Chapter 3

Theory

This chapter mainly deals with the methods on measuring microwave properties of microwave device. First, the scattering parameters (S-parameters) will be discussed to extract the unloaded quality factor (Q_u) of resonator from the measurements taken by network analyzer [121]. It is well known that the loaded quality factor (Q_l) can be estimated by simply taking the ratio of the central frequency to the 3 dB bandwidth of the transmission response of a resonator, and the unloaded quality factor can be calculated by simply taking the insertion loss measured at resonance into account. In particular, for a two-port measurement both symmetrical and asymmetrical coupling are considered. Throughout the discussion, criterions of how the accuracy of unloaded quality factor measurements could be obtained are given. In addition, some theoretical models which consider the transport properties of quasiparticles on the microwave measurement in *d*-wave superconductors with disorders, especially for Lee's model [80], Vishveshwara and Fisher's model [84], will be discussed.

3.1 S Parameter

The two-port resonator measured by a microwave vector network analyzer can be described in terms of scattering parameters by converting the ABCD matrix for the two-port circuit. There are the voltages and currents of the two-port network, in which the input and output are designated as 1 and 2, respectively. These relations can be represented as follows [122],

$$V_1 = AV_2 + BI_2 \quad \text{or} \quad \begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix} \tag{3.1}$$
$$I_1 = CV_2 + DI_2$$

For frequencies higher than 30 MHz, the measurement of voltages and currents for the circuit is difficult, because of parasitic effects on the circuits: the induced inductance of a short circuit and the induced capacitance of an open circuit. The parasitic effects can give rise to the measurement illegibility. Therefore, the above-mentioned methods are not suitable for measuring two-port circuit in high frequencies, and then the scattering parameters (S-parameters) are used to describe the relations between the incident and reflected waves in the system.

For a two-port network, the phasor quantities V^{in}, V^{out}, I^{in}, and I^{out} are represented as the forward and reverse voltage and current waves on the resonator, respectively. The characteristic impedance of the network is assumed to be Z_0. Then, we define

$$a = \frac{V^{in}}{\sqrt{Z_o}} = I^{in}\sqrt{Z_o}, \tag{3.2}$$

and

$$b = \frac{V^{out}}{\sqrt{Z_o}} = I^{out}\sqrt{Z_o}, \tag{3.3}$$

where a and b are the incident and reflected waves, respectively. So, the incident and reflected waves are related by the following equations,

$$\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned}, \quad \text{or} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{3.4}$$

where the set of S_{ij} are called the scattering parameters (S-parameters) and defined as

$$S_{11} \equiv \left.\frac{b_1}{a_1}\right|_{a_2 = 0} = \text{Port 1 Reflection Coefficient,} \tag{3.5}$$

$$S_{22} \equiv \left.\frac{b_2}{a_2}\right|_{a_1 = 0} = \text{Port 2 Reflection Coefficient,} \tag{3.6}$$

$$S_{21} \equiv \left.\frac{b_2}{a_1}\right|_{a_2 = 0} = \text{Forward Transmission Coefficient,} \tag{3.7}$$

$$S_{12} \equiv \left.\frac{b_1}{a_2}\right|_{a_1 = 0} = \text{Reverse Transmission Coefficient,} \tag{3.8}$$

respectively. Moreover, the insertion loss L_I can be obtained as $L_I = 20\log\frac{\mathrm{b}_2}{\mathrm{a}_1} = 20\log\frac{1}{|S_{21}|}$ with the measurement of S_{21}.

3.2 Losses of Microstrip Device

3.2.1 Conductor Loss

The Maxwell's equations in an isotropic and homogeneous medium can be written as

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{3.9}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3.10}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{3.11}$$

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{3.12}$$

where the magnetic flux density is related to the magnetic filed strength $\vec{B} = \mu\vec{H}$

and the displacement current is related to the electric field by $\vec{D} = \varepsilon\vec{E}$ in the linear response.

As an electromagnetic (EM) wave propagates in an infinite conductor, the wave equation for the electric and magnetic field can be derived from the Maxwell's equations as

$$\nabla^2\vec{E} + \omega^2\varepsilon\mu(1 - j\frac{\sigma}{\omega\varepsilon})\vec{E} = 0, \tag{3.13}$$

and

$$\nabla^2\vec{H} + \omega^2\varepsilon\mu(1 - j\frac{\sigma}{\omega\varepsilon})\vec{H} = 0, \tag{3.14}$$

where ε, μ, and σ are the dielectric constant, permeability and conductivity of the conductor, respectively. Both Eq. (3.13) and Eq. (3.14) can be simplified as:

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0, \tag{3.15}$$

and

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0. \tag{3.16}$$

We then define a complex propagation constant for the medium as

$$\gamma = j\omega\sqrt{\varepsilon\mu}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = \alpha + j\beta. \tag{3.17}$$

Assuming an electric field of the electromagnetic wave with only an $\hat{\boldsymbol{x}}$ component and uniform in x and y, and considering the boundary condition of the tangent electric field between the vacuum and an infinite conductor, the relation of the electric field can be taken as

$$\begin{aligned} E_x(z) &= E_x(0)\,e^{-\gamma z} \\ &= E_x(0)\,e^{-\alpha z}e^{-j\beta z}, \end{aligned} \tag{3.18}$$

where $E_x(0)$ is the electric field at boundary. It represents a wave traveling in the +z direction with a phase constant, β, and the attenuation constant, α, and indicates an exponential damping of the amplitude while it travels along the conductor. Furthermore, when considering the time harmonic variation in the system, then the electric wave becomes

$$\widehat{E}_x(z,t) = \widehat{E}_x(0,t)\, e^{-\alpha z} e^{j(\omega t - \beta z)}, \tag{3.19}$$

where β is the wave number of the propagating wave. By using Faraday's law

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \tag{3.20}$$

the associated magnetic field can be calculated as

$$H_y = \frac{j}{\omega\mu} \frac{\partial E}{\partial z} = \frac{-j\gamma}{\omega\mu} E_x\,. \tag{3.21}$$

This result shows that the magnetic field intensity vector lies in a plane normal to the direction of propagation, and that magnetic field is perpendicular to electric field. So, we can define the impedance as

$$Z = \frac{\vec{E}}{\vec{H}} = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma}. \tag{3.22}$$

If a good conductor has conductive current much greater than the displacement current, that is, $\sigma >> \omega\varepsilon$, then the propagation constant can be reduced as

$$\begin{aligned} \gamma &= \alpha + j\beta \\ &\cong j\omega\sqrt{\varepsilon\mu}\sqrt{\frac{\sigma}{j\omega\varepsilon}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}. \end{aligned} \tag{3.23}$$

The characteristic depth of penetration, or the skin depth δ_s, is defined as,

$$\delta_s = 1/\alpha = \sqrt{\frac{2}{\omega\mu\sigma}}\ . \tag{3.24}$$

Another meaning of the skin depth is that the surface current flows uniformly just below the surface within the depth. That is

$$J_x(z) = \sigma\, E_o\, e^{-\alpha z} = J_o\, e^{-\alpha z}, \tag{3.25}$$

and it can be simplified as

$$J_x(z) = J_o \quad \text{for} \quad 0 < z < \delta_s, \qquad (3.26)$$
$$= 0 \quad \text{for} \quad z > \delta_s.$$

The surface current is

$$\begin{aligned} J_s &= \int_0^\infty J_x(z)dz = \int_0^\infty \sigma E_o e^{-\alpha z} dz \\ &= \frac{\sigma E_o}{-\alpha} e^{-\alpha z}\Big|_0^\infty \\ &= \sigma E_o \delta_s = J_o \delta_s. \end{aligned} \qquad (3.27)$$

Moreover, the surface impedance, Z_S, of a good conductor can be obtained as

$$\begin{aligned} Z_s &= R_s + jX_s \\ &= (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}, \end{aligned} \qquad (3.28)$$

where R_s and X_s are the surface resistance and reactance, respectively.

3.2.2 Dielectric Loss

When the dielectric substrate interacts with an applied electric field, the dielectric material becomes polarized due to the displacements of bound charges in it. In fact, the inertia of the charged particles tends to counteract the opposite displacement and keep in phase with the field variations. Therefore, there is always a power loss in this condition, since work must be done to overcome the damping mechanism.

