.

It is obvious that parameter ρ causes the fluctuation from the standard Black-Scholes model. When the parameter equals to zero or large investor maintains his stock holdings without changing (i.e. $\alpha_t = 0$), the asset price dynamics follow the Black-Scholes model where volatility follows the famous Brownian motion. Then, we will provide the asset price dynamics with considering liquidity which is decided by large investor's trading strategy α_t depending on $S_t(\rho,\alpha)$.

(A4) If the trading strategy follows the previous assumptions, the stochastic differential equation is as follows:

$$dS_t = \sigma S_{t-} dW_t + \rho\lambda(S_{t-}) S_{t-} d\alpha_t^+ \tag{2.1}$$

where implies the left limit $\lim_{\substack{< \\ s\to t}} S_t$

In this stochastic differential equation, we assume σ to be positive and λ to be a continuous function. The parameter ρ must be greater than zero as the number of ρ increases, the less liquidity the market becomes. We use $1/(\rho\lambda(S_{t-})S_{t-})$ to represent the market depth at time t. From this equation, we realize how much shares could cause the asset price to move by one unit. According to the suggestion of Esser and Moench (2003), they indicate that continuous function λ does not have significant influence to explain the liquidity tendency. Thus, we do not incorporate λ in our model for the following pages. We transfer the stochastic differential equation transfers to equation (2.2).

$$dS_t = \sigma S_{t-} dW_t + \rho S_{t-} d\alpha_t^+ \tag{2.2}$$

The following shows an example providing a better understanding of the asset price assumptions. First, we set up a scenario that large investors hold the specific asset of K shares which K must over zero and less than $1/\overline{\rho}$. There is still another limitation of asset price. When the asset price drops under $\overline{S}$, large investors would sell all of the stock holdings. If there is no transaction obstacle (perfect liquid market), his portfolio value would be above or at least equals to $K\overline{S}$.

After the basic setting, we start to introduce a large investor trading strategy. The stopping time τ is set up as $\tau = \inf\{t > 0, S_t < \overline{S}\}$. The trading strategy α is set up as $\alpha_t = K, 0 \le t \le \tau$ *and* $\alpha_t = 0$ for t $> \tau$. It corresponds to the Assumption A1, A2 and A3. We can find out that α is left continuous because $\alpha_t = K, 0 \le t \le \tau$ and α^+ has a bounded downward jumps $\left(-K > -1/\overline{\rho}\right)$. Then, we can derive the portfolio value at time τ in illiquid markets. When investor's stock price becomes $S_\tau = S_{\tau-} - \rho S_{\tau-} K < S_{\tau-}$ his portfolio does not have the perfect protection, i.e. $KS_\tau = KS_{\tau-} - \rho S_{\tau-} K^2 < KS_{\tau-}$.

It is obvious that the difference between deterministic liquidity model and Black-Scholes model is whether implementing the liquidity pattern into the stochastic differential equation or not. To make the deterministic liquidity model useable, we need to find out how to determine the parameter ρ. As we known, there are two different avenues to determine the ρ. In this article, our approach uses the observed data to calculate the implied liquidity. Then, we would mention how to estimate the implied liquidity later.

2.2 Dynamic process of asset price

In this part, we combine the underlying asset's stochastic differential equation and the trading strategy α by using the $It\hat{o}'s$ formula. In addition, Frey and Patie assume that their stockholdings trading strategy is a smooth function ϕ of time and the current asset price. This assumption plays an important role of replicating in option valuation.

(A5) Trading strategy ϕ is a function of time and current asset.

$[0,T]\times\mathbb{R}^{+}\rightarrow\mathbb{R}$ is of class $C^{1,2}([0,T]\times\mathbb{R}^{+})$.

where $\phi(t,S_t)=\alpha_t$

Moreover, $\rho S\phi_s(t,S)<1$ for all $(t,S)\in[0,T]\times\mathbb{R}^{+}$

In the sequel, we derive the asset price dynamics by stochastic differential equation (2.2) and assumption A5. First, we implement $It\hat{o}'s$ formula into trading strategy to get the following

$$d\alpha_t=\phi_s(t,S_t)dS_t+\left(\phi_t(t,S_t)+\frac{1}{2}\phi_{ss}(t,S_t)v^2(t,S_t)S_t^{\,2}\right)dt \tag{2.3}$$

Hoping to get more realistic asset dynamics, we add the risk free rate γ into equation (2.2).

$$dS_t=rS_{t-}dt+\sigma S_{t-}dW_t+\rho S_{t-}d\alpha_t^{+} \tag{2.4}$$

Then, the further step is to put the equation (2.3) into equation (2.4).

$$dS_t=rS_tdt+\sigma S_tdW+\rho S_t\phi_s(t,S_t)dS_t+\rho S_t\left(\phi_t(t,S_t)+\frac{1}{2}\phi_{ss}(t,S_t)v^2(t,S_t)S_t^{\,2}\right)dt \tag{2.5}$$

After arrangement, we obtain

$$(1-\rho S_t\phi_s(t,S_t))dS_t=\sigma S_t dW+\left[\rho S_t\left(\phi_t(t,S_t)+\frac{1}{2}\phi_{ss}(t,S_t)v^2(t,S_t)S_t^{\ 2}\right)+rS_t\right]dt \quad (2.6)$$

By dividing the coefficient $(1-\rho S_t\phi_s(t,S_t))$, we get the new asset price dynamics.

$$dS_t=\frac{\sigma S_t}{(1-\rho S_t\phi_s(t,S_t))}dW+\frac{\left[\rho S_t\left(\phi_t(t,S_t)+\frac{1}{2}\phi_{ss}(t,S_t)v^2(t,S_t)S_t^{\ 2}\right)+rS_t\right]}{(1-\rho S_t\phi_s(t,S_t))}dt \quad (2.7)$$

In order to make the asset price dynamics more clearly, we take $v(t,S_t)$ as the volatility term of underlying asset and use $b(t,S_t)$ to represent the drift term of underlying asset. Finally, we derive the general form of the asset price process under market illiquidity.

$$dS_t=v(t,S_t)S_t dW_t+b(t,S_t)S_t dt \quad .(2.8)$$

$$v(t,S_t)=\frac{\sigma}{(1-\rho S_t\phi_s(t,S_t))} \quad (2.9)$$

$$b(t,S_t)=\frac{\rho}{(1-\rho S_t\phi_s(t,S_t))}\left(\phi_t(t,S_t)+\frac{1}{2}\phi_{ss}(t,S_t)v^2(t,S_t)S_t^{\ 2}\right)+\frac{r}{(1-\rho S_t\phi_s(t,S_t))} \quad (2.10)$$

There is something interesting after we derive this innovative asset price dynamics. The volatility term has transferred from constant into a function which depends on time and asset price, i.e. $v(t,S_t)$. This transition results from large investors' trading strategy. In trading markets, large investors have two general trading strategies, positive feedback strategy and contrarian strategy, to protect their position. When large investors follows positive feedback strategy (contrarian feedback strategy), their trading strategy is to buy (sell) additional shares of the assets while the asset price lifts up (drops down).

In other words, the new volatility becomes greater than the constant volatility if the representative investor chose the positive feedback strategy, i.e. $v(t,S_t)>\sigma$ if $\phi_s(t,S_t)>0$. On the contrary, the new volatility is smaller than the constant volatility if the representative investor chose the contrarian feedback strategy, i.e. $v(t,S_t)<\sigma$ if $\phi_s(t,S_t)<0$.

2.3 Derivatives with nonlinear PDE

The objection of this part is calculating the derivatives price. Having the new asset price dynamics helps us to replicate the derivatives such as call or put option. In recent research, there are at least three methods to estimate the value of derivatives including binomial tree, Monte Carol simulation and solving the partial differential equation (PDE). Each of approach has its own advantages and disadvantages. In this context, we choose PDE method as an instrument to estimate the price of derivatives in illiquid market.

In the beginning, we assume the derivative is path-independent and set its smooth payoff as $h(S_T)$. Before building up the new asset price dynamics (2.8), traders use the standard Black-Scholes strategy to hedge. However, Black-Scholes theory is only effective in perfectly liquid market. Consequently, traders would face a great loss when they adopt the Black-Scholes trading strategy to replicate the derivatives. Since Frey and Patie have built up the new theory considering market illiquidity, traders can obtain more reliable hedging strategy and value of derivatives by adopting the Frey and Patie theory.

