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Chapter 1

Introduction

The architecture of interconnection network is usually represented by a graph. A static interconnection network has point-to-point communication links among processors. Graph theory can be used to analyze the network reliability. Network reliability is one of the major factors in designing the topology of an interconnection network. It has been shown that a network is more reliable if it has higher connectivity [2, 3, 5, 9, 4, 21]. The classical Menger's Theorem [12] provided a useful viewpoint about connectivity. Furthermore, [13, 14] studied some strong properties of Menger's Theorem.

The collection of *n-cubes* is a popular class of interconnection networks. To verify that they have high potential to be used as a network, much research has been done on these topologies. For example, [1, 6, 7, 8, 15] studied these topological issues, [10, 11, 16, 19, 20] studied some fault tolerant issues when edge and/or vertex faults are present, and [9, 17] studied some conditional issues with faults.

Various networks are proposed by twisting some pairs of links in hypercubes. To make a unified study on these variants, [18] offered a class of graphs, called *a class of hypercube-like networks*. In this thesis, we show that all graphs in *the class of n-dimensional hypercube-like networks* have some strongly Menger-connected property, even if these graphs are with $n-2$ fault vertexs. Furthermore, if we restrict some conditions for each vertex having at least two fault-free adjacent vertices, the class of hypercube-like networks have the strongly Menger-connected property, even if these graph are with $2n-5$ fault vertexs. In the last chapter, we show that a more general class of graphs, called *Matching Composition Networks*, satisfying some conditions can have strongly Menger-connected property. Last, we

Chapter 2

Preliminary

The topology of a multiprocessor system can be modeled as an undirected graph $G = (V, E)$, where $V(G)$ represents the set of all processors and $E(G)$ represents the set of all connecting links between the processors. For a subset of vertices $F \subset V(G)$, the induced graph obtained by deleting the vertices of F from G is denoted by $G - F$. Let u be a vertex, we use $N(u)$ to denote the set of vertices adjacent to u, and use $deg(u)$ to denote the cardinality of $N(u)$. For every set of vertices V', the neighborhood of V' is defined as the set $N(V') = \{\bigcup_{v \in V'} N(v)\} - V'$. Let G be a graph with a set F of faulty vertices, the number of fault-free neighbors of u in $G - F$ is denoted by $deg_f(u)$.

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of vertices. A *one-to-one and onto connection* between $V(G_0)$ and $V(G_1)$ is defined as an edge set $M = \{(v, \phi(v)) \mid v \in V_0, \phi(v) \in V_1$ and $\phi : V_0 \to V_1$ is a bijection$\}$. We use $G_0 \oplus_M G_1$ to denote the graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup M)$. Different bijection functions ϕ lead to different operations $\oplus_M$ and generate different graphs.

A graph G is *r-regular* if the degree of every vertex in G is r. We say that a graph G is *connected* if there is a path between every pair of two distinct vertices. A subset S of $V(G)$ is a *cut set* if $G - S$ is disconnected. The *connectivity* of G, written as $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. We say that a graph G is *k-connected* if $k \leq \kappa(G)$. In addition, a graph has *connectivity* k if it is k-connected but not $(k+1)$-connected.

Some major issues about interconnection networks focus on the connectivity. Menger provided a useful point of view to illustrate the connectivity as following.

Theorem 1. *[12] If x,y are vertices of a graph G and $(x, y) \notin E(G)$, then the minimum size* *of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.(See Fig. 2.1.)*

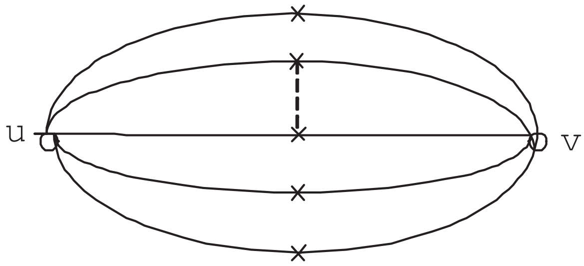

Figure 2.1: Illustration for Theorem 1

Following this theorem, OH et al. [13] gave a definition to extend the Menger's Theorem.

Definition 1. *[13] A n-regular graph G is strongly Menger-connected if for any copy $G - F$ of G with at most $n - 2$ vertices removed, each pair u and v of $G - F$ are connected by $min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths in $G - F$, where $deg_f(u)$ and $deg_f(v)$ are the degree of u and v in $G - F$, respectively.*

This property is called *strongly Menger-connected* property. OH et al. [13, 14] showed that an n-dimensional star graph S_n with at most $n-3$ vertices removed and an n-dimensional Hypercube graph Q_n with at most $n-2$ vertices removed still possess the strongly Menger-connected property.

The *hypercube* network is one of the popular topologies in interconnection networks. Various networks are proposed by twisting some pairs of links in hypercubes [1, 6, 7, 8]. To make a unified study on these variants, Vaidya et al. [18] offered a class of graphs, called *a class of hypercube-like networks.* We now give a recursive definition of the n-dimensional hypercube-like networks HL_n as follows:

(1)$HL_0 = K_1$, where K_1 is a trivial graph in the sense that it has only one vertex.

(2)$G \in HL_n$ if and only if $G = G_0 \oplus_M G_1$ for some $G_0, G_1 \in HL_{n-1}$

By the definitions above if G is a graph in HL_n, then G is a composition of $G_0 \oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, $n \geq 1$. Each vertex in G_0 has only one neighbor in G_1. The connectivity of an n-dimensional hypercube-like network HL_n is n [18]. So the connectivity of HL_{n-1} is $n - 1$ and both G_0 and G_1 are $(n - 1)$-connected.

