A Study on the Drawdown Solutions of Constant-head and

Constant-flux Tests

A Study on the Drawdown Solutions of Constant-head and
Constant-flux Tests

Student   Chih-Tse Wang

Advisor   Hund-Der Yeh

A Dissertation

Submitted to Institute of Environmental Engineering

College of Engineering

National Chiao Tung University

for the Degree of

Doctor of Philosophy

in Environmental Engineering

December, 2007

Hsinchu, Taiwan

Theis

Theis

Thiem

Thiem

(small p large t, SPLT)

Chen and Stone [1993]　　　　　SPLT

Thiem

Chen and Stone [1993]

Thiem　　　　　　　　　　　　　　Thiem

SPLT

A Study on the Drawdown Solutions of Constant-head and Constant-flux Tests

Student　Chih-Tse Wang　　　　　Advisor　Hund-Der Yeh

Institute of Environmental Engineering

National Chiao Tung University

Abstract

Theis equation is a non-equilibrium equation which can be used to predict the drawdown distribution during pumping or analyze drawdown data in determining the aquifer parameters. Physically, the aquifer drawdown changes with time at the early stage of pumping and approaches a constant value after a long period of pumping. However, the Theis equation can not reduce to Thiem equation mathematically when time approaches infinity. Also, the Thiem equation is not valid if the well radius approaches zero or the outer boundary goes to infinity. The main objectives of this dissertation are to derive the steady-state drawdown solution from the transient solutions of the constant-head and the constant-flux tests and to explain the use of mass balance concept in obtaining the steady-state solution. The result indicates that a flow system of a finite domain and a well of finite diameter are the necessary

conditions for obtaining the Thiem equation from the transient solutions. While

Thiem equation is employed, it implies that the problem has to be addressed within a

region instead of zero well radius and/or infinite outer boundary. In addition, the

dimensionless times criterion required to approximate the solutions of finite domain

by the infinite-domain solution and the Thiem equation are also presented. An

approximate solution is useful for practical applications if the corresponding

analytical solution is complicated and difficult to accurately evaluate. The second

objective of this dissertation is to examine the algorithm for obtaining a large-time

solution by using the Laplace transforms and the well-known "small p − large t"

relationship. In the past, the relationship was commonly applied to the Laplace

domain solution in developing a large-time solution in the groundwater area.

However, Chen and Stone [1993] pointed out that the use of this relationship might

fail to obtain a correct solution of the wellbore flux for the constant-head test problem.

This dissertation also shows that the relationship of small p versus large t is

appropriate for obtaining a large-time solution of the transient constant-head test

through the detailed mathematical development.

Keywords: Ground water; Steady state models; Large-time solution; Laplace

transforms; Well hydraulics.

206

Table of Contents

List of Tables

List of Figures

Notation

The following symbols are used in this dissertation:

I_0 = modified Bessel function of the first kind of order zero;

J_0 = Bessel function of the first kind of order zero;

J_1 = Bessel function of the first kind of order one;

K_0 = modified Bessel function of the second kind of order zero;

K_1 = modified Bessel function of the second kind of order one;

p = Laplace dummy variable;

q = $\sqrt{pS/T}$;

Q = wellbore flux for constant-head test or pumping rate for constant-flux test;

$\overline{Q}$ = Laplace domain wellbore flux or pumping rate;

r = radial coordinate;

r_w = well radius;

R = radius of influence;

s = drawdown;

$\overline{s}$ = Laplace domain drawdown;

s_w = drawdown in pumping well;

S = storativity of confined aquifer;

t = time;

T =   transmissivity of confined aquifer;

u = $r^2 S/(4Tt)$;

U = $R^2 S/(4Tt)$;

W =   well function;

Y_0 =   Bessel function of the second kind of order zero;

Y_1 =   Bessel function of the second kind of order one;

α_n =   roots of $J_0(r_w\alpha)Y_0(R\alpha) - Y_0(r_w\alpha)J_0(R\alpha) = 0$;

β_n =   roots of $J_1(r_w\beta)Y_0(R\beta) - Y_1(r_w\beta)J_0(R\beta) = 0$;

λ = $4T/(c^2 r_w^2 S)$ which $c = \exp(\gamma)$;

η = λt;

γ = $0.57722\ldots$ is the Euler's constant;

χ_n =   roots of Bessel function $J_0(\chi) = 0$; and

Φ =   coefficient used in Chen [1984].
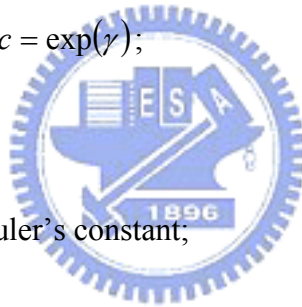
Chapter 1 Introduction

1.1 Statement of the problem

The spatial and temporal distributions of drawdown in an aquifer may change in response to a head change or pumping rate applied at the test well. The former holds a fixed drawdown in the test well and is referred to the constant-head test. The latter keeps a constant pumping rate in the test well and is referred to the constant-rate test. A given drawdown solution can be used to predict the drawdown distribution at any location and time if the aquifer parameters are known. It can also be used to determine the aquifer parameters if coupled an optimization approach in analyzing the observed drawdown data.
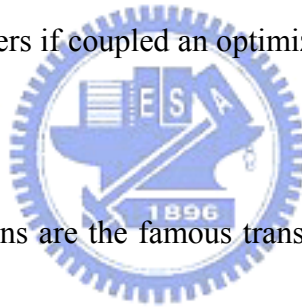
Theis and Thiem equations are the famous transient and steady-state drawdown solutions, respectively, of a constant-rate test conducted in confined aquifer. Both equations are easily used to predict the aquifer drawdown distribution in practical application. Theis equation is derived under the conditions of having an infinite extent confined aquifer and neglecting the effect of well radius. It, therefore, is not valid if the aquifer has a finite boundary. In addition, the Theis equation will give an infinite drawdown solution if the time approaches infinity or the radius approaches zero. Chen [1984] proposed a modified Theis equation for drawdown distribution in a finite confined aquifer and gave a time criterion when applying the Theis solution.

Thiem equation, the steady-state solution, can be derived either from the continuity equation coupled with Darcy's law [e.g., Todd and Mays, 2005] or the radial steady-state flow equation [e.g., Charbeneau, 2000]. Many researches have discussed the problem of steady-state flow and gave a warning of erroneous result when employing Thiem equation to problems of an infinite aquifer [e.g., Bear, 1979]. Zaadnoordijk [1998] proposed a superposition algorithm, includes Thiem equation and Theis equation, to simulate the transition between two given steady-state groundwater fields. His study indicates that the steady-state condition is directly related to the effects of well radius and finite boundary. However, the issue of deriving the Thiem equation directly from the transient drawdown solutions has never been addressed before.
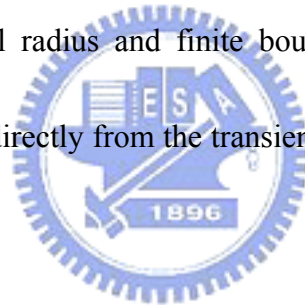
Contrary to the constant-flux test, the constant-head aquifer test is suitable in determining the hydraulic parameters of low-permeability aquifer. This test maintains a constant head on the pumping/injection well throughout the test period, and the change of flow rate across the wellbore versus time is recorded [e.g., Batu, 1998]. The head distribution for constant-head test in a confined aquifer is analogous to the heat distribution for constant temperature maintained at a bounded circular cylinder; thus, the head solutions in both Laplace domain and time domain can be obtained from Carslaw and Jaeger [1959]. The wellbore flux can then be

derived based on the solution of head distribution and Darcy's law [e.g., Yang and Yeh, 2002; Peng et al., 2002]. However, the time domain solution of the wellbore flux is complex and difficult to accurately evaluate. Therefore, it was common to derive the approximate solutions for small or large value of the time for the wellbore flux [e.g., Carslaw and Jaeger, 1959, p.336].
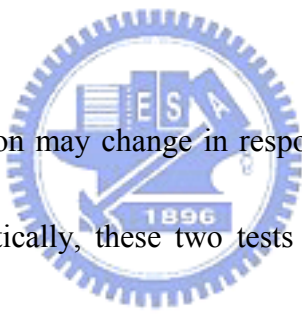
One way for obtaining the small- or large-time solution is to apply the relationship of large p versus small t (hereinafter referred to as LPST) or small p versus large t (hereinafter referred to as SPLT) to the Laplace domain solution, respectively. This concept is based on a symbolic relation between the derivative operator of time, i.e., d/dt, in the time domain and the dummy variable, p, in the Laplace domain [van Everdingen and Hurst, 1949]. Then, one may obtain a small- or large-time solution from taking the inverse Laplace transform on the reduced Laplace domain solution for a large or a small dummy variable, respectively. Some of the successful illustrations of applying this concept can be found in the groundwater literature. van Everdingen and Hurst [1949] used the relationships of LPST and SPLT to derive the pressure head of groundwater flow in a reservoir at small and large time, respectively. Accounting for the aquitard storage for the flow in a leaky aquifer system, Hantush [1960] obtained the small- and large-time drawdown solutions by applying those two relationships. Neuman and Witherspoon

[1969] studied the problem for flow in a confined two-aquifer system by considering the aquitard storage and drawdown in the unpumped aquifer. They used the LPST relationship to obtain the small-time solution for their problem. Singh and Sagar [1980] proposed approximate solutions of head to the linearized flow equation of slightly compressible fluids by using the relationships of LPST and SPLT. Javandel and Witherspoon [1983] and Butler and Liu [1993] provided a large-time solution for pumping-induced drawdown in a vertical and horizontal nonuniform aquifer, respectively, based on the SPLT relationship. Chakrabarty et al. [1993] provided a nonlinear pressure distribution of compressible liquid and the corresponding small- and large-time solutions obtained in a homogeneous formation using the relationships of LPST and SPLT, respectively. In addition, a number of approximate solutions at small and/or large time were derived based on these relationships in areas such as the solute transport problem [van Genuchten et al., 1984; Chen, 1985; 1986; Yates, 1990], dual-porosity media problem [Barker, 1985], and unsteady infiltration problem [Philip, 1986].

