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Two Doyen–Wilson theorems for maximum packings with triples

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Abstract

In this paper we complete the work begun by Mendelsohn and Rosa and by Hartman, finding necessary and sufficient conditions for a maximum packing with triples of order m MPT(m) to be embedded in an MPT(n). We also characterize when it is possible to embed an MPT(m) with leave L_1 in an MPT(n) with leave L_2 in such a way that $L_1 \subseteq L_2$.

1. Introduction

A *packing with triples* of *order* n (or a *partial Steiner triple system*) is an ordered triple (S,T,L) where T is a set of edge-disjoint copies of K_3 (*triples*) in K_n with vertex set S, and L is the set of edges in K_n belonging to no triple in T. L is known as the *leave* of the packing. A *maximum packing with triples* (or simply a *maximum packing*) of order n, denoted by MPT(n), is a packing with triples (S,T,L) of order n such that for any other packing with triples (S',T',L') of order n, $|T| \geqslant |T'|$ (or, if you prefer, $|L| \leqslant |L'|$). A maximum packing (S,T,L) of order n with $L = \emptyset$ is, of course, a *Steiner triple system*, STS(n). It has been known since 1847 [6] that an STS(n) exists if and only if $n \equiv 1$ or $3 \pmod 6$. It is also well known that for the other possible values of n, the subgraph induced by the leave is: a 1-factor if $n \equiv 0$ or $2 \pmod 6$; a *tripole* (that is, a spanning subgraph of K_n in which one vertex has degree 3 and the rest have degree 1) if $n \equiv 4 \pmod 6$; and a cycle of length 4 if $n \equiv 5 \pmod 6$. (The 3 adjacent edges in a tripole are called the *head* of the tripole.)

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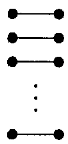
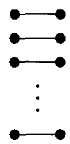
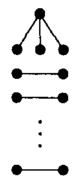
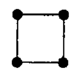
Table 1
Leaves of maximum packings

0 (mod 6)	1 (mod 6)	2 (mod 6)	3 (mod 6)	4 (mod 6)	5 (mod 6)
 1-factor	∅ Steiner triple system	 1-factor	∅ Steiner triple system	 tripole	 cycle of length 4

The packing (S_1, T_1, L_1) of order m is said to be *embedded* in the packing (S_2, T_2, L_2) if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. In 1973, Doyen and Wilson [2] set the standard for embedding problems by proving the following result.

Theorem 1.1 (Doyen and Wilson [2]). *Let $m, n \equiv 1$ or $3 \pmod 6$. Any STS(m) can be embedded in a STS(n) if $n \geqslant 2n + 1$.*

This lower bound on n is the best possible. Over the past 25 years much effort has been focussed on proving a similar theorem for embedding any *partial* STS(m) in a STS(n). The best result to date is that a partial STS(m) can always be embedded in a STS(n) for all $n \geqslant 4m + 1$ *and* $n \equiv 1$ or $3 \pmod 6$ [1]. The best possible result would be $n \geqslant 2m + 1$ and $n \equiv 1$ or $3 \pmod 6$.

In 1983, Mendelsohn and Rosa [7] considered the following generalization of Theorem 1.1. For which values of m and n can any maximum packing of order m be embedded in a maximum packing of order n? It is easy to see that the following are necessary conditions.

Lemma 1.2. *Let $n > m$. Suppose that any MPT(m) can be embedded in a MPT(n). Then*

(1) *if $m = 6$ then $n = 7$ or $n \geqslant 10$,*
(2) *if $m > 6$ and m is even then $n = m + 1$ or $n \geqslant 2m$, and*
(3) *if $m > 6$ and m is odd then $n \geqslant 2m$.*

Remark. There is no restriction on n if $m \leqslant 5$.

Mendelsohn and Rosa obtained a partial answer to this problem with the following theorem.

Theorem 1.3 (Mendelsohn and Rosa [7]). *Let $s, t \in \{0, 1, 2, 3, 4, 5\}$ such that $s \in \{4, 5\}$ if and only if $t \in \{4, 5\}$. Then the necessary conditions in Lemma 1.2 for the* *embedding of an* MPT(m) *in an* MPT(n) *are sufficient if*

(1) $m \leqslant 5$, *or*

(2) $m = 6$ *and* $n \in \{10, 11\}$, *or*

(3) m *is even and* $n = m + 1$, *or*

(4) $m \equiv s \pmod 6$ *and* $n \equiv t \pmod 6$.