In this thesis, we show that all graphs in *the class of n-dimensional hypercube-like networks* have the strongly Menger-connected property, even if these graphs are with $n - 2$

fault vertexs. Furthermore, if we restrict some conditions for each vertex having at least two fault-free adjacent vertices, the class of hypercube-like networks have the strongly Menger-connected property, even if these graph are with $2n-5$ fault vertexs. In the last chapter, we show that a more general class of graphs, called *Matching Composition Networks*, satisfying some conditions can have strongly Menger-connected property.

Chapter 3

Strong Menger Connectivity

In this chapter, we prove that all graphs in the class of *n-dimensional hypercube-like networks* are strongly Menger-connected, with a bounded amount of faults. Before proving this main result, we need the following lemma, essentially it says that every *n-dimensional hypercube-like network* with no more than $2n-3$ vertex faults, still contains a large connected component.

Lemma 1. *Let G be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leq 2n-3$, for $n \geq 2$. There exists a connected component C in $G - S$ such that $|V(C)| \geq 2^n - |S| - 1$.*

Proof. We prove this statement by induction on n. For $n = 2$, the result is trivial trace. Assume this lemma holds for $n-1$, for some $n \geq 3$,we will prove that it is true for n.

So, let $G \in HL_n$, $G = G_0 \oplus_M G_1$, and G_0, $G_1 \in HL_{n-1}$. Let S_0 and S_1 be subsets of set S in G_0 and G_1, respectively. Then $|S_0| + |S_1| = |S| \leq 2n-3$. Without loss of generality, we assume $|S_0| \leq |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leq |S_0| \leq 1$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geq 3$. All the vertices in $G_0 - S_0$ are connected and form a connected component C_0 with $|V(C_0)| = 2^{n-1} - S_0$. By definitions, all the vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Thus, $G - S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^n - |S| - 1$. (See Fig. 3.1. and Fig. 3.2)

Case 2: $|S_0| \geq 2$ and consequently $|S_1| \leq 2n-5$.

The only case for $|S_0| \leq |S_1|$ is $n \geq 4$. And $|S| \leq 2n-3$, so $|S_0| \leq n-2$, there is a

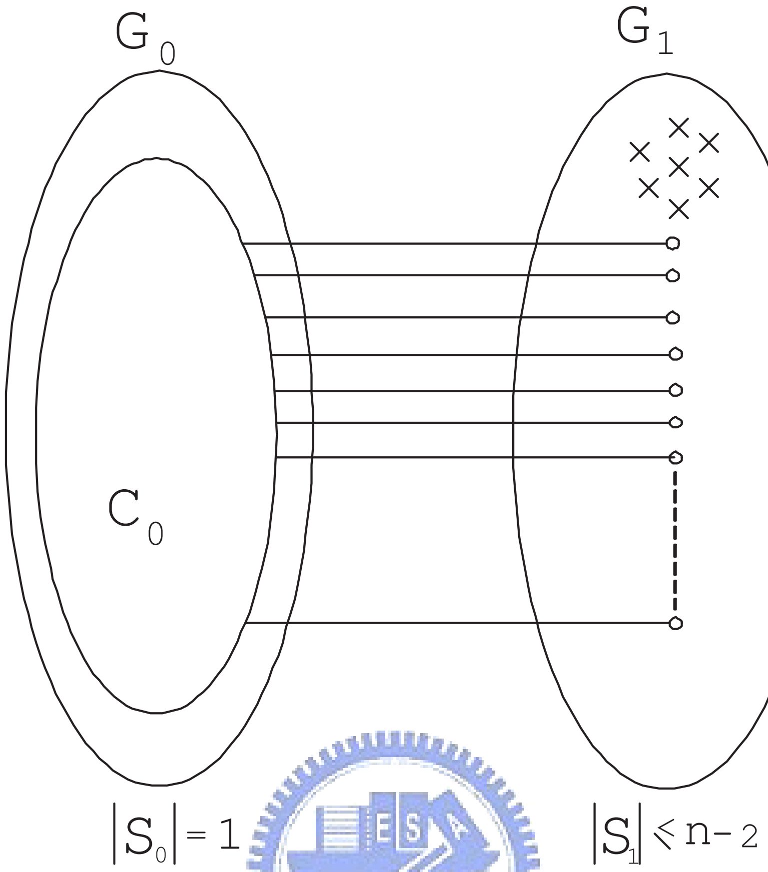

Figure 3.1: Illustration for Lemma1 case1.1

connected component C_0 in $G_0 - S_0$, and $|V(C_0)| \geq 2^{n-1} - |S_0|$. On the other hand,by induction hypothesis, there is a connected component C_1 in $G_1 - S_1$, and $|V(C_1)| \geq 2^{n-1} - |S_1| - 1$. Since $2^{n-1} - |S| \geq 1$ for $n \geq 4$, where $|S| \leq 2n - 3$. We can infer that $|V(C_0)| \geq 2^{n-1} - |S_0| \geq |S_1| + 1$. Then, there exists at least one edge connected to both C_0 and C_1. (See Fig. 3.3) Thus, there exists a connected component C such that $|V(C)| \geq 2^{n-1} - |S_0| + 2^{n-1} - |S_1| - 1 = 2^n - |S| - 1$. $\square$

By lemma 1, we have the following corollary.