For a dielectric material, an applied electric field $\vec{E}$ causes the polarization of the neutral atoms or molecules of the material to generate electric dipole moments that argue the total displacement flux, $\vec{D}$. This additional polarization vector is called the electric polarization $\vec{P}_e$, and one has

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}_e. \tag{3.29}$$

For a linear medium, the electric polarization vector is linear response with respect to the applied electric field as

$$\vec{P}_e = \varepsilon_0 \chi_e \vec{E}, \tag{3.30}$$

where χ_e is known as the electric susceptibility, which may be complex. Then, one has

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}_e = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon \vec{E}, \tag{3.31}$$

and,

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0 (1 + \chi_e), \tag{3.32}$$

where ε is the complex permittivity of the medium. The imaginary part of ε is accounted for the loss in the medium due to damping of the vibrating dipole moments. This dielectric loss may be considered as an equivalent loss of a conductor. For example, in a material with conductivity, σ, current density follows the Ohm's law:

$$\vec{J} = \sigma \vec{E}. \tag{3.33}$$

Maxwell's curl equation for $\vec{H}$ then becomes

$$\begin{aligned} \nabla \times \vec{H} &= j\omega \vec{D} + \vec{J} \\ &= j\omega\varepsilon \vec{E} + \sigma \vec{E} \\ &= j\omega\varepsilon' \vec{E} + (\omega\varepsilon'' + \sigma) \vec{E} \\ &= j\omega(\varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega}) \vec{E}, \end{aligned} \tag{3.34}$$

where it seems that loss due to dielectric damping $\omega\varepsilon''$ is indistinguishable from conductivity loss σ. The term $\omega\varepsilon''+\sigma$ could be considered as the total effective conductivity. A related quantity of interest is the loss tangent of dielectric, given by

$$\tan\delta = \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}. \tag{3.35}$$

It is the ratio between the real and the imaginary part of the total displacement current. If the loss tangent is not low enough, then the advantage of using a superconductor can be negated. By the way, the dielectric materials are usually characterized by specifying the real permittivity, $\varepsilon' = \varepsilon_r \varepsilon_0$, and by the loss tangent at a certain frequency. The dielectric loss can easily be introduced by replacing the real ε with a complex

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon'(1 - j\tan\delta). \tag{3.36}$$

3.2.3 Radiation Loss

Because the microstrip resonator is not fully enclosed, then it will radiate. It is important to investigate this radiation and consider whether it has any effect upon the loss mechanism. For example, the radiation loss from an open aperture can be equally, or more, important in contrast to other loss mechanisms. For the ultimate in performance of cavity resonator the radiation from the structure (or the absorption of energy due to the proximity of the box (sample housing) wall) must be considered and optimized. The radiation can be reduced by reducing the substrate thickness [123]. Box resonances must be kept well away from the resonance of interest, so as not to have any modal coupling. The box must also be made as large as possible in order to reduce wall losses. These two criteria may not be compatible and careful design is needed in order to minimize radiation loss.

3.3 Quality Factor (Q) of Resonator

3.3.1 Measured Quantities

If a certain amount of energy is stored in a resonator and left therein to dissipate, it would do so only after many circles of oscillation. The decay of the amplitude of the oscillations is a measure of the quality factor (Q) of the resonator. In other words, the measurements of the quality factor of the resonator are to extract the loss information about the resonator. It may be used to designate a guidepost for determining the parameters of the resonator for various application purposes. So, various types are involved in the Q-value in order to extract the microwave properties of the samples. Specifically, the unloaded quality factor Q_0 is of basic interest.

A resonator is able to sustain an oscillating electromagnetic field within it. In general, it has a number of distinct resonance frequencies or degenerate fundamental mode dependent upon the geometry of the resonator. The resonant electromagnetic field will gradually decay because of losses. These losses including the above-mentioned are mainly due to

(i) the finite conductivity of the resonator,

(ii) the loss tangent of the dielectric,

(iii) radiation loss.

The main reason for using high-T_c superconductors in the construction of a resonator is due to possess a function of reducing the conductor loss to a possible minimum. The decay (or loss) of the oscillating field can be expressed to be inversely proportional to the quality factor of the resonator. Quality factor, *Q*, is thus turned out an important quantitative parameter for clarifying a high quality resonator. The *Q* -factor for a resonator is then defined as

2π times the ratio of the time-averaged energy stored in the resonator to the energy loss per cycle.

That is,

$$Q \equiv 2\pi \frac{\text{Energy Stored}}{\text{Energy Loss per Cycle}} = \omega_o \frac{\text{Energy Stored}}{\text{Average Power Lost}}, \qquad (3.37)$$

where ω_o is the resonance frequency.

By the law of conservation of energy, the power dissipated in ohmic losses is the negative of the time rate of change of store energy U:

$$\frac{dU}{dt} = -\frac{\omega_o}{Q} U. \qquad (3.38)$$

The solution of Eq. (3.38) is

$$U(t) = U_o e^{-\omega_o t/Q}. \qquad (3.39)$$

For a resonator with finite Q, it decays away exponentially with a constant inversely proportional to Q. This implies that the resonator has a time-dependent field which is damping as

$$E(t) = E_o e^{-\omega_o t/2Q} e^{-j(\omega_0+\delta\omega)t}. \qquad (3.40)$$

The damping has a part of shift $\delta\omega$ to the dominant resonance frequency ω_o. Thus the field can be expressed in $\omega-$space as

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(\omega) e^{-j\omega t} d\omega, \qquad (3.41)$$

where

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} E_o e^{-\omega_o t/2Q} e^{j(\omega-\omega_o-\delta\omega)t} dt\ . \qquad (3.42)$$

The integral in Eq. (3.42) leads to a frequency distribution for the energy in the cavity resonator, resulting in a resonant line shape:

$$|E(\omega)|^2 = \frac{1}{2\pi} |E_o|^2 \frac{1}{(\omega-\omega_o-\delta\,\omega)^2 + (\omega_o/2Q)^2} \qquad (3.43)$$

The shape of the resonance peak is *Lorentzian*, which has a maximum at the frequency, $\omega_0+\delta\omega$, and has two frequencies at half-maximum with a frequency difference $\delta\omega$. The difference can be calculated as

$$\delta\omega = \frac{\omega_o}{Q}$$

$$\text{or} \quad Q = \frac{\omega_o}{\delta\omega}. \tag{3.44}$$

Eq. (3.44) gives the loaded Q factor extracted from the field response. In practice, the resonance frequency ω_0 was defined by the peak position on the curve of $E^2(\omega_0)$, while the Q-factor of the resonator was derived from the full bandwidth at half-maximum. Consequently, the maximum of the response in this system can be easily obtained from the definition of Q-factor. However, it is noted that the error of determining Q from the above equations will become significant for a low Q resonator, since the actual resonance frequency is $\omega_0+\delta\omega$ instead of ω_0, and $\delta\omega$ is quite pronounce for a poorly designed resonator.

3.3.2 Series Resonant Circuit

Microwave resonators made by high-T_c superconducting thin films can be used in a variety of applications, including oscillators, filters, tuned amplifiers, and frequency meters. Since the operation of microwave resonators is very similar to that of the lumped-element resonators of the circuit theory, we begin by modeling a series *RLC* lumped-element resonant circuit, and so we can derive some of the basic properties of such circuits. For a series *RLC* circuit, as shown in Fig. 3.1(a) [121], the input impedance is

$$Z_{in} = R + j\omega L - j\frac{1}{\omega C}, \tag{3.45}$$

and the power dissipated by the resistor R is

$$P_{loss} = \frac{1}{2}|I|^2 R. \tag{3.46}$$

The energy stored in the capacitor C and the inductor L are expressed as

$$W_C = \frac{1}{4}|V_C|^2 = \frac{1}{4}|I|^2\frac{1}{\omega^2 C}, \tag{3.47}$$

$$W_L = \frac{1}{4}|I|^2 L, \tag{3.48}$$

respectively.

Then the complex power delivered to the resonator can be written as

$$\begin{aligned} P_{in} &= \frac{1}{2}Z_{in}\,|I|^2 = \frac{1}{2}|I|^2\,(R + j\omega L - j\frac{1}{\omega C}) \\ &= P_{loss} + 2j\omega(W_L - W_C). \end{aligned} \tag{3.49}$$

Resonance occurs when the average stored magnetic and electric energies are equal, $W_C = W_L$, which implies that the resonance frequency can be defined as

$$\omega_o = \frac{1}{\sqrt{LC}} \quad , \tag{3.50}$$

and the input impedance at resonance is $Z_{in} = R$. The Q factor is a measure of the loss of the resonant circuit and is be expressed as

$$\begin{aligned} Q &= \omega_0\frac{(\textit{average}\text{ energy stored})}{(\text{energy loss / second})} \\ &= \omega_o\frac{Wc + W_L}{P_{loss}} = \omega_o\frac{2\frac{1}{4}|I|^2 L}{\frac{1}{2}|I|^2 R} \\ &= \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} \quad , \end{aligned} \tag{3.51}$$

which shows that Q increases with R decrease.