It is comparatively easy to understand the Frey and Patie PDE by introducing the Black-Scholes PDE at first because we are familiar with the classical theory. The Black-Scholes PDE is presented as follows:

$$u_t(t,S) + rSu_s(t,S) + \frac{1}{2}\sigma^2 S^2 u_{ss}(t,S) = ru(t,S) \tag{2.11}$$

where $u(T,S)$ is the derivative pice at time T, i.e. $h(S_T)$

To recall the equation (2.8), we find out the volatility term in equation (2.11) should be replaced as $v(t,S_t)$. Then, we obtain the extension of the Black-Scholes PDE as equation (2.12)

$$u_t(t,S) + rSu_s(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho S_t u_{ss}(t,S_t))^2} S^2 u_{ss}(t,S) = ru(t,S) \tag{2.12}$$

where $u_{ss}(t,S_t)=\phi_s(t,S_t)$

From the above equation, the Black-Scholes PDE becomes the special case of the Frey and Patie PDE. The $v(t,S_t)$ is transferred into σ while the parameter ρ equals zero in the Frey and Patie model. For this reason, the nonlinear PDE is the general form of the Black-Scholes PDE.

The nonlinear PDE provides us an avenue to estimate the value of derivatives, but it is easily to tell that we could not solve it directly. We must take advantage of numerical method to solve the equation such as finite difference method. The numerical method would be presented in the latter part of this article.

2.4 Basic concept of dynamic hedging

In order to understand the performance of different model's hedging strategy, traders must possess the fundamental knowledge of dynamic hedging. For this reason, it is essential to make detail explanations. We first assume that the representative hedger is a large investor and his initial stockholdings is α and the share of bond is β. The hedger uses these two assets to replicate the derivative which the expiration date is T.

When the market is not perfectly liquid, the value of the hedger's portfolio is hard to define. The hedger's position has two kinds of values. One is mark-to-market value and the other is liquidation value. If the stock market liquidity is perfect, these two kinds of value are the same. In other situation, mark-to-market value is higher than liquidation value. Taking the limit order for example, once the stock price declines below $\overline{S}$ the hedger sells all of his stockholdings. Theoretically, the value of this portfolio is $V_t^M=\alpha_t S_t(\rho,\alpha)+\beta_t$, i.e. mark-to market value. However, the hedger only receives the value (liquidation value) less than the mark-to-market value when the hedger sells his position to the market. Although liquidation value is in line with the real value of the hedger's position, it is difficult to define a precise liquidation cost.

Therefore, we adopt the mark-to-market value as the hedger's position value to the following analysis.

Corresponding to the stock price variation, the representative hedger adjusts his stockholdings and bonds to minimize the risk before the expiration date T. At time t, the mark-to-market value is $V_t^M = \alpha_t S_t(\rho,\alpha) + \beta_t$ and the initial value is V_0^M. Generally speaking, delta hedging strategy is widely used in financial market. If there is no external finance to support the hedger, the process of relocation position is called self-financing. During the period $(0 \sim T)$, the gains from relocation position are defined as $G_t = \int_0^T \alpha_s dS_s(\rho,\alpha)$. Throughout the self-financing strategy, the value of the portfolio at time T is $V_T^M = V_0 + G_t = V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)$.

2.5 Tracking error

Tracking error is an instrument to measure the performance of the self-financing. In other words, the difference between the derivative's payoff and the replication value at time T is called tracking error. The figure is positive when we made a loss from our hedging and vice versa. This instrument provides an easy understanding to capture whether the model we used is appropriate or not. Using the above hedging as an example, we define the tracking error (e_T^M) as:

$$e_T^M = h\left(S_T\left(\rho,\alpha\right)\right) - V_T^M = h\left(S_T\left(\rho,\alpha\right)\right) - (V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)) \qquad (2.13)$$

Therefore, we can compare the standard Black-Scholes hedging strategy and the innovative hedging strategy which considers the market is illiquid. If the former tracking error is greater than the latter, it proofs that the Frey and Patie model is a superior model. In the following proposition, we recall the nonlinear PDE (equation (2.12)) to proof the perfect hedging under market illiquidity. The tracking error equals to follows:

$$e_T^M = h(S_T) - V_T^M = h(S_T) - (V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)) \tag{2.14}$$

By using the $It\hat{o}'s$ formula we can obtain

$$\begin{aligned} h(S_T) &= u(T,S_T) \\ &= u(0,S_0) + \int_0^T u_S(t,S_t)dS_t + \int_0^T u_t(t,S_t) + \frac{1}{2}u_{SS}(t,S_t)v^2(t,S_t)S_t^2 dt \end{aligned} \tag{2.15}$$

We continuously transfer the left part pattern $\int_0^T u_t(t,S_t) + \frac{1}{2}u_{SS}(t,S_t)v^2(t,S_t)S_t^2 dt$ into $\int_0^T ru(t,S) - rSu_s(t,S)dt$ by using the nonlinear Black-Scholes PDE and then we derive the tracking error:

$$\begin{aligned} e_T^M &= h(S_T) - (V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)) \\ &= u(0,S_0) + \int_0^T u_S(t,S_t)dS_t + \int_0^T ru(t,S) - rSu_s(t,S)dt - (V_0 + \int_0^T \alpha_s dS_s(\rho,\alpha)) \\ &= \int_0^T ru(t,S) - rSu_s(t,S)dt \end{aligned} \tag{2.16}$$

If we assume that the risk-free interest rate equals to zero, the hedger who adopts self-financing hedging leads the tracking error to be zero. In other words, using the hedging strategy derived by nonlinear Black-Scholes PDE makes the perfectly hedging in illiquid market.

2.6 Numerical Method (Finite difference method)

As we known, partial differential equation (PDE) plays an important role in financial engineering and becomes the essential instrument for option pricing. However, the analytical solution is not always available by using the PDE. Taking the nonlinear Black-Scholes PDE for example, it could not be solved analytically. Therefore, numerical method is needed to estimate the value of the derivatives. In this way, we adopt the finite difference method which is widely used in option valuation.

When it comes to finite difference method, there are three kinds of approach such as explicit, implicit and Crank-Nicolson method. Explicit method uses the least time for computation, but it may face the instable problem. Implicit method is robustness for option valuation and provides the precise option value. However, it does not have the problem of instability. Last, Crank-Nicoleson approach is the hybrid between explicit and implicit method. Because we hope to have a more precise and robust option price, the implicit would be the optimum choice. Therefore, our first step is to make the nonlinear PDE become discrete. Throughout the discreteness, the original equation is transferred as below:

$$\frac{U_j^i - U_j^{i-1}}{\Delta t} + rS_j \frac{U_{j+1}^{i-1} - U_{j-1}^{i-1}}{2\Delta S} + \frac{1}{2}\left(v_j^i\right)^2 S_j^2 \frac{U_{j+1}^{i-1} - 2U_j^{i-1} + U_{j-1}^{i-1}}{\left(\Delta S\right)^2} = rU_j^{i-1} \qquad (2.17)$$

where i is the mesh grid of time. i=0,δt,2δt,...,Nδt=T

j is the mesh grid of asset price. j=0,δS,2δS,...,MδS=Smax

After arrangement, we have

$$\begin{aligned} U_j^i = &\left(0.5*r*j*\Delta t - 0.5*(v_j^i)^2 * j^2 * \Delta t\right)U_{j-1}^{i-1} + \left(1 + (v_j^i)^2 * j^2 * \Delta t + r*\Delta t\right)U_j^{i-1} \\ &+\left(\text{-}0.5*(v_j^i)^2 * j^2 * \Delta t - 0.5*r*j*\Delta t\right)U_{j+1}^{i-1} \end{aligned} \qquad (2.18)$$

That is equal to

$$\begin{aligned} U_j^i &= a_j U_{j-1}^{i-1} + b_j U_j^{i-1} + \mathrm{c}_j U_{j+1}^{i-1} \\ a_j &= \left(0.5*r*j*\Delta t - 0.5*(v_j^i)^2 * j^2 * \Delta t\right) \\ b_j &= \left(1 + (v_j^i)^2 * j^2 * \Delta t + r*\Delta t\right) \\ \mathrm{c}_j &= \left(\text{-}0.5*(v_j^i)^2 * j^2 * \Delta t - 0.5*r*j*\Delta t\right) \end{aligned} \qquad (2.19)$$