Corollary 1. *Let G be an n-dimensional hypercube-like network, $n \geq 2$, and let V' be a set of vertices in G with $|V'| = 2$. Then $|N(V')| \geq 2n - 2$.*

Now we show that the bounded amount of faults for an n-dimensional hypercube-like network being strongly Menger-connected is $n - 2$. Before proving it, we give an example to show that this bound is tight. Let (x, u) be an edge and v be a vertex different from x and u in an n-dimensional hypercube-like network. Suppose that all the $n - 1$ vertices adjacent to x (except for u) are faulty. Although the remaining degree of u and v are both n, the number

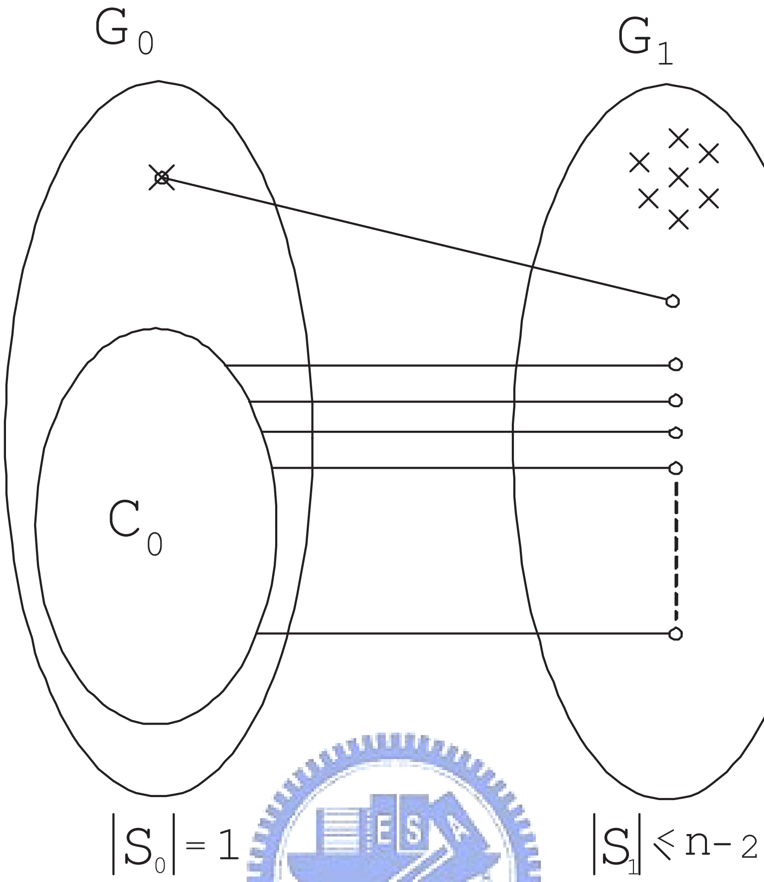

Figure 3.2: Illustration for Lemma1 case1.1

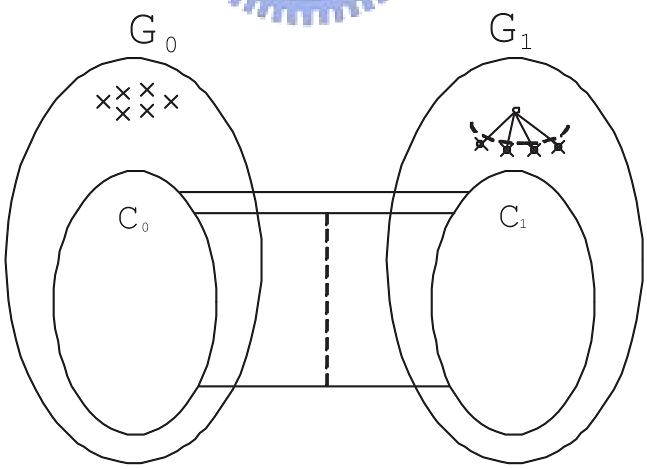

Figure 3.3: Illustration for Lemma1 case2

of vertex-disjoint paths between u and v is at most $n-1$, which implies that an n-dimensional hypercube-like network is not guaranteed to have the strongly Menger-connected property, if there are $n-1$ faulty vertices.(See Fig. 3.4.)

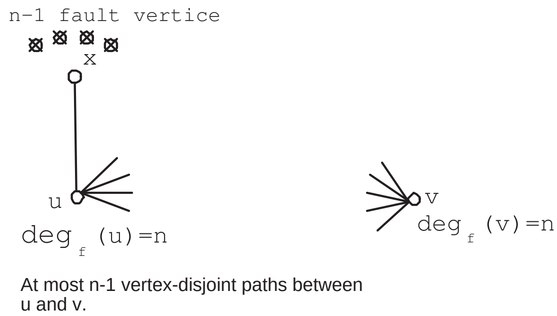

Figure 3.4: An example having n-1 faulty vertices

Theorem 2. *Consider an n-dimensional hypercube-like networks $G \in HL_n$, for $n \geq 2$. Let F be a set of faulty vertices with $|F| \leq n-2$. Then each pair of vertices u and v in $G-F$ are connected by $min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths, where $deg_f(u)$ and $deg_f(v)$ are the degree of u and v in $G-F$, respectively.*

Proof. Let G be an n-dimensional hypercube-like network, and u and v be two vertices in G. We first assume without loss of generality that $deg_f(u) \leq deg_f(v)$, so $min\{deg_f(u), deg_f(v)\} = deg_f(u)$. We now show that u is connected to v if the number of vertices deleted is smaller than $deg_f(u)-1$ in $G-F$, which implies that each pair of vertices u and v in $G-F$ are connected by $deg_f(u)$ vertex-disjoint fault-free paths, where $|F| \leq n-2$.