As the input impedance of the resonant circuit is near its resonance frequency, one lets $\omega = \omega_0 + \delta\omega$, where δω is small. The input impedance can then be rewritten as

$$
\begin{aligned}
Z_{in} &= R + j\omega L + j\frac{1}{\omega C} \\
&= R + j\frac{1}{\omega C}\left(\omega^2 LC - 1\right) \\
&= R + j\frac{LC\left(\omega_o^2 + 2\omega_o\delta\omega + \delta\omega^2\right) - 1}{\left(\omega_o + \delta\omega\right)C} \\
&\cong R + j2L\delta\omega
\end{aligned}
\qquad (3.52)
$$

This is a good approximation for most practical resonators because the loss is very small. The effect of loss can then be added to ω to replace ω_0, and input impedance becomes a complex form.

Consider a real microstrip resonator of length ℓ, short circuit at one end, which has the characteristic impedance Z_0, load impedance Z_L(= 0), propagation constant β, and attenuation constant α. As the frequency $\omega = \omega_0$, $\ell = n\lambda/2$, $\lambda = 2\pi / \beta$ and n = 1, 2, 3…, the input impedance of the transmission line is

$$
\begin{aligned}
Z_{in} &= Z_o\frac{Z_L + Z_o \tanh\gamma\ell}{Z_o + Z_L \tanh\gamma\ell} \\
&= Z_o \tanh\gamma\ell = Z_o \tanh(\alpha + j\beta)\ell \\
&= Z_o\frac{\tanh\alpha\ell + j\tan\beta\ell}{1 + j\tan\alpha\ell\tan\beta\ell}.
\end{aligned}
\qquad (3.53)
$$

Now let $\omega = \omega_0 + \delta\omega$, where $\delta\omega$ is small, then

$$
\begin{aligned}
\beta\ell &= \frac{\omega\ell}{v_p} = \frac{\omega_o\ell}{v_p} + \frac{\delta\omega\ell}{v_p} \\
&= \pi + \frac{\pi\delta\omega}{\omega_o}.
\end{aligned}
\qquad (3.54)
$$

Consequently, the loss tangent becomes

$$
\tan\beta\ell = \tan\left(\pi + \frac{\pi\delta\omega}{\omega_o}\right) = \tan\frac{\pi\delta\omega}{\omega_o} \cong \frac{\pi\delta\omega}{\omega_o}. \qquad (3.55)
$$

Inserting this result into Eq. (3.53) the impedance can be obtained as

$$Z_{in} \approx Z_o \frac{\alpha\ell + j(\pi\delta\omega/\omega_o)}{1 + j\alpha\ell\dfrac{\pi\delta\omega}{\omega_o}} \approx Z_o\left(\alpha\ell + j\pi\frac{\delta\omega}{\omega_o}\right). \tag{3.56}$$

Comparing with Eq. (3.52), we can then identify the resistance, inductance and capacitance of the equivalent circuit as

$$R = Z_o\alpha\ell, \tag{3.57}$$

$$L = \frac{\pi Z_o}{2\omega_o}, \tag{3.58}$$

$$C = \frac{1}{\omega_o^2 L}, \tag{3.59}$$

respectively. The Q factor of this microstrip resonator can be expressed as

$$Q = \omega_o\frac{L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}, \text{ for n = 1.} \tag{3.60}$$

3.3.3 Total Losses of Resonator

The losses in a resonator arise due to conductor loss, dielectric loss and radiation loss. Then the total loss of the resonator can be expressed as

$$\frac{1}{Q} \cong \frac{1}{Q_c} + \frac{1}{Q_d} + \frac{1}{Q_r}, \tag{3.61}$$

where Q_c, Q_d, and Q_r are the conductor, dielectric, and radiation quality factors, respectively. Using proper dielectric material with low loss tangent, we can reduce the dielectric loss significantly. The radiation loss can be decreased by using a housing well designed. Then the loss for the microstrip line is mainly due to the conductor loss.

The conductor quality factor Q_c is defined as

$$Q_c = \omega \frac{\text{Energy Stored in the resonator}}{\text{Average Power Lost due to ohmic dissipation}}$$

$$= \omega \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{conductor} \sigma_1 |E|^2 \, dv}, \tag{3.62}$$

where σ_1 is the normal conductivity. The term in numerator is integrated throughout the whole volume of the cavity resonator, and the denominator is integrated over the conductor volume only. Since $E = ZH$ by definition, the Eq. (3.62) becomes

$$Q_c = \frac{\omega\mu}{\sigma_1 |Z|^2} \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{conductor} |H|^2 \, dv}$$

$$= \frac{2\omega\mu}{\lambda\sigma_1 |Z|^2} \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{conductor} |H|^2 \, dS}. \tag{3.63}$$

The factor of $\lambda/2$ arises due to the integration of the exponentially decaying magnetic field distribution inside the conductor, and the integration in the denominator is over the surface of the conductor, now. This is a very good approximation because the penetration depth is small compared with the radius of curvature of the conductor. The Eq. (3.63) can also be reduced to

$$Q_c = \frac{\omega\mu}{R_s} \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{conductor} |H|^2 \, dS} = \frac{\Gamma}{R_s}, \tag{3.64}$$

where

$$\Gamma = \omega\mu \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{conductor} |H|^2 \, dS} \tag{3.65}$$

is the geometry factor of the resonator.

As the superconductor film was deposited on the surface of a dielectric substrate, there will be a loss of energy by dissipation in the dielectric material. This dissipation can be associated with a dielectric quality factor

$$\begin{aligned} Q_d &= \omega \frac{\text{Energy Stored in the resonator}}{\text{Average Power Lost in dielectric}} \\ &= \frac{1}{\tan\delta} \frac{\int_{resonator} \mu |H|^2 \, dv}{\int_{dielectric} \varepsilon |E|^2 \, dv}. \end{aligned} \tag{3.66}$$

The numerator is integrated throughout the whole volume of the resonator, and the denominator is the integration over $|E|^2$ within the volume of the dielectric material. The loss tangent of the dielectric is given by

$$\tan\delta = \frac{\sigma_d}{\omega\mu}, \tag{3.67}$$

where σ_d is the conductivity of the dielectric. As the dielectric completely fills in the cavity resonator, the dielectric Q_d is then simply

$$Q_d = \frac{1}{\tan\delta}. \tag{3.68}$$

If the radiation loss is considered in the resonator, then the radiation quality factor can be defined as

$$\begin{aligned} Q_r &= \omega \frac{\text{Energy Stored in the resonator}}{\text{Average Power radiated}} \\ &= \mu\omega \frac{\int_{resonator} |H|^2 \, dv}{\oint_{sphere} \frac{1}{2} \mathrm{Re}(E \times H) dv}. \end{aligned} \tag{3.69}$$

The average power radiated is given by the integration of the Poynting vector over the spherical surface surrounding the resonator. If the integration takes place in free space far away from the resonator, then the radiation quality factor can be reduced to

$$Q_r = 2\omega\sqrt{\mu\varepsilon}\,\frac{\int\limits_{resonator} |H|^2\, dv}{\oint\limits_{sphere} |H|^2\, dv}. \qquad (3.70)$$

We turn the attention to a real application system for the microwave device. When a resonator is coupled to an external circuit, additional power loss out of the coupling ports occurs. An external quality factor Q_e can be attributed to this. The combination of the external Q_e and the unloaded Q_u gives rise to the loaded quality factor of the cavity resonator Q_l. It follows that

$$\frac{1}{Q_l} = \frac{1}{Q_u} + \frac{1}{Q_e}. \qquad (3.71)$$

Here Q_l is the loaded quality factor and Q_u is the unloaded quality factor. The Q_u is a characteristic of the resonator itself and is not dependent on the coupling to the cavity resonator. In fact, the measurement of Q-factor for the microstrip resonator is no longer the characteristic of the resonator at all, but also includes the effect of the ensemble. There are actually three Q factors to be included. Namely,

$$\text{Unloaded } Q: Q_u \equiv \omega_o \frac{\text{Energy Stored in the Resonant Circuit}}{\text{Power Loss in the Resonant Circuit}} \qquad (3.72)$$

$$\text{External } Q: Q_e \equiv \omega_o \frac{\text{Energy Stored in the Resonant Circuit}}{\text{Power Loss in the External Circuit}} \qquad (3.73)$$

$$\text{Loaded } Q: Q_l \equiv \omega_o \frac{\text{Energy Stored in the Resonant Circuit}}{\text{Total Power Loss}}. \qquad (3.74)$$

3.3.4 Two Ports Measurements of Unloaded Quality Factor

In order to extract these values of Q from measurements of the transmission coefficients, an equivalent circuit model of a resonator will be considered. The transmission measurements are usually 2-ports measurements. The measurement methods are discussed below [121]. The transformers are assumed lossless in the derivations given below, and in most cases this is an excellent approximation.