This equation means that we need the three previous option prices to get the present option price. There are M option prices which belongs to different stock price needed to be solved simultaneously. Then, we have to solve the tridiagonal matrix as below:

$$A = \begin{pmatrix} b_1\ c_1 & & & & & \\ & a_2\ b_2\ c_2 & & & & \\ & & a_3\ b_3\ \ c_3 & & & \\ & & . & . & . & \\ & & & a_{M-2}\ b_{M-2}\ c_{M-2} & & \\ & & & & a_{M-1}\ b_{M-1} & \end{pmatrix} \tag{2.20}$$

Since we have already known the terminal value, the previous option value can be derived by backward method. In addition, we also apply Thomas algorithm which is designed to solve the tridiagonal matrix. In addition, Frey and Patie (2002) set up an artificial pattern to smooth the volatility term.

$$v^2 = \max\left\{\alpha_0, \frac{\sigma^2}{\left(1-\min\{\alpha_1, \rho Su_{SS}\}\right)^2}\right\} \tag{2.21}$$

$$where\ \alpha_0 = 0.02,\ \alpha_1 = 0.85$$

After incorporating this smooth pattern, the derivatives' payoff truly becomes smooth than before. However, it causes the volatility instable. If we assume that $\sigma = 0.4$ and $\alpha_1 > \rho Su_{SS}$, the volatility will turn into $\frac{0.4}{1-0.85} = 2.667$. It is definitely unreasonable for economic intuition. Consequently, we use bootstrap method to obtain the ninety-nine percentage of the volatility to control the volatility upper bound.

$$v^2 = \min\left\{\max\left[\alpha_0, \frac{\sigma^2}{\left(1-\min\{\alpha_1, \rho Su_{SS}\}\right)^2}\right], \hat{\sigma}_{99\%}\right\} \tag{2.22}$$

$where\ \alpha_0 = 0.02,\ \alpha_1 = 0.85, \hat{\sigma}_{99\%}$ is the 99% of the volatility distribution

The conversion of the volatility makes the Greeks of derivatives more stable. In addition, it also provides a better performance of option valuation. All of the above numerical methods help us to estimate the option value more feasible and reliable.

3. Numerical Results

In this section, we use a specific setting to show the hedging costs (option values) through different kinds of liquidity. From this experiment, we would show the tendency of hedging cost when the market liquidity is decreasing. In addition, we compare the Greeks of the derivative to show the difference between the Frey model and the improved Frey model which revises the volatility smooth function. Before analyzing, there is one thing needed to be informed that liquidity is the opposite of liquidity parameter. In other words, the higher (lower) the liquidity, the smaller (bigger) the liquidity parameter is.

3.1 Hedging cost in illiquidity

Six factors needed to estimate option values include the stock price, exercise price, risk free rate, maturity, sigma and liquidity. The underlying asset prices are between 0 and 100. The exercise price, risk free rate, maturity and sigma are 50, 0.05, 0.25 and 0.25 respectively. The range of the liquidity is among 0 ~ 1.

From figure 3-1, the hedging costs of call options have increased obviously when the liquidity decreases. This phenomenon indicates that the representative hedger has to pay additional cost because the low liquidity enlarges the volatility. According to the understanding of standard Black-Scholes formula, the option price goes up when the volatility goes up. Therefore, the reason of raising hedging cost is the decreasing liquidity which results in higher volatility. In addition, the hedging cost around the exercise price is higher than any other price.

[figure 3-1]

On the other hand, we find hedging cost of put option in the same situation from figure 3-2. The put option's hedging cost raises when the liquidity becomes worse. Consequently, the results show that the representative hedger has to pay more money than they are in the perfect liquidity.

[figure 3-2]

3.2 Greek letters in illiquid market

When it comes to dynamic hedging, the Greek letters are the most popular hedging strategies. Greek letters include delta, gamma, vega …etc. In this part, we are going show the influence of Greek letters resulting from various liquidities. Furthermore, we imply the volatility smooth function into the extension Black-Scholes PDE to get the steady Greek letters throughout the different liquidities. In the following subsection, we assume a situation when the representative hedger sells either a call or a put option. Using the simulation method observes the tendency and variation of Greek letters.

Delta ratio measures the sensitivity of the option price corresponding to the variation of underlying asset price. In figure 3-3, the call delta is increasing from 0 to 1. The representative hedger needs to buy additional shares of stock to protect his position while underlying asset price goes up. In particular, the call delta turns into flatness when the market becomes illiquid.

[figure 3-3]

In figure 3-4, the put delta is from -1 to 0 while the underlying asset price increases. The large investor could sell his shares of stock when the asset price is higher than strike price, and the put option would not be exercised. The put option turns into flatness when the market becomes illiquid. In other words, the illiquid underlying asset flattens the delta hedging strategy no matter what kinds of option are sold by the representative hedger. If the hedger does not adopt the correct liquidity parameter, he would face the over hedge or under hedge problem.

[figure 3-4]

Since the delta hedge plays an important role in dynamic hedging, investors use

Gamma hedging to measure the sensitivity of the delta ratio throughout the asset prices. Figure 3-5 and 3-6 represent call Gamma and put Gamma. Both of Gamma tendency shows that market liquidity lowers the height of Gamma and makes Gamma title to left. In the illiquid market, the representative hedger relocates stock shares before asset price approaches exercise price.

[figure 3-5]、 [figure 3-6]

If we use the original Frey model without improving the volatility function, the Gamma shows instability. Therefore, we implement the improved volatility smooth function into Frey model. Figure 3-7 and 3-8 shows that the delta ratio is originally stable in specific illiquidity (0-0.4). However, after we implement the smooth volatility function showing in figure 3-3 and 3-5, the delta and Gamma have become smooth and stable. After applying the improved volatility function, we release the limitation of liquidity to get the stable Gamma hedging and delta ratio.

[figure 3-7]、 [figure 3-8]

Vega is used to measure the sensitivity of sigma corresponding to asset price variation. According to equation (2.21), the volatility term increases because of the worse liquidity. The figure 3-9 is in line with the theory of volatility function.

[figure 3-9]

4. Empirical Results

The objection of this part is to investigate the performance of option valuation and dynamic hedging. We choose the standard Black-Sholes model as our benchmark and also compare the traditional Frey model.

4.1 Data selection

In order to select the representative data, we decide to use OptionMetrics as our database which belongs to Wharton university of USA. Because CBOE is famous for trading option, we choose the CBOE's top 50 equity options ranked by volume to be the research data. The data period is from 2000/01/01 to 2004/12/31 and risk free rate is the three years treasure bill of America. Option price is the average of the bid and ask. We also take the implied volatility as daily volatility of each asset. Furthermore, underlying asset price is the close price. Throughout the arrangement, we use these data to verify the performance of option pricing and dynamic hedging.

4.2 Least square method and loss function

In the improved Frey model, the only unknown factor is the liquidity parameter. Since there is no general definition of liquidity, we follow Bakshi, Cao and Chen (1997) to use the least square method to estimate the liquidity for each trading day. Before introducing the estimation procedure, we set up the daily close option price as $C_{i,j}$ which i is the ith option contract and j is the jth day. In addition, $\widetilde{C}$ represents the option value estimated by the improved Frey model. In the improved Frey model, the volatility smooth function needs a 99th of historical volatility to limit the upper bound of volatility. Therefore, we use the bootstrap method and repeat 100 times to calculate the volatility upper bound for each underlying asset.

$$\rho_N^* = \arg \min_{\rho} \sum_{j=1}^{N} \sum_{i=1}^{M_j} \left(\underbrace{\widetilde{C}(S_j, K_{i,j}, T_j, r_j, sigma_j^{implied}, \rho_{initial}, \alpha_0, \alpha_1, \widehat{\sigma}_{99th})}_{option \text{ price calculate from the improve Frey model}} - \underbrace{C_{i,j}^{obs}}_{observed \text{ option price}} \right)^2 \quad (4.1)$$