Furthermore, the smallest possible embedding when $n \geqslant 2m$ has been found in many cases by Mendelsohn and Rosa [7], by Hartman et al. [5], and in the remaining cases by Hartman [4], as the following theorem states.

Theorem 1.4 (Hartman [4], Hartman et al. [5] and Mendelsohn and Rosa [7]). *Let* $2m \leqslant n \leqslant 2m + 5$. *Any* MPT($m$) *can be embedded in an* MPT(n).

In this paper, we finish the proof of this problem, showing that the necessary conditions of Lemma 1.2 are sufficient for the embedding of an MPT(m) into an MPT(n). To do so, we need to prove the following.

Theorem 1.5. *Suppose that* $s \in \{0, 1, 2, 3\}$ *and* $t \in \{4, 5\}$ *or* $s \in \{4, 5\}$ *and* $t \in \{0, 1, 2, 3\}$. *Suppose also that* $m \equiv s \pmod 6$, $n \equiv t \pmod 6$, *and* $n > 2m$. *Then any* MPT(m) *can be embedded in an* MPT(n).

This result is proved in Sections 2 and 4. These results are all collected into one result in Section 5.

2. The case $s \in \{4, 5\}$ and $t \in \{0, 1, 2, 3\}$

For the purposes of this paper, a *difference triple* is a 3-element set $\{x, y, z\}$ of distinct positive integers such that $x + y = z$.

If $\{x, y\}$ is an edge in K_n with vertex set $\mathbb{Z}_n$, then $\{x, y\}$ is said to have *length* $\ell(x, y) = \min\{y - x \pmod n, x - y \pmod n\}$; so $1 \leqslant \ell(x, y) \leqslant n/2$. For any subset $L \subseteq \mathbb{Z}_{\lfloor n/2 \rfloor}$, let $G_n(L)$ be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{x, y\} \mid \ell(x, y) \in L\}$. The following is a special case of an extremely useful lemma of Stern and Lenz.

Lemma 2.1 (Stern and Lenz [8]). *If* $n/2 \in L$ *then there exists a* 1*-factorization of* $G_n(L)$.

The following lemma was essentially proved by Stern and Lenz [8] see also [3, Lemma 6.1]; the additional property that D can be defined so that it contains $\{1, 2, 3\}$ for the small values of L was proved by Fu et al. (see [3, Lemmas 6.5 and 6.7]).

Lemma 2.2. (i) *For all* $h \geqslant 2$, *the set* $\{1, 2, \ldots, 3h\} \setminus \{a, b, c\}$ *for some* $\{a, b, c\} \subseteq \{4, 5, \ldots, 3h\}$ *can be partitioned by a set* D *of difference triples, and if* $h \leqslant 10$ *then* D *can be defined so that* $\{1, 2, 3\} \in D$.

(ii) *For all* $h \geqslant 3$, *the set* $\{1,2,\dots,3h+1\} \backslash \{a,b,c,d\}$ *for some* $\{a,b,c,d\} \subseteq \{5,6,\dots,3h+1\}$ *can be partitioned by a set* D *of difference triples, and if* $h \leqslant 13$ *then* D *can be defined so that* $\{1,2,3\} \in D$.

The next two lemmas are the crucial ingredients used to prove Theorem 1.5.

Lemma 2.3. *Let* $h \geqslant 2$. *The edges in* $G_{6h+2}(\{1,2,3\})$ *can be partitioned into* 4 *matchings, each of which saturates all vertices except for the vertices* 3 *and* 10, *together with a set* T *of* $2h+2$ *triples.*

Proof. The proof consists of defining T and two cycles c_1 and c_2, each of which passes through each vertex except for 3 and 10. Then clearly c_1 and c_2 can each be partitioned into two matchings as required.

