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資訊科學與工程研究所

博 士 論 文

最大區域連通度與邊泛迴圈之條件式容錯度研究

Maximally Local-connected and Edge-bipancyclic Property with Conditional Faults on Interconnection Networks

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摘要

容錯度的問題已經是個相當廣泛討論的主題。在這篇論文當中，我們在幾個連結網路上研究了一些條件式容錯度的特性。首先，我們討論在*n*維度的超立方體網路中，當每個節點都必需要兩個好的節點與其相連時，任何 *2n-5* 個的邊壞掉的情形下，對任意一個邊都可以找到長度從 *6* 到 2^n 的迴圈通過這個邊。並且証明出此結果為最佳結果。

接著我們在類超立方體網路中對 *Menger* 定理做了研究。我們証明在 *n* 維度的類超立方體網路中，在壞掉 *n-2* 個節點之後，對任意一對點 *u* 跟 *v* 皆可以找到 min{*deg*(*u*), *deg*(*v*)}條 *vertex-disjoint* 的路徑連結這兩個點。若給予每個節點都必需要兩個好的節點與其相連的條件下，則容錯度可以上升到 *2n-5*。

最後我們定義了一對一及一對多兩種最大區域連通度的特性。我們証明了在 *k+1* 正規 *MCNs* 網路中，在壞掉容錯於小於 *k-1* 的情形下，都具有一對一及一對多的最大區域連通度的特性。進一步的我們也討論在 k+1 正規 *MCNs* 網路中，任意拔除 *f* 個點，對於任意獨立兩個大小為 *t* 的點集合中，可以找到 *t* 條 *vertex-disjoint* 的路徑連結這兩個點集合，其中 *f* 和 *t* 存在著 $f+t \leq 2k$ 的情形。我們也針對由 *transposition tree* 生成的 *Cayley graph* 做相關的研究。在 *n-1* 正規式 *transposition tree* 生成的 *Cayley* 網路中，容錯度在 *n-3* 的情形下皆有一對一及一對多的區域性連通度的特性。

關鍵字：連結網路；條件式容錯；容錯；泛迴圈；連通度；Menger 定理；區域性連通度；MCNs 網路；超立方體網路；類超立方體網路；星狀網路；泡泡性網路；Cayley 網路

Maximally Local-connected and Edge-bipancyclic Property with Conditional Faults on Interconnection Networks

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Abstract

The problem of fault-tolerance has been discussed widely. In this thesis, we study several properties with conditional fault on some interconnection networks. First of all, we show that for any set of faulty edges F of an n-dimensional hypercube Q_n with $F \leqq 2n\text{-}5$, each edge of the faulty hypercube Q_n -F lies on a cycle of every even length from 6 to 2^n with each vertex having at least two healthy edges adjacent to it, for $n \geqq 3$. Moreover, this result is optimal in the sense that there is a set F of $2n\text{-}4$ conditional faulty edges in Q_n such that Q_n -F contains no Hamiltonian cycle.

Second, we study the Menger property on a class of hypercube-like networks. We show that in all n-dimensional hypercube-like networks with $n\text{-}2$ vertices removed, every pair of unremoved vertices u and v are connected by min{$deg(u)$,$deg(v)$} vertex-disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively. Furthermore, under the restricted condition that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have the strong Menger property, even if there are up to $2n\text{-}5$ vertex faults.

The local connectivity of two vertices is defined as the maximum number of internally vertex-disjoint paths between them. Finally, we define two vertices to be maximally local-connected, if the maximum number of internally vertex-disjoint paths between them equals the minimum degree of these two vertices. We prove that a ($k+1$)-regular Matching Composition Network is maximally local-connected, even if there are at most ($k\text{-}1$) faulty vertices in it. Moreover, we introduce the one-to-many and many-to-many versions of connectivity, and prove that a ($k+1$)-regular Matching Composition Network is not only ($k\text{-}1$)-fault-tolerant one-to-many maximally local-connected but also f-fault-tolerant many-to-many t-connected (which will be defined subsequently) if $f+t=2k$. In the same issue, we show that an (n-1)-regular Cayley graph generated by transposition tree is maximally local-connected, even if there are at most (n-3) faulty vertices in it, and prove that it is also (n-1)-fault-tolerant one-to-many maximally local-connected.

Keywords: Interconnection networks; Fault-tolerant; Pancyclic; Conditional faults; Connectivity; Strong Menger connectivity; Local connectivity;
Matching Composition network; Hypercube; Hypercube-like networks;
Cayley graphs; Star graph; Bubble-sort graph;

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Chapter 1

Introductions and Motivations

The research about interconnection networks is important for parallel and distributed computer system. Many interconnection network topologies have been proposed in literature for the purpose of connecting a large number of processing elements and the designing of a parallel computing system. There are several requirements in designing a good topology for an interconnection network, such as connectivity and ring embedding. Many related works can be referred in recent research.

In practice, the processors or links in a network may be failure. Since failures are inevitable, fault tolerance is an important issue in multiprocessor systems. The connectivity is also related to the reliability and fault tolerance of a network. Many measures on fault tolerance of networks are related to the maximal size of the connected components of networks with faulty vertices/edges. In this dissertation, we consider some measures of conditional faults by restricting that every vertex has at least two fault-free neighboring vertices on some interconnection networks. Under this condition, the fault-tolerant capability is increased.

1.1 Basic Terms and Notations

The architecture of a multiprocessor system is usually modeled as an undirected graph. For the graph definitions and notations we follow [3]. Let $G = (V, E)$ be a graph, we use $V(G)$ and $E(G)$ to denote the vertex set V and the edge set E, respectively. The *connectivity* of a graph G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is k-*connected* if its connectivity is at least k. In addition, a graph has *connectivity* k if it is k-connected but not $(k+1)$-connected. The degree of a vertex x is the number of edges incident with it. We use $deg_G(x)$, or simply $deg(x)$ if there is no ambiguity, to denote the degree of vertex x in G; and use $\delta(G)$ to denote the minimum degree of all the vertices in G. We say that G is *maximally connected* if $\kappa(G) = \delta(G)$. Let u and v be two distinct vertices, a path P between them is a sequence of adjacent vertices, $< u, w_1, w_2, ..., w_k, v >$, where w_1, w_2, ..., w_k are distinct ones. The *local connectivity* between two distinct vertices u and v is the maximum number of internally disjoint $u - v$ paths.

Let G be a graph, and F be a subset of vertices, $F \subset V(G)$, the induced subgraph obtained by deleting the vertices of F from G is denoted by $G - F$. Let u be a vertex, we use $N_G(u)$, or simply $N(G)$ if there is no ambiguity, to denote the set of vertices adjacent to u in G. Let V' be a set of vertices, the neighborhood of V' is defined as the set $N_G(V') = \{ \bigcup_{v \in V'} N_G(v)\} - V'$. A graph G is k-*regular* if the degree of every vertex in G is k, and graph G is triangle-free if there is no cycle of length three. A *Hamiltonian cycle* is a cycle which includes every vertex of G. A path P is a sequence of adjacent vertices, written as $\langle v_0, v_1, ..., v_m \rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P.

1.2 Organization of the Thesis

In the follows, we describe the organization of this thesis. In Chapter 2, we discuss about the edge-bipancyclicity problem with conditional faults on hypercubes. We show that, for up to $|F| = 2n - 5$ faulty edges, each edge of the faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n with each vertex having at least two healthy edges adjacent to it, for $n \geq 3$.

In Chapter 3, we study the Menger property on a class of hypercube-like networks. We show that in all n-dimensional hypercube-like networks with $n-2$ vertices removed, every pair of unremoved vertices u and v are connected by $min\{deg(u), deg(v)\}$ vertex-disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively. Furthermore, under the restricted condition that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have the strong Menger property, even if there are up to $2n - 5$ vertex faults.

In Chapter 4, we focus on local connectivity problem. we define two vertices to be maximally local-connected, if the maximum number of internally vertex-disjoint paths between them equals the minimum degree of these two vertices. We prove that a $(k+1)$-regular Matching Composition Network is maximally local-connected, even if there are at most $(k-1)$ faulty vertices in it. Moreover, we introduce the one-to-many and many-to-many versions of connectivity, and prove that a $(k+1)$-regular Matching Composition Network is not only $(k-1)$-fault-tolerant one-to-many maximally local-connected but also f-fault-tolerant many-to-many t-connected (which will be defined subsequently) if $f + t = 2k$. Furthermore, we introduce the Cayley graphs generated by transposition tree, and show that an $(n-1)$-regular Cayley graph generated by transposition tree is maximally local-connected, even if there are at most $(n-3)$ faulty vertices in it, and prove that it is also $(n-1)$-fault-tolerant one-to-many maximally local-connected. At last, we present our conclusion in Chapter 5.

Chapter 2

Edge-bipancyclicity

2.1 The Edge-bipancyclic property

The ring embedding problem, which deals with all the possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks. A graph G is *pancyclic* if it contains a cycle of length l for each l satisfying $3 \leq l \leq |V(G)|$. The concept of pancyclic graphs is proposed by Bondy [3]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to cycles of even lengths. A bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$. There are some studies concerning ring embedding problem of some interconnection networks [9, 13, 14].

A bipartite graph is *k-edge-fault-tolerant edge-bipancyclic* if $G - F$ remains edge-bipancyclic for any set of faulty edges $F \subset E(G)$ with $|F| \leq k$. In this chapter, we discuss that for $|F| = 2n-5$ conditional faulty edges, each edges of $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, $n \geq 4$, provided not all edges in F are incident with the same vertex.

2.2 The Hypercube Networks

An n-dimensional hypercube is denoted by Q_n with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. Each vertex u of Q_n can be distinctly labeled by a n-bit binary strings, $u = u_{n-1}u_{n-2}...u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. In addition, we call e a *healthy edge* when e is fault-free in a graph. Let u and v be two adjacent vertices. If the binary labels of u and v differ in ith position, then the edge between them is said to be in ith dimension and the edge (u, v) is called an ith dimension edge. Let i be a fixed position, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n)|u_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n)|u_i = 1\}$. We say that Q_n is decomposed into Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcube of Q_n induced by the vertices with the ith bit position being 0 and 1 respectively. Q^0_{n-1} and Q^1_{n-1} are all isomorphic to Q_{n-1}. For each vertex $u \in V(Q^0_{n-1})$, there is exactly one vertex in Q^1_{n-1}, denoted by $u^{(1)}$, such that $(u, u^{(1)}) \in E(Q_n)$. Conversely, for each $u \in V(Q^1_{n-1})$, there is one vertex in Q^0_{n-1}, denoted by $u^{(0)}$, such that $(u, u^{(0)}) \in E(Q_n)$. Let D_i be the set of all edges with one end in Q^0_{n-1} and the other in Q^1_{n-1}. These edges are called crossing edges in the ith dimension between Q^0_{n-1} and Q^1_{n-1}. We also call D_i the set of all ith dimension edges. Consequently, $|D_i| = 2^{n-1}$ for all $0 \le i \le n-1$.

There are some properties on Q_n as follows.

Lemma 1 *[16] Q_n is edge-bipancyclic, and is (n-2)-edge-fault-tolerant edge-bipancyclic, for $n \ge 3$.*

Lemma 2 *[29] Each edge of $Q_4 - F$ lies on a cycle of every even length from 6 to $2^n = 16$*

for any $F \subset E(Q_4)$ with $|F| = 3$, provided not all the faulty edges in F are incident with the same vertex.

Lemma 3 *[29] Any two edges in Q_n are included in a Hamiltonian cycle, for $n \geq 2$.*

The above lemma can be improved; In addition, we have the following lemmas to simplify our proof.

Lemma 4 *Let $C_0 = \langle u, P_0, v, u\rangle$ be a cycle in Q^0_{n-1} with its even length from l_0 to 2^{n-1}, and $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$ be a cycle in Q^1_{n-1} with its even length from l_1 to 2^{n-1}. Then $C = \langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u\rangle$ is a cycle in Q_n with its even length from $l_0 + l_1$ to 2^n.*

Proof. The proof of this lemma is omitted. □

Lemma 5 *Let Q_n be an n-dimensional hypercube, $n \geq 2$, and let e_1 and e_2 be two edges in the same dimension i. Then there exists another dimension $j \neq i$ such that decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, we have (1) neither e_1 nor e_2 is a crossing edge, (2) not e_1 and e_2 are in the same subcube.*

Proof. Let $e_1 = (a, b)$ and $e_2 = (s, t)$ be two edges in the same dimension i. Let $a = a_n...a_i...a_1$ and $s = s_n...s_i...s_1$. Then $b = a_n...\overline{a_i}...a_1$ and $t = s_n...\overline{s_i}...s_1$. Since $e_1 \neq e_2$ and $n \geq 2$, there exists another dimension $j \neq i$, such that $a_j \neq s_j$. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then, e_1 and e_2 are not crossing edges and are in the different subcubes. □

Lemma 6 *Consider an n-dimensional hypercube Q_n, for $n \geq 4$. Let e_0, e_1 and e_2 be any three edges in Q_n, there is a cycle C containing e_1 and e_2 in $Q_n - \{e_0\}$ with the length $l(C) = 2^n$, $2^n - 2$ and $2^n - 4$.*

Proof. To prove this lemma, we consider the following two cases:

Case 1: *Both e_1 and e_2 are in the same dimension, say dimension i.* By Lemma 5, we can choose a dimension j such that e_1 and e_2 are in different subcubes. Without loss of generality, we assume that e_1 is in Q^0_{n-1} and e_2 is in Q^1_{n-1}. We then consider two cases:

1.1: *e_0 is not a crossing edge.* We assume without loss of generality that e_0 is in Q^0_{n-1}. By Lemma 1, in $Q^0_{n-1} - \{e_0\}$, there exists a cycle C_0 of every even length $4 \leq l(C_0) \leq 2^{n-1}$ going through e_1. Since $n \geq 4$, we can choose an edge (u, v) on cycle C_0 such that $(u, v) \neq e_1$, and $(u^{(1)}, v^{(1)}) \neq e_2$. By Lemma 3, in Q^1_{n-1}, there exists a cycle C_1 of length 2^{n-1} going through e_2 and $(u^{(1)}, v^{(1)})$. Thus, the conclusion follows according to Lemma 4.