However, Chen and Stone [1993] presented a calculation of the flow rate across the wellbore for the constant-head test problem and concluded that the SPLT relationship might fail to obtain a correct large-time solution in this case. Later, Mathias and Zimmerman [2003] indicated that a poor result was obtained by Gerke

and van Genuchten [1993] when using the relationship of small *p* versus large *t* to

derive the water transfer coefficient for the dual-porosity media problem.   The

relationships of LPST and SPLT have been widely used in the groundwater literature

for more than 40 years.   Previous researches indicated that the SPLT relationship

could successfully yield correct large-time solutions if applied to the related Laplace

domain solutions.   The contrary results of Chen and Stone [1993] and Mathias and

Zimmerman [2003], therefore, may depress the further study on this issue.

1.2 Objectives

The drawdown distribution may change in response to the constant-head or the

constant-flux test.   Mathematically, these two tests can be formulated as different

types of boundary value problem.   With or without considering the effect of well

radius, many studies have been devoted to developing analytical solutions for those

problems under various boundary conditions. The goal of this dissertation is to

investigate the problems regarding to the drawdown solutions of constant-head and

constant-flux tests in a homogeneous confined aquifer.   Toward this goal the

dissertation organization is as follows:

(1) To examine the drawdown solutions of constant-head and constant-flux tests in a

finite or infinite confined aquifer with or without the consideration of the effects of

well radius;

(2) To verify the drawdown solutions of these two tests in a finite domain converging to the Thiem equation after a long period of pumping;

(3) To present dimensionless time criteria required to approximate the solutions of finite domain by the infinite-domain solution or the Thiem equation;

(4) To resolve the dispute on the validity of the SPLT relationship raised by Chen and Stones [1993] and Mathias and Zimmerman [2003] through the detailed mathematical development; and

(5) To derive a correct large-time solution of the wellbore flux rate for the constant-head test based on the Laplace domain solution and the SPLT relationship.

Chapter 2 Drawdown Solutions for Constant-head Test

This chapter considers the constant-head test conducted in a homogeneous, isotropic confined aquifer of constant thickness as shown in Figure 1. The one-dimensional radial flow equation describing the drawdown in a confined aquifer can be written as [Batu, 1998, p.147]

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t} \tag{1}$$

where $s(r,t)$ is the observable drawdown corresponding to the radial distance r from the test well and the time variable t, S is the storativity, and T is the transmissivity. The drawdown is initially assumed zero before an aquifer test, i.e., $s(r,0) = 0$.
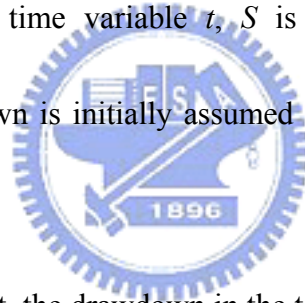
For the constant-head test, the drawdown in the test well is maintained a constant and denoted as s_w. The rim of the wellbore is selected as the inner boundary and the inner boundary condition for the drawdown is then denoted as $s(r_w,t) = s_w$ where r_w is the well radius. An outer boundary condition should be provided for solving the flow equation, Equation (1). Hereinafter, this study will present and discuss the solution for the outer boundary specified as a zero drawdown and located at either an infinite or a finite distance from the test well. In addition, the drawdown solution derived by considering or neglecting the effect of the well radius will also be

7

presented.

2.1 Infinite domain with a finite well radius

Analytical solution

It is assumed that the outer boundary is located at infinity and the outer boundary condition is expressed as $s(\infty, t) = 0$. The solution subject to the infinite domain assumption is referred to as an infinite-domain solution. By using the method of Laplace transforms, the general Laplace domain solution of Equation (1) can be obtained in terms of bases I_0 and K_0, which are modified Bessel functions of the first and second kinds of order zero, respectively. The function I_0 tends to infinity under the outer boundary condition and therefore must be excluded.

Application of the inverse Laplace transform under the inner boundary condition yields the Laplace domain and time domain drawdown distribution for the constant-head test, respectively, as [Carslaw and Jaeger, 1959, p.335]

$$\overline{s}(r,p) = \frac{s_w}{p} \frac{K_0(qr)}{K_0(qr_w)} \tag{2}$$

and
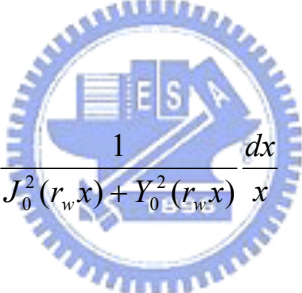
$$s(r,t) = s_w \left[1 - \frac{2}{\pi} \int_0^\infty \exp\left(-\frac{T}{S} x^2 t \right) \frac{Y_0(rx)J_0(r_w x) - J_0(rx)Y_0(r_w x)}{J_0^2(r_w x) + Y_0^2(r_w x)} \frac{dx}{x} \right] \tag{3}$$

where p is the Laplace variable; $q = \sqrt{pS/T}$; K_1 is modified Bessel function of the second kind of order one; and J_0 and Y_0 are the Bessel functions of the first and second kinds of order zero, respectively.

The flow rate across the wellbore at the test well, therefore, can be obtained by applying Darcy's law to Equations (2) and (3). Accordingly, the wellbore fluxes for Laplace domain and time domain are, respectively,

$$\overline{Q}(r_w, p) = 2\pi r_w T \frac{s_w}{p} \frac{q K_1(q r_w)}{K_0(q r_w)} \tag{4}$$

and

$$Q(r_w, t) = \frac{8 T s_w}{\pi} \int_0^\infty \exp\left(-\frac{T}{S} x^2 t\right) \frac{1}{J_0^2(r_w x) + Y_0^2(r_w x)} \frac{dx}{x} \tag{5}$$

Note that the negative sign of Q corresponds to withdrawal and the positive sign corresponds to injection.

Because the exponential term on the right-hand side of Equation (3) reduces to zero as time approaches infinity; thus, the steady-state drawdown is obtained as s_w. This result implies that the steady-state drawdown within the entire domain is kept the same as the fixed drawdown at test well. Also, Equation (5) shows that the wellbore flux decreases with the increasing time and approaches zero at the large time rather different from the Thiem equation. These results suggest that under the infinite outer

boundary condition there is not enough water to maintain a constant non-zero flow rate within the aquifer for the constant-head test. The solid line in Figure2 demonstrates the plot of dimensionless wellbore flux of Equation (5), $Q(r_w,t)/2\pi T s_w$, versus dimensionless time, $Tt/r_w^2 S$. It can be observed that the wellbore flux approaches zero as time becomes infinitely large.

__Approximate solution at large time__

The time domain solution of the wellbore flux, Equation (5), is complicate and difficult to evaluate accurately due to the problems of alternately oscillatory, slowly convergence, and a singularity at the origin [Peng et al., 2002]. Therefore, it is common to derive a large-time approximate solution of the wellbore flux by using the SPLT relationship. However, Chen and Stone [1993] and Mathias and Zimmerman [2003] indicate that erroneous results were obtained when applying the SPLT relationship. Based on SPLT, the approximate solution for large-time drawdown distribution of constant-head test under the assumptions of an infinite domain and a well of a finite radius can be derived successfully and is presented as follows. In addition, a detailed mathematical derivation using SPLT approach for deriving the water transfer coefficient involved in Mathias and Zimmerman [2003] is shown in Appendix A.

Using the limiting forms of $K_0(x) \cong -[\ln(x/2) + \gamma]$ and $K_1(x) \cong 1/x$, where

$\gamma = 0.57722\ldots$ is Euler's constant, for small value of x [Abramowitz and Stegun,

1970, p.375], the wellbore flux for small p, Equation (4), in Laplace domain can be

reduced to

$$\overline{Q}(r_w, p) \cong -4\pi T s_w \frac{1}{p \ln(p/\lambda)} \tag{6}$$

where $\lambda = 4T/(c^2 r_w^2 S)$ and $c = \exp(\gamma)$. The large-time wellbore flux is

subsequently obtained by taking the inverse Laplace transform of Equation (6).