Let

$$T = \{\{0,1,3\},\{2,3,4\},\{3,5,6\},\{7,8,10\},\{9,10,11\},\{10,12,13\}\} \cup \{\{3i+2,3i+3,3i+4\} \mid 4 \leqslant i \leqslant 2h-1\}.$$

Let $c_1 = (x_0,x_1,\dots,x_{6h-1})$ where $(x_0,x_1,\dots,x_{11}) = (0,2,5,4,6,7,9,8,11,13,15,12)$, and for $2 \leqslant i \leqslant h-1$ define $(x_{6i},x_{6i+1},\dots,x_{6i+5}) = (6i+2,6i+5,6i+4,6i+7,6i+9,6i+6)$, reducing sums modulo $6h+2$. Let $c_2 = (y_0,y_1,\dots,y_{6h-1})$ where $(y_0,y_1,\dots,y_{11}) = (0,6h+1,2,1,4,7,5,8,6,9,12,11)$, and for $2 \leqslant i \leqslant h-1$ define $(y_{6i},y_{6i+1},\dots,y_{6i+5}) = (6i+2,6i+1,6i+4,6i+6,6i+3,6i+5)$. □

Lemma 2.4. *Let* $h \geqslant 2$. *The edges in* $G_{6h+4}(\{1,2,3,4\})$ *can be partitioned into* 4 *matchings, each of which saturates all of the vertices except for vertices* 6 *and* 9, *together with a set* T *of* $4h+4$ *triples.*

Proof. Let

$$T = \{\{0,2,4\},\{1,3,5\},\{2,3,6\},\{4,6,8\},\{5,6,9\},\{6,7,10\},\{7,8,9\},\{9,10,11\},\{9,12,13\},\{11,12,15\},\{13,14,15\},\{14,16,17\}\} \cup \{\{6i+4,6i+6,6i+7\},\{6i+5,6i+6,6i+9\},\{6i+7,6i+8,6i+9\},\{6i+8,6i+10,6i+11\} \mid 2 \leqslant i \leqslant h-1\}.$$

Let c_1 be the $(6h+2)$-cycle $(x_0,x_1,\dots,x_{6h+1})$, where $(x_0,x_1,\dots,x_{13}) = (3,4,5,7,11,8,10,12,14,18,15,17,13,16)$ and for $3 \leqslant i \leqslant h$ $(x_{6i+2},x_{6i+3},\dots,x_{6i+7}) = (6i+2,6i+6,6i+3,6i+5,6i+1,6i+4)$, reducing sums modulo $6h+4$. Let c_2 be the $(6h-2)$-cycle $(y_0,y_1,\dots,y_{6h-3})$, where $(y_0,y_1,\dots,y_9) = (3,7,4,1,2,5,8,12,16,15)$ and for $3 \leqslant i \leqslant h$ $(y_{6i-8},6_{6i-7},\dots,y_{6i-3}) = (6i+1,6i-1,6i+2,6i,6i+4,6i+3)$. Finally, let c_3 be the 4-cycle $(10,13,11,14)$. Then c_2 and c_3 provide 2 matchings saturating each vertex except vertices 6 and 9, as does c_1. □

Lemma 2.5. *Let $n = 6h + 2$. For any t with $1 \leqslant t < h$, where if $t \leqslant 2$ then $h \leqslant 10$, there exists a partition of the edges of K_n into*

(i) 4 *matchings that each saturates all the vertices in K_n except for $u, v \in V(K_n)$;*

(ii) $6t + 1$ *1-factors of K_n; and*

(iii) *a set T of triples.*

Proof. Let $1 \leqslant t < h$ with $h \leqslant 10$ if $t \leqslant 2$. By Lemma 2.2(i), $\{1, 2, \ldots, 3h\} \backslash \{a, b, c\}$ with $\{1, 2, 3\} \cap \{a, b, c\} = \emptyset$ can be partitioned into a set D of $h - 1$ difference triples. Let D_1 be the set of difference triples in D that contain 1, 2 and 3 and let $\alpha = |D_1|$; then clearly $1 \leqslant \alpha \leqslant 3$, and by Lemma 2.2(i) if $t \leqslant 2$ then $D_1 = \{\{1, 2, 3\}\}$, so $\alpha = 1$. Let D_2 be a set of $t - \alpha$ difference triples in $D \backslash D_1$ (clearly $0 \leqslant t - \alpha \leqslant |D \backslash D_1|$). Let

$$L = \{a, b, c, 3h + 1\} \cup \{x \mid x \text{ is in a difference triple in } D_1 \cup D_2\} \backslash \{1, 2, 3\}.$$