In equation (4.1), the first step is using all of the option contracts in the first day to estimate the appropriate liquidity parameter by least square method. The range of the $\rho_{initial}$ is from 0 to 1. Pattern search algorithm chooses $\rho_{initial}$ corresponding to the minimum value of least square automatically. The parameter N represents the length of moving window and n represents the length of date. In this article, we choose N equals to 1 as daily estimation. In the successive days, the $\rho_{initial}$ is replaced by the ρ_T as shown in equation 4.2.

$$\rho_T^* = \arg \min_{\rho} \sum_{j=T-N+1}^{T} \sum_{i=1}^{M_j} \left(\underbrace{\widetilde{C}(S_j, K_{i,j}, T_j, r_j, sigma_j^{implied}, \rho_{T-1}, \alpha_0, \alpha_1, \widehat{\sigma}_{99th})}_{option \text{ price calculate from the improve Frey model}} - \underbrace{C_{i,j}^{obs}}_{observed \text{ option price}} \right)^2 \quad (4.2)$$

$$T = \{N, N+1, N+2, \ldots\ldots, n-1\}$$

As a result, we obtain series of liquidity throughout n days. Having the liquidity time series helps to calculate the at-the-money option contracts. In order to compare the performance of option pricing between the Black-Sholes model and improved Frey model, we illustrate different kinds of loss function such as \$MSE, %MSE and IVMSE. The definition of \$MSE, %MSE and IVMSE are displayed in equation 4.3, 4.4 and 4.5 respectively.

$$\$MSE = \frac{1}{n} \sum_{i=1}^{n} \left(\widetilde{C}_i - C_i^{obs} \right)^2 \quad (4.3)$$

$$\%MSE = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\widetilde{C}_i - C_i^{obs}}{C_i^{obs}} \right)^2 \quad (4.4)$$

$$IVMSE = \frac{1}{n} \sum_{i=1}^{n} \left(\widetilde{\sigma}_i - \sigma_i \right)^2 \quad (4.5)$$

where $\widetilde{\sigma}_i$ and σ_i are the implied volatility

$$\widetilde{\sigma}_i = BS^{-1}(\widetilde{C}_i, S_i, K_i, r_i, T_i)$$

$$\sigma_i = BS^{-1}(C_i^{obs}, S_i, K_i, r_i, T_i)$$

4.3 The performance of option pricing model

There are three tables comparing the performance of different models. Table 1 takes $MSE as the loss function. In table 1, most of the assets in the improved Frey model have the smallest values, and only four option contracts do not fit well. This result seems that the improved Frey model is the best option pricing model in this inspection. However, table 2 exhibits the top 50 options' %MSE, and the performance of improved Frey model do not defeat the traditional Frey model.

[Table 1]、[Table 2]

The difference between these two loss functions comes from the different criteria. In other words, we may have inconsistent solutions if we adopt the distinct loss function. According to Engle (1993), the choice of the loss function is important in model evaluation because it exits particular error structure. Taking $MSE for example, $MSE gives more weight for higher value option contracts such as in-the-money and long time-to-maturity contracts. On the other hand, %MSE also has the heteroskedastic error structure. Although %MSE has the advantage to give $1 error less weight in $100 option value than in $10 option value, it overcorrects the error structure. It faces the instable problem when the option contract is out-of-the money and short time-to-maturity. In Pan (2002), Pan takes IVMSE to instead of traditional loss functions. The advantage of IVMSE is that IVMSE does not have heteroskedastic error structure. Therefore, we apply IVMSE to investigate the performance of various models.

Table 3 shows that the improved Frey model seems better than the original Frey model. There are ten option contracts are not over the 95% t test but all of the figures are smaller than the traditional Frey model.

[Table 3]

4.4 Dynamic hedging

As a representative hedger, the performance of dynamic hedging is as important as option valuation. Since we have already known how to estimate the option value, it is naturally to implement the improved Frey model into hedging manipulation. The tracking error is a powerful instrument to verify the difference between the estimated option value and the observed option value. The tracking error method is shown in equation (2.14) and then we also compare the performance of the Black-Scholes and the traditional Frey model. In equation (2.14), we adopt the self-financing strategy to calculate the option value at time T (V_T^M). The below equation (4.6) shows the process of self-financing.

$$V_i^M : \widetilde{C}_i = \widetilde{C}_{i-1} + delta_{i-1}^{Frey} * \left(S(i) - S(i-1)\right) + r(i-1) * \left(\widetilde{C}_{i-1} - delta_{i-1}^{Frey} * S(i-1)\right) * \Delta t \qquad (4.6)$$

We also take the new pattern C_i^{obs} to replace the observed option value $h(S_T)$. For this reason, we have the total tracking error of specific option contract which time to maturity is n.

$$e_T^M = \sum_{i=2}^{T} h(S_i) - V_i^M = \sum_{2}^{T} C_i^{obs} - \widetilde{C}_i \qquad (4.7)$$

Therefore, we choose top 20 option contracts to test the improved Frey model dynamic hedging ability. The maturity time is around one year which is from 2000/01/01 to 2001/1/20. In particular, we only pick up the at-the-money option contracts to be our trading assets. As shown in table 4, the results of improved Frey model is better than Black-Scholes model in dynamic hedging. Through the statistic analysis, the difference between these two models is significant.

[Table 4]

From the above results, the improved Frey model not only provides more precise option values but also accurate dynamic hedging strategies.

5. Conclusion

This article follows Frey and Patie's (2002) research which incorporate the liquidity parameter into asset price dynamic process. The nonlinear PDE is derived from the stochastic process of underlying asset price. In order to solve the nonlinear partial difference equation, we adopt the finite difference method which is generally used to calculate the option price. Besides, due to the fact that Thomas algorithm is famous for tridiagonal computation, we apply it to reduce the computation time.

The most significant difference between the traditional Frey model and the improved Frey model is the improved volatility smooth function. Without correcting the volatility upper bound, option volatility could be irrational. The irrational volatility results in inconsistent and instable Greek letters. Thus, we release the limit of liquidity range for Greek letters. Originally, the Greek letters was turned into instability while liquidity parameter is over 0.4. Since we have improved the volatility smooth function, the limited liquidity problem no longer exists.

In empirical results, we use three different kinds of loss functions to investigate the performance of various models. The comparison shows that the improved Frey model is superior when we choose the reliable IVMSE as the loss function. On the other hand, tacking error exhibits the model's hedging ability. The improved Frey model still has the outstanding hedging performance.

In further research, we would use time series model to estimate the liquidity process. The time series of liquidity help market participants to forecast the market liquidity in advance. Therefore, option values are easily calculated by the improve Frey model. Investors take the estimated option value as the benchmark to deicide their trading strategy. Moreover, the exotic option value is also derived by the stochastic process of underlying asset.

References

Amihud, Y., and H. Mendelson. 1989. The effect of computer base trading on volatility and liquidity. *The challenge of information technology for the securities markets, liquidity, volatility, and global trading*: 59-85

Bakshi, G., C. Cao, and Z. Chen. 1997. Empirical performance of alternative option pricing models. *Journal of Finance* 52: 2003-2049.

Barndimarte, Paolo. 2002. Option valuation by finite difference methods. *Numerical Methods in Finance*, 347-364.

Brennan, J. Michael, and E. S. Schwartz. 1978. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *Journal of Financial and Quantitative Analysis* 13(Sep): 461-474

Bielecki, T., and M. Rutkowski. 2002. Credit Risk: Modeling, Valuation and Hedging. *Springer*.

Cetin, U., R. Jarrow, P. Protter, and M. Warachka. 2006. Pricing options in an extended Black Scholes economy with illiquidity: Theory and empirical evidence. *Oxford Journals* 19: 493-529

Chan, L. K. C., and J. Lakonishok 1995. The behavior of stock prices around institutional trades. *Journal of Finance* 50: 1147-1174

Esser, A. and B. Moench. 2003. Modeling feedback effects with stochastic liquidity. *Goethe University, Frankfurt am Main, Germany.*

Engle, R. 1993. A comment on Hendry and Clements on the limitations of comparing mean square forecast errors, *Journal of Forecasting* 12: 642-644

Frey, Rudiger. 2000. Market illiquidity as a source of model risk in dynamic hedging. *Model Risk*.

Frey, R., and P. Patie. 2002. Risk management for derivatives in illiquid markets: A simulation study. *RISK Publications.*

Henry-Labordere, P. 2004. The feedback effect of hedging in portfolio optimization. Working paper.