For two ports resonator it requires the calculation of four S-parameters, which are done by converting the ABCD matrix to an S-parameters matrix. The ABCD matrix for the circuit in Fig. 3.1(b) is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1/n_1 & 0 \\ 0 & n_1 \end{bmatrix} \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_2 & 0 \\ 0 & 1/n_2 \end{bmatrix} = \begin{bmatrix} n_2/n_1 & Z/n_1 n_2 \\ 0 & n_1/n_2 \end{bmatrix}. \quad (3.75)$$

The ABCD matrix can be translated to S-parameter matrix by the standard expression

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \frac{1}{Z_0 A + B + Z_0^2 C + Z_0 D} \begin{bmatrix} Z_0 A + B - Z_0^2 C - Z_0 D & 2Z_0(AD - BC) \\ 2Z_0 & -Z_0 A + B - Z_0^2 C + Z_0 D \end{bmatrix}. \quad (3.76)$$

Using Eq. (3.75) and Eq. (3.76), one gets

$$S_{11}(f) = \frac{(n_2^2 - n_1^2)Z_0 + Z}{(n_1^2 + n_2^2)Z_0 + Z}, \quad (3.77)$$

$$S_{11}(f) = \frac{(n_1^2 - n_2^2)Z_0 + Z}{(n_1^2 + n_2^2)Z_0 + Z}, \quad (3.78)$$

$$S_{12}(f) = S_{21}(f) = \frac{2Z_0 n_1 n_2}{(n_1^2 + n_2^2)Z_0 + Z}. \quad (3.79)$$

The impedance of the series resonance circuit can be written

$$Z = R[1 + jQ_0 \Delta_f], \quad (3.80)$$

where

$$Q_0 = \frac{\omega L}{R}, \quad (3.81)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \tag{3.82}$$

$$\Delta_f = \frac{f}{f_0} - \frac{f_0}{f} = \frac{f^2 - f_0^2}{f_0 f} \approx 2\frac{f - f_0}{f_0} = \frac{\delta f}{f_0}. \tag{3.83}$$

Here we let f ≈ f_0. Substitutions for Z into Eq. (3.77), Eq. (3.78), and Eq. (3.79) result in the values for the S-parameters given by

$$S_{11}(f) = \frac{1 + \beta_2 - \beta_1 + jQ_0\Delta_f}{1 + \beta_1 + \beta_2 + jQ_0\Delta_f}, \tag{3.84}$$

$$S_{22}(f) = \frac{1 + \beta_1 - \beta_2 + jQ_0\Delta_f}{1 + \beta_1 + \beta_2 + jQ_0\Delta_f}, \tag{3.85}$$

$$S_{12}(f) = S_{21}(f) = \frac{2\sqrt{\beta_1\beta_2}}{1 + \beta_1 + \beta_2 + jQ_0\Delta_f}, \tag{3.86}$$

where

$$\beta_1 = \frac{n_1^2 Z_0}{R} \text{ and } \beta_2 = \frac{n_2^2 Z_0}{R}. \tag{3.87}$$

At the resonance frequency ω_0, the above equations can reduce to

$$S_{11}(f_0) = \frac{1 + \beta_2 - \beta_1}{1 + \beta_1 + \beta_2}, \tag{3.88}$$

$$S_{22}(f_0) = \frac{1 + \beta_1 - \beta_2}{1 + \beta_1 + \beta_2}, \tag{3.89}$$

$$S_{12}(f_0) = S_{21}(f_0) = \frac{2\sqrt{\beta_1\beta_2}}{1 + \beta_1 + \beta_2}. \tag{3.90}$$

Substituting Eq. (3.90) into Eq. (3.84), Eq. (3.85), and Eq. (3.86), it gives

$$S_{11}(f) = \frac{S_{11}(f_0) + jQ_l\Delta_f}{1 + jQ_l\Delta_f}, \tag{3.91}$$

$$S_{22}(f) = \frac{S_{22}(f_0) + jQ_l\Delta_f}{1 + jQ_l\Delta_f}, \tag{3.92}$$

$$S_{12}(f) = S_{21}(f) = \frac{S_{12}(f_0)}{1 + jQ_l\Delta_f}. \tag{3.93}$$

The magnitude of the forward transmission coefficient S_{21} can be calculated as

$$|S_{21}(f)| = \frac{|S_{21}(f_0)|}{\sqrt{1 + Q_l^2\Delta_f^2}}. \tag{3.94}$$

The above curve of $|S_{21}(f)|$ is a conventional Lorentzian curve and can be fitted by using Lorentzian curve to reduce the noise and increase the experimental accuracy.

The loaded Q at the half-power point can be calculated by

$$\frac{|S_{21}^2(f_0)|}{|S_{21}^2(f)|} = 2. \tag{3.95}$$

Then the loaded Q is given by

$$Q_l = \frac{1}{|\Delta_f|} = \frac{f_0}{\delta f}, \tag{3.96}$$

where δf is the resonance bandwidth at the half-power points.

The external quality factor was calculated from Eq. (3.72) and Eq. (3.73). By considering the power dissipated both externally in ports and internally in the resonator:

$$Q_{e1} = Q_0\frac{P_0}{P_{e1}} = \frac{Q_0}{\beta_1}, \tag{3.97}$$

and

$$Q_{e2} = Q_0\frac{P_0}{P_{e2}} = \frac{Q_0}{\beta_2}, \tag{3.98}$$

thus the loaded quality factor can be calculated from Eq. (3.71) in the asymmetrical coupling case and be obtained as

$$Q_l = \frac{Q_0}{1 + \beta_1 + \beta_2}. \tag{3.99}$$

In the case of symmetric coupling, both ports are identically connected to the resonator, then $\beta_1 = \beta_2 = \beta$. Since the value of β can be determined from Eq. (3.90):

$$\beta = \frac{S_{21}(f_0)}{2[1 - S_{21}(f_0)]}. \tag{3.100}$$

Substitution of above equation into Eq. (3.99) one gets

$$Q_0 = \frac{Q_l}{1 - S_{21}(f_0)} = \frac{f_0}{\delta f(1 - S_{21}(f_0))}. \tag{3.101}$$

It is much simpler if the resonator is symmetrically coupled. If the resonator is weakly coupled and $S_{21}(f_0)$ is small, then $Q_l \approx Q_0$.

3.4 Complex Conductivity

Superconductivity is a condensed state of the conduction electron of the material. The condensed state is the formation of loosely associated pairs of electrons traveling adjacent to the Fermi surface. Due to the thermal fluctuations, some of the electron pairs are split, and still some normal electrons are always present under the transition temperature of superconductor. It is therefore possible to model the superconductor in terms of a complex conductivity $\sigma = \sigma_1 - j\sigma_2$ with the two fluid model [23]. Following the two-fluid model the carriers in a superconductor were assumed to consist of both superconducting and normal carriers. Under the influence of an electromagnetic field, the normal electrons are driven alternatively by the normal fluid and thus cause the resistive loss.

Considering the external force acted on an electron pair, the dynamic equation can be written as

$$m^* \frac{d\vec{v}_s}{dt} = e^* \vec{E} \text{ or } m \frac{d\vec{v}_s}{dt} = e\vec{E}, \tag{3.102}$$

where v_s is the velocity of the electron pair, e* is the effective charge of an electron pair, m* is the effective mass of an electron pair and $\vec{E}$ is the applied electric field.