Then $|L| = 4 + 3\alpha + 3(t - \alpha) - 3 = 3t + 1$. Since $3h + 1 \in L$, by Lemma 2.1 the edges in $G_{6h+2}(L)$ can be partitioned into $(6t + 1)$ 1-factors. By Lemma 2.3 the edges in $G_{6h+2}(\{1, 2, 3\})$ can be partitioned into 4 matchings that each saturates all the vertices in K_n except for two, together with a set T_1 of triples. Finally, the edges in K_n of lengths in the difference triples in $D \backslash (D_1 \cup D_2)$ are partitioned by the triples in

$$T_2 = \{\{i, x + i, x + y + i\} \mid \{x, y, x + y\} \in D \backslash (D_1 \cup D_2),\ i \in \mathbb{Z}_n\}.$$

So setting $T = T_1 \cup T_2$ provides the required partition. $\square$

Lemma 2.6. *Let $n = 6h + 4$. For any t with $1 \leqslant t < h$, where if $t \leqslant 3$ then $h \leqslant 13$, there exists a partition of the edges of K_n into*

(i) 4 *matchings that each saturates all the vertices in K_n except for $u, v \in V(K_n)$;*

(ii) $6t + 1$ *1-factors; and*

(iii) *a set T of triples.*

Proof. Let $1 \leqslant t < h$ with $h \leqslant 13$ if $t \leqslant 3$. If $h = 2$ (so $t = 1$), let $L = \{5, 6, 7, 8\}$ and $D = \phi$.

If $h \geqslant 3$ define L, D, D_1 and D_2 as follows. By Lemma 2.2(ii) $\{1, 2, \ldots, 3h + 1\} \backslash \{a, b, c, d\}$ with $\{1, 2, 3, 4\} \cap \{a, b, c, d\} = \emptyset$ can be partitioned into a set D of $h - 1$ difference triples. Let D_1 be the set of difference triples in D that contain 1, 2, 3 and 4, and let $\alpha = |D_1|$; then clearly $2 \leqslant \alpha \leqslant 4$, and by Lemma 2.2(ii) if $t \leqslant 3$ then $\alpha = 2$ (since $\{1, 2, 3\} \in D$). Let D_2 be a set of $t - \alpha$ difference triples in $D \backslash D_1$ (clearly $0 \leqslant t - \alpha \leqslant |D \backslash D_1|$). Let

$$L = \{a, b, c, d, 3h + 2\} \cup \{x \mid x \text{ is in a difference triple in } D_1 \cup D_2\} \backslash \{1, 2, 3, 4\}.$$

Then $|L| = 5 + 3\alpha + 3(t - \alpha) - 4 = 3t + 1$.

In any case, since $3h{+}2 \in L$, by Lemma 2.1, the edges in $G_{6h+4}(L)$ can be partitioned into $(6t + 1)$ 1-factors. By Lemma 2.4 the edges in $G_{6h+4}\{1, 2, 3, 4\}$ can be partitioned into 4 matchings that each saturates all the vertices in K_n except for two, together with a set T_1 of triples. Finally, the edges in K_n of lengths in the difference triples in $D\backslash(D_1 \cup D_2)$ are partitioned by the triples in

$$T_2 = \{\{i, x+i, x+y+i\} \mid \{x, y, x+y\} \in D\backslash(D_1 \cup D_2),\ i \in \mathbb{Z}_n\}.$$

So setting $T = T_1 \cup T_2$ provides the required partition. □

Proposition 2.7. *Let $m \equiv 5 \pmod 6$, $n \equiv 1$ or $3 \pmod 6$, and $n > 2m$. Any maximum packing of order m can be embedded in a* STS(n).

Proof. Let $m = 6t+5$ and $n - m \in \{6h+2, 6h+4\}$. Let (S_1, T_1, L_1) be a maximum packing of order m, and let S_2 be a set of size $n-m$ with $S_1 \cap S_2 = \emptyset$. By Theorem 1.3 we can assume that $m \geqslant 11$. So, since $n > 2m$, we have that $1 \leqslant t < h$.