For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_f(u)-1$. As a consequence, $|V_f| \leq n-1$ because of $deg_f(u) \leq deg(u) \leq n$. Then, the summation of the cardinality of these two sets F and V_f is $|F|+|V_f| = |S| \leq 2n-3$. By Lemma 1, there exists a connected component C in $G-S$ such that $|V(C)| \geq 2^n - |S| - 1$. It means that either $G-S$ is connected, or $G-S$ has two components, one of which is only one vertex. If $G-S$ has two components and one of which has only one vertex, the set V_f has to be the neighborhood of u and $|V_f| = deg_f(u)$, this contradicts to the fact that $|V_f| \leq deg_f(u)-1$.. Thus, u is connected to v when the number of vertices deleted is smaller than $deg_f(u)-1$ in $G-F$.

The proof is completed. □

Chapter 4

Strong Menger Connectivity with Conditional Faults

As we proved in last chapter, an n-dimensional hypercube-like network with at most $n-2$ faulty vertices is strongly Menger-connected. But the result can not be guaranteed, if there are $n-1$ faulty vertices and all these faulty vertices are adjacent to the same vertex .In most circumstances, the possibility of all the neighbors of a vertex being faulty simultaneity is very small. Motivated by the deficiency of traditional fault tolerance, we introduce a measure of conditional faults by claiming that, every vertex has at least two fault-free neighboring vertices.

Under this condition, we claim that for every n-dimensional hypercube-like network with at most $2n-5$ faulty vertices and $n \geq 5$, the resulting network is still strongly Menger-connected. We have an example to show that this result does not hold for $n = 4$. Consider the 4-dimensional HL_4, this network may not be strongly Menger-connected, if the number of conditional faults is 3. (See Fig. 4.1.) So we need the condition $n \geq 5$.

To prove our main result, we need some preliminary lemmas. In the following, we show that an n-dimensional hypercube-like network with at most $3n-6$ vertex faults S has a connected component having at least $2^n - |S| - 2$ vertices.

The proof is by induction, and the case for $n = 5$ is proved in the following two lemmas.

Lemma 2. *Let V' be a set of vertices in a 4-dimensional hypercube-like network with $|V'| = 3$. Then, $|N(V')| \geq 7$.*

Proof. Let G be in HL_4. G is a composition of two 3-dimensional hypercube-like networks G_0 and G_1. That is, $G = G_0 \oplus_M G_1$, for some operation $\oplus_M$. Let $V' = \{x, y, z\}$ in G. If x,

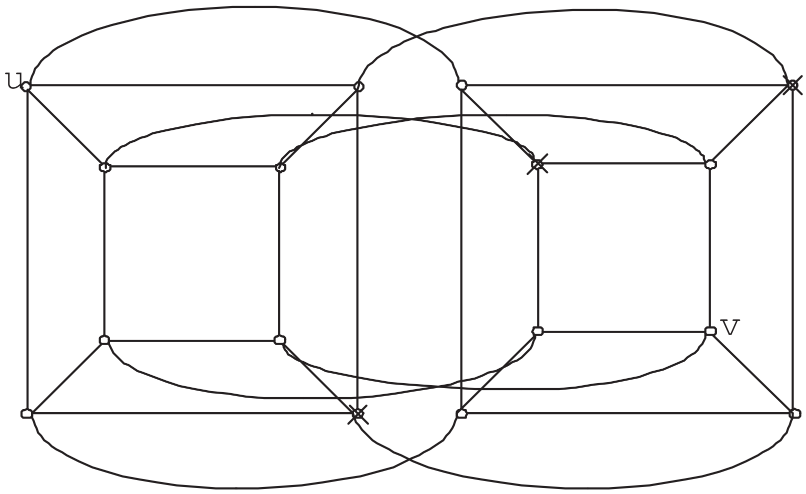

Figure 4.1: An example n=4(step1)

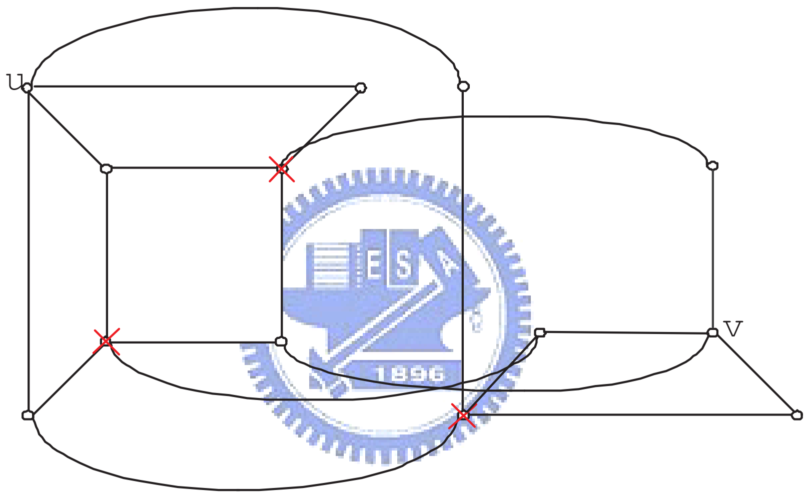

Figure 4.2: An example n=4(step2)

