國 立 交 通 大 學

應 用 數 學 系

博士論文

三級式克勞斯網路與多重對數網路的廣義不阻塞

Wide-Sense Nonblocking for 3-stage Clos
Network and Multi-$\log_d N$ Network

研 究 生: 郭君逸
指導教授: 黃光明 教授

中 華 民 國 九 十 四 年 六 月

三級式克勞斯網路與多重對數網路的廣義不阻塞

Wide-Sense Nonblocking for 3-stage Clos Network and Multi-$\log_d N$ Network

研 究 生 : 郭君逸　　Student: Junyi Guo
指導教授: 黃光明 教授　Advisor: Frank K. Hwang

國 立 交 通 大 學

應 用 數 學 系

博 士 論 文

A Dissertation
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy
in
Applied Mathematics
June 2005
Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 四 年 六 月

Abstract

The 3-stage network was first proposed by Clos and is one of the most basic multistage interconnecting network. Clos (1953) showed that the number of middle crossbar required for strictly nonblocking is $2n - 1$, where n is the number of inlets of an input crossbar. Beneš (1965) constructed an example to show that using packing routing strategy can make the number of middle crossbar required lower. This has remained the only example of wide-sense non-blocking 3-stage Clos network which is not strictly nonblocking.

In this thesis, we showed that the number of middle crossbar required for wide-sense nonblocking under several routing strategies: save the unused, packing, minimum index, cyclic static, and cyclic dynamic, which has been studied in the literature is the same as required for strictly nonblocking and extended them to asymmetric 3-stage Clos network. In particular, we prove the same conclusion for the multi-$\log_d N$ network and extend to a general class of network.

摘要

克勞斯 (Clos) 在1953年首先提出了「三級式網路」, 這也是最基本的多級式交換網路之一; 假設 n 是一個輸入交換器的進線個數, 克勞斯證明了這樣的網路只需要用到 $2n-1$ 個中繼交換器, 即能使得此網路達到絕對不阻塞。在1965年, 班尼斯 (Beneš) 舉了一個例子說明, 使用「優先選最忙的中繼交換器」的策略, 真的可以降低中繼交換器的使用數目, 而依然還是能讓此網路不阻塞。不過, 這個例子也是至今唯一個三級式克勞斯網路是廣義不阻塞, 而不是絕對不阻塞的例子。

在這篇論文裡, 我們證明了文獻中所提到的一些傳遞策略: 不用的最後(STU)、最忙錄的優先 (P)、編號小的優先 (MI)、從上次的編號開始 (CS) 及從下一個編號開始 (CD), 在這些策略之下, 要達到不阻塞所需要的中繼交換器個數與要達到絕對不阻塞所需要的一樣。而我們也將這個結果推廣到不對稱的情況, 甚至我們還在多重對數網路上也得到了同樣的結果。最後, 我們也將此結果推廣到了同類型的網路上。

誌謝

我最先要感謝的當然就黃光明老師。在還沒進到交大以前的我, 從沒有聽過「群試」這個名詞, 直到我聽了黃老師的第一門課之後, 對我產生極大的影響, 再加上看見了黃老師對研究的熱忱, 讓我對做研究產生了極大的興趣。另一方面, 也因為黃老師涉及很多領域, 也讓我學到了很多不同領域的東西。此外, 每次有什麼想法跟老師說, 或是有什麼問題去問老師, 他都非常用心的審視我提的問題, 儘速給我答案或是建議, 因此這幾年來有了黃老師的幫忙, 我才能如此順利畢業。

我也蠻感激劉昱琪學姊, 因為她極力推薦我黃老師很不錯, 才讓我開始對黃老師有了認識。陳秋媛老師也在這幾年來, 給了我很大的幫助, 她就像是我的好朋友一樣, 不僅是課業上, 生活上有什麼問題都可以找她聊聊; 她的對學生的關懷、奉獻, 真的很讓我感動。此外, 我還要感謝曾經指導過我的老師: 傅恆霖老師的圖論、組合設計、編碼, 他用生動的講解來取代課本上枯燥的證明, 第一次讓我覺得這些課程怎麼變的這麼有趣。黃大原老師啓發性的教學, 也很讓我印像深刻。翁志文老師對LaTeX熟悉, 常是我討論LaTeX的對像。在交大這幾年, 我也認識了不少朋友: 賓賓、飛黃、Robin、琲琪 . . . 等, 感謝他們陪我做研究、討論, 陪我一起打球, 甚至是打牌, 有了他們, 我才能這麼順利, 過得這麼快樂。在這裡, 我也認識了我的另一半, 惠蘭, 感謝她不厭其煩的聽我講述我做的結果, 給我建議, 也感謝她在我心情不好的時候, 能夠與我一起分擔。最重要的, 我還要感謝我的父母, 提供我這麼好的學習環境, 讓我能夠把心放在課業上、研究上, 他們的鼓勵、讚許, 一直給了我很大的信心及動力。