1.2: *e_0 is a crossing edge.* By Lemma 1, there exists a C_0 of every even length $4 \leq l(C_0) \leq 2^{n-1}$ going through e_1 in Q^0_{n-1}. We can choose an edge (u, v) on cycle C_0 such that (u, v) is not adjacent to e_0 and $(u, v) \neq e_1$ and $(u^{(1)}, v^{(1)}) \neq e_2$, since $n \geq 4$. By definition, $(u^{(1)}, v^{(1)})$ is an edge in Q^1_{n-1}. By Lemma 3, there exists a Hamiltonian cycle C_1 going through e_2 and $(u^{(1)}, v^{(1)})$ in Q^1_{n-1}. So the conclusion follows according to Lemma 4.

Case 2: *e_1 and e_2 are in different dimensions.* Suppose that e_0 is in the ith dimension. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension i. Then, e_0 is a crossing edge. Next, we consider two further cases:

2.1: *Either e_1 or e_2 is a crossing edge.* Without loss of generality, we assume that e_1 is a crossing edge, and e_2 is in Q^0_{n-1}. Let $e_1 = (u, u^{(1)})$, where $u \in V(Q^0_{n-1})$ and $u^{(1)} \in V(Q^1_{n-1})$. Since $n \geq 4$, there is a neighbor of u, say v, such that $(u, v) \neq e_2$ and (u, v) is not adjacent to e_0. By Lemma 3, there exists a Hamiltonian cycle C_0 going through (u, v) and e_2 in Q^0_{n-1}. By Lemma 1, in Q^1_{n-1}, there exists a cycle C_1 of every even length $4 \leq l(C_1) \leq 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$. By Lemma 4, the conclusion follows.

2.2: *Both e_1 and e_2 are not crossing edge.* If e_1 and e_2 are in different subcubes, this subcase is similar to case 1.2, and the proof is omitted. Otherwise, both e_1 and e_2 are in the same subcube. We assume without loss of generality that e_1 and e_2 are in Q^0_{n-1}. By Lemma 3, there exists a Hamiltonian cycle C_0 going through e_1 and e_2 in Q^0_{n-1}, and $l(C_0) = 2^{n-1}$. Since $n \geq 4$, there is a third edge (u, v) other than e_1 and e_2 on cycle C_0, and (u, v) is not adjacent to e_0. By Lemma 1, there exists a cycle C_1 of every even length $4 \leq l(C_1) \leq 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$ in Q^1_{n-1}. By Lemma 4, the conclusion follows. $\square$

2.3 The Conditional Fault-tolerance of Hypercube Networks

Chan and Lee [5] considered an injured n-dimensional hypercube where each vertex is incident with at least two healthy edges, and proved that it still contains a Hamiltonian cycle even it has $(2n-5)$ edge faults. Tsai [26] proved that such an injured hypercube Q_n contains a cycle of every even length from 4 to 2^n, even if it has up to $(2n-5)$ edge faults. Recently, Xu et al. [29] showed that for any set of faulty edges F of Q_n with $|F| \leq n-1$, each edge of $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, $n \geq 4$, provided not all faulty edges are incident with the same vertex. We observe that not all faulty edges are incident with the same vertex is equivalent to stating that each vertex has at least two healthy edges adjacent to it, if $|F| \leq n-1$. In this chapter, we consider a set of faulty edges satisfying the condition that each vertex of $Q_n - F$ is incident with at least two healthy edges. Such a set of faulty edges F is called a set of conditional faulty edges.

To prove our result, we need some preliminary lemmas.

Lemma 7 *Consider an n-dimensional hypercube Q_n, for $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n-5$. There are at most two vertices in Q_n incident with (n-2) faulty edges.*

Proof. If there are three vertices in Q_n incident with $(n-2)$ faulty edges, the number of faulty edge F is at least $3n-8$. However, $(3n-8) > (2n-5)$ for all $n \geq 4$ which is a contradiction. □

Let F be a set of faulty edges of Q_n. Suppose that we decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, and let $F_L = F \cap E(Q^0_{n-1})$, $F_R = F \cap E(Q^1_{n-1})$. Suppose that F is a set of conditional faulty edges of Q_n. If we arbitrarily decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by a dimension, F_L and F_R may not be conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. However, we will show that it is always possible to find some suitable dimension such that decomposing by this dimension, both F_L and F_R are conditional faulty sets in Q^0_{n-1} and Q^1_{n-1} respectively.

Lemma 8 *Consider an n-dimensional hypercube Q_n, $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n-5$. If there are two vertices x and y both incident with n-2* *faulty edges, then x and y are adjacent in Q_n and the edge (x,y) is a faulty edge. Suppose that (x,y) is in dimension j. Then decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, both F_L and F_R are sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Moreover, $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$.*

Proof. If there are two vertices x and y in Q_n incident with $(n-2)$ faulty edges, then these two vertices are connected by a faulty edge. Otherwise, $|F| = 2(n-2) = 2n-4 > 2n-5$ which is a contradiction. Suppose the edge (x, y) is in dimension j, we decompose Q_n into two subcubes. It is clearly that each vertex in Q^0_{n-1} and Q^1_{n-1} is still incident with at least two healthy edges, and both F_L and F_R are conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Then, $|F_L| = |F_R| = n-3 \leq 2n-6$, for $n \geq 4$. □

Lemma 9 *Consider an n-dimensional hypercube Q_n, for $n \geq 4$. Let F be a set of conditional faulty edges with $|F| = 2n-5$. Suppose that there exists exactly one vertex x having (n-2) faulty edges incident with it. Since $n-2 \geq 2$, let e_1 and e_2 be two faulty edges incident with x, and let e_1 and e_2 be jth and kth dimension edges respectively. Then decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by either one of these two dimensions j and k, F_L and F_R are still sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Moreover, $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$.*

Proof. If there exists only one vertex x having $(n-2)$ faulty edges incident with it, there are at least two faulty edges e_1 and e_2 incident with it, since $n \geq 4$. Obviously, these two faulty edges are in different dimensions. Without loss of generality, we may assume that e_1 is in dimension j and e_2 is in dimension k, for $j \neq k$. We can decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, and either e_1 or e_2 is a crossing edge.

Therefore, each vertex in these two subcubes is incident with at least two healthy edges and $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$. $\square$

Lemma 10 *Let Q_n be an n-dimensional hypercube, F be a set of faulty edges with $|F| \geq 2$, and e be a healthy edge, $n \geq 2$. Then there exists a dimension j, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by this dimension, such that e is not a crossing edge and not all the faulty edges are in the same subcube.*

Proof. Suppose that $e = (u, v)$ is in dimension i. If there is a faulty edge f not in dimension i, say in dimension j. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then f is a crossing edge but e is not, and all the faulty edges are not in the same subcube. Otherwise, all the faulty edges are in the same dimension i as e is in. We now choose any two faulty edges f_1 and f_2 in F. By Lemma 5, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by some dimension $j \neq i$ such that edges f_1 and f_2 are not in the same subcube, and e is not a crossing edge. $\square$

2.4 Edge-bipancyclicity of conditional faulty hypercube

In this section, we consider a set of faulty edges satisfying the condition that each vertex of $Q_n - F$ is incident with at least two healthy edges. Such a set of faulty edges F is called a set of conditional faulty edges and $Q_n - F$ is called a conditional faulty hypercube. We find that under this condition, the number of faulty edges can be much greater and the same result still holds. We show that, for up to $|F| = 2n-5$ conditional faulty edges, each edge of a faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geq 3$. We observe that, if $|F| < 2n-5$, we may arbitrarily delete some more edges to make a faulty edge set $F' \supseteq F$ and $|F'| = 2n - 5$. If our result holds for F', it holds for F. From now on, we shall assume $|F| = 2n - 5$.

The above result is optimal in the sense that the result can not be guaranteed, if there are $2n - 4$ conditional faulty edges. For example, take a cycle of length four in Q_n, let $\langle u_1, u_2, u_3, u_4 \rangle$ be the consecutive vertices on this cycle. Suppose that all the $(n-2)$ edges incident with vertex u_1 (respectively vertex u_3) are faulty except those two edges on the four cycle are healthy. There are $2(n - 2)$ conditional faulty edges. Then there does not exist a Hamiltonian cycle in this faulty Q_n, for $n \geq 3$.

We now prove our main result.

Theorem 1 *Let Q_n be an n-dimensional hypercube, and F be a set of conditional faulty edges with $|F| \leq 2n - 5$. Then each edge of the conditional faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geq 3$.*

Proof. We prove this theorem by induction on n. For $n = 3$, since $2n - 5 = n - 2$, by Lemma 1, the result is true. For $n = 4$, $2n - 5 = n - 1$, by Lemma 2, the result holds. Assume the theorem holds for $n - 1$, for some $n \geq 5$, we shall show that it is true for n.

As we mentioned before, we may assume $|F| = 2n - 5$. Let $e = (u, v)$ be an edge in $Q_n - F$. We shall find a cycle of every even length from 6 to 2^n passing through e in $Q_n - F$. Assume that e is an ith dimension edge, $e \in D_i$, for some $i \in \{1, 2, ..., n\}$. The proof is divided into three major cases:

Case 1: *There are two vertices x and y in Q_n incident with $(n - 2)$ faulty edges.* By Lemma 8, (x, y) is an edge in Q_n and is a faulty edge. We denote this edge by e_f. Suppose

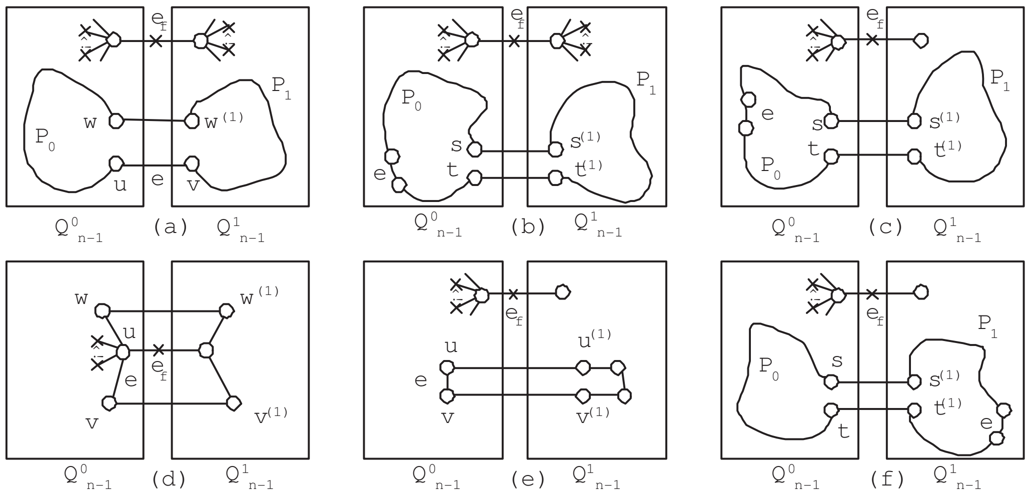

Figure 2.1: An illustration for Theorem 1.

that e_f is a jth dimension edge. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. We then consider two further cases:

1.1 *$e_f = (x, y)$ and $e = (u, v)$ are in the same dimension.* Thus, $j = i$ and $e_f \in D_i$. (Fig. 2.1-(a)) In this case, e is an edge crossing Q^0_{n-1} and Q^1_{n-1}. Without loss of generality, assume that $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Since $n \geq 5$, u has a neighboring vertex $w \in V(Q^0_{n-1})$, by the definition of hypercube, $w^{(1)}$ is a neighbor of v such that the edge $(w, w^{(1)})$ is a healthy edge and $(w, w^{(1)})$ is a crossing edge between Q^0_{n-1} and Q^1_{n-1}. By lemma 1, there exists a cycle C_0 in $Q^0_{n-1} - F_L$ passing through (u, w) of every even length $4 \leq l(C_0) \leq 2^{n-1}$ and a cycle C_1 in $Q^1_{n-1} - F_R$ going through $(v, w^{(1)})$ of every even length $4 \leq l(C_1) \leq 2^{n-1}$. We write C_0 as $\langle u, P_0, w, u\rangle$, and C_1 as $\langle v, P_1, w^{(1)}, v\rangle$. Thus, $\langle u, P_0, w, w^{(1)}, v, u\rangle$ is a cycle of length 6 with $l(P_0) = 3$. By Lemma 4, $\langle u, P_0, w, w^{(1)}, P_1, v, u\rangle$ can form a cycle of every even length from 8 to 2^n through e in $Q_n - F$.

1.2 *e_f and e are in different dimensions.* Thus, $j \neq i$ and $e_f \notin D_i$. (Fig 2.1-(b)) In this case, e is in Q^0_{n-1} or Q^1_{n-1}. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$.