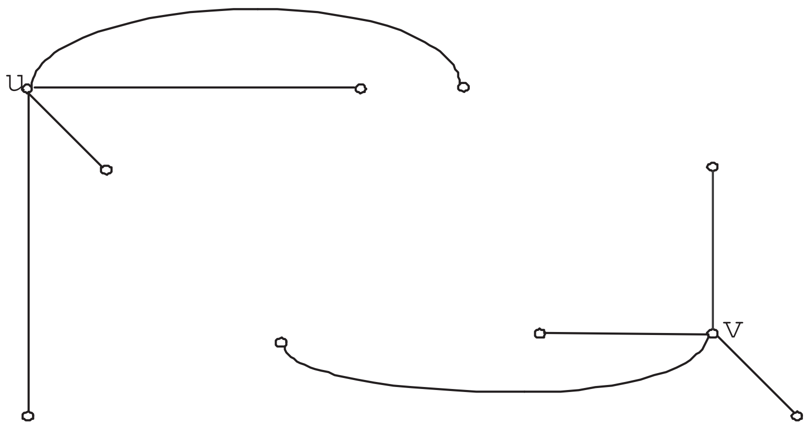

Figure 4.3: An example n=4(step3)

y, z are all in G_0, by Lemma 1, $\{x, y, z\}$ has at least 4 neighboring vertices in G_0. Besides, $\{x, y, z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geq 4+3 = 7$. If x, y are in G_0, and z is in G_1, by Lemma 1, $\{x, y\}$ has at least 4 neighboring vertices in G_0. In addition, $\{z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geq 4 + 3 = 7$. $\square$

Lemma 3. *Let G be a 5-dimensional hypercube-like network and S be a set of vertices with $|S| \leq 9$. There exists a connected component C in $G - S$ such that $|V(C)| \geq 2^5 - |S| - 2$.*

Proof. We first assume that $G = G_0 \oplus_M G_1$ with $G_0 \in HL_4$ and $G_1 \in HL_4$. Let S_0 and S_1 be sets of faulty vertices in G_0 and G_1, respectively. Without loss of generality, we assume $|S_0| \leq |S_1|$. We then consider three cases:

Case 1: $0 \leq |S_0| \leq 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n = 4$. There exists a connected component C_0 in $G_0 - S_0$ with $|V(C_0)| \geq 2^4 - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices of $G_0 = C \cup S_0$. Thus, the number of one connected component C in $G - S$ is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^5 - |S| - 2$.

Case 2: $|S_0| = 3$ and therefore $|S_1| \leq 6$.

We can find that $G_0 - S_0$ is connected, since G_0 is $(n-1)$-connected, for $n \geq 4$. Thus, the connected component C_0 existed in $G_0 - S_0$ with $|V(C_0)| = 2^4 - |S_0|$. Then, all vertices in G_1 are connected to component C_0, except for the three vertices in G_1 adjacent to the vertices in S_0. Since $|S_1| \leq 6$ and by Lemma 2, at least one of these three vertices is connected to component C_0. Then, a connected component C in $G - S$ exists with $|V(C)| \geq |V(G_0) - S_0| + |V(G_1) - S_1 - 2| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$.

Case 3: $|S_0| = 4$ and consequently $4 \leq |S_1| \leq 5$.

Since $5 \leq 2n - 3$, for $n \geq 4$. By Lemma 1, both $G_0 - S_0$ and $G_1 - S_1$ have connected components C_0 and C_1 such that $|V(C_0)| \geq 2^4 - |S_0| - 1$ and $|V(C_1)| \geq 2^4 - |S_1| - 1$, respectively. Thus, the number of one connected component C in $G - S$ is greater than $|V(G_0) - S_0 - 1| + |V(G_1) - S_1 - 1| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$. $\square$

Based on Lemma 3, the general case for $n \geq 5$ is stated as follows.

Lemma 4. *Let G be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leq 3n - 6$, for $n \geq 5$. There exists a connected component C in $G - S$ such that $|V(C)| \geq 2^n - |S| - 2$.*

Proof. We prove this statement by induction on n. By Lemma 3, the result holds for $n = 5$. Assume the lemma holds for $n - 1$, for some $n \geq 6$. We now show that it is true for n.

Let G be an n-dimensional hypercube-like network, and $G = G_0 \oplus_M G_1$, for some operation $\oplus_M$. Then $G_0, G_1 \in HL_{n-1}$. Let S_0 and S_1 be subsets of set S in G_0 and G_1, respectively. Therefore, $|S_0| + |S_1| = |S| \leq 3n - 6$. Without loss of generality, we assume $|S_0| \leq |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leq |S_0| \leq 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geq 6$. All vertices in $G_0 - S_0$ can be formed as a connected component C_0 and $|V(C_0)| \geq 2^{n-1} - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Thus, the number of one connected component C in $G - S$ is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^n - |S| - 2$.

Case 2: $|S_0| \geq 3$ and consequently $|S_1| \leq 3n - 9$.