Chen and Stone [1993] used a inverse Laplace transform formula given by the

Oberhettinger and Badii [1973] (formula 6.75, p.276 and 424) for the inverse Laplace

transform of term $1/[p \ln(p/\lambda)]$ in Equation (6) as

$$L^{-1}\left[\frac{1}{p \ln(p/\lambda)}\right] = \int_0^\infty \frac{(\lambda\, t)^x}{\Gamma(x+1)} dx \tag{7}$$

where $\Gamma(x)$ is the gamma function. They integrated Equation (7) and obtained the

large-time wellbore flux as $Q(r_w, t) = -\infty$. In addition, they showed that this

large-time wellbore flux was contradictory to the result obtained by applying the

Tauberian theorem (also called final-value theorem [Spiegel, 1965]) to Equation (6),

i.e., $Q(r_w, t \to \infty) = 0$. Based on their study, they concluded that the application of

the SPLT relationship should be used with care because of possible failure to obtain a

correct large-time solution.

In fact, the use of inverse Laplace transform formula of Equation (7) should be under a constraint of $p > \lambda$ which is proven in the Appendix A. For most confined aquifers, the value of the storativity falls in the range $10^{-5} \sim 10^{-3}$ and for sand and silt formations the value of hydraulic conductivity falls in the range of $10^{-2} \sim 10^{1}$ m/day [Todd and Mays, 2005, p.93]. Assuming that the thickness of confined aquifer is 10 m and the radius of the test well is 5 cm; then, the value of λ ranges from 10^{4}/day to 10^{9}/day. Notice that the Laplace variable p is required to be small in Equation (6) and λ should be small too. Yet, the value of λ is larger than 10^{4}/day in reality as demonstrated above; accordingly, Equation (7) does not hold at all.

Hereafter, this study proposes an alternative formula for the inverse Laplace transform of term $1/[p\ln(p/\lambda)]$ in Equation (6). Ritchie and Sakakura [1956] presented an article on expanding the solutions of the heat conduction equation in terms of an infinite series approximately in an internally bounded cylindrical solid. They gave the inverse Laplace transform for the term $1/[p\ln(p/\lambda)]$ when p is small as

$$L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = \sum_{s=0}^{N}\left[\left(\frac{-1}{\ln\eta}\right)^{s+1}\binom{-1}{s}\frac{d^s}{dv^s}\left(\frac{1}{\Gamma(1-v)}\right)\Bigg|_{v=0}\right] \qquad (8)$$

where the dimensionless variable $\eta = \lambda t$, column vector $(-1, s)^T$ is the binomial

coefficient, and *N* is the number of truncated terms depending on the values of the remainder.

The right hand side (RHS) of Equation (8) is a summation of products of the dimensionless variable $\ln \eta$ and the constant value of the Gamma function. For a large value of time, the terms of $1/(\ln \eta)^5$ and higher order terms may be truncated since η is proportional to the dimensionless time. Therefore, the large-time wellbore flux can be obtained based on Equations (6) and (8). Appendix B shows the detailed expansion of Equation (8) and the large-time solution for the wellbore flux is obtained approximately as

$$Q(r_w,t) \cong 4\pi T s_w \left[\frac{1}{\ln \eta} - \frac{\gamma}{(\ln \eta)^2} + \frac{\gamma^2 - \frac{\pi^2}{6}}{(\ln \eta)^3} - \frac{\frac{\pi^2}{2}\gamma + 2\xi(3)}{(\ln \eta)^4} \right] \qquad (9)$$

The numerators of the RHS terms of Equation (9) are all constants and the denominators are the function of dimensionless time, $\eta = 4Tt/(c^2 r_w^2 S)$. The value of $\ln \eta$ reaches infinity if *t* approaches infinity; therefore, Equation (9) becomes zero, i.e., the steady-state solution for the wellbore flux is zero. Notice that Jaeger [1943] and Carslaw and Jaeger [1959, p.336] gave a large-time wellbore flux which has the first three terms and first two terms of Equation (9), respectively.

2.2 Infinite domain with neglecting the well radius

For the case of neglecting the effect of test well radius, that is, $r_w \to 0$, the basis K_0 of the general Laplace domain solution tends to infinity and therefore there is no solution for this case.   The transient and steady-state solutions of the drawdown for the constant-head test with and without considering the effect of well radius under the infinite domain assumption are listed in Table 1.

2.3 Finite domain with a finite well radius

In this section, a well of a finite radius is considered and a finite distance, *R,* is selected to represent the outer boundary if the drawdown beyond *R* is zero or negligible.   Generally speaking, the radius of influence which can be estimated by an empirical relationship [Bear, 1979, p.306] is finite.   Thus, the solution under a finite outer boundary condition denoted as $s(R,t)=0$ is referred to as a finite-domain solution.   Carslaw and Jaeger [1959, p.332] gave the solution for this problem under Cauchy boundary conditions.   By assuming the flux of the Cauchy boundary being equal to zero, a new drawdown solution for an aquifer of a finite domain under the assumption of a finite well radius can be derived as

$$s(r,t) = s_w \left[\frac{\ln(R/r)}{\ln(R/r_w)} - \pi \sum_{n=1}^{\infty} \exp\left(-\frac{T}{S}\alpha_n^2 t\right) \frac{J_0(r_w\alpha_n)Y_0(r\alpha_n) - Y_0(r_w\alpha_n)J_0(r\alpha_n)}{\left[J_0^2(r_w\alpha_n) - J_0^2(R\alpha_n)\right]/J_0^2(R\alpha_n)} \right] \quad (10)$$

where α_n are the roots of $J_0(r_w\alpha)Y_0(R\alpha) - Y_0(r_w\alpha)J_0(R\alpha) = 0$. A detailed derivation for Equation (10) is given in Appendix C. The flow rate at wellbore can then be determined as

$$Q(r_w, t) = 2\pi T s_w \left[\ln\frac{r_w}{R} + 4\sum_{n=1}^{\infty} \frac{\exp(-T\alpha_n^2 t/S)}{\left[J_0^2(r_w\alpha_n) - J_0^2(R\alpha_n)\right]/J_0^2(R\alpha_n)} \right] \qquad (11)$$

Obviously, the steady-state wellbore flux of Equation (11) for time approaching infinity is $Q(r_w) = 2\pi T s_w / \ln(R/r_w)$ which indeed is the Thiem equation. Figure 2 shows that the curve of Equation (11), i.e., the finite-domain solutions, coincides with that of Equation (9), i.e., the infinite-domain solutions, at the early stage of the pumping test and asymptotically approaches Thiem equation after a long period of time. In other words, the infinite-domain solution can approximate the finite-domain solution when time is less than the boundary-effect time criterion, denoted as t_1, implying that the finite boundary has no affect on the wellbore flux. In addition, the finite-domain solution can be reduced to Thiem equation when time is larger than the steady-state time criterion, denoted as t_2, implying that the wellbore flux can be considered at the steady state. It is worth noting that both the Thiem equation and the infinite-domain solution has the advantage of computing the drawdown solution more easily over the finite-domain solution.

Figure 2 indicates that both t_1 and t_2 increase with the dimensionless distance

R/r_w .   The time criteria of t_1 and t_2 can be obtained as $R^2 S/10T$ and $4R^2 S/T$,

respectively, for the constant-head test where the absolute difference of dimensionless

wellbore flux between Equations (9) and (11) is about less than 10^{-5} .   Table 2 gives

a list of time criteria for the finite-domain solution.

2.4 Finite domain with neglecting the well radius

In this section, the outer boundary of radial flow system is located at finite

distance from the test well and the inner boundary of well radius is negligible.   Note

that I_0 and K_0 are the bases of the general Laplace domain drawdown solution for

the constant-head test as mentioned in section 2.1.   Therefore, the base K_0 has to

be excluded from the general solution as $r_w \to 0$ and the other base I_0 can not

satisfy the boundary conditions of fixed drawdown at wellbore and zero drawdown at

the finite distance simultaneously.   Therefore, there is no solution for this case.

Table 1 also lists the transient and steady-state drawdown solutions of the

constant-head test when the outer boundary is finite with and without considering the

effect of well radius.

Chapter 3 Drawdown Solutions for the Constant-flux Test

This chapter considers a constant-flux test conducted at a confine aquifer. Equation (1) is the one-dimensional radial flow equation describing the drawdown distribution. The initial condition for the drawdown before pumping is assumed zero. The drawdown solutions are presented in the following sections for the cases of the outer boundary of zero drawdown located at either an infinite or a finite domain with or without considering the effect of the inner boundary of well radius.

3.1 Infinite domain with a finite well radius

The constant-flux test at a constant pumping rate, says Q, is conducted at the test well. Thus, the inner boundary condition for the constant-flux test may be written as

$$\left.\frac{\partial s}{\partial r}\right|_{r=r_w} = -\frac{Q}{2\pi r_w T} \tag{12}$$

The drawdown solution for the constant-flux test subject to Equation (12) and zero drawdown at infinity can be obtained by the application of the Laplace transforms as [Carslaw and Jaeger, 1959, p.338]

$$s(r,t) = \frac{Q}{\pi^2 r_w T}\int_0^\infty \left[1-\exp\left(-\frac{T}{S}x^2 t\right)\right]\frac{Y_0(rx)J_1(r_w x)-J_0(rx)Y_1(r_w x)}{J_1^2(r_w x)+Y_1^2(r_w x)}\frac{dx}{x^2} \tag{13}$$

where J_1 and Y_1 are the Bessel functions of the first and second kinds of order one, respectively. Yeh et al. [2003] presented a closed-form solution with detailed

17

numerical evaluations for a radial two-zone drawdown equation for groundwater under constant-flux pumping in a finite-radius well. Their solution reduces to Equation (13) if the transmissivities and storativities for the skin zone and formation zone are the same.