By Lemma 1, there exists a cycle C in $Q^0_{n-1} - F_L$ going through the edge e of every even length l, $6 \le l \le 2^{n-1}$. Let C_0 be a cycle of length $2^{n-1} - 2$ or 2^{n-1} passing through e in $Q^0_{n-1} - F_L$. Since $n \ge 5$, there exists an edge (s,t) on C_0 such that neither s nor t is adjacent to e_f and $(s,t) \ne e$. By definition, $(s^{(1)}, t^{(1)})$ is an edge in Q^1_{n-1}, and $(s, s^{(1)})$, $(t, t^{(1)})$ are healthy edges. By Lemma 1, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ through $(s^{(1)}, t^{(1)})$ of every even length $4 \le l(C_1) \le 2^{n-1}$. Thus, the conclusion follows according to Lemma 4.

Case 2: *There is exactly one vertex in Q_n incident with $(n-2)$ faulty edges.* Let x be the vertex having $(n-2)$ faulty edges incident with it. Let f_1 and f_2 be two faulty edges incident with x, so f_1 and f_2 are in different dimensions j and k. By Lemma 9, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, both $F_L = F \cap E(Q^0_{n-1})$ and $F_R = F \cap E(Q^1_{n-1})$ are sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1} respectively. Between dimension j and k, we choose one to decompose Q_n into Q^0_{n-1} and Q^1_{n-1}, say dimension j, such that the required edge e is not a crossing edge. Therefore, there is an faulty edge crossing Q^0_{n-1} and Q^1_{n-1}, we denote this edge by e_f, and $e_f \in F \cap D_j$ is incident with x. Without loss of generality, we may assume that $x \in V(Q^0_{n-1})$.

2.1: *Suppose $|F_L| \le 2n-7$ and $|F_R| \le 2n-7$.* (Fig. 2.1-(c)) Without loss of generality, we further assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C in $Q^0_{n-1} - F_L$ of every even length $6 \le l(C) \le 2^{n-1}$ passing through e. Let C_0 be a cycle of length $2^{n-1} - 4 \le l(C_0) \le 2^{n-1}$ through e in $Q^0_{n-1} - F_L$. Since $|C_0 - e| \ge 2^{n-1} - 4 - 1 > 2(2n-5) \ge 2|F \cap D_j|$, for all $n \ge 5$. There exists an edge (s,t) on C_0 such that (s,t) is not e, and both $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By induction hypothesis, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ of every even length $6 \le l(C_1) \le 2^{n-1}$ passing through

$(s^{(1)}, t^{(1)})$. By Lemma 4, the conclusion follows.

2.2: $|F_L| = 2n - 6$. In this case, $|F \cap D_j| = 1$ and $|F \cap E(Q^1_{n-1})| = |F_R| = 0$.

2.2.1: *e is in subcube Q^0_{n-1}*. To find a cycle of length 6 passing through $e = (u, v)$, we discuss the case that whether e is incident with x or not. If e is incident with x, without loss of generality, we assume that $u = x$. (Fig. 2.1-(d)) Thus, $(v, v^{(1)})$ is a healthy edge. Since F_L is a set of conditional faulty edges in Q^0_{n-1}, vertex $u = x$ has two healthy edges incident with it. Let w be a neighbor of u in Q^0_{n-1} such that (w, u) and $(w, w^{(1)})$ are healthy edges and $w \neq v$. Thus, $\langle u, v, v^{(1)}, u^{(1)}, w^{(1)}, w, u\rangle$ is a cycle of length 6 in $Q_n - F$. Otherwise, e is not incident with x, then $(u, u^{(1)})$ and $(v, v^{(1)})$ are healthy edges. (Fig. 2.1-(e)) By Lemma 1, there exists a cycle $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$ of length four in Q^1_{n-1} through the edge $(u^{(1)}, v^{(1)})$. Thus, $\langle u, u^{(1)}, P_1, v^{(1)}, v, u\rangle$ is a cycle of length 6 in $Q_n - F$, where $l(P_1) = 3$.

Let e_1 be a faulty edge in Q^0_{n-1} that is not adjacent to e_f. Though e_1 is a faulty edge, we treat it as a healthy edge temporarily, then the total number of faulty edge in Q^0_{n-1} is $2n - 7$. By induction hypothesis, there exists a cycle C_0 of every even length $6 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - \{F_L - \{e_1\}\}$. If C_0 passes e_1, we choose e_1 , or else, we choose any one edge other then e on C_0 which is not adjacent to e_f. Let the chosen edge be denoted by (s, t). We write cycle C_0 as $\langle s, P_0, t, s\rangle$. Since $|F \cap D_j| = 1$ and $|F_R| = 0$, $(s, s^{(1)})$, $(t, t^{(1)})$ and $(s^{(1)}, t^{(1)})$ are all healthy edges. Thus, $\langle s, P_0, t, t^{(1)}, s^{(1)}, s\rangle$ is a cycle of length 8 in $Q_n - F$ if $l(P_0) = 5$. Suppose that $10 \leq l \leq 2^n$ and l is even. By Lemma 1, in Q^1_{n-1}, there exists a cycle C_3 of length $4 \leq l(C_3) \leq 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. We write C_3 as $\langle s^{(1)}, P_3, t^{(1)}, s^{(1)}\rangle$. By Lemma 4, $\langle s, P_0, t, t^{(1)}, P_3, s^{(1)}, s, t\rangle$ is a cycle of length l through e in $Q_n - F$.

2.2.2: *e is in subcube* Q^1_{n-1}. (Fig. 2.1-(f)) By Lemma 1, there exists a cycle C of every even length $4 \leq l \leq 2^{n-1}$ passing through e in Q^1_{n-1}. Suppose that $2^{n-1}+2 \leq l \leq 2^n$ and l is even. Since F_L is a set of conditional faulty edges, there are at most $(n-3)$ faulty edges adjacent to e_f in Q^0_{n-1}. For $n \geq 5$, $n-3 \geq 2$, we can choose a faulty edge $e_2 = (s,t)$ in Q^0_{n-1} such that e_2 is not adjacent to e_f and $(s^{(1)}, t^{(1)})$ is not e. Treating the edge e_2 as a healthy edge, by induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ going through e_2 in $Q^0_{n-1} - \{F_L - \{e_2\}\}$. We observe that $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By Lemma 6, there exists a cycle C_1 of every length $2^{n-1}-4$, $2^{n-1}-2$, or 2^{n-1} through $(s^{(1)}, t^{(1)})$ and e in Q^1_{n-1}. By Lemma 4, the conclusion follows.

Case 3: *Every vertex in* Q_n *is incident with at most* $(n-3)$ *faulty edges.* In this case, suppose that $e = (u,v)$ is in dimension i. By Lemma 10, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by a dimension j different from i such that e is not a crossing edge and not all the faulty edges are in the same subcube. Then $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$. Next, we consider two further cases:

3.1: *At least one faulty edge is a* j*th dimension edge.* Thus, $|F \cap D_j| \neq 0$.

We then consider two cases: (a)$|F_L| \leq 2n-7$ and $|F_R| \leq 2n-7$, and (b)$|F_L| = 2n-6$ or $|F_R| = 2n-6$. The proof of this subcase is exactly the same as that of case 2.

3.2: *None of the faulty edges is a* j*th dimension edge.* Thus, $|F \cap D_j| = 0$.

3.2.1: $|F_L| \leq 2n-7$ and $|F_R| \leq 2n-7$. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C of every even length $6 \leq l(C) \leq 2^{n-1}$ in $Q^0_{n-1} - F_L$ passing through e. Let C_0 be a cycle of every even length $2^{n-1}-4 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - F_L$. There exists an edge (s,t) other than e in C_0. Since $|F \cap D_j| = 0$, $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. We write

C_0 as $\langle s, P_0, t, s\rangle$. By induction hypothesis, there exists a cycle C_1 of every even length $6 \leq l(C_1) \leq 2^{n-1}$ in $Q^1_{n-1} - \{F_R - (s^{(1)}, t^{(1)})\}$ through $(s^{(1)}, t^{(1)})$. Thus, the conclusion follows according to Lemma 4.

3.2.2: *Suppose* $|F_L| = 2n-6$ *or* $|F_R| = 2n-6$*, say the former case.* In this case, $|F_R| = 1$. We then consider two cases: (a) e is in subcube Q^0_{n-1}, and (b) e is in subcube Q^1_{n-1}.

(a) $e = (u, v)$ *is in subcube* Q^0_{n-1}. Since $|F \cap D_j| = 0$, both $(u, u^{(1)})$ and $(v, v^{(1)})$ are healthy edges. Let l be an even number with $6 \leq l \leq 2^{n-1}$. By Lemma 1, there exists a cycle C_1 of every even length from 4 to 2^{n-1} passing through $(u^{(1)}, v^{(1)})$ in $Q^1_{n-1} - \{F_R - (u^{(1)}, v^{(1)})\}$. We write C_1 as $\langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$. No matter $(u^{(1)}, v^{(1)})$ is healthy or not, $\langle u, u^{(1)}, P_1, v^{(1)}, v, u\rangle$ forms a cycle of length l through e in $Q_n - F$. Suppose that $2^{n-1} + 2 \leq l \leq 2^n$. Let e_1 be a faulty edge in Q^0_{n-1}. We may treat e_1 as a healthy edges temporarily. By induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ going through e in $Q^0_{n-1} - \{F_L - \{e_1\}\}$. If C_0 passes the edge e_1, we choose e_1 to be deleted. Otherwise, we choose another edge other than e on cycle C_0. Let the chosen edge be denoted by (s, t). We write the cycle C_0 as $\langle s, P_0, t, s\rangle$. Treating $(s^{(1)}, t^{(1)})$ as a healthy edge, by Lemma 1, there exists a cycle C_3 of every even length from 4 to 2^{n-1} passing through $(s^{(1)}, t^{(1)})$ in $Q^1_{n-1} - \{F_R - (s^{(1)}, t^{(1)})\}$. By Lemma 4, the conclusion follows.

(b): e *is in subcube* Q^1_{n-1}. Let e_1 be the only faulty edge in Q^1_{n-1}. By Lemma 1, there exists a cycle C of every even length from 6 to 2^{n-1} through e in $Q^1_{n-1} - \{e_1\}$. Suppose that $2^{n-1} + 2 \leq l \leq 2^n$, and l is even. Let $e_0 = (s, t)$ be a faulty edge in Q^0_{n-1} such that $(s^{(1)}, t^{(1)}) \neq e$ and $(s^{(1)}, t^{(1)}) \neq e_1$. By induction hypothesis, there exists a cycle C_0 of length $6 \leq l(C_0) \leq 2^{n-1}$ in $Q^0_{n-1} - \{F_L - \{e_0\}\}$ going through e_0. If $(s^{(1)}, t^{(1)}) = e_1$, treat e_1 as a healthy edge temporarily, by Lemma 6, there exists a cycle C_1 of length $2^{n-1} - 4$, $2^{n-1} - 2$, or 2^{n-1} respectively going through both $(s^{(1)}, t^{(1)})$ and e in Q^1_{n-1}. By Lemma 4, the conclusion follows. Otherwise, if $(s^{(1)}, t^{(1)}) \neq e_1$, by Lemma 6, there exists a cycle C_3 of length 2^{n-1}, $2^{n-1} - 2$, or $2^{n-1} - 4$, respectively, going through both e and $(s^{(1)}, t^{(1)})$ in $Q^1_{n-1} - \{e_1\}$. Thus, the conclusion follows according to Lemma 4. $\square$

Chapter 3

Strong Menger-connectivity

The architecture of an interconnection network is usually denoted as an undirected graph G. Among all fundamental properties for interconnection networks, the (vertex) connectivity is a major parameter widely discussed for the connection status of networks. A basic definition of the connectivity of a graph G is defined as the minimum number of vertices whose removal from G produces a disconnected graph. In contrast to this concept, Menger [17] provided a local point of view, and define the connectivity of any two vertices as the minimum number of internally vertex-disjoint paths between them.

In this chapter, we study the Menger property on a class of hypercube-like networks [27], which is a variation of the classical hypercube network by twisting some pairs of links in it. We show that in all n-dimensional hypercube-like networks with some vertices removed, every pair of unremoved vertices u and v are connected by $min\{deg(u), deg(v)\}$ vertex-disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively. This concept is firstly applied on hypercubes and stars by Oh and Chen [18, 19, 20]. Furthermore, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have this strong Menger property, even if there are up to $2n-5$ vertex faults. The bound of $2n-5$ is sharp.

3.1 Menger-connectivity and Strong Menger-connectivity

In this section, we discuss the strong Menger-connected property. A classical theorem about connectivity was provided by Menger as follows.

Theorem 2 *[17] Let x and y be two nonadjacent vertices of a graph G. The minimum size of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.*

Following this theorem, OH et al. [19] gave a definition to extend the Menger's Theorem.

Definition 1 *[19] A k-regular graph G is strongly Menger-connected if for any subgraph $G-F$ of G with at most $k-2$ vertices removed, each pair of vertices u and v in $G-F$ are connected by $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ vertex-disjoint fault-free paths in $G-F$, where $deg_{G-F}(u)$ and $deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.*

By Definition 1, OH et al. [18, 19, 20] showed that an n-dimensional star graph S_n (respectively, an n-dimensional hypercube Q_n)with at most $n-3$ (respectively, $n-2$) vertices removed is strongly Menger-connected. In order to be consistent with Definition 1, we say that a graph G possess the strongly Menger-connected property with respect to a vertex set F if, after deleting F from G, there are $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ vertex-disjoint fault-free paths connecting u and v, for each pair of vertices u and v in $G-F$. Throughout this paper, we shall call a graph "strongly Menger-connected", and omit the description of the remaining structure $G-F$ of the graph, if there is no ambiguous.