要感謝的人很多, 感謝的心情, 實在難以形容, 我想, 我還要感謝上天, 讓我一路走來, 能夠平平穩穩、順順利利。感謝各位老師、同學的支持與鼓勵, 讓君逸能夠有如今的成果。謝謝你們, 謝謝。

Contents

List of Figures

Chapter 1

Introduction

The need of a switching network first came from the need to interconnect pairs of telephones. Later, it was reinvented for parallel computer to interconnect a set of processors with a set of memories. Currently, it is intended for many other applications, data transmission, conference calls, satellite communication .... One frequently discussed topic in switching networks is its nonblocking property. There are different levels of nonblockingness: strictly nonblocking, wide-sense nonblocking, and rearrangeable nonblocking. We will discuss more detail in Section 1.1. Because wide-sense nonblocking networks use an algorithm to route requests, the cost of it is expected to be less than strictly nonblocking networks. In these thesis, we study several routing strategies which have been studied in the literature [9] and gave an amazing result that the cost of these networks are the same with strictly nonblocking networks in 3-stage Clos network and multi-log N network.

1.1 Preliminaries

The basic components of switching network are crossbar switches, or just *crossbars*, and *links* which connect crossbars. A crossbar with n inlets and m outlets, denoted by X_{nm}, is said of size $n \times m$(See Figure 1.1(a)). (For convenience, we draw a two side crossbar without exposing its internal wiring (See Figure 1.1(b))). Inlets(outlets) on the same crossbar are called *co-*

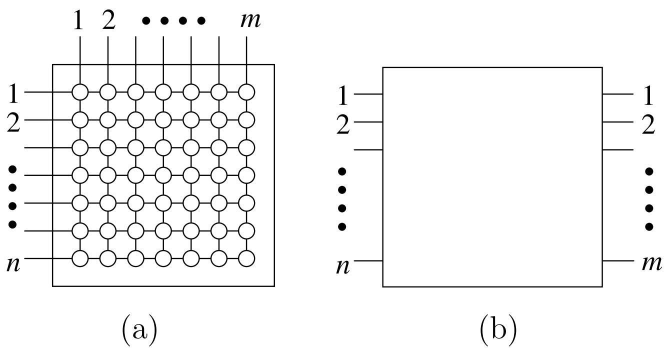


Figure 1.1: X_{nm}

inlets(*co-outlets*). In a crossbar, each inlet and each outlet are connected by a crosspoint. Therefore, there are $m \times n$ crosspoints in X_{nm}. Each crosspoint determine some inlet a and some outlet b is connected or not. In general, we assume one inlet(outlet) can only connect to only one outlet(inlet) and any matching between the inlets and the outlets is routable.

In an s-stage interconnection network, the crossbars are lined up into s columns, each called a *stage*. Crossbars in the same stage have the same size and links exist only between crossbars in adjacent stages(See Figure 1.2). The inlets(outlets) of the first(last) stage are called inputs(outputs) of the

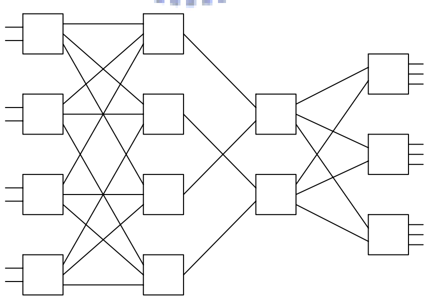
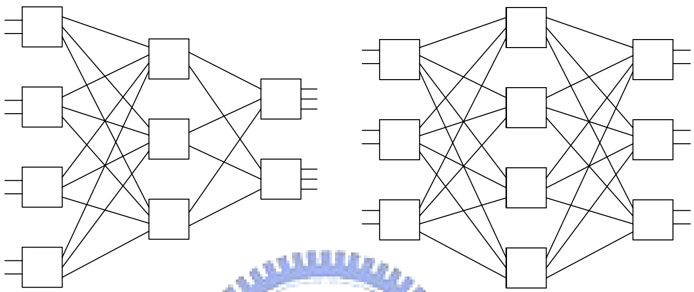


Figure 1.2: A 4-stage interconnection network

network.