Figure 3 gives a graphical representation of dimensionless drawdown of Equation (13) for the dimensionless time ranging form 10^2 to 10^9. The plot of drawdown distribution demonstrated that the drawdown increases with time and diverges as time approaches infinity. It implies that there is not enough water at the outer boundary to balance the continuous well pumping and the aquifer is overdrawdown when time goes infinitely large. This result indicates that the steady-state drawdown solution of the constant-flux test in an infinite domain by considering the effect of the well radius does not reduce to the Thiem equation.

3.2 Infinite domain with neglecting the well radius

The solution for the constant-flux test neglecting the effect of well radius conducted in an infinite domain is analyzed in this section. By applying the Laplace transforms and the asymptotic form of modified Bessel function [Abramowitz and Stegun, 1970], one obtains the drawdown equation as

$$s(r,t) = \frac{Q}{4\pi T} \int_u^\infty \frac{\exp(-x)}{x} dx \qquad (14)$$

where $u = r^2 S/(4T t)$ is inversely proportional to time.

Equation (14) is the famous Theis equation. The integration in Equation (14) is called the well function and expressed as $W(u)$ which tends to infinity when time approaches infinity. Thus the drawdown of the Theis equation becomes infinity. In addition, Cooper-Jacob's solution, a special case of the Theis equation under the condition of $u < 0.01$, should also give an infinite drawdown solution when u approaches zero.

The transient and steady-state drawdown solutions for a constant-flux test conducted in an aquifer with an infinite outer boundary are compared in Table 1. The results for the constant-head and constant-flux tests indicate that these two tests in an infinite domain do not have steady-state solutions; thus, Equations (3), (13), and Theis equation can not reduce to the Thiem equation.

3.3 Finite domain with a finite well radius

Similar to the development of Equation (10), the drawdown solution for Equation (1) subject to Equation (12) and a zero drawdown at finite distance can be obtained as [Carslaw and Jaeger, 1959, p.334]

$$s(r,t) = \frac{Q}{2\pi T}\left[\ln\frac{R}{r} - \frac{\pi}{r_w}\sum_{n=1}^{\infty}\exp\left(-\frac{T}{S}\beta_n^2 t\right)\frac{J_1(r_w\beta_n)Y_0(r\beta_n) - Y_1(r_w\beta_n)J_0(r\beta_n)}{\beta_n\left[J_1^2(r_w\beta_n) - J_0^2(R\beta_n)\right]/J_0^2(R\beta_n)} \right] \quad (15)$$

where β_n represent the roots of $J_1(r_w\beta)Y_0(R\beta) - Y_1(r_w\beta)J_0(R\beta) = 0$. The steady-state drawdown solution can be easily obtained by setting the time to be infinity in Equation (15) which is indeed the Thiem equation if the distance r is equal to the well radius.

Figure 3 shows the distribution of dimensionless drawdown of Equation (15) versus dimensionless time for R/r_w equal to 10^3 and 10^4 with $r/r_w = 10^2$. It also demonstrates that the curve of Equation (15), i.e., finite-domain solutions, can be approximated by that of Equation (13), i.e., infinite-domain solutions, at early period of time and reduce to Thiem equation at a large period of time. The former indicates the drawdown solution can be treated as that for an infinite aquifer before the boundary effect acts. The latter implies that the drawdown almost reaches its steady-state solution and the Thiem equation is applicable if the time is larger than the steady-state time criterion. When the absolute difference of dimensionless drawdown is less than 10^{-5}, the dimensional time criteria of t_1 and t_2 for the constant-flux test considering well radius are $3R^2S/100T$ and $8R^2S/T$, respectively, as shown in Table 2.

3.4 Finite domain with neglecting well radius

In this section, the outer boundary is some finite distance away and the well radius is negligible. Chen [1984] obtained the drawdown solution for this case as

$$s(r,t) = \frac{Q}{4\pi T}\left[W(u) - W(U) + 2\Phi\right] \tag{16}$$

and

$$\Phi = \sum_{n=1}^{\infty} \frac{J_0\left[(u/U)^{1/2}\chi_n\right]}{\chi_n J_1(\chi_n)} \cdot \int_0^1 \exp\left[\frac{-U}{x} - \frac{\chi_n^2(1-x)}{4U}\right]\frac{dx}{x} \tag{17}$$

where U is $R^2 S/(4Tt)$ and χ_n represents the roots of Bessel function $J_0(\chi_n) = 0$.

Both the arguments u and U in Equation (11) are inversely proportional to time. The functions $W(u)$ and $W(U)$ can be expressed in terms of an infinite series and approximated as $-(0.5772 + \ln u)$ and $-(0.5772 + \ln U)$, respectively, as $u \to 0$ and $U \to 0$ after a large period of time [Todd and Mays, 2005, p.167]. The difference of $W(u)$ and $W(U)$ reduces to $2\ln(R/r)$ where r is a finite distance from the test well. Moreover, the exponential term in the integrand of Equation (17) approaches zero as $U \to 0$ and the value of Φ is negligible. As a result, the drawdown of Equation (16) approaches Thiem equation when time approaches infinity. The results of transient and steady-state drawdown solutions of the constant-flux test under a finite outer boundary condition by considering or neglecting

the effect of well radius are also shown in Table 1.

Again, Equation (16), i.e., a finite-domain solution, can be approximated by the Theis equation, Equation (14), when $U \geq 4$ [Chen, 1984] and reduce to the Thiem equation when $U \leq 10^{-5}$ if the absolute difference of dimensionless drawdown is less than 10^{-5}. The time criteria of t_1 and t_2 for the constant-flux test by neglecting the effect of well radius are $R^2 S/16T$ and $10^5 R^2 S/4T$, respectively, as shown in Table 2.

Chapter 4 Conclusions

The uses of the drawdown solution in predicting the drawdown distribution or determining the formation properties have been intensively studied in hydrology literature. Two well-known tests, the constant-head and constant-flux tests, are routinely used toward those goals. However, the derivations from the transient-state solution to the Thiem equation and the approximate solution at large period of time obtained from Laplace domain solution have not ever been thoroughly or correctly studied. Two main contributions from this dissertation can be summarized as follows.

First, the steady-state solutions developed from transient drawdown solutions for constant-head and constant-flux tests in a finite or infinite domain and with or without considering the effect of well radius have been presented. A new transient drawdown solution is derived for the constant-head test by considering the effect of well radius and maintaining a zero drawdown at a finite boundary. The results show that the finite domain condition is necessary for obtaining a steady-state solution from a transient solution for a groundwater flow problem. Such an imposed condition ensures that the mass balance is satisfied and the flow can reach steady state condition within a bounded domain. In addition, the time criteria are provided for the approximation of the finite-domain solution by the infinite-domain solution or Thiem

equation. The infinite-domain solutions can be used to determine the drawdown distribution or the aquifer parameters if the time is smaller than the boundary-effect time criterion. Similarly, the Thiem equation is valid if the time is greater than the steady-state time criterion.

Second, in regard to the large-time solution of wellbore flux in constant-head problem, this study found that the inverse Laplace transform formula in Oberhettinger and Badii [1973] adopted to invert Laplace domain solution by Chen and Stone [1993] should be under the constraint $p > \lambda$. One will obtain an erroneous solution if applying that inverse Laplace transform formula without satisfying this necessary constraint. Therefore, the inconsistent results obtained by Chen and Stone [1993] from the SPLT relationship and the Tauberian theorem that arose from a violation of the constraint from applying the inverse Laplace transform formula rather than using the SPLT relationship. In addition, a new large-time solution of the wellbore flux for the constant-head test is presented based on the SPLT relationship and the work of Ritchie and Sakakura [1956].

References

Abramowitz, M., and I.A. Stegun (1970), *Handbook of Mathematical Functions*, Dover Publications, New York.

Barker, J.A. (1985), Block-geometry functions characterizing transport in densely fissured media, *J. Hydrol.*, *77*, 263-279.

Batu, V. (1998), *Aquifer Hydraulics*, John Wiley, New York.

Bear J. (1979), *Hydraulics of groundwater*, McGraw-Hill, New York.

Butler, J.J., Jr., and W. Liu (1993), Pumping tests in nonuniform aquifers: The radially asymmetric case, *Water Resour. Res., 29*, 259-269.

Carslaw, H.S., and J.C. Jaeger (1959), *Conduction of Heat in Solids*, 2nd ed., Oxford University Press, London.

Charbeneau R.J. (2000), *Groundwater hydraulics and pollutant transport*, Prentice Hall, New York.

Chakrabarty, C., S.M.F. Ali, and W.S. Tortike (1993), Analytical solutions for radial pressure distribution including the effects of the quadratic-gradient term, *Water Resour. Res., 29*, 1171-1177.