3.2 The Class of Hypercube-like Networks

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of vertices. A one-to-one connection between $V(G_0)$ and $V(G_1)$ is defined as an edge set $M = \{(v, \phi(v)) \mid v \in V_0, \phi(v) \in V_1 \text{ and } \phi : V_0 \rightarrow V_1 \text{ is a bijection}\}$. We use $G_0 \oplus_M G_1$ to denote the graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup M)$. Different bijection functions ϕ lead to different operations $\oplus_M$ and generate different graphs.

The *hypercube* network is one of the popular topologies in interconnection networks. Several variants of hypercubes are proposed by twisting some pairs of links in hypercubes, including twisted cubes [1, 12], Möbius cubes [8], and crossed cubes [11], to name a few. To make a unified study on these variants, Vaidya et al. [27] proposed a class of graphs, called *a class of hypercube-like networks.* We now give a recursive definition of the n-dimensional hypercube-like networks HL_n as follows: (1)$HL_0 = K_1$, where K_1 is a trivial graph in the sense that it has only one vertex; and (2)$G \in HL_n$ if and only if $G = G_0 \oplus_M G_1$ for some $G_0, G_1 \in HL_{n-1}$. By the definitions above if G is a graph in HL_n, then G is a composition of $G_0 \oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, $n \geq 1$. Each vertex in G_0 has exactly one neighbor in G_1.

It is known that the connectivity of an n-dimensional hypercube-like network HL_n is n [27]. To extend the connectivity result of HL_n further, we study the strongly Menger-connected property of HL_n with at most $n-2$ vertices deleted. Moreover, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, HL_n still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

3.3 Strong Menger Connectivity on the Class of Hypercube-like Networks

There are some preliminaries on the class of Hypercube-like Networks.

Lemma 11 *Let $G \in HL_n$ be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leq 2n-3$, for $n \geq 2$. There exists a connected component C in $G-S$ such that $|V(C)| \geq 2^n - |S| - 1$.*

Proof. We prove this statement by induction on n. For $n=2$, HL_2 is a cycle of length four, the result is trivially true. Assume this lemma holds for $n-1$, for some $n \geq 3$, we will prove that it is true for n.

Let G be an n-dimensional hypercube-like network, $G = G_0 \oplus_M G_1$, and G_0, $G_1 \in HL_{n-1}$. Let S be a set of vertices with $|S| \leq 2n-3$, for $n \geq 3$, and let S_0 and S_1 be subsets of set S in G_0 and G_1, respectively. Then $|S_0| + |S_1| = |S| \leq 2n-3$. Without loss of generality, we assume $|S_0| \leq |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leq |S_0| \leq 1$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geq 3$. All the vertices in $G_0 - S_0$ are connected and form a connected component C_0 with $|V(C_0)| = 2^{n-1} - S_0$. By definition, all the vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Thus, $G-S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^n - |S| - 1$. (See Fig. 3.1.)

Case 2: $|S_0| \geq 2$ and consequently $|S_1| \leq 2n-5$.

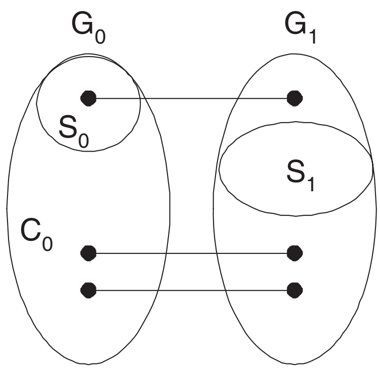

Figure 3.1: The illustration of the proof of Case 1 in Lemma 11.

Since $2 \le |S_0| \le |S_1| \le 2n-5$, so $|S_0| \le n-2$ and $n \ge 4$. By induction hypothesis, there exists a connected component C_1 in $G_1 - S_1$, and $|V(C_1)| \ge 2^{n-1} - |S_1| - 1$. Since the connectivity of G_0 is $n-1$ and $|S_0| \le n-2$, $G_0 - S_0$ is connected. Then $G - S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + (|V(G_1) - S_1| - 1) = |V(G)| - |S| - 1 = 2^n - |S| - 1$. □

By Lemma 11, we have the following corollary.

Corollary 1 *Let G be an n-dimensional hypercube-like network, $n \ge 2$, and let V' be a set of vertices in G with $|V'| = 2$. Then $|N(V')| \ge 2n - 2$.*

In the following, we show that with up to $n-2$ vertex faults, an n-dimensional hypercube-like network has strongly Menger-connected property. Referring to the relative study proposed by OH et al. [18], the strong Menger connectivity of regular hypercube networks has been proved. Here we provide a significantly simpler proof for the general hypercube-like networks.

We now prove our main result.

Theorem 3 *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \ge 2$.*

Let F be a set of faulty vertices with $|F| \leq n-2$. Then each pair of vertices u and v in $G-F$ are connected by $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ vertex-disjoint fault-free paths, where $deg_{G-F}(u)$ and $deg_{G-F}(v)$ are the remaining degree of u and v in $G-F$, respectively.

Proof. Let G be an n-dimensional hypercube-like network, and u and v be two fault-free vertices in $G-F$. We first assume without loss of generality that $deg_{G-F}(u) \leq deg_{G-F}(v)$, so $min\{deg_{G-F}(u), deg_{G-F}(v)\} = deg_{G-F}(u)$. We now show that u is connected to v if the number of vertices deleted is smaller than $deg_{G-F}(u) - 1$ in $G-F$. By Theorem 2, this implies that each pair of vertices u and v in $G-F$ are connected by $deg_{G-F}(u)$ vertex-disjoint fault-free paths, where $|F| \leq n-2$.

For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{G-F}(u) - 1$. As a consequence, $|V_f| \leq n-1$ because of $deg_{G-F}(u) \leq deg(u) \leq n$. Then, the summation of the cardinality of these two sets F and V_f is $|F| + |V_f| \leq 2n-3$. Let $S = F \cup V_f$. By Lemma 11, there exists a connected component C in $G-S$ such that $|V(C)| \geq 2^n - |S| - 1$. It means that (**i**) either $G-S$ is connected, or (**ii**) $G-S$ has two components, one of which contains only one vertex. If $G-S$ is connected, it contradicts to the assumption that u and v are disconnected. Otherwise, if $G-S$ has two component and one of which contains only one vertex x. Since we assume that u and v are separated, one of u and v is the vertex x, say $u = x$. Thus, the set V_f must be the neighborhood of u and $|V_f| = deg_{G-F}(u)$, which is also a contradiction. Then, u is connected to v when the number of vertices deleted is smaller than $deg_{G-F}(u) - 1$ in $G-F$.

The proof is completed. $\square$

3.4 Strong Menger Connectivity with Conditional Faults on the Class of Hypercube-like Networks

As proved in the previous section, an n-dimensional hypercube-like network with at most $n-2$ faulty vertices is strongly Menger-connected. But the result can not be guaranteed, if there are $n-1$ faulty vertices and all these faulty vertices are adjacent to the same vertex. In most circumstances, the possibility of all the neighbors of a vertex being faulty simultaneously is very small. Motivated by the deficiency of traditional fault tolerance, we consider a measure of conditional faults by restricting that every vertex has at least two fault-free neighboring vertices.

Under this condition, we claim that for every n-dimensional hypercube-like network with at most $2n-5$ faulty vertices and $n \geq 5$, the resulting network is still strongly Menger-connected. We have an example to show that this result does not hold for $n = 4$. Consider a 4-dimensional HL_4, this network may not be strongly Menger-connected, if the number of conditional faults is 3. (See Fig. 3.2. The remaining degrees of nodes u and v are both four, with three vertices deleted as indicated in the graph. But the number of vertex-disjoint paths between u and v is three.) So we can only expect the result holds for $n \geq 5$.

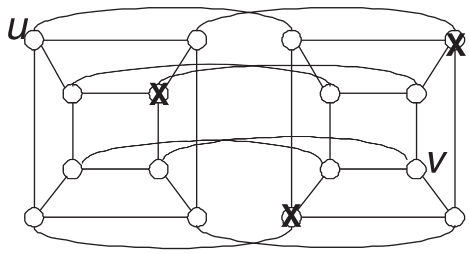

Figure 3.2: An example showing that an HL_4 is not strongly Menger-connected.

To prove this result, we need some preliminary lemma. In the following, we show that an n-dimensional hypercube-like network with at most $3n-6$ vertex faults S has a connected component having at least $2^n - |S| - 2$ vertices.

The proof is by induction, and the case for $n = 5$ is proved in the following two lemmas.

Lemma 12 *Let V' be a set of vertices in a 4-dimensional hypercube-like network with $|V'| = 3$. Then, $|N(V')| \geq 7$.*

Proof. Let G be a 4-dimensional hypercube-like network. G is a composition of two 3-dimensional hypercube-like networks G_0 and G_1, $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Without loss of generality, let V' be a subset of $V(G)$ containing three vertices $\{x, y, z\}$. If x, y, z are all in G_0, by Lemma 11, $\{x, y, z\}$ has at least 4 neighboring vertices in G_0. Besides, $\{x, y, z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geq 4+3 = 7$. If x, y are in G_0, and z is in G_1, by Lemma 11, $\{x, y\}$ has at least 4 neighboring vertices in G_0. In addition, $\{z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geq 4+3 = 7$.
□

Lemma 13 *Let G be a 5-dimensional hypercube-like network and S be a set of vertices with $|S| \leq 9$. ($3n - 6 = 9$, for $n = 5$.) There exists a connected component C in $G - S$ such that $|V(C)| \geq 2^5 - |S| - 2$.*

Proof. Let G be a 5-dimensional hypercube-like network, $G_0, G_1 \in HL_4$, and $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leq 3n-6 = 9$, for $n = 5$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Without loss of generality, we assume $|S_0| \leq |S_1|$. (Note that $|S| \leq 9$, so $|S_0| \leq 4$.) We then consider three cases:

Case 1: $0 \leq |S_0| \leq 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n = 4$. So $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices of $G_0 = C \cup S_0$. Let C be the connected component of $G - S$ containing C_0. Then the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^5 - |S| - 2$.

Case 2: $|S_0| = 3$ and therefore $|S_1| \leq 6$.

$G_0 - S_0$ is connected by the fact that G_0 is $(n-1)$-connected, for $n \geq 4$. Thus, $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. Then, all vertices in G_1 are connected to component C_0, except for the three vertices in G_1 adjacent to the vertices in S_0. Since $|S_1| \leq 6$ and by Lemma 12, at least one of these three vertices is connected to component $G_1 - S_1$. So at least $2^4 - |S_1| - 2$ vertices are connected to component C_0. Let C be the connected component of $G - S$ containing C_0. Then, the number of vertices in C is $|V(C)| \geq |V(G_0) - S_0| + |V(G_1) - S_1 - 2| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$.

Case 3: $|S_0| = 4$ and consequently $4 \leq |S_1| \leq 5$.

Since $5 \leq 2n - 3$, for $n \geq 4$. By Lemma 11, there exists a connected components C_0 (respectively, C_1) in $G_0 - S_0$ (respectively, $G_1 - S_1$) such that $|V(C_0)| \geq 2^4 - |S_0| - 1$ (respectively, $|V(C_1)| \geq 2^4 - |S_1| - 1$). Thus, there exists a connected component C in $G - S$ such that $|V(C)| \geq |V(G_0) - S_0 - 1| + |V(G_1) - S_1 - 1| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$.
□

Based on Lemma 13, the general case for $n \geq 5$ is stated as follows.

Lemma 14 *Let G be an n-dimensional hypercube-like network, and S be a set of vertices* *with* $|S| \leq 3n-6$, *for* $n \geq 5$. *There exists a connected component* C *in* $G-S$ *such that* $|V(C)| \geq 2^n - |S| - 2$.

Proof. We prove this statement by induction on n. By Lemma 13, the result holds for $n=5$. Assume the lemma holds for $n-1$, for some $n \geq 6$. We now show that it is true for n.

Let G be an n-dimensional hypercube-like network, $G_0, G_1 \in HL_{n-1}$, and $G = G_0 \oplus_M G_1$, for some matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leq 3n-6$, for $n \geq 6$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Therefore, $|S_0| + |S_1| = |S| \leq 3n-6$. Without loss of generality, we assume $|S_0| \leq |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leq |S_0| \leq 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geq 6$. Let $C_0 = G_0 - S_0$, C_0 is a connected component with $|V(C_0)| \geq 2^{n-1} - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Let C be the connected component of $G-S$ containing C_0. The number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geq 2^n - |S| - 2$.

Case 2: $|S_0| \geq 3$ and consequently $|S_1| \leq 3n-9$.