Let $X \times Y$ to denote a network structure in which each crossbar of the last stage of X has a link to each crossbar of the first stage of Y. The 3-stage network $X_{n_1m} \times X_{r_1r_2} \times X_{mn_2}$ was first proposed by Clos(1953) and is now known as the 3-stage Clos network. For convenience, we use the notation $C(n_1, r_1, m, n_2, r_2)$ to denote $X_{n_1m} \times X_{r_1r_2} \times X_{mn_2}$ and label the crossbars of the first(second, third) stage from $I_1(M_1,\ O_1)$ to $I_{r_1}(M_m,\ O_{r_2})$. See Figure 1.3(a). Then the number of inputs $N_1 = n_1r_1$ and the number of outputs $N_2 = n_2r_2$. If $n_1 = n_2$ and $r_1 = r_2$, then we call this 3-stage Clos network symmetric and denoted by $C(n, m, r)$. See Figure 1.3(b).



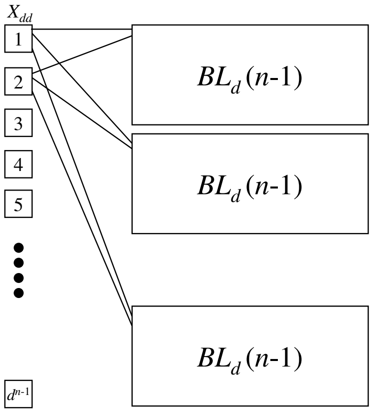
(a) $C(2, 4, 3, 3, 2)$ (b) $C(2, 4, 3)$

Figure 1.3: (a) asymmetric (b) symmetric

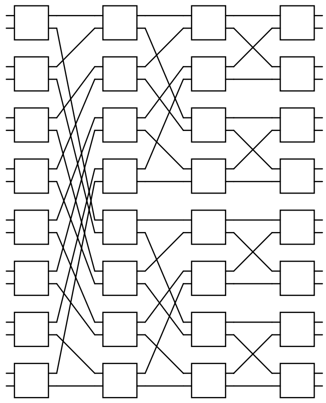
A d-nary baseline network of order n denoted by $BL_d(n)$ has d^n inputs, d^n outputs and n stages, each stage contains d^{n-1} $d \times d$ crossbars. The linking pattern of $BL_d(n)$ can be obtained by $n-1$ recurrent constructions(See Figure 1.4(a)). Figure 1.4(b) gives the example of $BL_2(4)$.

Two networks are called equivalent if the crossbars can be labeled such that the linking functions of the two networks become identical. There are many networks equivalent to baseline network, see [3], such as banyan, Omega, ... We call this class of network $\log_d N$ network, where $N = n^d$. In this thesis, only the baseline architecture will be considered.

For convenience of analysis, we transform a $\log_d N$ network to a digraph by converting each link, including the inputs and the outputs, to a node, while a crosspoint connecting two links in the network becomes an arc in the digraph(See Figure 1.5). Nodes are arranged in $n+1$ stages labeled $0, 1, \ldots, n$



(a) Linking pattern



(b) $BL_2(4)$

Figure 1.4: Linking pattern of the baseline network and an example.

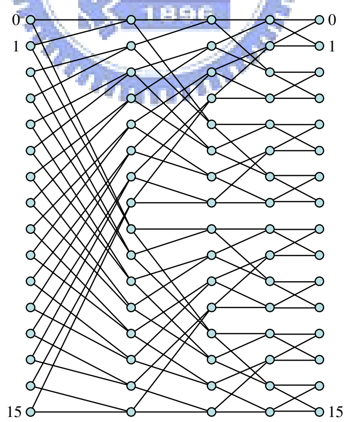


Figure 1.5: The graph model of $BL_2(4)$

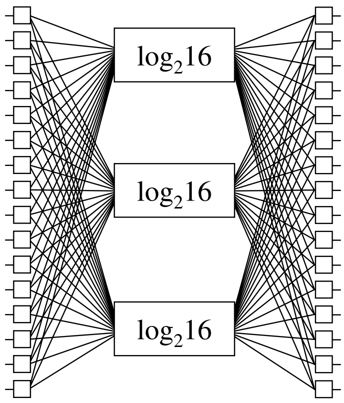


Figure 1.6: Multi-$\log_d N$ Network

from left to right. The nodes in stage 0 correspond to inputs and the nodes in stage n correspond to outputs.