Hull, John .C. 2006. Wiener Process and Ito's Lemma. *Options, Futures, and Other Derivatives*, 263-277.

Kyle, A. S. 1985. Continuous auctions and insider trading. *Journal of Finance* 53: 1315-35

Lyukov, A. 2004. Option Pricing with Feedback Effects. *International Journal of Theoretical and Applied Finance.*

Merton, C. R. 1975. Option pricing when underlying stock return are discontinuous. *Journal of Economics* 3: 125-144.

Pritsker, M. 2002. Large investors, implications for equilibrium asset returns, shock absorption, and liquidity. mimeo. Federal Reserve Board.

Sanfelici, S. 2006. Calibration of a nonlinear feedback option pricing model. Working paper.

Schwartz, R. A. 1988. Equity markets: Structure, trading, and performance. New York:Harper and Row, inc.

Schonbucher, P., and P. Wilmott. 2000. The Feedback Effect of Hedging in illiquid Markets. *SIAM Journal on Applied Mathematics* 61: 232-272.

Vijh, A. M. 1990. Liquidity of the CBOE Equity Options. *Journal of Finance* 45: 1157-79.

Tables

Table 1. $MSE of top 50 option contracts

	BS $MSE	Frey $MSE	SFrey $MSE	Std	T statistic		BS $MSE	Frey $MSE	SFrey $MSE	Std	T statistic
BAC	0.6462	0.1028	0.0586	0.2585	6.0327	BGEN	0.2499	0.1920	0.1065	1.2797	2.1128
CE	0.0799	0.0460	0.0054	0.4354	2.9986	CIEN	0.3258	0.3107	0.0190	2.0548	4.9374
Citigroup	0.4749	0.0729	0.0514	0.0915	8.2656	CNXT	0.4773	0.4505	0.4154	4.2624	0.2637
CSCO	0.1728	0.0925	0.0114	0.7412	3.8628	CY	0.0548	0.0427	0.0046	0.3131	4.3111
DELL	0.1674	0.0306	0.0133	0.0658	9.2973	F	0.3221	0.2535	0.0614	1.7475	3.8721
EP	0.1230	0.0226	0.0086	0.0708	6.9729	GLW	1.0047	0.5415	0.1921	3.3707	3.6430
GE	0.3543	0.0343	0.0060	0.0884	11.3110	JDSU	0.4287	0.3933	0.1085	4.1517	2.4262
GM	0.4106	0.0438	0.0222	0.1052	7.2409	JNPR	4.4466	2.2138	1.1251	7.0796	5.4415
HPQ	0.1513	0.0968	0.0113	0.7791	3.8703	KO	0.4183	0.0616	0.0454	0.0967	5.9051
IBM	1.4252	0.2776	0.1284	0.9060	5.8101	LSI	0.5545	0.1708	0.1189	0.3956	4.6454
INTC	0.2007	0.0573	0.0079	0.3757	4.6343	MOT	0.2139	0.1609	0.0894	1.1449	2.2096
JPM	0.2574	0.0426	0.0209	0.1501	5.0863	MRK	0.7456	0.1042	0.0777	0.3236	2.8984
MO	0.4706	0.0666	0.0473	0.1136	6.0048	NOK	0.8593	0.1705	0.0591	1.3737	2.8710
MSFT	0.9610	0.0917	0.0618	0.1469	7.1862	NT	1.5271	0.8537	0.7427	1.2478	3.0815
ORCL	0.0799	0.0694	0.0114	0.4593	4.4566	NVLS	0.1029	0.0659	0.0130	0.2959	6.3384
QCOM	0.3350	0.1899	0.0324	1.1130	4.9897	NXTL	0.3046	0.0864	0.0207	0.6391	3.6428
TWX	0.2235	0.0408	0.0103	0.1483	7.2501	PALM	0.2100	0.1596	0.1160	4.9042	0.2998
WMT	0.5858	0.1010	0.0667	0.1727	6.9888	PFE	0.3186	0.0431	0.0293	0.0623	7.8249
XMSR	0.0371	0.0194	0.0112	0.0779	3.4163	Q	0.0476	0.0347	0.0261	0.4398	0.6861
YHOO	1.3244	0.3440	0.0575	3.4969	2.8907	TMX	2.6397	2.2634	2.2261	3.4922	0.3700
AMAT	0.1528	0.1426	0.0595	0.9712	3.0283	TXN	0.1566	0.1048	0.0143	0.8391	3.8169
AMCC	0.9761	0.5449	0.1604	2.4070	5.5336	TYC	0.3295	0.1167	0.0861	0.4394	2.4659
AMD	0.1044	0.0892	0.0071	0.7409	3.9249	VRTS	1.2432	1.1119	0.6541	3.8894	4.1632
AMZN	0.5425	0.0840	0.0327	0.4201	4.3242	XLNX	0.1238	0.0920	0.0216	0.6157	4.0473
BMY	0.2618	0.0510	0.0241	0.2786	3.4197	XRX	0.0307	0.0076	0.0030	0.0217	7.3817

Table 2.　%MSE of top 50 option contracts

	BS %MSE	Frey %MSE	SFrey %MSE	Std	T statistic		BS %MSE	Frey %MSE	SFrey %MSE	Std	T statistic
BAC	4.8578	0.9935	0.7814	1.7760	4.1370	BGEN	0.0636	0.0404	0.0105	0.3270	2.8962
CE	0.6891	0.1548	0.0969	1.4218	1.3073	CIEN	1.3462	0.5409	0.4422	3.2971	1.0469
Citigroup	2.2746	0.4884	0.4811	2.6146	0.0926	CNXT	3.0225	3.3146	3.2971	3.7965	0.1222
CSCO	1.8752	0.2883	0.2521	2.4506	0.5184	CY	0.1997	0.0481	0.0135	0.3175	3.8589
DELL	0.6919	0.0955	0.0837	0.9362	0.4483	F	7.6450	4.6748	4.5001	2.0530	2.1062
EP	0.4258	0.0801	0.0373	0.4625	3.2608	GLW	3.9990	0.6790	0.1215	2.9232	6.7119
GE	6.6343	1.3486	0.9224	11.4000	1.3224	JDSU	1.0806	0.5012	0.4213	0.4320	6.4038
GM	0.9656	0.1390	0.1113	0.8188	1.1897	JNPR	1.7166	0.4091	0.0673	3.2655	3.7095
HPQ	1.6184	0.2403	0.2271	1.7321	0.2644	KO	3.9112	0.5617	0.4310	4.0797	1.1379
IBM	1.3129	0.2419	0.2213	1.7488	0.4125	LSI	13.4340	4.5765	1.6858	13.1750	7.7731
INTC	0.7016	0.0594	0.0591	0.4783	0.0224	MOT	8.4932	6.3928	4.3343	18.0287	3.9553
JPM	1.8600	0.3501	0.3024	2.7427	0.6099	MRK	2.3723	0.4813	0.2280	2.2138	4.0548
MO	4.5312	0.7967	0.5740	7.2656	1.0847	NOK	11.0760	2.5643	0.1311	17.7740	4.8518
MSFT	5.8402	1.3535	1.3390	7.2604	0.0651	NT	0.8694	0.3591	0.1399	1.5942	4.3812
ORCL	0.4689	0.0764	0.0391	0.3342	3.9310	NVLS	0.1538	0.1207	0.0113	3.7652	1.0287
QCOM	0.0178	0.0142	0.0104	0.1402	0.9576	NXTL	2.5488	0.3918	0.1085	3.2336	3.1064
TWX	0.7891	0.1601	0.0961	1.7236	1.3079	PALM	5.4323	3.4556	3.0202	4.2098	3.5827
WMT	5.4230	2.8050	2.6561	10.6040	0.4864	PFE	1.9348	0.3154	0.3110	1.1111	0.1326
XMSR	0.1107	0.0430	0.0344	0.1114	2.4963	Q	0.5569	0.3315	0.2888	3.3644	0.4518
YHOO	0.6596	0.0928	0.0235	0.6080	4.0264	TMX	3.5201	0.6696	0.5760	6.7867	0.4915
AMAT	0.1764	0.0186	0.0082	0.0529	6.9188	TXN	0.3653	0.0376	0.0224	0.3972	1.3515
AMCC	1.3001	1.1951	1.1161	6.1653	0.4591	TYC	19.2760	9.4093	7.9265	15.2347	3.3716
AMD	0.0862	0.0398	0.0103	0.2283	4.5669	VRTS	1.3394	0.6759	0.6279	1.0503	1.5845
AMZN	0.4214	0.0364	0.0236	0.1614	2.7968	XLNX	0.0851	0.0126	0.0045	0.0442	6.4591
BMY	5.6361	0.9886	0.9101	6.1192	0.4585	XRX	0.5667	0.0957	0.0564	0.3518	3.9497