By induction hypothesis, there are two connected components C_0 and C_1 in $G_0 - S_0$ and $G_1 - S_1$, and $|V(C_0)| \geq 2^{n-1} - |S_0| - 2$ and $|V(C_1)| \geq 2^{n-1} - |S_1| - 2$, respectively. Without loss of generality, we assume that $|V(C_0)| \geq |V(C_1)|$. Now we focus on the number of vertices in the component C_1, and discuss two situations. First, suppose $|V(C_1)| = 2^{n-1} - |S_1| - 2$. By Corollary 1, $|S_1| \geq 2(n-1) - 2 = 2n-4$. So $|S_0| = |S| - |S_1| \leq n-2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected. $G_0 - S_0$ has only one connected component C_0 and $|V(C_0)| = 2^{n-1} - |S_0|$. Let C be the connected component containing C_0. Then $|V(C)| = |V(C_0)| + |V(C_1)| \geq 2^{n-1} - |S_0| + 2^{n-1} - |S_1| - 2 \geq 2^n - |S| - 2$. Second, suppose that $|V(C_1)| \geq 2^{n-1} - |S_1| - 1$. Since $|V(C_0)| \geq |V(C_1)| \geq 2^{n-1} - |S_1| - 1$, there exists a connected component C containing C_0 such that $|V(C)| = |V(C_0)| + |V(C_1)| \geq 2^{n-1} - |S_0| - 1 + 2^{n-1} - |S_1| - 1 \geq 2^n - |S| - 2$. □

Corollary 2 *Let G be an n-dimensional hypercube-like network, $n \geq 5$, and let V' be a set of vertices in G with $|V'| = 3$. Then $|N(V')| \geq 3n - 5$.*

For the next theorem, we define a set of vertices F_c in graph G to be a *conditional* faulty vertex set if, in the induced subgraph $G - F_c$, every vertex has at least two fault-free neighboring vertices. We also call the subgraph $G - F_c$ a *conditional* faulty graph.

Theorem 4 *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geq 5$. Let F_c be a set of conditional faulty vertices with $|F_c| \leq 2n - 5$. Then each pair of vertices u and v in $G - F_c$ are connected by $min\{deg_{G-F_c}(u), deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $deg_{G-F_c}(u)$ and $deg_{G-F_c}(v)$ are the degree of u and v in $G - F_c$, respectively.*

Proof. Without loss of generality, we assume $deg_{G-F_c}(u) \leq deg_{G-F_c}(v)$, and therefore $min\{deg_{G-F_c}(u)\ , deg_{G-F_c}(v)\} = deg_{G-F_c}(u)$. We want to prove that each pair of vertices u and v in $G - F_c$ are connected by $deg_{G-F_c}(u)$ vertex-disjoint fault-free paths, for $|F_c| \leq 2n - 5$. We are going to show that u is connected to v if the number of vertices deleted is smaller than $deg_{G-F_c}(u) - 1$ in $G - F_c$, where $|F_c| \leq 2n - 5$.

Suppose on the contrary that u and v are separated by deleting a set of vertices V_{f_c}, where $|V_{f_c}| \leq deg_{G-F_c}(u) - 1$. By $deg_{G-F_c}(u) \leq deg(u) \leq n$, we have $|V_{f_c}| \leq n - 1$. We sum up the cardinality of these two sets F_c and V_{f_c}. Since $|F_c| \leq 2n-5$ and $|V_{f_c}| \leq n-1$, then $|F_c| + |V_{f_c}| \leq 3n-6$. Let $S = F_c \cup V_{f_c}$. By Lemma 14, there exits a connected component C in $G-S$ such that $|V(C)| \geq 2^n - |S| - 2$ and $|S| \leq 3n-6$. It means that there are at most two vertices in $G-S$ not belonging to C. We then consider three cases:

Case 1: $|V(C)| = 2^n - |S|$. It means that all vertices in $G-S$ are connected, which contradicts to the assumption that u and v are disconnected.

Case 2: $|V(C)| = 2^n - |S| - 1$. Only one vertex is disconnected to $G-S$. Since $|V_{f_c}| \leq deg_{G-F_c}(u) - 1 \leq deg_{G-F_c}(v) - 1$, neither u nor v can be the only one disconnected vertex, a contradiction.

Case 3: $|V(C)| = 2^n - |S| - 2$. Let a and b be the two vertices in $G-S$ not belonging to C. We consider two situations. (**i**) Suppose first that $u \in C$. If $v \in C$, then u and v are connected, a contradiction. If $v \in \{a, b\}$, since $|V_{f_c}| \leq deg_{G-F_c}(v) - 1$, v is connected to at least one vertex in component C, a contradiction. (**ii**) Suppose $u \in \{a, b\}$. We without loss of generality let $u = a$, and consider the adjacency between a and b.

Subcase 1: Suppose that a is not adjacent to b. By the assumption that u and v are separated by deleting a set of vertices V_{f_c} with $|V_{f_c}| = deg_{G-F_c}(u) - 1$. Let V_{f_c} be a subset of the neighborhood of u, that is, $V_{f_c} \subset N(u)$. Since $|V_{f_c}| < |N(u)|$, vertex u and component C are connected, which is a contradiction.

Subcase 2: Suppose that a is adjacent to b. Let $V_{f_c} = N(u) - \{b\}$. Since $G - F_c$ is a conditional faulty graph, one of the neighbors of b is in C. Then, b is connected to C, which is a contradiction.

Therefore, vertex u and v are still connected with up to $deg_{G-F_c}(u) - 1$ vertex faults.

By Theorem 2, this implies that each pair of vertices u and v in $G - F_c$ are connected by $min\{deg_{G-F_c}(u), deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $|F_c| \leq 2n - 5$. The proof is complete. $\square$

Chapter 4

Maximal Local-connectivity

One of the central issue in various interconnection networks is studying the value of connectivity. A basic definition of the connectivity of a graph is defined as the minimum number of vertices whose removal results in a disconnected or trivial graph. In contrast to this concept, Menger [17] provided a local point of view, and define the connectivity of any two vertices as the minimum number of vertex-disjoint paths between them. Following this concept, Volkman [28] discussed some issues on it, Oh et al. [19][20] and Shih et al. [23][24] investigated some related properties on the star graph and the class of hypercube-like networks, respectively.

In this chapter, we define two vertices to be maximally local-connected, if the maximum number of internally vertex-disjoint paths between them equals the minimum degree of these two vertices. Moreover, we introduce the one-to-many and many-to-many versions of connectivity.

4.1 Local-connectivity and Maximal Local-connectivity

The local connectivity of two vertices is defined as the maximum number of internally vertex-disjoint paths between them. A pair of vertices x and y is *maximally local-connected* if the local connectivity of x and y equals $\min\{deg(x), deg(y)\}$, and a graph G is *maximally local-connected* if every pair of vertices in G are maximally local-connected.

Now we give the definition of a graph to be f-fault-tolerant maximally local-connected.

Definition 2 *A graph G is f-fault-tolerant maximally local-connected, abbreviated as f-maximally local-connected, if for a set of faulty vertices F, $|F| \leq f$, each pair of vertices x, y of $G-F$ are connected by $\min\{deg_{G-F}(x), deg_{G-F}(y)\}$ vertex-disjoint fault-free paths, where $deg_{G-F}(x)$ and $deg_{G-F}(y)$ are the degrees of x and y in $G-F$, respectively.*

In the previous definition, we discuss the maximal local-connectivity, indicating that for every pair of vertices in a graph with a reasonable number of faulty vertices, there is an amount of vertex-disjoint paths between them, where the amount depends on the minimum remaining degree of the two vertices. Now we shall extend this concept to a "one-to-many" version. In this approach, we consider a vertex (as a source) and a set of vertices (as destinations). Under some constraints we prove that there exists a set of disjoint paths between the source and the destinations.

A classical theorem about the one-to-many connectivity was provided by Dirac[10] as follows.

Definition 3 *[10] Given a vertex x and a set U of vertices, an x,U-fan of size k is a set of k-paths from x to U such that any two of them share only one vertex x.*

We note that the cardinality of U is necessarily greater than k, $|U| \geq k$.

Theorem 5 *[10] A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.*

We then give the definition of a graph to be *one-to-many f-fault-tolerant maximally local-connected*.

Let G be a graph and F be a set of faulty vertices. In graph $G-F$, pick an arbitrary vertex x (as a source) and a set U of vertices (as destinations) with $|U| = t$, $x \notin U$. We want to find a set of t paths from x to U such that each pair of them share only the vertex x. In order to do so, the set U must satisfy some necessary conditions: (i) the cardinality of U is not greater than the remaining degree of x, that is, $|U| \leq deg_{G-F}(x)$, and (ii) the set U cannot contain any vertex and all its neighbors, that is, $\{v\} \cup N_{G-F}(v)$ is not contained in U for each $v \in U$.

We call a set of vertices U in $G-F$ satisfying the above two conditions a *conditional terminal set* with respect to x and F, abbreviated as a *conditional terminal set* if there is no ambiguity. As a short remark, we note that $U \subseteq V(G-F)$, $x \notin U$, $|U| \leq deg_{G-F}(x)$, and $U \not\supseteq \{v\} \cup N_{G-F}(v)$ for each $v \in U$.

Definition 4 *A graph G is one-to-many f-fault-tolerant maximally local-connected, abbreviated as one-to-many f-maximally local-connected, if given any set of faulty vertices F with $|F| \leq f$ and $x \in G-F$, let $t \leq deg_{G-F}(x)$, there is a set of t paths from x to U such that each pair of them share only the vertex x, for each conditional terminal set U with $|U| = t$.*

For a vertex x and a vertex set U, $x \notin U$, a *separator* of x and U is defined to be a set of vertices S, $x \notin S$, whose removal results in the disconnection between x and $U - S$. The separator S is trivial if either $S = U$ or $S = N(x)$.

In the previous two definitions, we discussed the local connectivity in two directions: a "one-to-one" version and a "one-to-many" version. Now we shall study the concept of a "many-to-many" version. Given any two vertex set U_1 and U_2 with $|U_1| = |U_2| = t$ in a graph G such that $U_1 \cap U_2 = \phi$, we are concerned about many-to-many disjoint paths $P_1, P_2, ..., P_t$ connecting U_1 and U_2 in G, such that $V(P_i) \cap V(P_j) = \phi$ for all $i \neq j$. We call such a set of t disjoint paths to be many-to-many vertex-disjoint paths between U_1 and U_2. Park et al. [21] discussed some related properties on the class of hypercube-like networks. In the discussion of the connectivity, a classical theorem has made some extension to prove the existence of a number of disjoint paths between two sets of vertices: in a graph of connectivity k, there are k vertex-disjoint paths between every two disjoint sets of vertices both with k vertices. In the following, we shall strengthen this idea. If there are some constraints on the two sets such that each set cannot contain any vertex and all its neighbors, the number of vertex-disjoint paths can be increased to almost double the size of k. We need some terminologies to describe such constraint. Let U be a set of vertices in a graph G, the set U is defined to be a *conditional selected set* if $\{v\} \cup N_G(v)$ is not contained in U for every $v \in U$. With this condition, our goal is to show the following result: in a graph of connectivity k with some good properties, for every pair of conditional selected vertex sets U_i with $|U_i| = t$, for $i = 1, 2$, there is a set of t vertex-disjoint paths between them, where $t \leq 2k - 2$.

Before showing this result, we first explain why the number $2k-2$ is the best possibility. Let (u, v) be an edge in a k-connected k-regular graph G. Let U_1 be the set of $N_G(u) \cup$

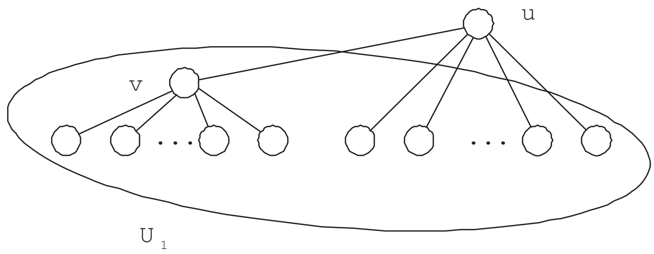

Figure 4.1: An example showing that in a k-connected k-regular graph, there are at most $2k-2$ vertex-disjoint paths between two conditional selected vertex sets.

$N_G(v)\backslash\{u\}$. Clearly, the size of U_1 is $2k-1$, and U_1 is a conditional selected set. Arbitrarily choosing another conditional selected set U_2 of size $2k-1$, then it is not hard to see that there do not exist $2k-1$ vertex disjoint paths connecting U_1 and U_2. As a consequence, the size $2k-2$ is the upper bound for our result. (See Fig. 4.1.)

In the following, we formulate the above concept as a "many-to-many" version of connectivity. Furthermore, even in a graph with some faulty vertices, such concept can also be applied on the graph. Below is the generalized definition of the fault-tolerant many-to-many connectivity.

Definition 5 *A graph G is f fault-tolerant many-to-many t-connected, abbreviated as many-to-many t/f-connected, if given any set of faulty vertices F with $|F| \leq f$, for each pair of conditional selected vertex set U_1 and U_2 with $|U_1| = |U_2| = t$ in $G - F$, there is a set of vertex-disjoint t paths from U_1 to U_2.*

For two vertex sets U_1 and U_2, a *separator* of U_1 and U_2 is defined to be a set of vertices S whose removal results in the disconnection between $U_1 - S$ and $U_2 - S$. The separator S is trivial if either $S = U_1$ or $S = U_2$.

4.2 The Matching Composition Networks

The Matching Composition Network (MCN) [6][15], a recursively constructed topology, is a family of interconnection networks. The construction of an MCN is to join two graphs G_0 and G_1 of the same number of vertices by adding a perfect matching between the vertices of G_0 and G_1. Many well-known interconnection networks are special cases of the MCN family, such as the Hypercube, [22] the Crossed cubes [11], the Twisted cubes [1][12], and the Möbius cubes [8].