Chen C.S. (1984), A reinvestigation of the analytical solution for drawdown distributions in a finite confined aquifer, *Water Resour. Res., 20*, 1466-1468.

Chen, C.S. (1985), Analytical and approximate solutions to radial dispersion from an

injection well to a geological unit with simultaneous diffusion into adjacent strata, *Water Resour. Res., 21*, 1069-1076.

Chen, C.S. (1986), Solutions for radionuclide transport from an injection well into a single fracture in a porous formation, *Water Resour. Res., 22*, 508-518.

Chen, C.S., and W.D. Stone (1993), Asymptotic calculation of Laplace inverse in analytical solutions of groundwater problems, *Water Resour. Res., 29*, 207-210.

Crank, J. (1956), Mathematics of Diffusion, Oxford Univ. Press, New York.

Doetsch, G. (1961), Guide to the Applications of Laplace Transforms, Van Nostrand Reinhold, New York.

Dykhuizen, R.C. (1990), A new coupling term for dual-porosity models, *Water Resour. Res., 26*, 351-356.
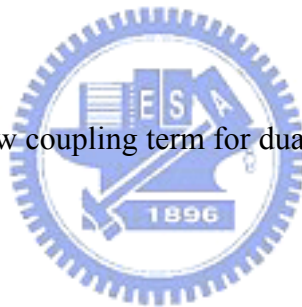
Gerke, H.H., and M.T. van Genuchten (1993), Evaluation of first-older water transfer term for variably saturated dual-porosity flow modes, *Water Resour. Res., 29*, 1225-1238.

Hantush, M.S. (1960), Modification of the theory of leaky aquifers, *J. Geophys. Res., 65*, 3713-3725.

Jaeger, J.C. (1943), Heat flow in the region bounded internally by a circular cylinder, *Proc. Roy. Soc. Edinburgh*, *61*, 223-229.

Javandel I., and P.A. Witherspoon (1983), Analytical solution of a partially

penetrating well in a two-layer aquifer, *Water Resour. Res., 19*, 567-578.

Mathias, S.A., and R.W. Zimmernan (2003), Laplace transform inversion for late-time behavior of groundwater flow problems, *Water Resour. Res., 39*, 1283, doi: 10.1029/2003WR002246.

Neuman, S.P., and P.A. Witherspoon (1969), Theory of flow in a confined two-aquifer system, *Water Resour. Res., 5*, 803-817.

Oberhettinger, F., and L. Badii (1973), *Tables of Laplace Transforms*, Springer-Verlag, New York.

Peng, H. Y., H. D. Yeh, and S. Y. Yang (2002), Improved numerical evaluation for the radial groundwater flow equation, *Adv. Water Resour., 25*, 663-675.

Philip, J. R. (1986), Linearized unsteady multidimensional infiltration, *Water Resour. Res., 22*, 1717-1727.

Ritchie, R.H., and A.Y. Sakakura (1956), Asymptotic expansions of solutions of the heat conduction equation in internally bounded cylindrical geometry, *Appl. Phys., 27*, 1453-1459.

Singh, S.R. and B. Sagar (1980), On Jacob's approximation in flow through porous media, *Water Resour. Res., 16*, 377-380.

Spiegel, M.R. (1965), *Theory and Problems of Laplace Transforms*, Schaum Pub. Co., New York.

Todd, D.K., and L.M. Mays (2005), *Groundwater Hydrology*, John Wiely & Sons.

van Everdingen, A.F., and W. Hurst (1949), The application of the Laplace transformation to flow problems in reservoirs, *Trans. Am. Inst. Min. Metall. Eng., 186,* 305-324.

van Genuchten, M.T., D.H. Tang, and R. Guennelon (1984), Some exact solutions for solute transport through soils containing large cylindrical macrospores, *Water Resour. Res., 20*, 335-346.

Yang, S.Y., and H.D. Yeh (2002), Solution for flow rates across the wellbore in a two-zone confined aquifer, *J. Hydraulic Eng. ASCE, 128*, 175-183.

Yates, S.R. (1990), An analytical solution for one-dimensional transport in heterogeneous porous media, *Water Resour. Res., 26*, 2331-2338.

Yeh H.D., S.Y. Yang, H.Y. Peng (2003), A new closed-form solution for a radial two-layer drawdown equation for groundwater under constant-flux pumping in a finite-radius well, *Adv. Water Resour., 26*, 747-757.

Zaadnoordijk W.J. (1998), Transition from transient Theis wells to steady Thiem wells, *Hydrological Sciences J, 43*, 859-873.

Appendixes

Appendix A: Examination of SPLT for dual-pore problem
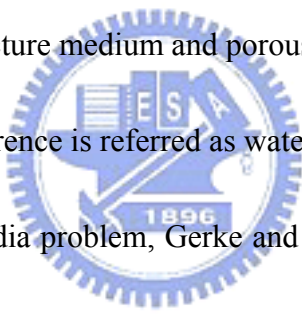
Appendix B: Derivation of Equation (7)

Appendix C: Derivation of Equation (9)

Appendix D: Derivation of Equation (10)

Appendix A: Examination of SPLT for dual-pore problem

A fractured aquifer consisted of fractures, fissures, cracks, and macropores can be simulated as two separate but connected medium, one is associated with fracture medium and the other involves porous matrix. Such a fractured aquifer system is called dual-porosity system. Mathematically, a dual-porosity system involves two flow equations which are coupled by means of a transfer term of water flow [Dykhuizen, 1990]. Gerke and van Genuchten [1993] proposed a first-order model to account this transfer term which is assumed to be proportional to the difference in pressure head between the fracture medium and porous matrix. The specific value of water transferred to head difference is referred as water transfer coefficient.

For the dual-porosity media problem, Gerke and van Genuchten [1993] solved a large-time water transfer coefficient by comparing two Laplace domain solutions, one was the "first-order" flux equation between fracture and matrix and the other was Richards's equation within the matrix block, while p became small. Moreover, Mathias and Zimmerman [2003] also obtained a large-time water transfer coefficient for the dual-porosity media problem based on the time domain approach. Their solution differs from the one obtained by Gerke and van Genuchten [1993] using the Laplace domain approach based on the SPLT relationship. When comparing both Laplace- and time domain solutions with the exact solution, they implicitly indicated

that the SPLT relationship did not hold.   Therefore, the following context will go

through the detailed mathematical derivations involved in the issues of Gerke and van

Genuchten [1993] and resolve the argument on the validity of the SPLT relationship.

If the pressure head of the fracture pore is considered to be a constant in time, the

water transfer coefficient can be obtained by comparing two rearranged flow

equations in Laplace domain as [Gerke and van Genuchten, 1993]

$$\overline{h_m}(p) = \left(\frac{h_f - h_{m,i}}{p} \right) \frac{\tanh(\xi)}{\xi} + \frac{h_{m,i}}{p} \tag{A1}$$

and

$$\overline{h_m}(p) = \frac{h_f}{p} \left(\frac{1}{1+\zeta} \right) + \frac{h_{m,i}}{p} \left(\frac{1}{1+\zeta^{-1}} \right) \tag{A2}$$

where $\overline{h_m}$ and $h_{m,i}$ are the Laplace domain head and the initial head, respectively,

in the matrix block; h_f is the imposed head at the fracture boundary; and variables

$\xi = a\big[(1-w_f)C_m p / K_a\big]^{0.5}$ and $\zeta = (1-w_f)C_m p / \alpha_w$ while a is the characteristic

half width of the matrix block, w_f is the fracture porosity, K_a is the hydraulic

conductivity of the matrix block near the fracture/matrix interface, C_m is the specific

water capacity at the matrix, and α_w is the water transfer coefficient.

Mathias and Zimmerman [2003] applied the Laurent-type expansion to Equations

(A1) and (A2) and transferred those series to the time domain by the asymptotic

formula of Doetsch [1961] when time is large. Based on the time domain approach, they obtained an exact large-time water transfer coefficient of $\alpha_w = \pi^2 K_a / 4a^2 (\cong 2.47 K_a / a^2)$ which differs from the Laplace domain approach of $\alpha_w = 3K_a / a^2$ obtained by Gerke and van Genuchten [1993] derived from Equations (A1) and (A2) based on the SPLT relationship.

Once the water transfer coefficient is obtained, the normalized head difference for large value of time is [Mathis and Zimmerman, 2003]

$$\frac{h_m(t) - h_f}{h_{m,i} - h_f} = \exp\left[\frac{-\alpha_w t}{(1 - w_f)C_m}\right] \tag{A3}$$

where h_m is the time domain head. Equation (A3) can be verified by comparing it with the exact solution given in Crank [1956, p.48] which was shown in Mathis and Zimmerman [2003] as

$$\frac{h_m(t) - h_f}{h_{m,i} - h_f} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left[\frac{-(2n+1)^2 \pi^2 K_a t}{(2a)^2 (1 - w_f)C_m}\right] \tag{A4}$$

It is clear that the normalized head difference approaches zero while dimensionless time $K_a t / \left[(2a)^2 (1 - w_f)C_m\right]$ on the RHS of Equation (A4) goes to a very large value.