Lea(1990) introduced the multi-$\log_d N$ network composed of p copies of $\log_d N$ network connected in parallel. In each copy, there is exactly one link between an arbitrary input and output. See Figure 1.6.

A request is an (input, output) pair seeking connection. A set of requests can be routed if there exists link-disjoint paths connecting them. A switching network is said to be strictly nonblocking (SNB) if a request can always be routed regardless how the previous requests are routed. It is said to be wide-sense nonblocking (WSNB) with respect to a routing strategy $\mathbb{A}$ if every request is routable under $\mathbb{A}$. The restraint that no two paths in the original network competes for the same link is translated to that no two paths in the graph model competes for the same node. In this thesis we make the common assumption that a crossbar is SNB.

For the 3-stage Clos network, a routing strategy deals with the choice of a middle switch to route the request when many are available. We review the five routing strategies proposed in the literature [9]:

(i) Save the unused (STU). Do not route through an empty middle crossbar unless there is no choice.

(ii) Packing (P). Choose a busiest, yet available, middle crossbar.

(iii) Minimum index (MI). For each request, route in the order M_1, M_2,..., until the first available one emerges.

(iv) Cyclic dynamic (CD). If M_k was used last, try $M_{k+1}, M_{k+2}, \ldots$, until the first available one emerges.

(v) Cyclic static (CS). If M_k was used last, try copy $M_k, M_{k+1}, \ldots$, until the first available one emerges.

For multi-$\log_d N$ network, we translate these routing strategies by replacing "choosing a middle crossbar" to "choosing a copy (of $\log_d N$)". Note that P $\Rightarrow$ STU. So WSNB under STU $\Rightarrow$ WSNB under P since P is a choice of STU. On the other hand, not WSNB under P implies not WSNB under STU.

1.2 Literature review and thesis overview

The notion of wide-sense nonblocking in switching networks is a fascinating idea to computer scientists. It suggests that the hardware can be reduced through intelligent software (routing) without affecting the nonblocking property of the network. The existence of a WSNB network was first demonstrated by Beneš(1965) for the symmetric 3-stage Clos network. He proved that $C(n, m, 2)$ is WSNB under packing if and only if $m \geq \lfloor 3n/2 \rfloor$ which is the only positive result. Lots of efforts have been spent to expand this result, but without success.

Smith[14] proved that $C(n, m, r)$ is not WSNB under P or MI if $m < \lfloor 2n - \frac{n}{r} \rfloor$. Du *et al.* [7] improved to $\lfloor 2n - \frac{n}{2^r-1} \rfloor$ which was extended to cover CS in Hwang [9]. For P, Yang and Wang [18] gave a linear programming formulation of the problem and ingeniously found the closed-from solution $m \geq \lfloor 2n - \frac{n}{F_{2r-1}} \rfloor$ where F_{2r-1} is the $2r-1^{st}$ Fibonacci number, as a necessary condition for $C(n, m, r)$ to be WSNB. Actually, there was an earlier stronger result of Du *et al.* reported in the 1998 look of Hwang [9] that $m \geq 2n - 1$

is necessary and sufficient for $C(n, m, r)$, $n \geq 3$, to be WSNB under P. This result for $r \geq 3$ together with Beneš result for $r = 2$ gave a definitive answer to the WSNB property of $C(n, m, r)$ under P. Finally, Tsai, Wang and Hwang [15] proved that for all n, there exists r large enough such that $C(n, m, r)$ is not WSNB under any algorithm.