Let G_0 and G_1 be two graphs with the same number of vertices, and M be an arbitrary perfect matching between $V(G_0)$ and $V(G_1)$. We use $G(G_0, G_1; M)$ to denote the Matching Composition Network composed of G_0 and G_1 by M, which has the vertex set $V(G) = V(G_0) \bigcup V(G_1)$ and the edge set $E(G) = E(G_0) \bigcup E(G_1) \bigcup M$.

Lemma 15 *Let $G = (V, E)$ be a k-regular and triangle-free graph, and every two vertices in G have at most two common neighboring vertices. For every subset V' of V with $|V'| = 2$, the number of neighbors of V' is at least $2k - 2$. That is, $|N_G(V')| \geq 2k - 2$.*

Proof. Let $V' = \{v_1, v_2\}$. If v_1 and v_2 are adjacent, the number of neighboring vertices of $\{v_1, v_2\}$ is $2(k-1)$ because graph G is triangle-free. Otherwise, since every two vertices have at most two common neighbors, the number of neighbors of V' is at least $2(k-2)+2$. So the result holds. □

Lemma 16 *Let G be a k-regular and triangle-free graph with n vertices. Then $n \geq 2k$.*

Proof. Let $e = (u, v)$ be an edge of $E(G)$ and $V' = \{u, v\}$. Since G is triangle-free and the degrees of u and v are both k, the number of vertices of G is at least

$|N_G(V') \cup \{u, v\}| = 2k.$ □

Below is a lemma stating the structural properties of a matching composition network. It shows that an MCN constructed by two k-regular subgraphs is quite fault resistant, that is, even with up to $2k-1$ vertex faults present and the resulting graph disconnected, it will have a large connected component and exactly one small component, which is an isolated vertex.

Lemma 17 *Let G_0 and G_1 be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, and let M be an arbitrary perfect matching between G_0 and G_1. Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of G_0 and G_1, and let T be a set of vertices in G with $|T| \leq 2k - 1$. Assume that every two vertices in G_i, $i = 0, 1$, have at most two common neighboring vertices, for all $k \geq 1$. Then $G - T$ satisfies that either (1) $G - T$ is connected or (2) $G - T$ has two connected components, one of which is a trivial component.*

Proof. Let $T_0 = T \cap V(G_0)$ and $T_1 = T \cap V(G_1)$, respectively. By assumption, $T_0 \cap T_1 = \phi$ and $|T_0| + |T_1| = |T| \leq 2k - 1$. Without loss of generality, we suppose that $|T_0| \geq |T_1|$. Then $|T_1| \leq k - 1$. Since G_1 is k-connected, $G_1 - T_1$ is connected. We then consider two cases:

Case I: $G_0 - T_0$ is connected. By Lemma 16, the number of vertex in G_0 is at least $2k$. Since $|V(G_0)| \geq 2k$ and $|T| \leq 2k - 1$, there is at least one vertex in $G_0 - T_0$ connecting to $G_1 - T_1$. Thus, $G - T$ is connected.

Case II: $G_0 - T_0$ is disconnected. Suppose that $G_0 - T_0$ is divided into m disjoint connected components, say $C_1, ..., C_m$, where $m \geq 2$. In the following subcases, we consider the number of vertices of C_i for each $i \in 1, 2, ..., m$.

Subcase II.1: For those components C_i containing two or more vertices, $|V(C_i)| \geq 2$. Let (a, b) be an edge in C_i and $V' = \{a, b\}$. Since $|N_{G_0}(V') \cup \{a, b\}| = 2k$ and $|T| \leq 2k-1$, there exists at least one edge with both ends fault-free remaining between C_i and $G_1 - T_1$. So every component containing two or more vertices is connected to $G_1 - T_1$.

Subcase II.2: For those components C_i containing only one vertex, $|V(C_i)| = 1$. Suppose that there is only one trivial component in $G_0 - T_0$. If its adjacent vertex in G_1 is fault-free, $G - T$ is connected; otherwise, this trivial component in $G_0 - T_0$ is isolated in $G - T$. Suppose that there are at least two trivial components in $G_0 - T_0$. We arbitrarily choose two of them, say u and v, to form a subset V'. By Lemma 15, the number of neighbors of V' is at least $2k - 2$. So $|T_0| \geq 2k - 2$, $|T_1| \leq 1$, and at most one of $\{u, v\}$ is not connected to $G_1 - T_1$. Therefore, there is at most one trivial component in $G - T$.

This proves the lemma. □

In the above lemma, changing the condition of T slightly by replacing a faulty vertex by a faulty edge, the connection status of an MCN remains the same.

Lemma 18 *Let G_0 and G_1 be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, and let M be an arbitrary perfect matching between G_0 and G_1. Assume that every two vertices in G_i, $i = 0, 1$, have at most two common neighboring vertices, for all $k \geq 1$. Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of G_0 and G_1, and e_f be an edge in G and T_v be a set of vertices in G with $|T_v| \leq 2k - 2$. Then $G - T_v - \{e_f\}$ satisfies either that (1) $G - T_v - \{e_f\}$ is connected or that (2) $G - T_v - \{e_f\}$ has two connected components, one of which is a trivial component.*

Proof. By Lemma 17, under the constraint that $|T_v| \leq 2k-2$, $G-T_v$ is either connected or with two components one of which is trivial. Now we classify the situations into two cases.

Case I: *$G-T_v$ is connected.* If $G-T_v-\{e_f\}$ is connected, we are done. Otherwise, let C_0, C_1 be two connected components of $G-T_v-\{e_f\}$, and $e_f=(v_0,v_1)$ where $v_0 \in C_0$, $v_1 \in C_1$. Without loss of generality, assume that $|V(C_0)| \leq |V(C_1)|$. If $|V(C_0)|=1$, we are done. If $|V(C_0)|=2$, there is another vertex, say w, adjacent to v_0 in C_0. Let $V'=\{v_0,w\}$. Since G is a $(k+1)$-regular graph, by Lemma 15, $|N_G(V')| \geq 2k$ and $T_v \supseteq N_G(V')-\{v_1\}$. So $|T_v|+|\{e_f\}| \geq |N_G(V')-\{v_1\}|+|\{e_f\}| \geq 2k$. It is a contradiction to the assumption that $|T_v|+|\{e_f\}| \leq 2k-1$. So $|V(C_0)| \geq 3$. Deleting the vertex v_0 from $G-T_v$ results in a subgraph with less number of edges than that of $G-T_v-\{e_f\}$. Since $|T_v \cup \{v_0\}| \leq 2k-1$, $G-T_v-\{v_0\}$ also contain one trivial graph and one connected component. Consequently, $G-T_v-\{e_f\}$ does so. So this case holds.

Case II: *$G-T_v$ has two connected components, and one of which is a trivial graph.* Let u be the trivial graph and C the connected component. After deleting the edge e_f from $G-T_v$, if C is still connected, we are done. Otherwise, the component C is divided into two components C_0 and C_1 and $e_f=(v_0,v_1)$ where $v_0 \in C_0$, $v_1 \in C_1$. We without loss of generality assume that $|V(C_0)| \leq |V(C_1)|$. If $|V(C_0)|=1$, let $V'=\{v_0,u\}$. Again, since $|N_G(V')-\{v_1\}|+|\{e_f\}| \geq 2k$, it is a contradiction to the assumption that $|T_v|+|\{e_f\}| \leq 2k-1$. Otherwise, the situations about $|V(C_0)|=2$ and $|V(C_0)| \geq 3$ are similar to Case I and the proofs are similar.

This completes the proof. □

We make some remarks concerning the above lemmas. If both graphs G_0 and G_1 have the properties that (1) each one is triangle-free and (2) every pair of distinct vertices in each graph share at most two common neighbors, then the constructed MCN $G = (G_0, G_1; M)$ also has properties (1) and (2). Therefore the result can be applied recursively. We observe that many interconnection networks have these two properties. For example, the hypercube-like graphs and the star graphs do.

4.3 Cayley Graphs Generated by Transposition Trees

Let Γ be a group, and let $H \subseteq \Gamma$ be a set of group elements such that the identity element $I \notin H$. The Cayley graph associated with (Γ, H) is then defined as the directed graph having one vertex associated with each group element and directed edge (u, v) whenever $uv^{-1} \in H$. The Cayley graph may depend on the choice of a generating set, and is connected if and only if H generates Γ. That is, the set H are group generators of Γ. In this paper, we choose the finite group to be Γ_n, the symmetric group on $\{1, 2, ..., n\}$, and the generating set H to be a set of transpositions. The vertices of the corresponding Cayley graph are permutations. Since H only contains transpositions, there is an arc from vertex u to v if and only if there is an arc from v to u. So we can regard these Cayley graphs as undirected graphs. A way to represent H is via a graph with vertex set $\{1, 2, ..., n\}$ where there is an edge between i and j if and only if the transposition (ij) belongs to H. This graph is called the *transposition generating graph of* (Γ_n, H), abbreviated as *(transposition) generating graph* if it is clear from the context.

We require the transposition generating graph to be connected, since an interconnection network need to be connected. In this paper, we restrict the graphs obtained from transposition generating graph that are trees. For convenience, we call the corresponding

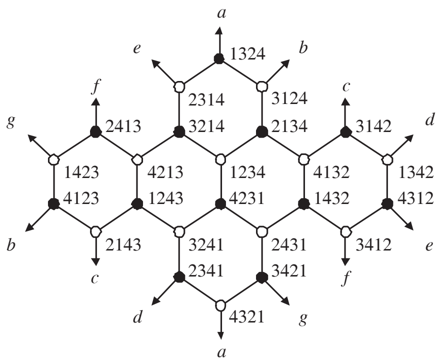

Figure 4.2: The Star Graph: $H = \{(12), (13), (14)\}$.

transposition generating graph a *transposition tree*. So the Cayley graphs obtained by these transposition trees are $(n-1)$-regular and have $n!$ vertices. This includes the star graph whose generating tree is $K_{1,n-1}$ and the bubble-sort graph whose generating tree is a path. To avoid trivial cases, we assume that a generating tree has at least three vertices. This class of graphs is a popular generalization of the star graphs and the bubble sort graphs [2] studied in [25]. The star graph and the bubble-sort graph are illustrated in Figs. 4.2 and 4.3 for case $n = 4$.

4.4 Maximal Local-Connectivity on the Matching Composition Networks

In this section, we investigate the property of local connectivity on the Matching Composition Network. Let G be a graph, x and y be two distinct vertices in G and k= $\min\{deg(x), deg(y)\}$. We say that x and y are maximally local-connected, if there exist k vertex-disjoint paths connecting x and y. A graph G is maximally local-connected if every pair of vertices in G are maximally local-connected. A regular MCN is actually

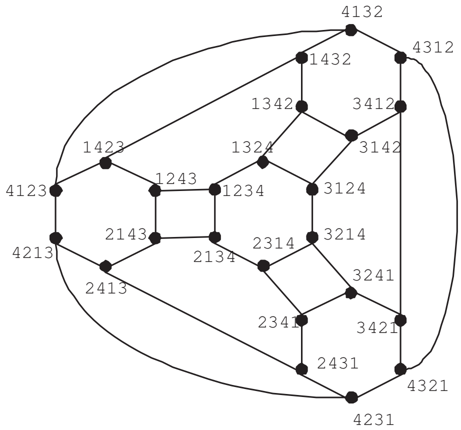

Figure 4.3: The Bubble-sort Graph: $H = \{(12), (23), (34)\}$.

maximally local-connected. Moreover, even with a set of faulty vertices, we will propose a strong fault-tolerant version of maximal local-connectivity, and prove that a $(k+1)$-regular MCN is $(k-1)$-fault-tolerant maximally local-connected. So far the concept of local connectivity can be referred as a one-to-one type of connectivity, since we only consider the maximum number of vertex-disjoint paths between two vertices. In classical theory, there is also a one-to-many version of connectivity. We extend this concept to a fault-tolerant version, called one-to-many f-fault-tolerant maximally locally connected property. At last, we discuss a fault-tolerant many-to-many version of connectivity. All the definitions of these three types of connectivity are given in a strong fault-tolerant version. We will prove that a $(k+1)$-regular MCN is not only $(k-1)$-fault-tolerant one-to-many maximally local-connected, but also f-fault-tolerant many-to-many t-connected (which will be defined subsequently) if $f + t \leq 2k$.

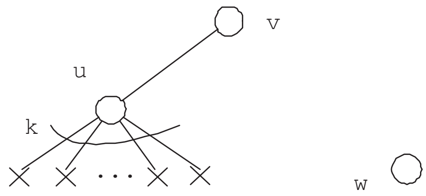

Figure 4.4: An example showing that a $(k+1)$-regular MCN is not k-maximally local-connected.

4.4.1 One-to-One Maximal Local-connectivity

In this section, we are going to prove that an MCN composed of two k-regular graphs with some additional properties is $(k-1)$-maximally local-connected. Note that the MCN here is $(k+1)$-regular. This result is optimal in the sense that the result cannot be guaranteed if there are k faulty vertices. We give an example to show it (as illustrated in Fig. 4.4), let (u, v) be an edge in the MCN. Suppose that all the k vertices adjacent to u except v are faulty. Choose a vertex w different from u and v, and $deg_{G-F}(v) = deg_{G-F}(w) = k+1$. However, there are at most k vertex-disjoint paths between v and w. So the $(k+1)$-regular MCN is not k-maximally local-connected. Before proving the main result, we make some simple observations.