A similar equation can be written with respect to the normal electrons and traveling at velocity $\vec{v}_n$:

$$m\frac{d\langle\vec{v}_n\rangle}{dt}+m\frac{\langle\vec{v}_n\rangle}{\tau_n}=-e\vec{E}, \tag{3.103}$$

where τ_n is the relaxation time of the normal electrons and $\langle\vec{v}_n\rangle$ is the slowly varying part of the velocity. This is called Drude model for metals. If the velocities of the pairing and un-pairing electrons are proportional to $e^{j\omega t}$ with applied microwave, where ω is the microwave frequency, Eq. (3.102) and Eq. (3.103) can be represented by

$$j\omega m\vec{v}_s=-e\vec{E}, \tag{3.104}$$

and

$$\left(j\omega m+\frac{m}{\tau_n}\right)\langle\vec{v}_n\rangle=-e\vec{E}, \tag{3.105}$$

respectively. Then, the total current gives

$$\begin{aligned}\vec{J}&=\vec{J}_s+\vec{J}_n\\&=n_s(-e)\vec{v}_s+n_n(-e)\langle\vec{v}_n\rangle\\&=\left[\frac{n_n e^2\tau_n}{m(1+\omega^2\tau_n^2)}-j\left(\frac{n_s e^2}{\omega m}+\frac{\omega n_n e^2\tau^2}{m(1+\omega^2\tau_n^2)}\right)\right]\vec{E}\\&=\sigma\vec{E}=(\sigma_1-j\sigma_2)\vec{E},\end{aligned} \tag{3.106}$$

where n_n and n_s are the normal and paired electron densities respectively. For $\omega^2\tau_n^2<<1$ and the real and the imaginary part of the complex conductivity σ can be reduced as

$$\sigma_1=\frac{n_n e^2\tau_n}{m}=\sigma_n\frac{n_n}{n} \quad\text{and}\quad \sigma_2=\frac{1}{\mu_o\omega\lambda_0^2}, \tag{3.107}$$

where $\sigma_n = \tau_n e^2 n/m$ is the conductivity in normal state and $\lambda_0 = \sqrt{m/\mu_o e^2 n_s}$ is the London penetration depth in the local limit ($\xi << \lambda_0$, ξ is the coherence length). It is accessible to measure the complex conductivity of a superconductor and this measured value can be used for elucidating the intrinsic properties of a microwave device.

Since the surface impedance of a good conductor is $Z_s = (j\omega\mu_o/\sigma_c)^{1/2}$, similarity for a superconductor the surface impedance can be obtained by using the complex conductivity as Eq. (3.107) instead of σ_c, thus

$$
\begin{aligned}
Z_s^{s.c.} &= \sqrt{\frac{j\omega\mu_o}{\sigma_1 - j\sigma_2}} = \left[\frac{j\omega^2\mu_o^2\lambda^2}{\frac{n_n}{n}\sigma_n\mu_o\omega\lambda^2 - j}\right]^{1/2} \\
&= j\mu_o\omega\lambda\left[1 + j\frac{n_n}{n}\sigma_n\mu_o\omega\lambda^2\right]^{-1/2} \\
&\cong j\mu_o\omega\lambda\left[1 - j\frac{n_n}{2n}\sigma_n\mu_o\omega\lambda^2\right]^{1/2} = \frac{\omega^2\mu_o^2\lambda^3 n_n\sigma_n}{2n} + j\mu_o\omega\lambda \\
&= R_s + jX_s,
\end{aligned}
\qquad (3.108)
$$

where R_s and X_s are the surface resistance and reactance of the superconductor, respectively. The surface resistance of a superconductor increases more rapidly ($\propto \omega^2$) than that in normal metals in frequency. The surface reactance can be seen that energy storage in the surface of the superconductor takes place via two mechanisms: kinetic energy storage and the energy storage due to the penetration of the field. If both R_s and X_s have been determined by measurements, it is often useful to determine σ_1 and σ_2 from their values. Equation (3.108) can be rearranged to give

$$
\sigma_1 - j\sigma_2 = \frac{\omega\mu}{X_s^4 + 2R_s^2X_s^2 + R_s^4}(2R_sX_s - j(X_s^2 - R_s^2)). \qquad (3.109)
$$

From Eq. (3.109), the real part σ_1 and imaginary part σ_2 of the complex conductivity

$\sigma = \sigma_1 - j\sigma_2$ can be found as [121]

$$\sigma_1(T) = 2\mu_0\omega \frac{R_s X_s}{(R_s^2 + X_s^2)^2}, \tag{3.110}$$

$$\sigma_2(T) = \mu_0\omega \frac{X_s^2 - R_s^2}{(R_s^2 + X_s^2)^2}. \tag{3.111}$$

If required, this can be simplified further using $R_s << X_s$.

Based on two-fluid model, the σ_1 relates to the quasiparticle's relaxation time to be

$$\sigma_1(T) = \frac{n_n(T)e^2}{m} \frac{\tau(T)}{1 + \omega^2\tau^2(T)}, \tag{3.112}$$

where τ(T) is the relaxation time of the quasiparticles. The paired electron (superfluid) resulting in the imaginary part of the complex conductivity is

$$\sigma_2(T) = \frac{n_s(T)e^2}{m\omega} = \frac{1}{\mu_0\omega\lambda^2}. \tag{3.113}$$

From Eq. (3.112) and Eq. (3.113) we can get

$$\frac{\sigma_1(T)}{\sigma_2(T)} = \frac{n_n(T)}{n_s(T)} f(\omega\tau), \tag{3.114}$$

$$f(\omega\tau) = \frac{\omega\tau(T)}{1 + \omega^2\tau^2(T)}. \tag{3.115}$$

Note that f(ωτ) is only a function of ωτ and it has the following properties [55].

1. An linear slope, f(ωτ) ≈ ωτ for the condition of ωτ << 1,
2. a broad peak centered at ωτ = 1 with maximum value of 1/2, and
3. f(ωτ) ≈ 1/ωτ for ωτ >> 1.

Moreover, a small value of f(ωτ) is usually occurring for temperature near T_c, and the value is increasing with respect to temperature decreasing. According the Eq. (3.114), the $\sigma_1(T)/\sigma_2(T)$ remains finite values while $n_n(T)/n_s(T)$ is going to zero at zero temperature. Therefore the $[\sigma_1(T)/\sigma_2(T)]/[n_n(T)/n_s(T)]$ diverges at zero temperature. This question was a long-standing problem and was already studied in detail in conventional superconductors [119]. If the divergence of $[\sigma_1(T)/\sigma_2(T)]/[n_n(T)/n_s(T)]$ could be attributed to a residual normal electron density fraction x_n which depends on the defects in the sample, then the temperature-dependence of ratio of the normal electrons to the pairing electrons is

$$\frac{n_n(T)}{n_s(T)} = \frac{1}{(1-x_n)(\lambda^2(5K)/\lambda^2(T))} - 1. \tag{3.116}$$

From the measurements of $\sigma_1(T)$ and $\sigma_2(T)$, x_n can be determined from Eq. (3.114) if $f(\omega\tau)$ is known.

Using Eq. (3.114), Eq. (3.115) with Eq. (3.116) one can calculate the scattering rate τ^{-1} as

$$\frac{1}{\tau} = \frac{2a\omega}{1-\sqrt{1-4a^2}}, \tag{3.117}$$

where

$$a = \frac{\sigma_1(T)/\sigma_2(T)}{1/(1-x_n)(\lambda^2(5K)/\lambda^2(T)) - 1}. \tag{3.118}$$

Here we have used the generalized two-fluid model to analyze the real and imaginary part of the conductivity obtained from surface resistance and penetration depth. It turns out that the generalized, $f(\omega\tau)$, can be used to extract the temperature dependence of scattering rate τ^{-1}.

3.5 Penetration Depth

To understand the electrodynamics of HTS one must first know the London penetration depth, which is the length scale to reveal the properties within which the supercurrent screens out external fields from the surface into the superconductor. In the local limit, the London penetration depth is given by

$$\lambda = \sqrt{\frac{\mu_0 n_s e^2}{m^*}}, \tag{3.119}$$

where μ_0, n_s and m* are permeability of vacuum, superfluid density and effective mass, respectively. Herein a rich information about the ratio between the superfluid density and the effective mass of paired electrons, n_s/m^* can be obtained. However, it could not account for the behavior of the penetration depth of the Sn and Pb due to the non-local effect resulted from the fact that the penetration depth λ is shorter than coherence length ξ in the superconducting state. When the dimension of ξ is smaller than λ, the London's model can be applied to the system.