Suppose first that if $t \leqslant 2$ and $n - m = 6h+2$ then $h \leqslant 10$, and if $t \leqslant 3$ and $n - m = 6h+4$ then $h \leqslant 13$. Then by Lemmas 2.5 and 2.6, there exists a partition of the edges of K_{n-m} with vertex set S_2 into 4 matchings M_1, M_2, M_3 and M_4 that each saturates all vertices in S_2 except for $u, v \in S_2$; $6t+1$ 1-factors $F_1, F_2, \dots, F_{6t+1}$ of K_{n-m}; and a set T of triples. Let $L_1 = \{\{a_1,a_2\},\{a_2,a_3\},\{a_3,a_4\},\{a_1,a_4\}\}$ and let $\phi\colon S_1\backslash\{a_1,a_2,a_3,a_4\} \to \{1,2,\dots,6t+1\}$ be a 1–1 mapping. Define

$$T_2 = T_1 \cup T \cup \{\{a_1,a_2,u\},\{a_3,a_4,u\},\{a_1,a_4,v\},\{a_2,a_3,v\}\}$$
$$\cup \{\{a_i,x,y\} \mid 1 \leqslant i \leqslant 4, \{x,y\} \in M_i\} \cup \{\{s,x,y\} \mid s \in S_1, \{x,y\} \in F_{\phi(s)}\}.$$

Then clearly $(S_1 \cup S_2, T_2)$ is an STS(n), and (S_1, T_1, L_1) is embedded in this STS(n).

Now suppose that either $1 \leqslant t \leqslant 2$, $n - m = 6h+2$ and $h \geqslant 11$ (so $n = 6t+5+6h+2 \geqslant 79$), or $1 \leqslant t \leqslant 3$, $n - m = 6h+4$ and $h \geqslant 14$ (so $n \geqslant 99$). If $n - m = 6h+2$ then the previous case shows that (S_1, T_1, L_1) can be embedded in a STS(37) (that is, $37 = 6t+5+6h'+2$, where $h' = 4$ or 3 if $t = 1$ or 2, respectively), and by Theorem 1.1 this STS(37) can be embedded in a STS(n) since $n \geqslant 75$. If $n - m = 6h+4$ then we have shown that (S_1, T_1, L_1) can be embedded in a STS(49), which by Theorem 1.1 can be embedded in a STS(n) since $n \geqslant 99$.

Thus, in any case (S_1, T_1, L_1) has been embedded in a STS(n). □

Proposition 2.8. *Let $m \equiv 5 \pmod 6$ and $n \equiv 0$ or $2 \pmod 6$, or $m \equiv 4 \pmod 6$ and $n \equiv 0, 1, 2$ or $3 \pmod 6$. Let $n > 2m$. Then any maximum packing of order m can be embedded in a maximum packing of order n.*

Proof. Let (S_1, T_1, L_1) be a maximum packing of order m.

If $m \equiv 5 \pmod 6$ then by Proposition 2.7, (S_1, T_1, L_1) can be embedded in a STS$(n+1)$ (S_2, T_2). For any $s \in S_2\backslash S_1$, $(S_2\backslash\{s\}, \{t \mid t \in T_2$ and $s \notin t\})$ is a maximum packing of order n that contains (S_1, T_1, L_1).

If $m \equiv 4 \pmod 6$ then L_1 is a tripole, say $L_1 = \{\{s_1,s_2\},\{s_1,s_3\},\{s_1,s_4\}\} \cup \{\{s_{2i-1},s_{2i}\} \mid 3 \leqslant i \leqslant m/2\}$. Then (S_1, T_1, L_1) is embedded in the maximum packing $(S_1 \cup \{s\}, T_2, L_2)$ of order $m+1$, where $T_2 = \{\{s,s_1,s_4\}\} \cup \{\{s,s_{2i-1},s_{2i}\} \mid 3 \leqslant i \leqslant m/2\} \cup T_1$ and $L_2 =$