If an MCN $G = G(G_0, G_1; M)$ is $(k-1)$-maximally local-connected, the number of vertices in each component G_i, $i = 0, 1$, has to be large enough. More precisely, each component G_i has to contain at least $2k$ vertices.

Intuitively, if each component G_i contains only $2k-1$ or less vertices. Then there are at most $2k-1$ "bridges" connecting G_0 and G_1 in the MCN $G = (G_0, G_1; M)$. If there are $k-1$ faulty vertices to destroy $k-1$ "bridges", there are only k "bridges" left between G_0 and G_1. Pick a vertex u in G_0 and another vertex v in G_1, each with degree $k+1$.

Then it is intuitively clear that there are no $k+1$ vertex-disjoint paths connecting u and v. A formal proof is given below.

Lemma 19 *Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of two k-regular graphs G_0 and G_1 both with the same number of vertices n, where $k \neq 2$. If G is $(k-1)$-maximally local-connected, then $n \geq 2k$.*

Proof. Before proving this Lemma, we explain why the lemma does not hold if $k = 2$. When $k = 2$, graphs G_0 and G_1 are cycles of the same length. It is straightforward but tedious that the MCN generated here is 1-fault-tolerant maximally local-connected. However, it is not necessarily that $n \geq 2k$. For example $n = 3$, both G_0 and G_1 are triangles, and the number $n(= 3)$ is less than $2k(= 4)$.

When $k = 1$, graphs G_0 and G_1 are both one edge incident with two vertices. The MCN is indeed 0-fault-tolerant maximally local-connected and it also holds that $n \geq 2k$.

Now we consider the situation that $k \geq 3$. Suppose on the contrary that $n \leq 2k-1$. If one of the two subgraphs G_0 and G_1 is a complete graph, since each subgraph is k-regular, the number of vertices in each subgraph is $k+1$. Then both G_0 and G_1 are complete graph. Hence, the cardinality of the perfect matching M between $V(G_0)$ and $V(G_1)$ is $k+1$. Let x be a vertex in G_0, and y be the adjacent vertex of x in G_1. We choose a set of $k-1$ vertices $V_f = \{f_1, f_2, ..., f_{k-1}\}$ not containing x and y, where f_1 is an arbitrary vertex in G_0, and $f_2, f_3, ..., f_{k-1}$ are other vertices in G_1 not adjacent to f_1. In the induced subgraph of $V(G) - V_f$, the number of edges with one end in G_0 and the other end in G_1 is two, and the remaining degrees of x and y are k and three, respectively. There are only two fault-free edges between G_0 and G_1. Therefore, it is easy to see that there does not exist three vertex-disjoint paths between x and y, and the graph G is not $(k-1)$-fault-tolerant maximally local-connected.

Now, suppose that neither G_0 nor G_1 is a complete graph. Let x be a vertex in G_0, and y be the adjacent vertex of x in G_1. We index the adjacent vertices of x in G_0 as $u_1, u_2, ..., u_k$. The other ones left in G_0 are indexed as $u_{k+1}, u_{k+2}, ..., u_{n-1}$, in any arbitrary order. For each u_i in G_0, the corresponding (adjacent) vertex in G_1 is named as v_i, for $1 \leq i \leq k$. Since G_1 is not a complete graph and G_1 is k-regular, there exist two vertices $v_p, v_q \in \{v_1, v_2, ..., v_k\} \cup \{y\}$, such that $(v_p, v_q) \notin E(G_1)$. Without loss of generality, let $v_q \neq y$. Recall that $n \leq 2k-1$. Now we pick vertices $\{u_i \mid k+1 \leq i \leq n-1\}$ and v_q to form a vertex set V_f, which has cardinality at most $k-1$. The cardinality of V_f can be verified as $[(n-1)-k]+1 \leq [((2k-1)-1)-k]+1 = k-1$. In the induced subgraph of $V(G) - V_f$, the remaining degrees of x and v_p are both $k+1$. However, in this induced subgraph, there are only k edges connecting the vertices in $G_0 - V_f$ and $G_1 - V_f$, which results in that there are no $k+1$ vertex-disjoint paths between x and v_p. So the graph G is not $(k-1)$-maximally local-connected.

The proof is complete. □

Therefore, to study the $(k-1)$-fault-tolerant maximally local-connectivity of MCN, we need a k-regular graph containing at least $2k$ vertices. Recall that Lemma 16 states that every k-regular and triangle-free graph contains at least $2k$ vertices.

Now, we are ready to present our first main result.

Theorem 6 *Let G_0 and G_1 be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, for $k \geq 1$, and let M be an arbitrary perfect matching between G_0 and G_1. Assume that any two vertices in G_i, $i = 0, 1$, have at most* *two common neighboring vertices. The Matching Composition Network $G = (G_0, G_1; M)$ is $(k-1)$-maximally local-connected.*

Proof. By Lemma 16, the number of vertices in G_i is greater than $2k$, for $i = 0, 1$. Let F be a set of faulty vertices with $|F| \leq k-1$, and let x and y be two fault-free vertices in $G-F$. We assume without loss of generality that $deg_{G-F}(x) \leq deg_{G-F}(y)$, so $\min\{deg_{G-F}(x), deg_{G-F}(y)\} = deg_{G-F}(x)$. We now show that after deleting $deg_{G-F}(x)-1$ arbitrary vertices in $G-F$, vertex x is still connected to y. By Theorem 2, this implies that each pair of vertices x and y are connected by $deg_{G-F}(x)$ vertex-disjoint fault-free paths, where $|F| \leq k-1$. We now consider two cases:

Case I: *x and y are not adjacent in $G-F$.* We then show that x is connected to y if the number of vertices deleted is smaller than $deg_{G-F}(x) - 1$. For the sake of contradiction, suppose that x and y are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{G-F}(x)-1$. As a consequence, $|V_f| \leq k$ because of $deg_{G-F}(x) \leq deg(x) \leq k+1$. Then, the summation of the cardinality of these two sets F and V_f is $|F|+|V_f| \leq 2k-1$. Let $T = F \cup V_f$. By Lemma 17, either $G-T$ is connected, or $G-T$ has two components, one of which contains only one vertex. If $G-T$ is connected, it contradicts to the assumption that x and y are disconnected. Otherwise, if $G-T$ has two components and one of which contains only one vertex u. Since we assume that x and y are separated, one of x and y is the vertex u, say $x = u$. Thus, the set V_f must be the neighborhood of x and $|V_f| = deg_{G-F}(x)$, which is also a contradiction. Then, x is connected to y when the number of vertices deleted is smaller than $deg_{G-F}(x) - 1$ in $G-F$.

Case II: *x and y are adjacent in $G-F$.* We need to show that x is connected to y if the number of vertices deleted is smaller than $deg_{G-F}(x)-2$ in $G-F-\{(x,y)\}$. Suppose on the contrary that in $G-F-\{(x,y)\}$, x and y are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{G-F}(x)-2$. Since $deg_{G-F}(x) \leq k+1$, we get $|V_f| \leq k-1$. Then the union set T of F and V_f has cardinality $|T| = |F| + |V_f| \leq 2k-2$. In this circumstance, Lemma 18 implies either that $G-T-\{(x,y)\}$ is connected or that $G-T-\{(x,y)\}$ has two components one of which is trivial. For the first situation, $G-T-\{(x,y)\}$ is connected, this contradicts to the assumption that x and y are separated in $G-F-\{(x,y)\}$. For the second situation, the trivial graph must be x or y, and we without loss of generality let x be such trivial graph. In $G-F-\{(x,y)\}$, in order to make x a trivial graph, the number of vertices deleted must be greater or equal to $deg_{G-F}(x)-1=k$. However, $|V_f| \leq k-1$, which is a contradiction. So x and y are connected when the number of vertices deleted is smaller than $deg_{G-F}(x)-2$ in $G-F-\{(x,y)\}$.

The proof is complete. □

4.4.2 One-to-Many Maximal Local-connectivity

As the one-to-one case, if a $k+1$-regular Matching Composition Network is one-to-many $(k-1)$-maximally local-connected, each component has to contain large enough vertices. The next lemma states this claim.

Lemma 20 *Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of two k-regular graphs G_0 and G_1 both with n vertices, where $k \neq 2$. If G is one-to-many $(k-1)$-maximally local-connected, then $n \geq 2k$.*

Proof. Since one-to-many $(k-1)$-maximally local connected property implies one-to-one $(k-1)$-maximally local connected property, by Lemma 19, $n \geq 2k$. □

We need a result, which is essentially the classical Menger's theory, to prove our next theorem as following.

Lemma 21 *Let G be a graph, x be a vertex and U be a vertex set with $x \notin |U|$. The minimum size of a separator of x and U equals the maximum number of pairwise internally disjoint paths from x to U such that each pair of the paths share only the vertex x.*

Proof. We construct G' from G by adding a new vertex y adjacent to all vertices of U. By Theorem 2, the minimum size of a separator of x and y equals the maximum number of pairwise internally disjoint paths from x to y in G'. The result holds after deleting the vertex y in G'. □

Now the one-to-many version of the maximally local-connectivity of an MCN is given below.

Theorem 7 *Let G_0 and G_1 be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, for $k \geq 1$, and let M be an arbitrary perfect matching between G_0 and G_1. Assuming that any two vertices in G_i, $i = 0, 1$, have at most two common neighboring vertices, the Matching Composition Network $G = (G_0, G_1; M)$ is one-to-many $(k-1)$-maximally local-connected.*

Proof. Let x be a vertex and U be a conditional terminal set in $G - F$ such that $t \leq deg_{G-F}(x)$ and $|U| = t$. We want to show that x and U do not separate if the number of vertices deleted is smaller than $t - 1$ in $G - F$, where $|F| \leq k - 1$. It implies that there are $deg_{G-F}(x)$ vertex-disjoint fault-free paths from x to U such that each pair of the paths share only the vertex x according to Lemma 21. Suppose not, let x

and $U - V_f$ are separated by deleting a set of vertices V_f, where $|V_f| \leq t - 1$. Since $t \leq deg_{G-F}(x) \leq deg(x) \leq k + 1$, so $|V_f| \leq t - 1 \leq k$. We now count the summation of the cardinality of these two sets F and V_f, which is $|F| + |V_f| \leq k - 1 + k = 2k - 1$. Let $T = F \cup V_f$, By Lemma 17, either $G - T$ is connected, or $G - T$ has two components, one of which is only one vertex. If $G - T$ is connected, it contradicts to the assumption that x and $U - V_f$ are disconnected. So we consider that $G - T$ has two components and one of which has only one vertex u. If $x = u$, the set V_f has to be the neighborhood of x and $|V_f| = deg_{G-F}(x)$, which contains a contradiction. Otherwise, $u \in U$. Since U is a conditional terminal set, $\{u\} \cup N_{G-F}(u)$ is not contained in U for each $u \in U$, either there are at least two remaining vertices of $U - V_f$, or the only one vertex in $U - V_f$ is not trivial in $G - T$, which contains a contradiction. Thus, there is a set of t paths from x to U such that any two of them share only the vertex x when the number of vertices deleted is smaller than $deg_{G-F}(x) - 1$ in $G - F$.

This proof is complete. □

So far we know that an MCN constructed from two k-regular graphs is $(k-1)$-maximally local-connected, and we will prove that it is also "one-to-many" $(k-1)$-maximally local-connected. In fact, the "one-to-many" result is stronger than the fundamental "one-to-one" result. For example, let x and y be two distinct vertices, and $\{x_1, x_2, ..., x_m\}$ and $\{y_1, y_2, ..., y_n\}$ be the neighbors of x and y, respectively. Assume that $m > n$. In the "one-to-one" result, it only says that there exists n vertex-disjoint paths joining x and y. In the "one-to-many" result, we can select any n vertices from the neighbors $\{x_1, x_2, ..., x_m\}$ of x, and there are n vertex-disjoint paths from y to all these n vertices. So the "one-to-many" version of fault-tolerant maximal local-connectivity implies the "one-to-one" version.

4.4.3 Many-to-Many Maximal Local-connectivity

In the following, we consider a $(k+1)$-regular MCN composed of two k-regular graphs both with connectivity k. Note that in this MCN, the connectivity is $k+1$, and there are at most $2k$ vertex-disjoint paths between two given conditional selected vertex sets.

Moreover, our result provides a generalized fault-tolerant version of connectivity, instead of being restricted in a fault-free graph. Notice that in the $(k+1)$-regular MCN with at most f faulty vertices, the value f should be limited to at most $2k-1$. By Lemma 17, the resulting graph would be connected or be two components one of which is trivial. We observe that the possible isolated vertex is not a conditional selected set.

We need the following lemma, essentially it is the classical Menger's theory, to prove our main theorem.

Lemma 22 *Let G be a graph, and U_1, U_2 be two arbitrary vertex sets with $|U_1| = |U_2|$. The minimum size of a separator of U_1 and U_2 equals the maximum number of pairwise internally disjoint paths from U_1 to U_2.*

Proof. We construct G' from G by adding two new vertices x and y adjacent to all vertices of U_1 and U_2, respectively. By Theorem 2, the minimum size of a separator of x and y equals the maximum number of pairwise internally disjoint paths from x to y in G'. The result follows after deleting the vertices x and y in G'. □

Now, we are ready to introduce the main theorem.