By induction hypothesis, there are two connected components C_0 and C_1 in $G_0 - S_0$ and $G_1 - S_1$, and $|V(C_0)| \geq 2^{n-1} - |S_0| - 2$ and $|V(C_1)| \geq 2^{n-1} - |S_1| - 2$, respectively. Without loss of generality, we assume that $|V(C_0)| \geq |V(C_1)|$. Now we focus on the number of vertices in the component C_1, and discuss two situations. First, suppose $|V(C_1)| = 2^{n-1} - |S_1| - 2$. By Corollary 1, $|S_1| \geq 2(n-1) - 2 = 2n - 4$. So $|S_0| = |S| - |S_1| \leq n - 2$. Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected. The connected component C_0 existed of $G_0 - S_0$, where $|V(C_0)| = 2^{n-1} - |S_0|$. Thus, there exists a connected component C such that $|V(C)| = |V(C_0)| + |V(C_1)| \geq 2^{n-1} - |S_0| + 2^{n-1} - |S_1| - 2 \geq 2^n - |S| - 2$. Second, suppose that $|V(C_1)| \geq 2^{n-1} - |S_1| - 1$. Since $|V(C_0)| \geq |V(C_1)| \geq 2^{n-1} - |S_1| - 1$, there exists a connected component C such that $|V(C)| = |V(C_0)| + |V(C_1)| \geq 2^{n-1} - |S_0| - 1 + 2^{n-1} - |S_1| - 1 \geq 2^n - |S| - 2$. $\square$

Corollary 2. *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geq 5$. Let V' be a set of vertices in G with $|V'| = 3$. Then $|N(V')| \geq 3n - 5$.*

Now we show that the bounded amount of conditional faults for an n-dimensional hypercube-like network being strongly Menger-connected is $2n-5$. Before proving it, we give an example to show that this bound is tight(See Fig. 4.2.).

F is the set of faulty set with $|F| = 2n - 4$. Let $x, y, z, u, v \in G - F$, and $deg_f(u) = deg_f(v)$=n. The neighborhoods of x and y are all faulty except vertices u, v. Thus the number of vertex disjoint fault-free paths between u and v are at most $n - 1$. So $G - F$ can not guaranteed to have strongly Menger-connected property, for $|F| = 2n - 4$.

Theorem 3. *Consider an n-dimensional hypercube-like networks $G \in HL_n$, for $n \geq 5$. Let F_c be a set of conditional faulty vertices with $|F_c| \leq 2n - 5$. Then each pair of vertices u and*

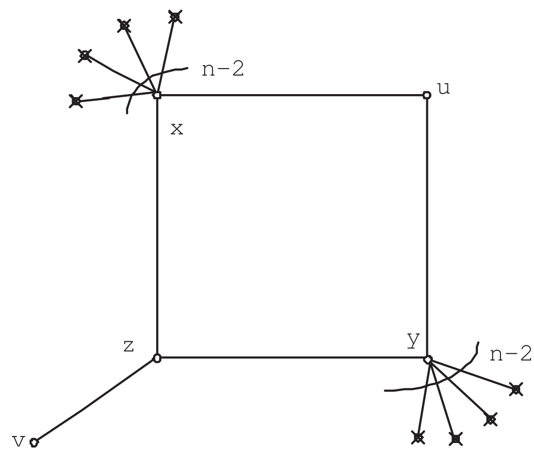

Figure 4.4: An example having 2n-4 faulty vertices

v in $G - F_c$ are connected by $min\{deg_{f_c}(u), deg_{f_c}(v)\}$ vertex-disjoint fault-free paths, where $deg_{f_c}(u)$ and $deg_{f_c}(v)$ are the degree of u and v in $G - F_c$, respectively.

Proof. Without loss of generality, we assume $deg_{f_c}(u) \leq deg_{f_c}(v)$, and therefore $min\{deg_{f_c}(u)$ $, deg_{f_c}(v)\} = deg_{f_c}(u)$. We want to prove that each pair of vertices u and v in $G - F_c$ are connected by $deg_{f_c}(u)$ vertex-disjoint fault-free paths, for $|F_c| \leq 2n - 5$. So we show that u is connected to v if the number of vertices deleted is smaller than $deg_{f_c}(u) - 1$ in $G - F_c$, where $|F_c| \leq 2n - 5$.

Suppose on the contrary that u and v are separated by deleting a set of vertices V_{f_c}, where $|V_{f_c}| \leq deg_{f_c}(u) - 1$. By $deg_{f_c}(u) \leq deg(u) \leq n$, we get $|V_{f_c}| \leq n - 1$. We sum the cardinality of these two sets F_c and V_{f_c}. Since $|F_c| \leq 2n - 5$ and $|V_{f_c}| \leq n - 1$, then $|F_c| + |V_{f_c}| = |S| \leq 3n - 6$. By Lemma 4, there exits a connected component C in $G - S$ such that $|V(C)| \geq 2^n - |S| - 2$ and $|S| \leq 3n - 6$. It means that there are at most two vertices, denoted by a and b, not belonging to C and S. Without loss of generality, let $a = u$. We then consider two cases:

case 1: Suppose a is not adjacent to b. By the assumption that u and v are separated by deleting a set of vertices V_{f_c} with $|V_{f_c}| = deg_{f_c}(u) - 1$. Let V_{f_c} be a subset of the neighborhood of u, i.e. $V_{f_c} \subset N(u)$. Since $|V_{f_c}| < |N(u)|$, vertex u and component C are connected, which is a contradiction.

case 2: Suppose a is adjacent to b. Let $V_{f_c} = N(u) - \{b\}$. Since $G - F_c$ is conditional, one of the neighbors of b is in C. Then, b is connected to C. Consequently, a is connected to C, which is a contradiction.