We consider a microstrip line resonator, which was successfully fabricated using double-sided YBCO or Ca-YBCO films deposited on $LaAlO_3$ (LAO) substrates. Assuming that the magnetic field is zero just at outside resonator and neglecting the edge effects, the magnetic field can be calculated in the following way.

The field solution of the superconducting microstrip line structure must satisfy Maxwell's equations and the London equation. We assume that the superconducting currents flow in the z-direction in the strip line so that vector potential A has only the z-component A_z. For a wide superconducting strip line it is reasonable that the normal magnetic field B_n vanishes along the dielectric-superconductor boundary. So to a good approximation, the normal magnetic field along the boundary may be assumed to be zero (assuming no vortex),

$$B_n = \frac{\partial A_z}{\partial t} = 0, \tag{3.120}$$

where t indicates tangential direction. Integrating Eq. (3.120) in tangential direction, we find that A_z is constant along the dielectric-superconductor boundary. The relationship of vector potential A_z satisfies with the Laplace equation in the dielectric region. Thus the magnetic field distribution for A_z in the dielectric region can be solved in term of the Laplace equation with the boundary conditions

$A_z = 1$ at the strip-line surface,

$A_z = 0$ at the ground-plane surface. (3.121)

The two constants can be chosen arbitrarily, but are chosen to be 0 and 1 for convenience. For the thin strip over a ground plane, the boundary conditions are $B_x(b) = B_o = \mu_o I/w$ and $B_x(0) = 0$, where I is the total current, w and b are the width and thickness of the strip, respectively. Then one can obtain

$$B_x(y) = (\mu_o I/w)\, sinh(y/\lambda)/sinh(b/\lambda). \tag{3.122}$$

The total storage energy per unit length includes the magnetic field energy and the kinetic energy, which can be expressed in terms of inductance as

$$\frac{LI^2}{2} = \int_V \left(\frac{B^2}{2\mu_o} + \frac{L_k I^2}{2} \right) \mathrm{dv} \tag{3.123}$$

From the right hand side of the Eq. (3.123), the first term presents the storage energy due to the magnetic field and the second term is the kinetic energy of the supercurrent flowing in the superconductor. When the supercurrent in the London local limit, then we have

$$J_s = j\left(\frac{n_s^* e^{*2}}{\omega m^*} \right) E, \tag{3.124}$$

where E is the electric field. Comparing with the inductance of an effective circuit, it shows

$$L = \frac{1}{j\omega}\frac{V}{I} = \frac{1}{j\omega}\frac{E\cdot\ell}{J\cdot A} = \frac{1}{j\omega}\frac{\ell}{A\sigma} \tag{3.125}$$

where A and ℓ are the cross-section area and the length of the transmission line, respectively, and σ is the conductivity. Then the kinetic inductance of the superconductor is

$$L_k = \frac{m^*}{n_s^* e^{*2}}\frac{\ell}{A} = \mu_o\lambda^2\frac{\ell}{A} \tag{3.126}$$

, since $\lambda^2 = m^*/\mu_o n_s^* e^{*2}$. In turn, the kinetic energy of the supercurrent per unit volume is

$$\frac{I^2 L_k}{2V} = \frac{(J_s\cdot A)^2}{2(A\cdot\ell)}\frac{\mu_o\lambda^2\ell}{A} = \frac{\mu_o\lambda^2 J_s^2}{2}. \tag{3.127}$$

Inserting above equation into Eq. (3.123), the total energy can be expressed as

$$\frac{LI^2}{2} = \int_V\left(\frac{B^2}{2\mu_o} + \frac{\mu_o\lambda^2}{2}J_s^2\right)\mathrm{dv}. \tag{3.128}$$

The integration of the volume is performed including three parts:

(i) the upper strip with thickness b_1,

(ii) the ground plane with thickness b_2,

(iii) the intervening space with thickness h.

Combining the Eq. (3.122) with $\vec{J} = \nabla\times(\vec{B}/\mu_o)$, one can obtain

$$J_{sz}(y) = \frac{I}{j\omega}\frac{\cosh(y/\lambda)}{\sinh(b/\lambda)}. \tag{3.129}$$

Therefore, the storage energy of the upper strip per unit length is

$$\frac{1}{2\mu_o}\int_{S_1}\left(\frac{\mu_o^2 I^2}{w^2}\frac{\sinh^2(y/\lambda_1)}{\sinh^2(b_1/\lambda_1)}+\mu_o^2\lambda_1^2\frac{I^2}{\lambda_1^2 w^2}\frac{\cosh^2(y/\lambda_1)}{\sinh^2(b_1/\lambda_1)}\right)ds_1$$

$$=\frac{\mu_o^2 I^2/w^2}{2\mu_o\sinh^2(b_1/\lambda_1)}\int_0^{b_1}\left(\sinh^2(y/\lambda_1)+\cosh^2(y/\lambda_1)\right)w\cdot dy \tag{3.130}$$

$$=\frac{\mu_o I^2}{2w}\lambda_1\coth(b_1/\lambda_1).$$

Similarly, the storage energy of the ground plane per unit length is

$$\frac{1}{2\mu_o}\int_{S_2}\left(\frac{\mu_o^2 I^2}{w^2}\frac{sinh^2(y/\lambda_2)}{sinh^2(b_2/\lambda_2)}+\mu_o^2\lambda^2\frac{I^2}{\lambda_2^2 w^2}\frac{cosh^2(y/\lambda_2)}{sinh^2(b_2/\lambda_2)}\right)ds_2$$

$$=\frac{\mu_o I^2}{2w}\lambda_2\,coth(b_2/\lambda_2). \tag{3.131}$$

Finally, we assume that the magnetic field is uniform in the intervening dielectric, and then the stored magnetic energy per unit length is obtained as

$$\frac{1}{2\mu_o}\int_{S_3}\left(\frac{\mu_o^2 I^2}{w^2}\right)ds_3=\frac{1}{2\mu_o}\int_0^h\frac{\mu_o^2 I^2}{w^2}w\cdot dy$$

$$=\frac{\mu_o I^2}{2w}h. \tag{3.132}$$

In combination with Eq. (3.130), Eq. (3.131), Eq. (3.132) and Eq. (3.128), the total inductance of the microstrip transmission line is given by

$$L=\frac{\mu_o h}{w}\left(1+\frac{\lambda_1}{h}coth(b_1/\lambda_1)+\frac{\lambda_2}{h}coth(b_2/\lambda_2)\right). \tag{3.133}$$

It is evident that the inductance of microstrip line resonator is expressed by the function of the penetration depths [124]. Furthermore, the above equation is not available if the contribution of the fringe field due to the finite dimension of the strip line ($w/h=1$ in our microstrip line resonator) takes into account. A detailed formula was derived by Chang [125-127], who considered the inductance of a finite-width of the superconducting microstrip line. The result is given by Eq. (3.134). The first term in the equation represents the contribution due to the external magnetic field, the second and the third terms are the contributions from the strip line, and the fourth term is the contribution from the ground plane.

$$L = \frac{\mu_o}{wK(w,h,b)}\{h + \lambda_1[\coth(b_1/\lambda_1) + g\,csch(b_1/\lambda_1)] + \lambda_2 \coth(b_2/\lambda_2)\}, \quad (3.134)$$

where the fringing field factor K is

$$K = \frac{h}{w}\frac{2}{\pi} ln\left(\frac{2r_b}{r_a}\right), \quad (3.135)$$

and

$$g = \frac{2p^{1/2}}{r_b}, \quad (3.136)$$

$$r_b = r_{bo} - [(r_{bo} - 1)(r_{bo} - p)]^{1/2} + (p+1)\,tanh^{-1}\left(\frac{r_{bo} - p}{r_{bo} - 1}\right)^{1/2} - 2p^{1/2}\,tanh^{-1}\left(\frac{r_{bo} - p}{p(r_{bo} - 1)}\right)^{1/2} + \frac{\pi w}{2h}p^{1/2}, \quad \text{for } 5 > \frac{w}{h} \geq 1, \quad (3.137)$$

$$r_{bo} = \eta + \frac{p+1}{2}\ln(\Delta), \quad (3.138)$$

$$p = 2\beta^2 - 1 + \left[(2\beta^2 - 1)^2 - 1\right]^{1/2}, \quad (3.139)$$

$$\Delta = \text{larger value}, \eta \text{ or } p \quad (3.140)$$

$$\eta = p^{1/2}\left\{\frac{\pi w}{2h} + \frac{p+1}{2p^{1/2}}\left[1 + \ln\left(\frac{4}{p-1}\right)\right] - 2\tanh^{-1}\left(p^{-1/2}\right)\right\}, \quad (3.141)$$

$$p = 2\beta^2 - 1 + \left[(2\beta^2 - 1)^2 - 1\right]^{1/2}, \quad (3.142)$$

$$\beta = 1 + \frac{b_1}{h}, \quad (3.143)$$

$$\ln r_a = -1 - \frac{\pi w}{2h} - \frac{p+1}{p^{1/2}}\tanh^{-1}\left(p^{-1/2}\right) - \ln\left(\frac{p-1}{4p}\right), \quad (3.144)$$

and the effective capacitance per unit length is

$$C = \frac{\varepsilon_r \varepsilon_o w}{h} K(w, h, b_1), \tag{3.145}$$

where ε_r is the dielectric constant of the intervening material.