Theorem 8 *Let G_0 and G_1 be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, for $k \geq 1$, and let M be an arbitrary per-*

fect matching between G_0 and G_1. Assume that any two vertices in G_i, for $i = 1, 2$, have at most two common neighboring vertices. Let F be a set of vertices with $|F| = f$, and U_1, U_2 be two arbitrary conditional selected sets with $|U_1| = |U_2| = t$. The Matching Composition Network $G = (G_0, G_1; M)$ is many-to-many t/f-connected if $f + t \leq 2k$.

Proof. Let U_1 and U_2 be two conditional selected sets with $|U_1| = |U_2| = t$ in $G - F$ where $|F| = f$. Suppose $f + t \leq 2k$, we want to show that there are t vertex-disjoint paths connecting U_1 and U_2. By Lemma 22, we prove this by showing that U_1 and U_2 do not separate if the number of vertices deleted is at most $t - 1$. Suppose on the contrary that U_1 and U_2 are separated by deleting a set of vertices V_f, where $|V_f| \leq t - 1$. Let T be the union set of F and V_f, $T = F \cup V_f$, and we have $|T| = |F| + |V_f| \leq 2k - 1$. By Lemma 4.1, either $G - T$ is connected, or $G - T$ has two components one of which is only one vertex. We only consider the second case because the first case is a contradiction to the assumption that $U_1 - V_f$ and $U_2 - V_f$ are disconnected. Let u be the trivial graph in $G - T$. Certainly, either $u \in U_1 - T$ or $u \in U_2 - T$, and we without loss of generality assume that $u \in U_1 - T$. We observe that $G - T$ has only two components one of which is the single vertex u, $U_1 - V_f$ and $U_2 - V_f$ are disconnected in $G - T$, and $u \in U_1 - V_f$. So $U_1 - V_f$ contains only one vertex, namely u. Since $|U_1| = t$, $|V_f| \leq t - 1$ and $|U_1 - V_f| = 1$, we have $V_f \subseteq U_1$. However, U_1 is a conditional selected set in $G - F$, U_1 does not contain all the neighboring vertices of u. Thus vertex u cannot be isolated by deleting V_f from $G - F$. This contradicts to the assumption that vertex u is an isolated vertex in $G - T$. Therefore, by Lemma 22, there are t vertex-disjoint paths from U_1 to U_2 in $G - F$ with $|F| = f$ if $t + f \leq 2k$.

This proof is complete. □

4.5 Maximal Local-Connectivity on Cayley graphs generated by transposition trees

To prove our main result, we need the following lemmas. Cheng et al. studied some result on Cayley graphs generated by Transposition Tree.

Theorem 9 *[7] Let G be a Cayley graph generated by a transposition tree H on $\{1, 2, ..., n\}$ where $n \geq 3$. If T is a set of vertices with $|T| \leq k(n-1) - \frac{k(k+1)}{2}$, where $1 \leq k \leq n-2$, then $G - T$ has one large (connected) component, and the remaining small components has at most $k-1$ vertices in total.*

We can rewrite this Theorem 9 with respect to the next lemma if the number k equals 2.

Lemma 23 *Let G be a Cayley graph generated by a transposition tree H on $\{1, 2, ..., n\}$, and T be a set of vertices in G with $|T| \leq 2n-5$, where $n \geq 3$. There exists a connected component C in $G - T$ such that $|V(C)| \geq n! - |T| - 1$.*

Proof. This Lemma follows from Theorem 9 when $k = 2$. □

There is a similar result by replacing a faulty vertex by a faulty edge as following.

Lemma 24 *Let G be a Cayley graph generated by a transposition tree H on $\{1, 2, ..., n\}$, where $n \geq 3$. Let e_f be an edge in G and T_v be a set of vertices in G with $|T_v| \leq 2n-6$. There exists a connected component C in $G - T_v - \{e_f\}$ such that $|V(C)| \geq n! - |T_v| - 1$.*

Proof. By Lemma 23, under the constraint that $|T_v| \leq 2n-6$, $G - T_v$ has a connected component C containing at least $n! - |T_v| - 1$ vertices. Now suppose on the contrary that each connected component of $G - T_v - \{e_f\}$ has at most $n! - |T_v| - 2$ vertices. Then the maximum connected component C of $G - T_v$ is divided into two components, denoted C_0 and C_1, and $|V(C_i)| \leq n! - |T_v| - 2$ for $i = 1, 2$. Let $e_f = (u, v)$. Without loss of generality, we assume that $u \in C_0$, $v \in C_1$ and $|V(C_0)| \geq |V(C_1)|$. Then choose $T = T_v \cup \{u\}$. By Lemma 23, there is a connected component with at least $n! - |T| - 1$ vertices in $G - T$. It means that either C_0 has at least $n! - |T|$ vertices or C_1 has at least $n! - |T| - 1$ vertices, contradicting to the fact that $|V(C)| \leq n! - |T_v| - 2$, for $i = 1, 2$. So the result holds. □

4.5.1 One-to-One Maximal Local-connectivity

Let G be a Cayley graph obtained from a transposition tree H on $\{1, 2, ..., n\}$ where $n \geq 3$. In this section, we are going to prove that G with some additional properties is $(n-3)$-fault-tolerant maximally local-connected. Note that G is $(n-1)$-regular. This result is optimal in the sense that the result cannot be guaranteed if there are $n-2$ faulty vertices. We give an example to show this. Let (x, w) be an edge in G. Suppose that all the $n-2$ vertices adjacent to w except x are faulty. Choose a vertex y different from x and w, and $deg_{G-F}(x) = deg_{G-F}(y) = n-1$. Then there are at most $n-2$ vertex-disjoint paths between x and y. So G is not $(n-2)$-maximally local-connected.

We now show our first main result.

Theorem 10 *Let G be a Cayley graph obtained from a transposition tree H on $\{1, 2, ..., n\}$ where $n \geq 3$. Then G is $(n-3)$-fault-tolerant maximally local-connected.*

Proof. Let F be a set of faulty vertices with $|F| \leq n-2$, and let x and y be two fault-free vertices in $G - F$. We assume, without loss of generality, that $deg_{G-F}(x) \leq deg_{G-F}(y)$, so

$\min\{deg_{G-F}(x), deg_{G-F}(y)\} = deg_{G-F}(x)$. We then show that vertex x is still connected to y after deleting $deg_{G-F}(x) - 1$ vertices in $G - F$. It implies that each pair of vertices x and y are connected by $deg_{G-F}(x)$ vertex-disjoint fault-free paths with $|F| \leq n - 3$ according to Theorem 2. We now consider two cases:

Case I: *x and y are not adjacent.* We now show that x is still connected to y if the number of vertices deleted is smaller than $deg_{G-F}(x) - 1$. Suppose on the contrary that x and y are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{G-F}(x) - 1$. Since $deg_{G-F}(x) \leq deg(x) \leq n - 1$, we have $|V_f| \leq n - 2$. Then, we sum up the cardinality of these two sets F and V_f. Since $|F| \leq n - 3$ and $|V_f| \leq n - 2$, then $|F| + |V_f| \leq 2n - 5$. Let $T = F \cup V_f$. By Lemma 23, there exists a connected component C in $G - T$ such that $|V(C)| \geq n! - |T| - 1$ and $|T| \leq 2n - 5$. In other words, either $G - T$ is connected or $G - T$ has two components, one of which contains only one vertex. It contradicts to the assumption that x and y are disconnected if $G - T$ is connected. So we consider the case that $G - T$ has two components and one of which contains only one vertex. Since we assume that x and y are separated, one of x and y is the trivial graph. We without loss of generality let x be such trivial graph. Thus, the set V_f has to be the neighborhood of x and $|V_f| = deg_{G-F}(x)$, which is also a contradiction. Then, x is connected to y when the number of vertices deleted is smaller than $deg_{G-F}(x) - 1$ in $G - F$.

Case II: *x and y are adjacent.* We need to show that x is connected to y if the number of vertices deleted is smaller than $deg_{G-F}(x) - 2$ in $G - F - \{(x, y)\}$. For the sake of contradiction, suppose that x and y are separated by deleting a set of vertices V_f, where $|V_f| \leq deg_{G-F}(x) - 2$ in $G - F - \{(x, y)\}$. We get $|V_f| \leq n - 3$ since $deg_{G-F}(x) \leq n - 1$. Then the union set T of F and V_f has cardinality $|T_v| = |F| + |V_f| \leq 2n - 6$. Lemma 24 implies that there exists a connected component C in $G - T_v - \{(x, y)\}$ such that

$|V(C)| \geq n! - |T_v| - 1$ and $|T_v| \leq 2n-5$. It means that either $G - T_v - \{(x,y)\}$ is connected or that $G - T_v - \{(x,y)\}$ has two components one of which is trivial. For the first situation, $G - T_v - \{(x,y)\}$ is connected, this contradicts to the assumption that x and y are separated in $G - F - \{(x,y)\}$. For the second situation, let u be the trivial graph in $G - F - \{(x,y)\}$. One of x and y is the vertex u, say $x = u$. So the number of vertices deleted must be greater or equal to $deg_{G-F}(x) - 1 = n-2$. However, $|V_f| \leq n-3$, which is a contradiction. So x and y are connected when the number of vertices deleted is smaller than $deg_{G-F}(x) - 2$ in $G - F - \{(x,y)\}$.

The proof is complete. □

4.5.2 One-to-Many Maximal Local-connectivity

Now the one-to-many version of the maximally local-connectivity in Cayley graphs generated by transposition trees is given below.

Theorem 11 *Let G be a Cayley graph obtained from a transposition tree H on $\{1,2,...,n\}$ where $n \geq 3$. Then G is one-to-many $(n-3)$-fault-tolerant maximally local-connected.*

Proof. Let x be a vertex and U be a conditional terminal set in $G - F$ such that $t \leq deg_{G-F}(x)$ and $|U| = t$. We are going to show that x and U do not separate if the number of vertices deleted is smaller than $t-1$ in $G-F$, where $|F| \leq n-3$. Suppose not, let x and $U - V_f$ are separated by deleting a set of vertices V_f, where $|V_f| \leq t-1$. Since $t \leq deg_{G-F}(x) \leq deg(x) \leq n-1$, so $|V_f| \leq t-1 \leq n-2$. We now sum up the cardinality of these two sets F and V_f, which is $|F| + |V_f| \leq 2n-5$. Let $T = F \cup V_f$, By Lemma 23, there exists a connected component C in $G - T$ such that $|V(C)| \geq n! - |T| - 1$. Thus, either $G-T$ is connected, or $G-T$ has two components, one of which is contains only one vertex. If $G-T$ is connected, it contradicts to the assumption that x and $U-V_f$ are disconnected. So we consider that $G-T$ has two components and one of which has only one vertex u. If $x=u$, the set V_f has to be the neighborhood of x and $|V_f| = deg_{G-F}(x)$, this contradicts to the assumption $|V_F| \leq t-1 \leq deg_{G-F}(x)-1$. So $u \in U-V_f$. Since U is a conditional terminal set, $\{v\} \cup N_{G-F}(v)$ is not contained in U for each $v \in U$, either there are at least two remaining vertices of $U-V_f$, or the only one vertex in $U-V_f$ is not trivial in $G-T$, which contains a contradiction. Therefore, there is a set of t paths from x to U such that any two of them share only the vertex x when the number of vertices deleted is smaller than $deg_{G-F}(x)-1$ in $G-F$.

This completes the proof. $\square$

Chapter 5

Conclusion and Future Work

In this thesis, we study several properties with conditional fault on some interconnection networks. Under the restricted condition that each vertex has at least two fault-free adjacent vertices, we propose two optimal results. We first show that for any set of faulty edges F of an n-dimensional hypercube Q_n with $|F| \leq 2n-5$, each edge of the faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geq 3$. Then we study the Menger property. We prove that in all n-dimensional hypercube-like networks with $2n-5$ vertices removed, every pair of unremoved vertices u and v are connected by $min\{deg(u), deg(v)\}$ vertex-disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively.

The local connectivity of two vertices is defined as the maximum number of internally vertex-disjoint paths between them. The main concept of local connectivity can be referred as a one-to-one type of connectivity, since we only consider the maximum number of vertex-disjoint paths between two vertices. In classical theory, there is also a one-to-many version of connectivity. In this thesis, we extend this concept to a fault-tolerant version, called one-to-many f-fault-tolerant maximally locally connected property. Moreover, we introduce the one-to-many and many-to-many versions of connectivity. In these issues, we prove that a $(k+1)$-regular Matching Composition Network is $(k-1)$-maximally local-connected, and $(k-1)$-fault-tolerant one-to-many maximally local-connected. We also show that a $(k+1)$-regular Matching Composition Network is f-fault-tolerant many-to-many t-connected if $f+t=2k$. Besides, we show the similar result on Cayley graphs generated by transposition trees. We prove that an $(n-1)$-regular Cayley graph generated by transposition tree is maximally local-connected, even if there are at most $(n-3)$ faulty vertices in it, and prove that it is also $(n-1)$-fault-tolerant one-to-many maximally local-connected.

We show our result on the hypercube, hypercube-like, Matching Composition Network and the Cayley graphs generated by transposition trees. In addition to the graphs introduced in this thesis, there are other interesting graphs with different construction. In fact, many well-known systems may have some similar properties as defined in this thesis. We would like to extend our results to other graphs and hopefully to find more new properties.

Based on the generalized versions of edge-bipancyclicity and connectivity proposed in this thesis, the fault-tolerant capability may be increased if we add some restrictions on these networks. More precisely, if we add some conditions to the faulty vertices, the upper bound of fault-tolerance may possibly be increased. These are issues worth studying.

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