This completes the proof. □

Chapter 5

Matching Composition Networks

In the previous chapter, we prove that strongly Menger-connected property holds in *the class of hypercube-like networks*. In this chapter, we provide a similar result of Theorem 2, but it is not limited to *the class of hypercube-like networks*. In fact, this result can be applied to a certain general class of graphs, called *matching composition networks*, we define the *Matching Composition Network (MCN)* as follows: Let G_1 and G_2 be two graphs with the same number of nodes. Let M be an arbitrary perfect matching between the nodes of G_1 and G_2, i.e., M is a set of edges connecting the nodes of G_1 and G_2 in a one to one function; the resulting composition graph is called a *Matching Composition Network (MCN)*. For convenience, G_1 and G_2 are called the components of the *MCN*. Formally, we use the notation $G(G_1, G_2; M)$ to denote an *MCN*. We need two preliminary lemmas. In the following, we first show that a graph $G(G_1, G_2; M)$ is n-regular and $\kappa(G) = n$, where G_1,G_2 is $(n-1)$-regular graph with $\kappa(G_1) = \kappa(G_2) = n-1$, containing no triangle, $|V(G_1)| = |V(G_2)| \geq 2n-2$. Subsequently, we show that $G = G(G_1, G_2; M)$ with no more than $2n-3$ faulty vertices, still contains a large connected component, where G_1,G_2 is k-regular graph with $\kappa(G_1) = \kappa(G_2) = n-1$, containing no triangle, $|V(G_1)| = |V(G_2)| \geq 2n-2$.

Lemma 5. *Let G_1,G_2 be a $(n-1)$-regular graph with $\kappa(G_1) = \kappa(G_2) = n-1$, containing no triangle. Then $G = G(G_1, G_2; M)$ is n-regular graph and $\kappa(G) = n$, where n is an positive integer.*

Proof.

It is trivial that $G = G(G_1, G_2; M)$ is n-regular graph.

Let F is the set of faulty vertices with $|F| = n-1$, $F_1 = G_1 \cap F$, $F_2 = G_2 \cap F$, and $|F| = |F_1| + |F_2|$. Without loss of generality, let $|F_1| \leq |F_2|$. We now show that $G - F$ is connected, which implies $\kappa(G) = n$. We then consider two cases:

case 1: $|F_1| = 0$ and consequently $|F_2| = n - 1$

G_1 is fault-free, all the vertices in $G_2 - F_2$ are adjacent to the vertices in G_1. Then $G - F$ is connected.

case 2: $0 < |F_1| \leq |F_2| \leq n - 2$

Since G_1 is $(n-1)$-connected, $G_1 - F_1$ is connected. All the vertices in $G_1 - F_1$ are connected and form a connected component C_1. On the other hand, since G_2 is $(n-1)$-connected, $G_2 - F_2$ is connected. All the vertices in $G_2 - F_2$ are connected and form a connected component C_2. Since G_1 and G_2 contain no triangle ,then $|V(G_1)| = V(G_2) \geq 2n-2$, we can infer $|V(C_1)| = |V(G)| - |F_1| \geq 2(n-1) - (n-2) = n \geq |F_2| + 1$. Then, there exists at least one crossing edge connected to both C_1 and C_2. Thus, $G - F$ is connected. $\square$

Lemma 6. *Let G_1,G_2 be a $(n-1)$-regular graph with $\kappa(G_1) = \kappa(G_2) = n-1$, containing no triangle. $G = G(G_1, G_2; M)$. Then exists a connected component C in $G - S$ such that $|V(C)| \geq |V(G)| - |S| - 1$, where S is the set of faulty vertices in G and $|S| \leq 2n-3$, for $n \geq 2$.*

Proof.

Let $S_1 = G_1 \cap S$, $S_2 = G_2 \cap S$, and $|S| = |S_1| + |S_2|$. We consider the following two cases:

case 1: u has at least fault-free vertex, for any $u \in V(G)$.

We claim that $G - S$ is connected for every vertex subset S of G. Since $|S| \leq 2n-3$ and $S_1 \cap S_2 = \emptyset$, $|S_1| \leq n-2$ or $|S_2| \leq n-2$. We may without loss of generality assume that $|S_1| \leq n-2$. Since $\kappa(G_1) = n-1$, $G_1 - S_1$ is connected. A crossing edge of G is an edge such that one endpoint is in G_1 and the other one is in G_2.

case 1.1: $G_2 - S_2$ is also connected.

We have know $|V(G_1)| = |V(G_2)| \geq 2n-2$($G_1$,$G_2$ contains no triangle). The number of crossing edges between G_1 and G_2 is at least $2n-2$, which is greater than $|S|$. So there exists a crossing edge (u, v) such that $u \in V(G_1 - S_1)$ and $v \in V(G_2 - S_2)$. Then, $G - S$ is connected.

case 1.2: $G_2 - S_2$ is disconnected.

We may without loss of generality assume that $G_2 - S_2$ is divided into t disjoint connected components, say H_1, H_2, . . . , H_t, where $t \geq 2$. That is, $V(H_i) \cap V(H_j) = \emptyset$ for every $i \neq j$ and $H_1 \cup H_2 \cup$ . . . $\cup H_t = G_2 - S_2$. Now, we shall prove that there exists an edge (a_i, b_i) such that $a_i \in G_1 - S_1$, $b_i \in H_i$, and $a_i \notin S$ for $i = 1, 2,$ . . . ,t. In the following two cases, we consider the number of vertices of H_i with $i = 1, 2,$ . . . ,t.

case 1.2.1: $|V(H_i)| = 1$.

Let u_h be the vertex in H_i and (u_h, u_g) be its corresponding crossing edge between H_i and G_1. And we have know that u_h has at least fault-free vertex. So $u_g \notin S$. Then, u_h is connected with every vertex in $G_1 - S$.

case 1.2.2: $|V(H_i)| \geq 2$.