Table 3. IVMSE of top 50 option contracts

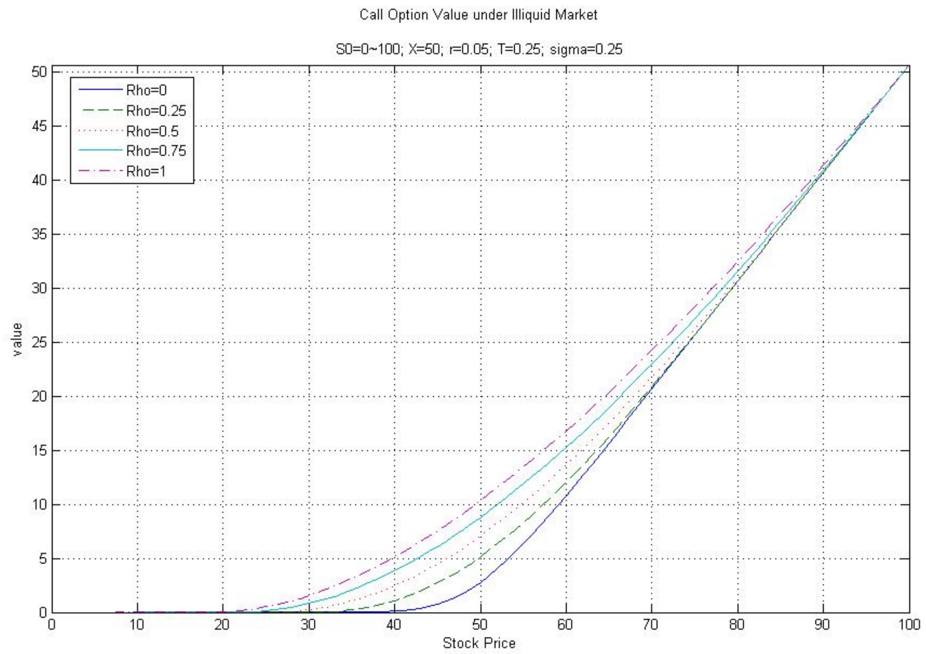
	BS_IVMSE	Frey_IVMSE	SFrey_IVMSE	Std	T statistic
BAC	0.0255	0.0038	0.0028	0.0040	8.5148
CE	0.0202	0.0082	0.0021	0.0310	6.3222
Citigroup	0.0313	0.0050	0.0039	0.0047	8.5596
CSCO	0.0307	0.0071	0.0030	0.0171	8.5861
DELL	0.0218	0.0038	0.0021	0.0060	10.4730
EP	0.0284	0.0113	0.0083	0.0671	1.5692
GE	0.4560	0.0387	0.0051	0.9117	1.2766
GM	0.0254	0.0027	0.0020	0.0044	5.6405
HPQ	0.0307	0.0052	0.0030	0.0090	8.4035
IBM	0.0257	0.0042	0.0030	0.0054	7.7903
INTC	0.0234	0.0037	0.0016	0.0072	10.0160
JPM	0.0254	0.0037	0.0027	0.0071	5.1765
MO	0.0357	0.0052	0.0042	0.0065	5.4128
MSFT	0.0595	0.0093	0.0082	0.0056	6.9705
ORCL	0.0249	0.0061	0.0030	0.0204	5.4018
QCOM	0.3340	0.0589	0.0344	0.3259	2.6431
TWX	0.0305	0.0051	0.0027	0.0141	5.9668
WMT	0.0339	0.0059	0.0049	0.0057	5.6724
XMSR	0.0731	0.0606	0.0466	0.1375	3.3046
YHOO	0.0895	0.0070	0.0059	0.0235	1.5719
AMAT	0.0133	0.0038	0.0015	0.0097	8.6067
AMCC	0.0579	0.0401	0.0346	0.0915	2.1170
AMD	0.0159	0.0092	0.0047	0.0316	5.0388
AMZN	0.0631	0.0104	0.0047	0.0290	6.9593
BMY	0.0330	0.0059	0.0044	0.0182	2.9514

	BS_IVMSE	Frey_IVMSE	SFrey_IVMSE	Std	T statistic
BGEN	0.0181	0.0061	0.0026	0.0202	5.5510
CIEN	0.0461	0.0185	0.0050	0.0549	8.5637
CNXT	0.2317	0.0932	0.0764	0.8845	0.6726
CY	0.0153	0.0058	0.0018	0.0162	8.7490
F	0.2148	0.1987	0.1063	0.6634	4.4998
GLW	0.1453	0.0333	0.0271	0.1221	1.7969
JDSU	0.1080	0.0243	0.0136	0.0758	5.3311
JNPR	0.4542	0.3966	0.3228	0.4667	5.4778
KO	0.0281	0.0041	0.0037	0.0033	4.4664
LSI	0.2597	0.0523	0.0122	0.0729	19.5150
MOT	0.0200	0.0076	0.0033	0.0382	3.9597
MRK	0.0446	0.0082	0.0041	0.0194	7.6500
NOK	0.1493	0.0251	0.0042	0.0460	16.1130
NT	0.2156	0.1864	0.1251	0.7437	2.8553
NVLS	0.0121	0.0040	0.0016	0.0225	3.7463
NXTL	0.0534	0.0101	0.0063	0.0369	3.6554
PALM	0.5290	0.4697	0.4594	0.3654	0.9765
PFE	0.0362	0.0051	0.0043	0.0038	6.9891
Q	0.4222	0.2132	0.2107	1.0535	0.0771
TMX	0.4910	0.4425	0.4269	0.4880	1.2062
TXN	0.0180	0.0032	0.0011	0.0058	13.0120
TYC	0.0437	0.0189	0.0169	0.1025	0.6955
VRTS	0.0343	0.0166	0.0138	0.0559	1.7829
XLNX	0.0094	0.0038	0.0017	0.0223	3.4277
XRX	0.0277	0.0092	0.0065	0.0281	3.3986

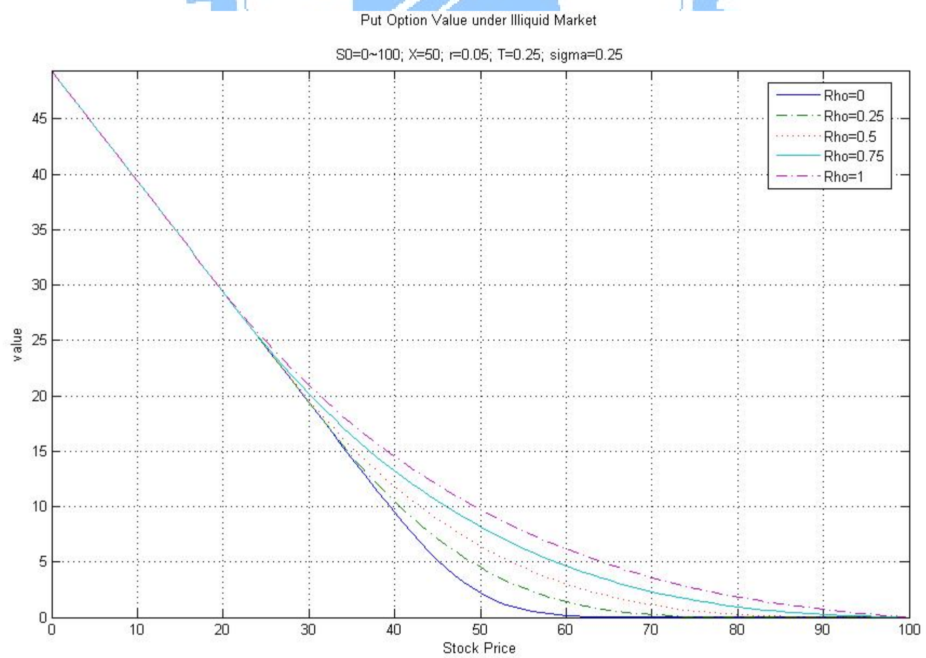
Table 4. Hedging performances of top 20 option contracts