After comparing with the exact solution given by Crank [1956], Mathias and Zimmerman [2003] concluded that the discrepancy of the water transfer coefficient

from the time domain approach and Laplace domain approach arises from the use of

the SPLT relationship.   However, this study finds that this relationship is indeed

correct and the defect in Gerke and van Genuchten [1993] is mainly caused by

ignoring the convergent requirements of series expansion for Equations (A1) and

(A2).

Gerke and van Genuchten [1993] expanded $\tanh(\xi)$ and $1/(1+\zeta)$ in Equations

(A1) and (A2) in terms of a series, respectively.   Those two equations were then

respectively expressed as

$$
\overline{h_m}(p) = \frac{h_f}{p}\left[1 - \frac{a^2(1-w_f)C_m p}{3K_a} + \frac{2a^4(1-w_f)^2 C_m^2 p^2}{15K_a^2} - \frac{17a^6(1-w_f)^3 C_m^3 p^3}{315K_a^3} + \dots\right] +
$$
$$
\frac{h_{m,i}}{p}\left[\frac{a^2(1-w_f)C_m p}{3K_a} - \frac{2a^4(1-w_f)^2 C_m^2 p^2}{15K_a^2} + \frac{17a^6(1-w_f)^3 C_m^3 p^3}{315K_a^3} - \dots\right]
$$

(A5)

and

$$
\overline{h_m}(p) = \frac{h_f}{p}\left[1 - \frac{(1-w_f)C_m p}{\alpha_w} + \frac{(1-w_f)^2 C_m^2 p^2}{\alpha_w^2} - \frac{(1-w_f)^3 C_m^3 p^3}{\alpha_w^3} + \dots\right] +
$$
$$
\frac{h_{m,i}}{p}\left[\frac{(1-w_f)C_m p}{\alpha_w} - \frac{(1-w_f)^2 C_m^2 p^2}{\alpha_w^2} + \frac{(1-w_f)^3 C_m^3 p^3}{\alpha_w^3} - \dots\right] \qquad (A6)
$$

The water transfer coefficient can then be obtained as $\alpha_w = 3K_a/a^2$ if one

compares Equations (A5) with (A6) and sets the second term on RHS of these two

equations are equal.   Similarly, the water transfer coefficients obtained from the third

and fourth terms are $\alpha_w = \sqrt{15/2}\,K_a\big/a^2$ $(\cong 2.74\,K_a\big/a^2)$ and $\alpha_w = \sqrt[3]{315/17}\,K_a\big/a^2$

$(\cong 2.65K_a\big/a^2)$, respectively. It seems that the estimated water transfer coefficient

appears monotonously decreasing from 3 and asymptotically approaches to 2.47.

Notice that Gerke and van Genuchten [1993] truncated the third and remaining higher

order terms of p in Equations (A5) with (A6) because of small p.

In fact, the series expansion for $\tanh(\xi)$ in Equation (A1) and $1/(1+\zeta)$ in

Equation (A2) should be restricted to the convergent criteria $|\xi| < \pi/2$ and $|\zeta| < 1$

[Abramowitz and Stegun, 1970, p.15 and 85], respectively. The series expansion of

$\tanh(\xi)$ in Equation (A1) can be expressed as [Abramowitz and Stegun, 1970, p.85]

$$\tanh\xi = \xi - \frac{\xi^3}{3} + \frac{2}{15}\xi^5 - \frac{17}{315}\xi^7 + ... + \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\xi^{2n-1} + ...,\quad n = 1,2,...$$

(A7)

where B_{2n} is the nth Bernoulli number and the convergent criterion of $\tanh(\xi)$ is

$|\xi| < \pi/2$. Subsequently, one uses the Fourier expansion of the Bernoulli number

[Abramowitz and Stegun, 1970, p.805] and obtains

$$B_{2n} = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}},\quad n = 1,2,...$$

(A8)

Therefore, the nth term of water transfer coefficient is obtained by letting

Equation (A5) equal to Equation (A6) as

$$\alpha_{w,n} = \left[\frac{(-1)^{n-1}(2n)!}{2^{2n}(2^{2n}-1)B_{2n}} \right]^{\frac{1}{n-1}} \frac{\mathrm{K}_a}{a^2} = \left[\frac{\pi^{2n}}{(2^{2n+1}-2)\sum\limits_{k=1}^{\infty}\frac{1}{k^{2n}}} \right]^{\frac{1}{n-1}} \frac{\mathrm{K}_a}{a^2}, \quad n = 2,3,...$$

(A9)

The limit of α_w for $n \to \infty$ is

$$\alpha_w = \lim_{n\to\infty} \left[\frac{\pi^{2n}}{(2^{2n+1}-2)\sum\limits_{k=1}^{\infty}\frac{1}{k^{2n}}} \right]^{\frac{1}{n-1}} \frac{\mathrm{K}_a}{a^2} = \frac{\pi^2}{4}\frac{\mathrm{K}_a}{a^2}$$

(A10)

where the limit of Riemann Zeta function, $\lim\limits_{n\to\infty}\sum_{k=1}^{\infty}k^{-2n}$, equals 1.

The result shows that the water transfer coefficient has to be $\alpha_w = \pi^2 K_a / 4a^2$. Obviously, the coefficient of $\alpha_w = 3K_a / a^2$ obtained from the Laplace domain approach based on the SPLT relationship is exactly the same as that of Mathias and Zimmerman [2003] derived from the time domain approach. Therefore, the dispute in the discrepancy that was calculated using Laplace domain approach by making use of the SPLT relationship and those found by working in the time domain is clearly resolved.

Appendix B: Derivation of Equation (7)

By applying the Laplace transform, the integral function of Equation (7) can be expressed as

$$L\left[\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dx\right] = \int_0^\infty e^{-pt}\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dx\,dt \tag{B1}$$

The RHS of (B1) is a double integral and can be rearranged as

$$\int_0^\infty e^{-pt}\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)}dx\,dt = \frac{1}{p}\int_0^\infty \frac{(\lambda/p)^x}{\Gamma(x+1)}\left[\int_0^\infty e^{-pt}(pt)^x\,d(pt)\right]dx \tag{B2}$$

The Gamma function is defined as [Abramowitz and Stegun, 1970, p.255]

$$\Gamma(x+1) = \int_0^\infty e^{-u}u^x\,du \tag{B3}$$

Replacing the second integral in Equation (B2) by Gamma function, the RHS of Equation (B2) after the integration gives

$$\frac{1}{p}\int_0^\infty \left(\frac{\lambda}{p}\right)^x dx = -\lim_{z\to\infty}\frac{1}{p}\frac{1}{\ln(p/\lambda)}\left(\frac{p}{\lambda}\right)^{-x}\Bigg|_{x=0}^{x=z} \tag{B4}$$

If $p > \lambda$, the term $(p/\lambda)^{-z}$ approaches to zero as $z \to \infty$, then Equation (B4) reduces to

$$L\left[\int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)} dx\right] = \frac{1}{p \ln(p/\lambda)} \tag{B5}$$

This derivation shows that Equation (7) is valid only under the condition that $p > \lambda$.

Appendix C: Derivation of Equation (9)

The first four terms of Equation (8) can be rewritten using the notation of Ritchie and Sakakura [1956] as

$$
L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = -\left[\frac{B_0^{0,-1}}{\ln\eta} + \frac{B_1^{0,-1}}{(\ln\eta)^2} + \frac{B_2^{0,-1}}{(\ln\eta)^3} + \frac{B_3^{0,-1}}{(\ln\eta)^4}\right] \tag{C1}
$$

and

$$
B_s^{0,-1} = (-1)^s \binom{-1}{s}\frac{d^s}{dv^s}\left[\frac{1}{\Gamma(1-v)}\right]\Bigg|_{v=0}, \quad s = 0,1,2,3 \tag{C2}
$$

where the coefficient $B_s^{0,-1}$ is related to the Gamma function. The selected properties of binomial coefficient, Gamma function, and Polygamma function applied to the following derivation are, respectively, [Abramowitz and Stegun, 1970]

$$
\binom{-1}{s} = (-1)^s \tag{C3}
$$

$$
\Gamma(1-v) = -v\Gamma(-v) \tag{C4}
$$

and

$$
\frac{d^s}{dv^s}\psi(v) = \frac{d^{s+1}}{dv^{s+1}}[\ln\Gamma(v)] \tag{C5}
$$

The values of $B_s^{0,-1}$ for s equal to 0, 1, 2, and 3 can be derived, respectively, as

$$\mathrm{B}_0^{0,-1} = \binom{-1}{0}(-1)^0 \frac{1}{\Gamma(1)} = \frac{1}{\Gamma(1)} = 1 \tag{C6}$$

$$\mathrm{B}_1^{0,-1} = \binom{-1}{1}(-1)^1 \frac{d}{d\nu}\left[\frac{1}{\Gamma(1-\nu)}\right]_{\nu=0} = \frac{\Gamma'(1)}{\Gamma^2(1)} = -\gamma \tag{C7}$$

$$\mathrm{B}_2^{0,-1} = \binom{-1}{2}(-1)^2 \frac{d^2}{d\nu^2}\left[\frac{1}{\Gamma(1-\nu)}\right]_{\nu=0} = -\frac{\Gamma''(1)}{\Gamma^2(1)} + 2\frac{\left(\Gamma'(1)\right)^2}{\Gamma^3(1)} = \gamma^2 - \frac{\pi^2}{6} \tag{C8}$$

and

$$\begin{aligned}
\mathrm{B}_3^{0,-1} &= \binom{-1}{3}(-1)^3 \frac{d^3}{d\nu^3}\left[\frac{1}{\Gamma(1-\nu)}\right]_{\nu=0} \\
&= \frac{\Gamma'''(1)\Gamma^2(1) - 2\Gamma''(1)\Gamma'(1)\Gamma(1)}{\Gamma^4(1)} - 2\frac{2\Gamma''(1)\Gamma'(1)\Gamma^3(1) - 3\left(\Gamma'(1)\right)^2\Gamma^2(1)\Gamma(1)}{\Gamma^6(1)} \\
&= -\gamma^3 + \frac{\pi^2}{2}\gamma - 2\xi(3)
\end{aligned} \tag{C9}$$

where the Riemann Zeta function $\xi(3) = 1.2020569032$.