Since the resonance frequency, $f_0(T)$, is proportional to the inverse square root of the product of inductance and capacitance, and the inductance is function of penetration depth, one thus can extract $\lambda(T)$ from the measurement of $f_0(T)$. If the dielectric constant ε_r is insensitive to the temperature dependence, then the change of the capacitance with respect to the temperature should be much smaller than that of the inductance. So, the variation of the resonance frequency is mainly due to the change of the inductance, and one can obtain the relation

$$\frac{f_o(T)}{f_o(T_0)} = \sqrt{\frac{L(\lambda(T_0))}{L(\lambda(T))}}. \tag{3.146}$$

Substituting Eq. (3.134) into Eq. (3.146), we have

$$\frac{f_o(T)}{f_o(T_0)} = \sqrt{\frac{h + \lambda(T_0)[\coth(b_1/\lambda(T_0)) + g\,\mathrm{csch}(b_1/\lambda(T_0))] + \lambda(T_0)\coth(b_2/\lambda(T_0))}{h + \lambda(T)[\coth(b_1/\lambda(T)) + g\,\mathrm{csch}(b_1/\lambda(T))] + \lambda(T)\coth(b_2/\lambda(T))}}, \tag{3.147}$$

where T_0 is an arbitrary temperature (taken as 5 K here) and g is the conductor asymmetry factor. So, the relation of the penetration depth and temperature can be estimated by the Eq. (3.147) in the microstrip line resonators. For the ring resonator, we will also use the above equation as a sound approximation since the mean radius of the ring is much greater than the width of the strip. When the penetration depth is known, the other microwave properties, such as surface impedance, can be extracted immediately.

As mentioned previously, from the unloaded quality factor Q_u and the resonance frequency $f(T)$, the surface resistance Rs and reactance Xs consisting of surface impedance $Zs(T)=Rs(T)+iXs(T)$ are calculated with the aid of the formula:

$$R_s(T) = \Gamma / Q_u(T), \quad (3.148)$$

$$Xs(T)-Xs(T_0)=-2\Gamma[f(T)-f(T_0)]/f(T), \quad (3.149)$$

where Γ is the geometry factor of the resonator and T_0 is the lowest temperature in the experiment (T_0 = 5K). Actually, Γ depends [121] on the penetration depth λ and geometry parameters (such as the microstrip line width w, substrate thickness h, and thin film thickness b) of the superconducting resonator if b/λ is not much larger than 1. The value of Γ can be calculated by the method of incremental frequency rules [128-130] via

$$\Gamma=\mu_0\omega^2/4\{-\Delta\omega/\Delta\lambda\}^{-1}, \quad (3.150)$$

where Δω/Δλ is the derivative of the angular resonance frequency (ω=2π*f*) with respect to the penetration depth λ(*T*). Using Chang's formula for the inductance and capacitance of a superconducting strip line the angular resonance frequency is given by:

$$\omega (T)=(c/\varepsilon_r^{1/2}L)\{1+\lambda(T)/d[2\coth(b/\lambda(T))+g\mathrm{csch}(b/\lambda(T))]\}^{-1/2}, \quad (3.151)$$

where ε_r is the dielectric constant of the substrate, d is the effective substrate thickness, L is the length of the strip line, g is the conductor asymmetry factor, and c is the velocity of the light in vacuum. By substituting Δω/Δλ obtained from Eq. (3.151) into Eq. (3.150), Γ can be derived as

$$\Gamma=(1/2)\mu_0\omega d\{1+\lambda/d[2\coth(b/\lambda)+g/\sinh(b/\lambda)]\}/$$
$$\{\coth(b/\lambda)+(b/\lambda)/\sinh^2(b/\lambda)+g[1+(t/\lambda)\coth(b/\lambda)]/[2\sinh(b/\lambda)]\}, \quad (3.152)$$

where g is the conductor asymmetry factor accounting for the fringe field effect as the aspect ratio w/d of the stripline near or less than 1. On the other hand, we have the relation between the surface reactance Xs(*T*) and the penetration depth λ(*T*), i.e.,

$$Xs(T)=\mu_0\omega\lambda(T). \quad (3.153)$$

Here we have used Chang's inductive formula and the generalized two-fluid model to analyze the magnetic penetration, from which the surface resistance and reactance can be obtained. It turns out that the real and imaginary part of the conductivity can be extracted by use of Eq. (3.110) and Eq. (3.111).

3.6 Some Models in Disordered d-wave Superconductors (Lee's model, Vishveshwara and Fisher's model)

From theoretical viewpoint, a number of theories on the oxide d-wave superconductors give strong indications that the effect of disorder playing an essential role in these materials, especially for Lee's model [80], Vishveshwara and Fisher's model [84] that we will utilize these two models to analyze our experimental results. In Lee's model, impurity scatters in the unitary limit produce low energy quasiparticles in a two-dimensional d-wave superconductor. They argued that if the impurity concentration is small so that the wave functions in the normal state are essentially extended, the quasiparticles in the superconducting state becomes strongly localized for a short coherence length d-wave superconductor. An effective mobility gap then leads to thermally activated behavior for the microwave conductivity and possibly for the London penetration depth. Their model also stressed that the quasiparticles are localized in a localization length ξ_L with energy $E < \gamma_0$, where γ_0 is the impurity band. And these quasiparticles have caused the universal conductance $\sigma_0 = \frac{e^2}{2\pi\hbar}\frac{\xi_0}{a}$, where $\xi_0 = \frac{v_F}{\pi\Delta_0}$ is the coherence length and a is lattice constant. It should be noted that in this case σ_0 is independent of scattering rate $1/\tau$.

In Lee's model, two models of disorder were considered. In model Ⅰ, he assumed a δ correlated random potential $U(\vec{r})$ such that $< U(\vec{r})U(0) >= u_0^2\delta(\vec{r})$. In the normal state, the Born approximation leads to an isotropic scattering rate

$\tau^{-1} = 2\pi\rho_0 u_0^2$, where ρ_0 is the density of states. In model Ⅱ, he assumed a dilute density n_i of strong scatterers, each with phase shift δ_0. For simplicity, he specialized to the unitary limit $\delta_0 = \pi/2$, in which case the normal state scattering rate is $\tau^{-1} = \Gamma \equiv n_i/\pi\rho_0$. Note that in contrast to the Born approximation τ^{-1} is inversely proportional to ρ_0. In both models, he assumed that $\varepsilon_F\tau >> 1$, so that the normal state is a good metal.

In particular, we consider Lee's model with model Ⅱ, in which the low-lying quasiparticles are localized. The self energy is given by

$$\Sigma(i\omega_n) = \Gamma g_0(\omega_n)/[c^2 - g_0^2(\omega_n)], \tag{3.154}$$

where $\Gamma = n_i/\pi\rho_0$, $c = \cot\delta_0$, and $g_0(\omega_n) = (\pi\rho_0)^{-1} M \times \Sigma_k G(k,\omega_n)$. G is the Green function near each zero of d-wave energy gap, $\Delta_{\vec{k}} = \Delta_0(\cos k_x a - \cos k_y a)$. If we focus around the point $\vec{k}_0$ on the Fermi surface in the (1,1) direction, it lets us introduce a coordinate system parallel ($\hat{k}_1$) and perpendicular ($\hat{k}_2$) to the Fermi surface at $\vec{k}_0$. One obtains $\xi_{\vec{k}} = v_F k_2$ and $\Delta_{\vec{k}} = v_1 k_1$ where $k_1 = (k_x - k_y)/\sqrt{2}$, $k_2 = (k_x + k_y)/\sqrt{2} - |\vec{k}_0|$, and $v_1 = \sqrt{2}\Delta_0 a \sin(k_{0x} a)$. Then, G is given by

$$G(k,\omega_n) = (-i\tilde{\omega}_n + v_F k_2)/(\tilde{\omega}_n^2 + v_1^2 k_1^2 + v_F^2 k_2^2). \tag{3.155}$$

In the unitary case, we have $\delta_0 = \pi/2$ or $c = 0$. The solution of Eq. (3.154) in the limit $\omega \to 0$ yields an estimate for $\gamma_0 = i\Sigma(\omega \to 0)$ to be

$$\gamma_0 = (\Gamma\pi k_F v_1/2)^{1/2} \approx \Delta_0(\Delta_0\tau)^{-1/2}, \tag{3.156}$$

where we have ignored a small logarithmic correction of order $\ln(\Delta_0/\Gamma)$. For unitary scatterers, the scattering rate $\gamma'(\omega)$ of a quasipartilce with energy ω is estimated to be $\gamma(\omega) \approx \Gamma\Delta_0/\omega$ for $\gamma_0 < \omega < \Delta_0$.