	BS	Improved_Frey	Std	T	P		BS	Improved_Frey	Std	T	P
BAC	0.5630	0.5541	0.0309	4.1085	0.00006	INTC	1.3557	1.3212	0.1558	3.5956	0.00039
CE	0.2116	0.1128	0.3738	4.3070	0.00003	JPM	1.9558	1.9443	0.0505	3.7037	0.00026
Citigroup	2.2183	2.2129	0.0181	4.7660	0.00000	MO	0.3734	0.3692	0.0172	3.9696	0.00009
CSCO	5.2731	4.5691	2.3742	4.8261	0.00034	MSFT	0.9737	0.9448	0.1000	4.7002	0.00000
DELL	0.6709	0.6486	0.0530	6.8437	0.00000	ORCL	6.6402	6.5256	0.3659	5.0891	0.00000
EP	0.4940	0.4857	0.0260	5.1133	0.00000	QCOM	1.9295	1.6568	0.5394	8.2168	0.00000
GE	1.1087	1.1042	0.0236	3.0769	0.00231	TWX	1.4726	1.3915	0.2329	5.5221	0.00000
GM	0.8154	0.8102	0.0179	4.7172	0.00000	WMT	0.7225	0.7092	0.0383	5.6426	0.00000
HPQ	1.6919	1.6312	0.1687	5.8378	0.00000	XMSR	0.3280	0.3131	0.0619	3.1040	0.00225
IBM	1.2586	1.2374	0.1103	3.1343	0.00192	YHOO	2.9279	2.3269	1.6026	6.0930	0.00000

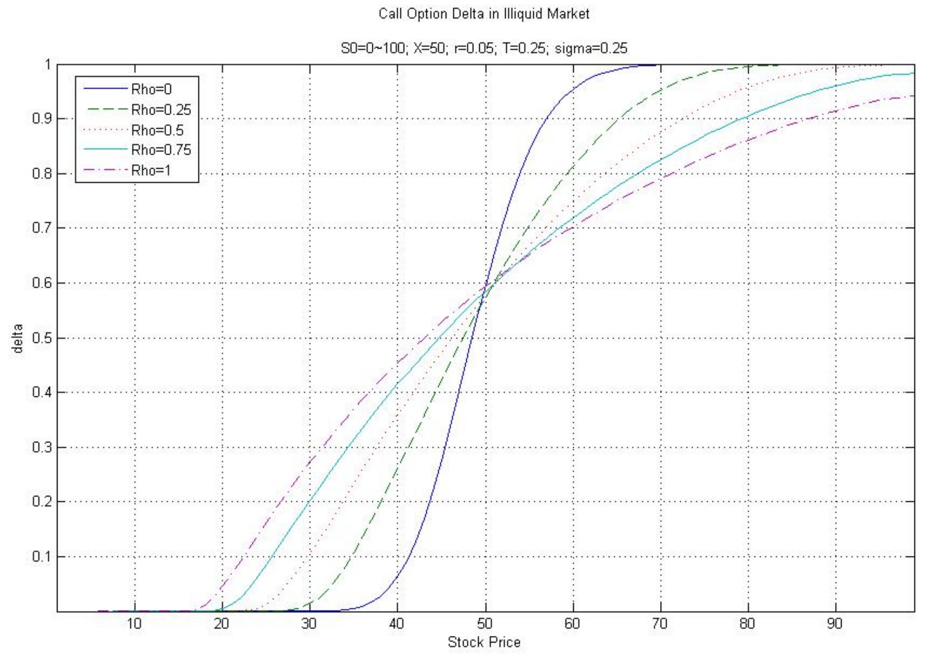
Figures



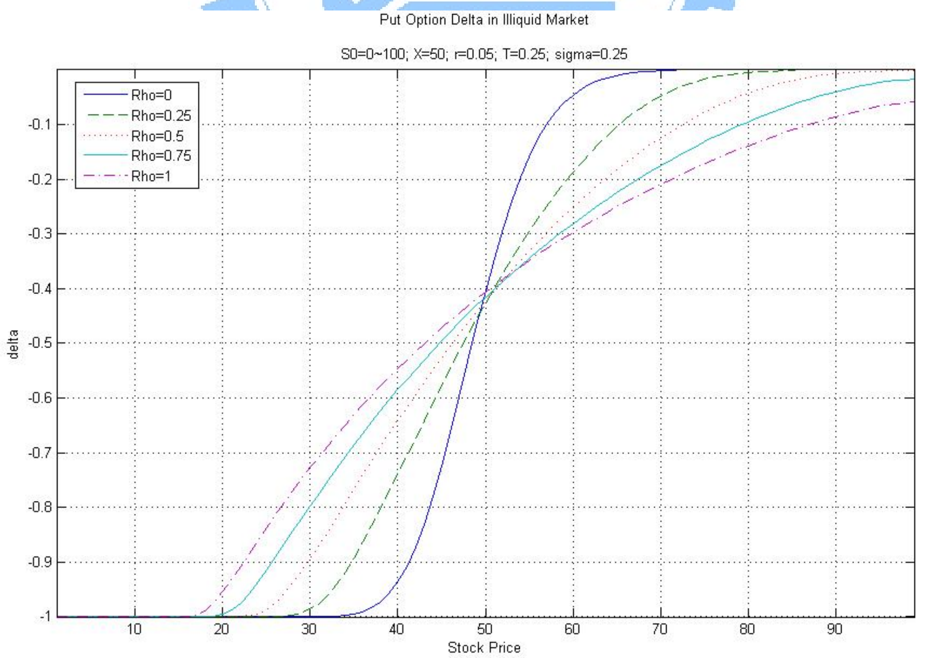
[figure 3-1]　Call option prices under different illiquid markets



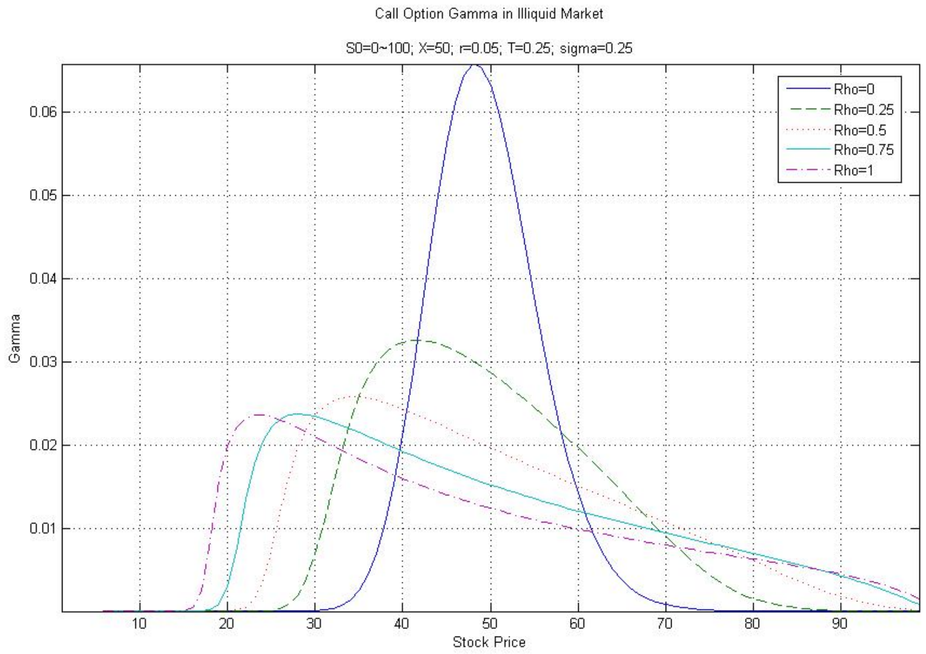
[figure 3-2]　Put option prices under different illiquid markets