Substituting Equations (C6) - (C9) into Equation (C1) gives

$$L^{-1}\left[\frac{1}{p\ln(p/a)}\right] = -\left[\frac{1}{\ln\eta} - \frac{\gamma}{(\ln\eta)^2} + \frac{\gamma^2 - \frac{\pi^2}{6}}{(\ln\eta)^3} - \frac{\gamma^3 - \frac{\pi^2}{2}\gamma + 2\xi(3)}{(\ln\eta)^4}\right] \tag{C10}$$

Thus, the inverse Laplace transform of Equation (6) results in Equation (9) when

truncating high-order terms of Equation (8).

Appendix D: Derivation of Equation (10)

General solutions of one-dimensional radial heat conduction equation in analogy to the groundwater drawdown equation subject to Cauchy boundary condition at the edges of hollow cylinder were given in Carslaw and Jaeger [1959]. The Cauchy boundary conditions in terms of drawdown were expressed as

$$k_1 \frac{\partial s}{\partial r} - k_2 s = k_3 , \quad r = a \tag{D1}$$

and

$$k_1^{'} \frac{\partial s}{\partial r} - k_2^{'} s = k_3^{'} , \quad r = b \tag{D2}$$

where $k_1, k_2, k_3, k_1^{'}, k_2^{'}, k_3^{'}$ are constant; a and b are the radial distance of inner boundary and outer boundary in the considered region, respectively. Equations (D1) and (D2) represent the combination of the constant-head and constant-flux boundary conditions.

By using the Laplace transforms, the drawdown solutions based on Equations (D1) and (D2) is

$$s(r,t) = A_1 - \pi \sum_{n=1}^{\infty} \exp\left(-\frac{T}{S}\alpha_n^2 t\right) \frac{\left[k_1^{'}\alpha_n J_1(b\alpha_n) - k_2^{'} J_0(b\alpha_n)\right] C_0(r,\alpha_n)}{F(\alpha_n)} A_2(\alpha_n) \tag{D3}$$

where

$$A_1 = \frac{-ak_3\left[k_1^{'} - bk_2^{'}\ln(b/a)\right] + bk_3^{'}\left[k_1 + ak_2\ln(r/a)\right]}{ak_2k_1^{'} + bk_1k_2^{'} + abk_2k_2^{'}\ln(b/a)}$$

$$A_2 = k_3\left[k_1^{'}\alpha_n J_1(b\alpha_n) - k_2^{'}J_0(b\alpha_n)\right] - k_3^{'}\left[k_1\alpha_n J_1(a\alpha_n) + k_2 J_0(a\alpha_n)\right]$$

$$C_0(r,\alpha_n) = J_0(r\alpha_n)\left[k_1\alpha_n Y_1(a\alpha_n) + k_2 Y_0(a\alpha_n)\right] - Y_0(a\alpha_n)\left[k_1\alpha_n J_1(a\alpha_n) + k_2 J_0(a\alpha_n)\right]$$

$$F(\alpha_n) = \left(k_1^{'2}\alpha_n^2 + k_2^{'2}\right)\left[k_1\alpha_n J_1(a\alpha_n) + k_2 J_0(a\alpha_n)\right]^2$$
$$- \left(k_1^2\alpha_n^2 + k_2^2\right)\left[k_1^{'}\alpha_n J_1(b\alpha_n) - k_2^{'}J_0(b\alpha_n)\right]^2$$

and α_n are the positive roots of

$$\left[k_1\alpha J_1(a\alpha) + k_2 J_0(a\alpha)\right]\left[k_1^{'}\alpha Y_1(b\alpha) - k_2^{'}Y_0(b\alpha)\right]$$
$$- \left[k_1\alpha Y_1(a\alpha) + k_2 Y_0(a\alpha)\right]\left[k_1^{'}\alpha J_1(b\alpha) - k_2^{'}J_0(b\alpha)\right] = 0$$

For the constant-head test with a finite well radius and outer boundary, the constants in Equations (D1) and (D2) can be replaced by $k_1=0$, $k_2= -1$, $k_3=s_w$, $k_1^{'}=0$, $k_2^{'}=1$, $k_3^{'}=0$, $a=r_w$, and $b=R$.   By carefully substitution, the drawdown solution, i.e., Equation (10) can be obtained from Equation (D3).

Table 1   Transient and steady-state drawdown solutions for constant-head and constant flux tests

State	Constant-head test		Constant-flux test	
	Infinite domain	Finite domain	Infinite domain	Finite domain
Considering well radius				
Transient	Equation (3)	Equation (10)	Equation (13)	Equation (15)
Steady	s_w	Thiem equation	Infinity	Thiem equation
Neglecting well radius				
Transient	No solution	No solution	Theis equation	Equation (16)
Steady	No solution	No solution	Infinity	Thiem equation

Table 2 The boundary-effect and steady-state time criteria for the finite-domain

solutions

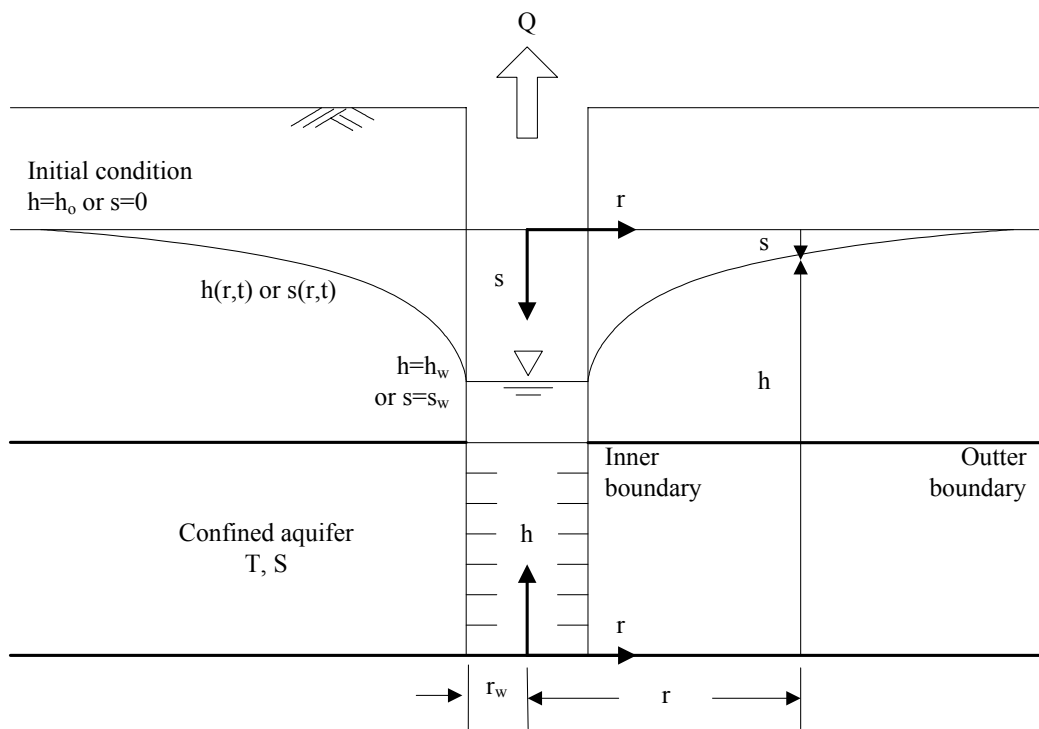| Solution type	Boundary-effect time criterion, t_1	Steady-state time criterion, t_2
Constant-head test	$\dfrac{1}{10}\tau$	4τ
Constant-flux test	$\dfrac{3}{10^2}\tau$	8τ
Constant-flux test when neglecting well radius	$\dfrac{1}{16}\tau$	$\dfrac{10^5}{4}\tau$

Note: $\tau = R^2 S/T$.