The proof of the $m \geq 2n - 1$ result by Du *et al.* is quite difficult to check and the proof of Yang and Wang is also complicated. In Chapter 2 we give a much simpler proof which not only works for P (hence STU), but also for CD, CS and MI. We also extend all these results to the asymmetric 3-stage Clos network $C(n_1, n_2, m, r_1, r_2)$. In Chapter 3, we prove a similar conclusion these strategies require the same number of copies as SNB does. In Chapter 4, we extend our results to a more general class.

Chapter 2

3-stage Clos networks

In this chapter, we study the five strategies, CD, CS, STU, P, and MI, in the 3-stage clos network. And we give a sequence of requests to force each of them using the maximum number of middle crossbars, i.e., the same number of crossbars as required by the SNB network.

2.1 Strictly nonblocking

First, we give classical SNB result on symmetric and asymmetric 3-stage Clos network. Then we can compare it to the result in Section 2.2 and Section 2.3.

Theorem 2.1.1 (Clos [6]). *Assuming* $\min\{r_1, r_2\} \geq 2$, $C(n_1, r_1, m, n_2, r_2)$ *is SNB if and only if* $m \geq \min\{n_1 + n_2 - 1, n_1r_1, n_2r_2\}$.

Proof. Without loss of generality, assume the new request γ is from I_1 to O_1. Clearly, $m = \min\{n_1r_1, n_2r_2\}$ is sufficient since only $\min\{n_1r_1, n_2r_2\}$ requests can be generated, and we can route them each through a distinct middle crossbar in worst case. Furthermore, if the busy co-inlets and co-outlets are each routed through a distinct middle crossbar, then at most $(n_1-1)+(n_2-1)$ middle crossbars are taken; so n_1+n_2-1 middle crossbars also suffice.

Next we prove necessity. Suppose $m = n_1 + n_2 - 2$ and neglect the boundary condition. If we connect $n_1 - 1$ co-inlets to O_2 and $n_2 - 1$ co-outlets from I_2, and let these $n_1 + n_2 - 2$ requests route through distinct middle crossbar, then γ is blocked. Hence m must be greater than n_1+n_2-2. Then we prove the theorem. □

Corollary 2.1.2. *If $r \geq 2$, then $C(n, m, r)$ is SNB if and only if $m \geq 2n-1$.*

Proof. Because $r \geq 2$, $N = nr \geq 2n > 2n - 1$. Hence $\min\{2n - 1, N\} = 2n - 1$. □

2.2 Wide-sense nonblocking in symmetric case

The only positive result in about WSNB is Beneš result [1]. He demonstrated the following theorem on $C(n, m, 2)$:

Theorem 2.2.1. *$C(n, m, 2)$ is WSNB under STU if $m \geq \lfloor 3n/2 \rfloor$.*

Proof. Let s be a state and let $n(s)$ denote the set of busy middle crossbars (carrying at least one connection) in s. Let $n_{ij}(s)$ denote the set of middle crossbars carrying a connection from I_i to O_j, $i, j = 1, 2$. We will prove the theorem by induction on the number of steps to reach s (from the initial empty state):

(i) $|n(s)| \leq \lfloor 3n/2 \rfloor$,

(ii) $|n_{11}(s) \cup n_{22}(s)| \leq n$,

(iii) $|n_{12}(s) \cup n_{21}(s)| \leq n$.

All three claims are trivially true at empty state. Consider a general step from state s' to s. If s is obtained from s' by deleting a connection, the three claims obviously remain true. So assume s is obtained from s' by adding a connection. Without loss of generality, assume it is from I_1 to O_1. If $n_{22}(s') \setminus n_{11}(s') \neq \varnothing$, then a crossbar belong to $n_{22}(s') \setminus n_{11}(s')$ will carry the new request under STU. Thus $|n_{11}(s) \cup n_{22}(s)| = |n_{11}(s') \cup n_{22}(s')| \leq n$ and the claims remain true. Therefore, we assume $n_{22}(s') \setminus n_{11}(s') = \varnothing$.

(i) Since I_1 and O_1 can each be engaged in at most $n-1$ connections,

$$|n_{11}(s') \cup n_{12}(s')| \leq n-1,$$

$$|n_{11}(s') \cup n_{21}(s')| \leq n-1.$$

Using the induction hypothesis (iii)

$$|n_{12}(s') \cup n_{21}(s')| \leq n.$$

Adding them up, we obtain

$$2|n(s')| \leq 3n-2, \text{ or } |n(s')| \leq \lfloor 3n/2 \rfloor - 1.$$

Route the new request through an unused middle crossbar. Then $|n(s)| \leq \lfloor 3n/2 \rfloor$.