The localization length ξ_L was then be estimated for unitary scatterers. The conclusion that the dimensionless conductance $g \approx v_F / v_2 \approx \xi_0 / a \approx \varepsilon_F / \Delta$ is not changed and is of order unity for the oxide superconductors. The mean free path, i.e., the distance scale at which diffusion begins, is now given by v_F / γ_0 and we estimate the localization length to be

$$\xi_L \approx (v_F / \Delta_0)(\Delta_0 \tau)^{1/2} \exp(\varepsilon_F / \Delta_0), \tag{3.157}$$

which is reasonably short for a short coherence length superconductor even in the clean limit $\Delta_0 \tau \approx 100$.

To ascertain the physical significance of the localization length we estimate the typical energy level spacing between states within a localization length, i.e., $\Delta W = (\rho \xi_L^2)^{-1}$. Using Eq. (3.157) we find

$$\Delta W \approx \gamma_0 (\Delta_0 / \varepsilon_F) \exp(-2\varepsilon_F / \Delta_0). \tag{3.158}$$

For the oxide superconductors, $\Delta_0 / \varepsilon_F \approx 1$ and we find that ΔW is a reasonable fraction of γ_0. The significance of ΔW is that for $T < \Delta W$ the localized states begin to decouple and conductivity becomes activated.

The microwave conductivity can be calculated via the formula, $\sigma_1 \approx v_F^2 \sum_k G_+(\vec{k}) G_-(\vec{k})$, where $G_\pm = (\xi_k \pm i / 2\tau)^{-1}$, for normal metals or superconductors. For a d-wave superconductor, the density of states is linear in energy $N(\varepsilon) \propto \varepsilon$, so that crudely we expect $\sum_k \to \int d\xi \xi$. The extra factor of ξ changes the power counting so that σ_1 is independent of τ. After being careful calculations performed by Lee, the microwave conductivity at $T \to 0$ and $\omega \to 0$ was obtained as $\sigma_1 \approx \frac{e^2}{2\pi\hbar} \frac{\xi_0}{a}$ when considering the low-lying quasiparticles are localized in model Ⅱ. In fact, this result of $\sigma_1 \approx \frac{e^2}{2\pi\hbar} \frac{\xi_0}{a}$ at $T \to 0$ and $\omega \to 0$ was also hold in model Ⅰ.

From the above discussion, we have known that the presence of localization of impurity-induced low-energy quasiparticle states in a two dimensional d-wave superconductor is shown for a small amount of disorder in the limit of unitary scatterers. Recently, the role of imperfections in d-wave superconductors has also been considered by Hirschfeld et al. [131], Durst and Lee [81], Vishveshwara and Fisher [84], Balatsky and Salkola [85], and subsequently by others [132, 133]. In particular, Vishveshwara and Fisher theoretically explored the quasiparticle transport properties which are greatly influenced by the disordered state in the samples. They started from that qausiparticle excitations about the superconducting ground state are well described within the framework of the Bogoliubov-de Gennes (BdG) Hamiltonian for a spin-singlet paired superconductor. To repeat the example offered of the dirty d-wave superconductor, they consider a system of impure superconducting sheets with d-wave pairing coupled to one another. For low interplane coupling strength or high impurity concentration, they found that the qausiparticle states to be localized, and upon increasing the coupling or lowing the disorder, they found that the quasiparticle states are of accessible a critical point beyond which these states become extended. A detail calculation and numerical analysis was reference to Ref. [84]. In a word, they conclude that the low-energy quasiparticles can either be delocalized and free to move through the sample as an extended state, or can be localized by the disorder. These two possibilities correspond to two distinct superconducting phases –the thermal metal with delocalized quasiparticle excitations and the thermal insulator with localized quasiparticles- and the critical point between them.

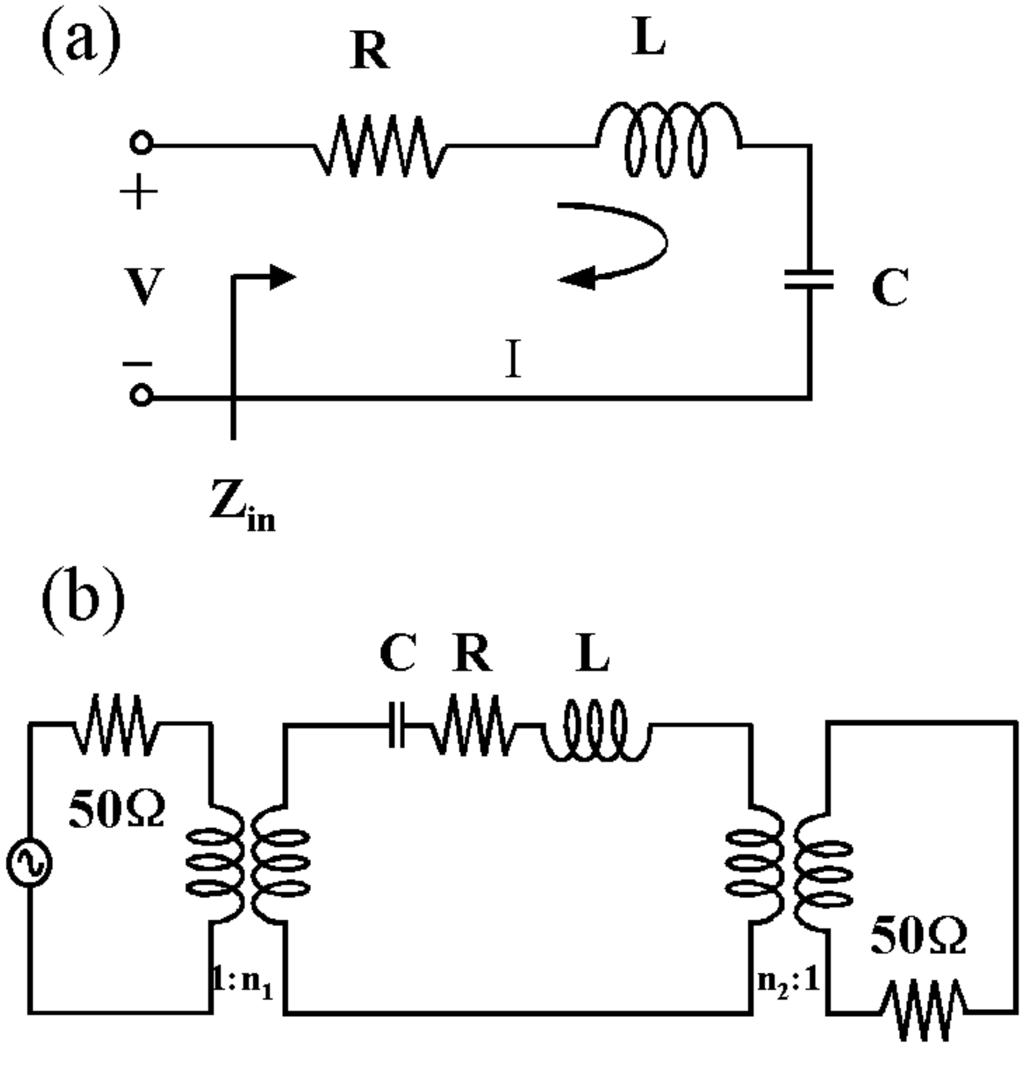

Fig. 3.1. (a) A series RLC circuit. (b) Equivalent circuit of a cavity resonator for two-port measurements.