$\{\{s,s_2\},\{s_2,s_1\}, \{s_1,s_3\},\{s,s_3\}\}$. Since $m+1 \equiv 5 \pmod 6$ and since $n > 2m$, by Proposition 2.7, $(S_1 \cup \{s\}, T_2, L_2)$ can be embedded in: a STS(n) (S_3, T_3) if $n \equiv 1$ or $3 \pmod 6$, thus providing the required embedding; and in a STS($n+1$) (S_3, T_3) if $n \equiv 0$ or $2 \pmod 6$, in which case $(S_3 \backslash \{s\}, \{t \mid t \in T_3 \text{ and } s \notin t\}, \{\{a,b\} \mid \{s,a,b\} \in T_3\})$ provides the required embedding. □

3. Another embedding problem

There is a second generalization of Theorem 1.1 that makes sense to consider, and that is to ensure that the leave of the MPT(m) is preserved during the embedding. More specifically, the second problem is to find the integers m and n for which any MPT(m) (S_1, T_1, L_1) can be embedded in an MPT(n) (S_2, T_2, L_2) so that $L_1 \subseteq L_2$. This extra requirement that $L_1 \subseteq L_2$ restricts more severely the integers n for which such an embedding is possible. It is easy to see that the following conditions are necessary.

Lemma 3.1. *Let $n > m$. Suppose that any* MPT(m) (S_1, T_1, L_1) *can be embedded in an* MPT(n) (S_2, T_2, L_2) *with $L_1 \subseteq L_2$. Then*

(1) *if $m \equiv 0$ or $2 \pmod 6$ then n is even,*

(2) *if $m \equiv 4 \pmod 6$ then $n \equiv 4 \pmod 6$,*

(3) *if $m \equiv 5 \pmod 6$ then $n \equiv 5 \pmod 6$, and*

(4) *$n \geqslant 2m$, with strict inequality if $m \equiv 0, 2, 4$ or $5 \pmod 6$.*

Proof. (1)–(3) follow directly from the requirement that $L_1 \subseteq L_2$. The requirement that $n \geqslant 2m$ follows from (1)–(3) and Lemma 1.2.

To see that if $m = 0, 2, 4$ or $5 \pmod 6$ then $n \neq 2m$, consider the following. If $n = 2m$ then (1)–(3) require m to be even, so the leave of (S_1, T_1, L_1) is a spanning graph. Therefore, all the edges joining vertices in S_1 to vertices in $S_2 \backslash S_1$ (except for at most two such edges that could occur in the head of the tripole when $n \equiv 4 \pmod 6$) must occur in triples that contain one vertex in S_1 and two vertices in $S_2 \backslash S_1$. It is easy to check that there are not enough edges joining vertices in $S_2 \backslash S_1$ for this to be possible. Hence, in these cases $n > 2m$. □

It turns out that in many cases, the embeddings of Mendelsohn and Rosa have this additional property.

Theorem 3.2 (Mendelsohn and Rosa [7]). *Let $n \geqslant 2m$, with strict inequality if $m \equiv 0, 2, 4$ or $5 \pmod 6$. Suppose that $m, n \equiv 0$ or $2 \pmod 6$, or $m \equiv n \equiv 4$ or $5 \pmod 6$, or $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 0, 1, 2$ or $3 \pmod 6$. Then any* MPT(m) (S_1, T_1, L_1) *can be embedded in an* MPT(n) (S_2, T_2, L_2) *with $L_1 \subseteq L_2$.*

Therefore, to prove that the necessary conditions of Lemma 3.1 are sufficient, we need only consider the case where $m \equiv 0$ or $2 \pmod 6$ and $n \equiv 4 \pmod 6$, and the case where $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 4$ or $5 \pmod 6$, with $n > 2m$ in each case. In the next section we will consider these cases as we complete the proof of Theorem 1.5.

4. The case $s \in \{0, 1, 2, 3\}$ and $t \in \{4, 5\}$

Two of these cases have already been settled with the following result.

Proposition 4.1 (Fu et al. [3]). *Let $m \equiv 0$ or $2 \pmod 6$ and $n \equiv 4 \pmod 6$ with $n > 2m$. Then any MPT(m) (S_1, T_1, L_1) can be embedded in an MPT(n) (S_2, T_2, L_2) in which $L_1 \subseteq L_2$, and in which the edges in $L_2 \backslash L_1$ all join pairs of vertices that are both in $S_2 \backslash S_1$.*

We can use Proposition 4.1 to obtain the following result.