Figure1 Schematic diagram of drawdown distribution under constant-head or constant-flux test in a confined aquifer.
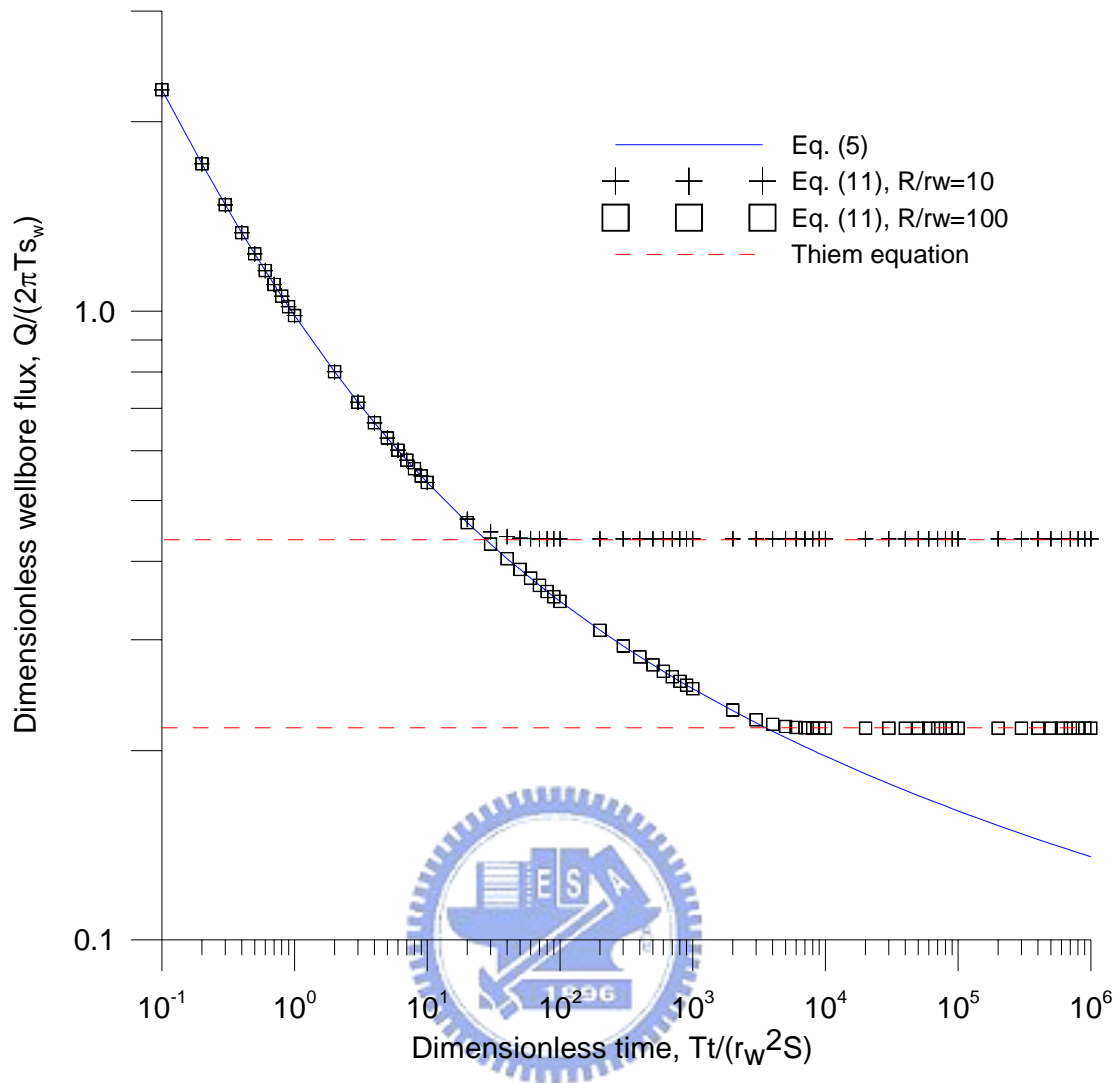
Figure2 Dimensionless wellbore flux versus dimensionless time for the finite-domain

solution, Equation (11) with R/r_w = 10 and 10^2, the infinite-domain solution,

Equation (5), and the Thiem equation.   Note that both Equations (5) and (11) are
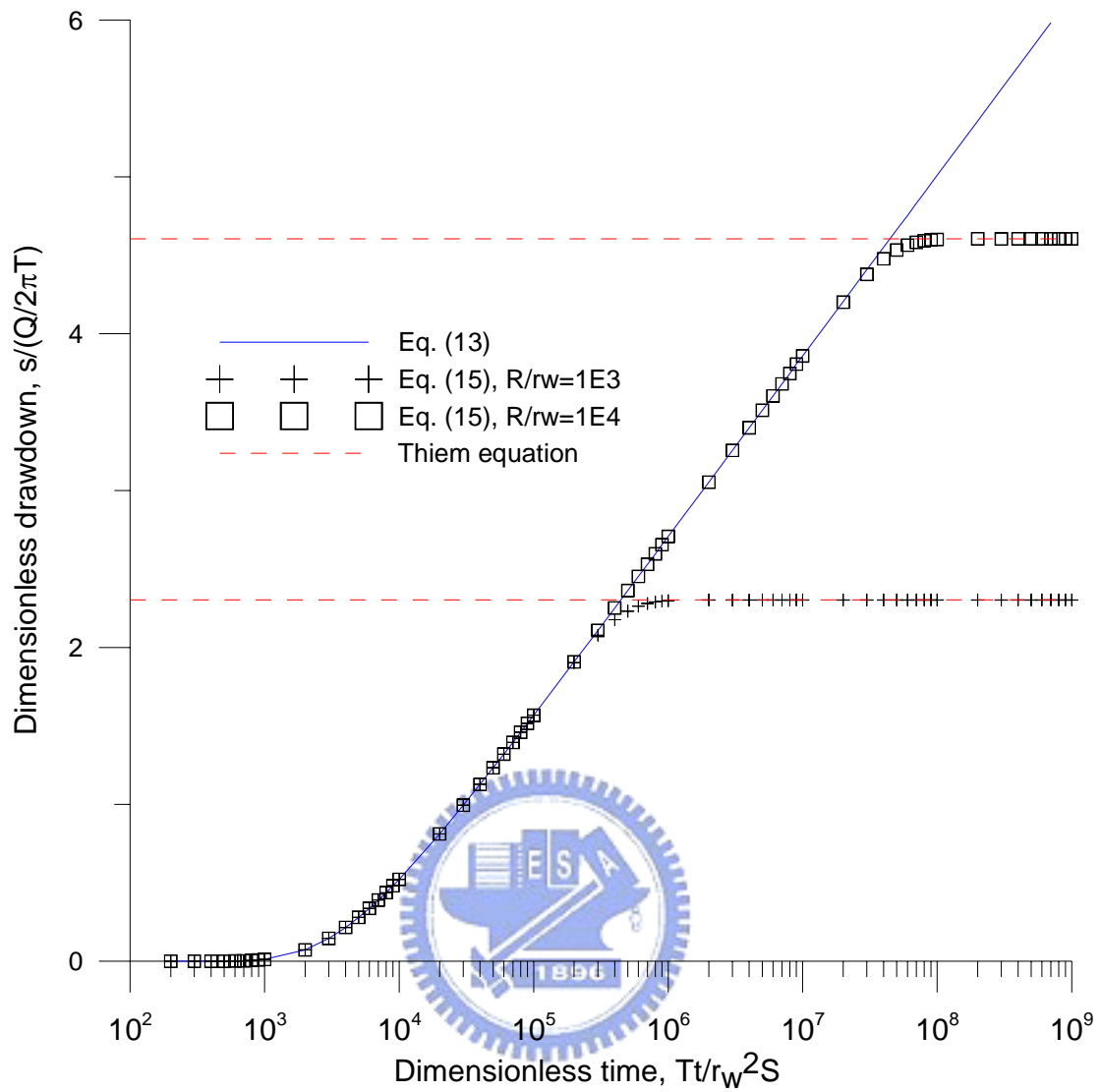
solved for the constant-head test.

Figure3 Dimensionless drawdown versus dimensionless time for the finite-domain

solution, Equation (15), with $R/r_w = 10^3$ and 10^4 and $r/r_w = 10^2$, the infinite-domain

solution, Equation (13), and the Thiem equation.   Note that both Equations (13) and

(15) are solved for the constant-flux test.

(Chih-Tse Wang)

1961   08   23

(2001-2007)
(1988-1990)
(1980-1984)

(2001-2003)
(1997-2001)

106   91-14

0960289264

ctwang.ev90g@nctu.edu.tw

Yeh, H.D., and <u>C.T. Wang</u> (2007), Large-time solutions for groundwater flow problems using the relationship of small p versus large t, *Water Resources Research*, *43*(6), W06502, doi:10.1029/2006WR005472.

<u>Wang, C.T.</u>, and H.D. Yeh (2007), Obtaining the steady-state drawdown solutions of constant-head and constant-flux tests, *Hydrological Processes 21*, doi:10.1002/hyp.6950.

<u>Wang, C.T.</u>, and H.D. Yeh (2007), A Study for Residual Drawdown Solution with Considering the Wellbore Storage, *American Geophysical Union (AGU) 2007 Fall Meeting*, Poster H13B-1244, San Francisco, CA, December 10-14.

————— (2006)

97   320-330

————— (2006)

96   312-319

————— (2005)

453   15

————— (2005)

————— (2005)   Levin   102

(II)109-112

————— (2004)

244   1515-1522

————— (2004)

F217-F221

————— (2004)

85-93