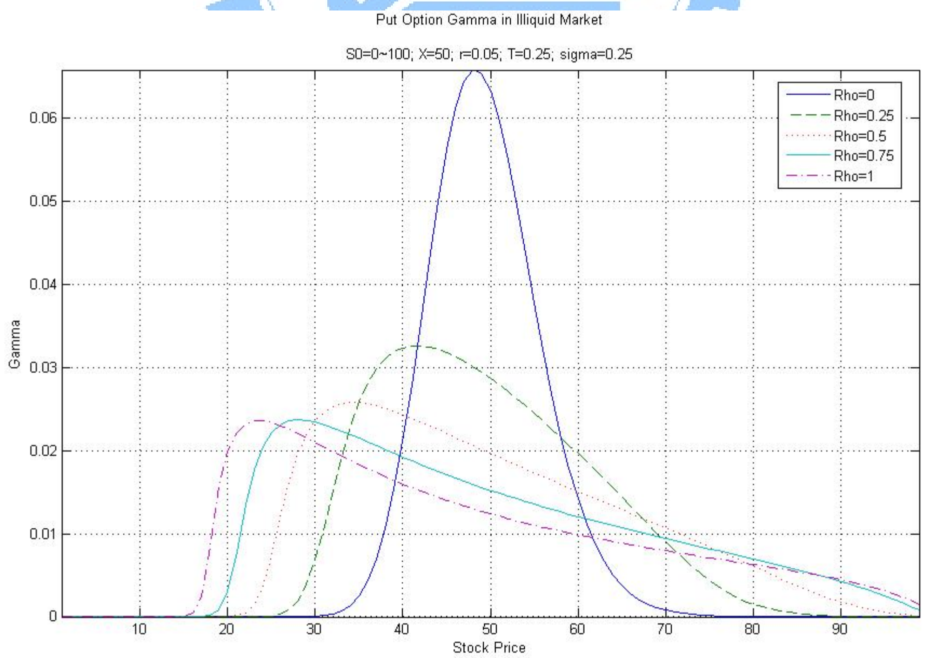
[figure 3-3]　Call option deltas under different illiquid markets



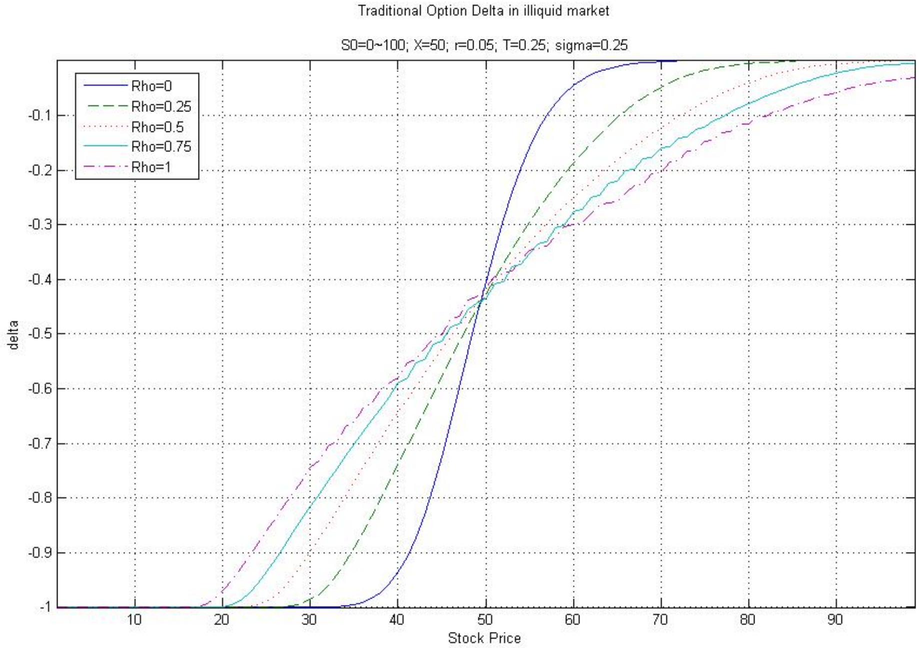
[figure 3-4]　Put option deltas under different illiquid markets



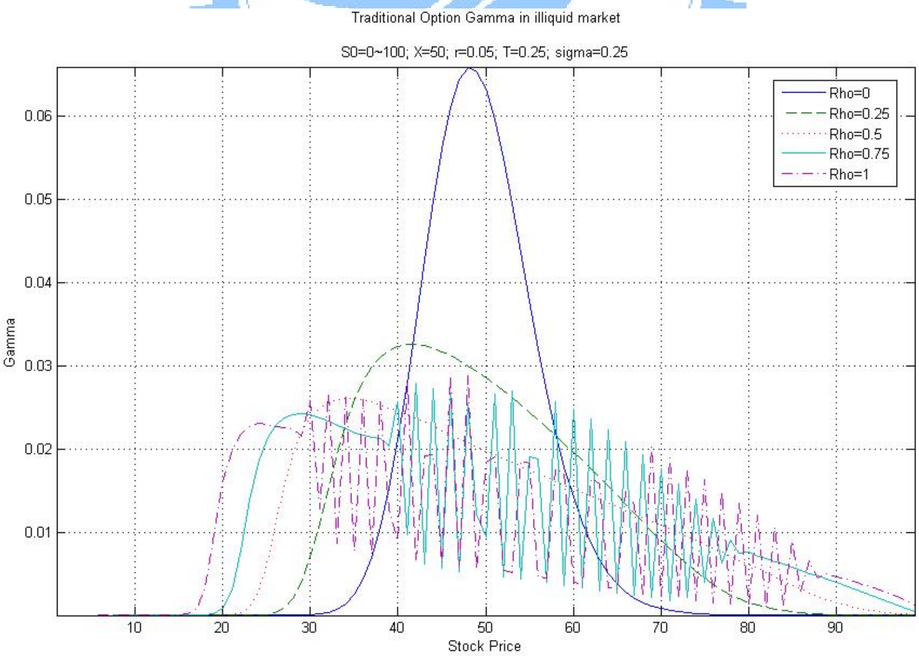
[figure 3-5]   Call option Gammas under different illiquid markets



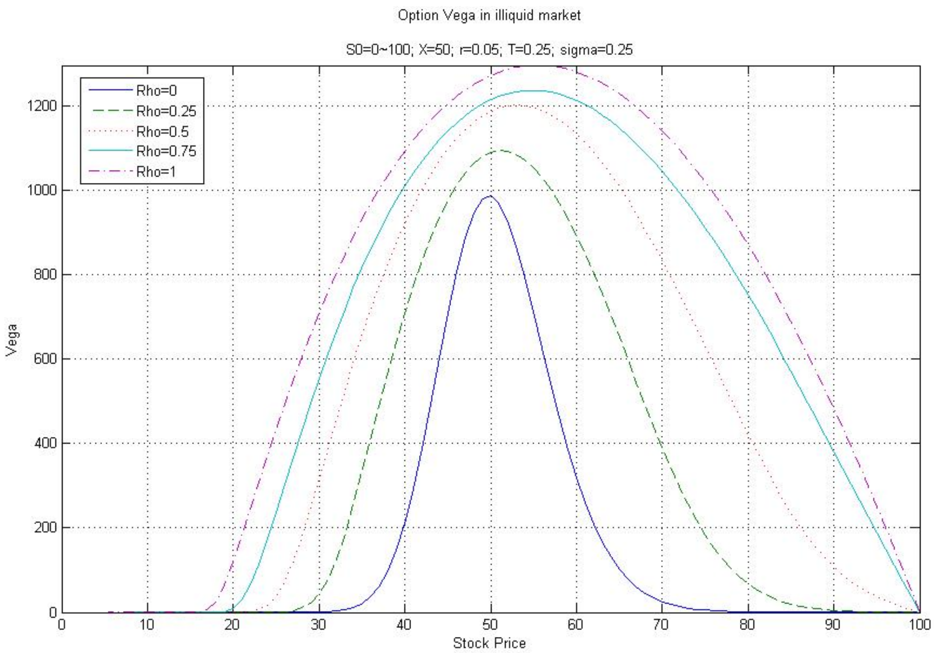
[figure 3-6]   Put option Gammas under different illiquid markets



[figure 3-7]　Traditional option deltas under illiquid markets



[figure 3-8]　Traditional option Gammas under illiquid markets



[figure 3-9] Option Vegas under different illiquid markets