Let (u_h, v_h) be an edge in H_i. Since there is no triangle in G, $|N_2(u_h) \cup N_2(v_h)| = n-1+n-1 = 2n-2$, where $N_2(u)$ is the neighborhood of vertex u in G_2. Let $|V(H_i)| = m \geq 2$. We have the following inequality.

$|\bigcup_{v \in V(H_i)} N_2(v)| - |V(H_i)| \geq 2n - 2 - m$

Suppose each crossing edge between G_1 and H_i has at least one faulty vertex, then $|S| \geq (2n-2-m) + m = 2n-2 > 2n-3$. It's a contradiction to our assumption that $|S| \leq 2n-3$. So there exists a crossing edge (a_i, b_i) such that $a_i \in V(G_1)$, $b_i \in V(H_i)$, and $a_i \notin S$. Therefore, $G - F$ is connected.

case 2: There exists a vertex $u \in G$, and all neighborhood of vertex u are faulty.

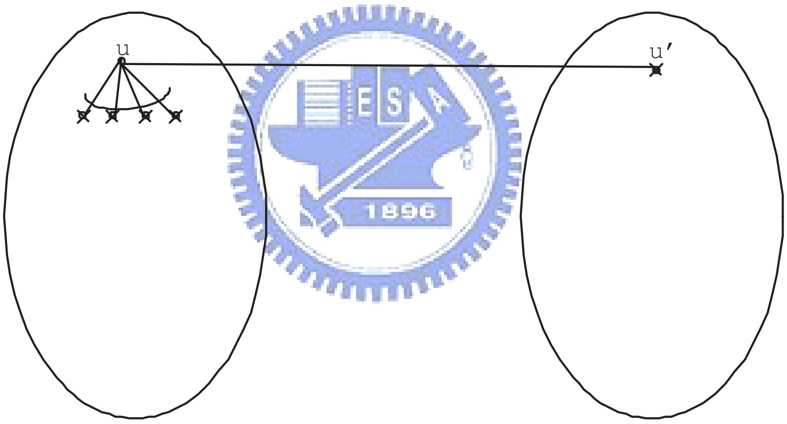

Figure 5.1: all neighbor of u are faulty

We claim that there exists a connected component C with $|C| \geq |V(G)| - |S| - 1$. Without of loss generality, assume $u \in G_1$, $u' \in G_2$ and u' is adjacent to u. Then, we know $|S_1| \geq n-1$ and $|S_2| \leq n-2$. Since $\kappa(G_2) = n-1$, $G_2 - S_2$ is connected.

case 2.1: $|S_1| \leq 2n-4$ and consequently $|S_2| = 1$.

Since every vertices in $G_1 - \{u\} - S_1$ are adjacent to $G_2 - \{u\}$, there exists a connected component C such that $|V(C)| \geq |V(G)| - |S| - 1$.

case 2.2: $|S_1| \leq 2n-5$ and consequently $|S_2| \geq 2$.

Since $\forall x \in G_1 - \{u\}$, $N(x)$ in $G_1 - \{u\}$ is at least $k-1$, and by definition, G contains no triangle, which means $\forall x, y \in G_1 - \{u\}$, x, y have no common neighbors. Thus We know $G_1 - \{u\}$ is connected if removed the number of vertices is less than k-3.

We have know $|V(G_1)| = |V(G_2)| \geq 2n-2$. The number of crossing edges between G_1 and G_2 is at least $2n-2$, which is greater than $|S|$. So there exists a crossing edge (u, v) such that $u \in V(G_1 - \{u\} - S_1)$ and $v \in V(G_2 - S_2)$. Then, $G - S$ is connected. There exists a connected component C such that $|V(C)| \geq |V(G)| - |S| - 1$. $\square$

Now, we have the following theorem.

Theorem 4. *Let G_1,G_2 be a $n-1$-regular graph with $\kappa(G_1) = \kappa(G_2) = n-1$, containing no triangle. $G = G(G_1, G_2; M)$. Let F be a set of faulty vertices with $|F| \leq n-2$. Then each pair of vertices u and v in $G - F$ are connected by $min\{deg_f(u), deg_f(v)\}$ vertex-disjoint fault-free paths, where $deg_f(u)$ and $deg_f(v)$ are the degree of u and v in $G - F$, respectively.*

Proof.

Let u and v be two vertices in G. We first assume without loss of generality that $deg_f(u) \leq deg_f(v)$, so $min\{deg_f(u), deg_f(v)\} = deg_f(u)$. We now show that u is connected to v if the number of vertices deleted is smaller than $deg_f(u) - 1$ in $G - F$, which implies that each pair of vertices u and v in $G - F$ are connected by $deg_f(u)$ vertex-disjoint fault-free paths, where $|F| \leq n-2$.

For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_f(u) - 1$. As a consequence, $|V_f| \leq n-1$ because of $deg_f(u) \leq deg(u) \leq n$. Then, the summation of the cardinality of these two sets F and V_f is $|F| + |V_f| = |S| \leq 2n-3$. By Lemma 6, there exists a connected component C in $G - S$ such that $|V(C)| \geq |V(G)| - |S| - 1$. It means that either $G - S$ is connected, or $G - S$ has two components, one of which is only one vertex. If $G - S$ has two components and one of which has only one vertex, the set V_f has to be the neighborhood of u and $|V_f| = deg_f(u)$, this contradicts to the fact that $|V_f| \leq deg_f(u) - 1$. Thus, u is connected to v when the number of vertices deleted is smaller than $deg_f(u) - 1$ in $G - F$.

The proof is completed. $\square$

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