(ii) Because $n_{22}(s') \subseteq n_{11}(s')$ and $|n_{11}(s')| \leq n-1$, we obtain $|n_{11}(s) \cup n_{22}(s)| \leq n$.

(iii) $|n_{12}(s) \cup n_{21}(s)| = |n_{12}(s') \cup n_{21}(s')| \leq n$.

□

Note that Theorem 2.2.1 is for the almost trivial network $r = 2$. Lots of efforts have been spent to expand this result, but without success. In the following section, we will give the unexpected results that for $r \geq 3$, not only packing and STU, but also CS, CD, and MI do not save any middle crossbars.

A state of $C(n, m, r)$ can be represented by an $r \times r$ matrix where cell (i, j) consists of the set of the labels of middle crossbars carrying a connection from I_i to O_j. Then each row or column can have at most n entries and the entries must be all distinct. The n-uniform state is the matrix where each diagonal cell contains $\{1, \ldots, n\}$ and all other cells are empty. The $\lfloor 2n - \frac{n}{2^{r-1}} \rfloor$ result was actually proved [9] for all algorithms which can reach the n-uniform state, which, as shown in [9], includes P, STU, MI and CS. Hung (private communication) observed that CD can also reach the n-uniform state. Chang et al. [4] give a stronger result based on his method.

Lemma 2.2.2. *CD can reach any state s from any state s'.*

Proof. Since we can disconnect all paths in s' to reach the empty state, it suffices to prove for s' the empty state. We prove this by adding each M_k in s to it's proper cell one by one. Suppose M_k is in cell (i, j). Consider a request γ. Suppose CD assigns M_h to connect γ. If $h \neq k$, disconnect γ and reconnect it immediately. Then CS would assign M_{h+1} to connect the pair. Repeat this until M_k is assigned. Since M_k is arbitrary, s can be reached. □

Corollary 2.2.3. *CD can reach the the n-uniform state.*

For CS we prove a weaker property. Let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$ if $i \leq j$, and the empty set if $i > j$.

Lemma 2.2.4. *Let state s be obtained from s' by adding $[i, j]$, $i < j$, to a cell C. Then s can be reached from s' under CS.*

Proof. Suppose the last assignment is M_k in s'. Since $i < j$, we can add at least two connections in C. Then M_k and M_{k+1} will be assigned. If $k \neq i$, disconnect the connection through M_k and regenerate a connection in C, for which M_{k+2} will be assigned. Continue this until M_i and M_{i+1} are assigned. Then add $j - i - 1$ connections to C for which $M_{i+2}, \ldots, M_j$ will be assigned. □

Theorem 2.2.5. *$C(n, m, r)$ for $r \geq 2$ is WSNB under CD and CS if and only if $m \geq 2n - 1$.*

Proof. The "if" part is trivial since $C(n, 2n-1, r)$ is SNB, hence WSNB. To prove the "only if" part, we claim that if $m = 2n - 2$, then there exists a blocking state.

It is well known [9] that it suffices to prove for the minimum r which is 2 here. By Lemma 2.2.2 and 2.2.4, the state in which cell $(1, 1)$ contains

$[1, n-1]$ and cell $(2,2)$ contains $[n, 2n-2]$ can be reached. But a new request in cell $(1,2)$ is blocked. Hence m must be grater than $2n-2$. $\square$

Theorem 2.2.6. *For P, hence STU, $C(n,m,r)$, $r \geq 3$, is WSNB if and only if $m \geq 2n-1$.*

Proof. The "if" part is trivial. We prove the "only if" part by showing that for $r = 3$ there exists a sequence of calls and disconnections forcing the use of $2n-1$ middle switches:

$$\begin{array}{c|c|c} [1,n] & & \\ \hline & [1,n] & \\ \hline & & \end{array} \longrightarrow \begin{array}{c|c|c} n & & \\ \hline n+1 & [1,n-1] & \\ \hline & & \end{array}$$

$$\longrightarrow \begin{array}{c|c|c} n & & n+1 \\ \hline n+1 & [1,n-1] & \\ \hline & & \end{array} \longrightarrow \begin{array}{c|c|c} n & & n+1 \\ \hline & [1,n-1] & \\ \hline n+1 & & \end{array}$$