Proposition 4.2. *Let $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 4$ or $5 \pmod 6$ with $n > 2m$. Then any MPT(m) $(S_1, T_1, L_1 = \emptyset)$ can be embedded in an MPT(n) (S_2, T_2, L_2) in which trivially $L_1 \subseteq L_2$, and in which the edges in $L_2 \backslash L_1$ all join vertices in $S_2 \backslash S_1$.*

Proof. Suppose first that $n = 6h + 5$. Let $s \in S_1$. By Proposition 4.1 we can embed the MPT($m - 1$) $(S_1', T_1', L_1') = (S_1 \backslash \{s\}, \{t \mid t \in T_1,\ s \notin t\}, \{\{x, y\} \mid \{s, x, y\} \in T_1\})$ in an MPT($n-1$) (S_2', T_2', L_2') in which $s \notin S_2'$ and $L_1 \subseteq L_2'$. Let $L_2' = \{\{x_i, y_i\} \mid 1 \leqslant i \leqslant 3h\} \cup \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}\}$, where by Proposition 4.1 we can assume that $\{s_1, s_2, s_3, s_4\} \subseteq S_2 \backslash S_1$. Then $(S_1, T_1, L_1 = \emptyset)$ is embedded in the MPT(n) $(S_2, T_2, L_2) = (S_2' \cup \{s\}, \{\{s, x_i, y_i\} \mid 1 \leqslant i \leqslant 3h\} \cup \{\{s, s_1, s_2\} \cup T_2'\}, \{\{s, s_3\}, \{s, s_4\}, \{s_1, s_3\}, \{s_1, s_4\}\})$. Trivially $L_1 \subseteq L_2$.

Also, (S_1, T_1, L_1) is embedded in the MPT($6h + 4$) $(S_2 \backslash \{s_1\}, \{t \mid t \in T_2,\ s_1 \notin t\}, \{\{x, y\} \mid \{s_1, x, y\} \in T_2)$, so the result is proved. □

Proposition 4.3. *Let $m \equiv 0$ or $2 \pmod 6$ and let $n \equiv 5 \pmod 6$ with $n > 2m$. Then any MPT(m) can be embedded in an MPT(n).*

Proof. Let $n = 6h + 5$. Let (S_1, T_1, L_1) be an MPT(m). Then by Theorem 1.4 if $n - 1 = 2m$ and by Proposition 4.1 otherwise, we can embed this in an MPT($n - 1$), which can itself be embedded in an MPT(n) in the same way that (S_2', T_2', L_2') was embedded in (S_2, T_2, L_2) in the previous proof. □

5. Summary

We now collect together these results.

Theorem 5.1. *Let $n > m$. Any* MPT(m) *can be embedded in an* MPT(n) *if and only if*

(1) *if $m = 6$ then $n = 7$ or $n \geqslant 10$,*
(2) *if $m > 6$ and m is even then $n = m + 1$ or $n \geqslant 2m$, and*
(3) *if $m > 6$ and m is odd then $n \geqslant 2m$.*

Proof. The necessity comes from Lemma 1.2. The sufficiency follows from Theorems 1.3–1.5. Theorem 1.5 is proved by Propositions 2.7, 2.8 and 4.1–4.3. □

Theorem 5.2. *Let $n > m$. Any* MPT(m) (S_1, T_1, L_1) *can be embedded in an* MPT(n) (S_2, T_2, L_2) *such that $L_1 \subseteq L_2$ if and only if*

(1) *if $m \equiv 0$ or $2 \pmod 6$ then n is even,*
(2) *if $m \equiv 4 \pmod 6$ then $n \equiv 4 \pmod 6$,*
(3) *if $m \equiv 5 \pmod 6$ then $n \equiv 5 \pmod 6$, and*
(4) *$n \geqslant 2m$, with strict inequality if $m \equiv 0, 2, 4$ or $5 \pmod 6$.*

Proof. The necessity follows from Lemma 3.1. The sufficiency is proved by Theorem 3.2 and Propositions 4.1 and 4.2. □

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