$$\longrightarrow \begin{array}{c|c|c} n & & n+1 \\ \hline n+2 & [1,n-1] & \\ \hline n+1 & & \end{array} \longrightarrow \begin{array}{c|c|c} n & & [n+1,n+2] \\ \hline n+2 & [1,n-1] & \\ \hline n+1 & & \end{array}$$

$$\longrightarrow \begin{array}{c|c|c} n & & [n+1,n+2] \\ \hline & [1,n-1] & \\ \hline [n+1,n+2] & & \end{array} \longrightarrow \cdots$$

$$\longrightarrow \begin{array}{c|c|c} n & & [n+1,2n-2] \\ \hline & [1,n-1] & \\ \hline [n+1,2n-2] & & \end{array}$$

$$\longrightarrow \begin{array}{c|c|c} n & & [n+1,2n-2] \\ \hline 2n-1 & [1,n-1] & \\ \hline [n+1,2n-2] & & \end{array}$$

$\square$

Note that this proof is much more elementary than the proof in [7].

For MI, we first prove a lemma.

Lemma 2.2.7. *Consider a state s in $C(n,m,2)$ consisting of x requests from I_1 to O_1 carried by the set X of middle switches, and y requests from*

I_2 to O_2 carried by the set Y of middle switches such that $X \cap Y = \varnothing$, $X \cup Y = \{1, \ldots, x+y\}$, i.e. cell $(1,1)$ is X and cell $(2,2)$ is Y. Then a state s' can be obtained from s, where s' is same as s except that x becomes x', and y becomes $y' = x + y - x'$.

Proof. Without loss of generality, assume $x' > x$(otherwise we work with y). Disconnect $x' - x$ requests whose indices are smallest in Y from s. Add $x' - x$ new requests in cell $(1,1)$. By the MI rule, these new requests must be carried by S. Thus s' is obtained. □

Theorem 2.2.8. *$C(n, m, r)$ for $r \geq 2$ is WSNB under MI if and only if $m \geq 2n - 1$.*

Proof. It suffices to prove that $m = 2n-1$ is necessary for WSNB for $r = 2$.

By induction on n, suppose $m = 2n-3$ is necessary for $C(n-1, m, 2)$ to be WSNB. Therefore there exists a state

X	$2n-3$
	Y

in $C(n, 2n-1, 2)$, such that $x = y = n-2$, $X \cup Y = \{1, \cdots, 2n-4\}$. Therefore we can obtain a state s' from s by adding $2n-2$ to the $(1,2)$ cell. Delete the four calls carried by $[2n-7, 2n-4]$ in the $(1,1)$ and $(2,2)$ cells, and use Lemma 2.2.7 to rebalance the members of calls carried by them, i.e., each carrying $n-4$ calls. Assign $[2n-7, 2n-4]$ to cell $(2,1)$.

Next we delete $[2n-11, 2n-8]$ from cells $(1,1)$ and $(2,2)$, do the balancing and assign $[2n-11, 2n-8]$ to cell $(1,2)$. Repeatedly doing so, eventually (the last step may delete only two calls) we reach a state consisting of $2n-2$ distinct indices in cells $(1,2)$ and $(2,1)$. Thus a new $(1,1)$ request must be carried by M_{2n-1}. □

Example 1. The following example shows that $C(n, m, 2)$ can be routed through $2n-1$-th middle crossbar under MI for $n \leq 6$. "⇒" means to do the balancing.

$$n=1: \begin{array}{c|c} 1 & \\ \hline & \end{array}$$

$$n=2: \rightarrow \begin{array}{c|c} 1,2 & \\ \hline & \end{array} \rightarrow \begin{array}{c|c} 2 & \\ \hline & 1 \end{array} \rightarrow \begin{array}{c|c} 2 & 3 \\ \hline & 1 \end{array}$$

$$n=3: \rightarrow \begin{array}{c|c} 2 & 3,4 \\ \hline & 1 \end{array} \rightarrow \begin{array}{c|c} & 3,4 \\ \hline 1,2 & \end{array} \rightarrow \begin{array}{c|c} 5 & 3,4 \\ \hline 1,2 & \end{array}$$

$$n=4: \rightarrow \begin{array}{c|c} 5,6 & 3,4 \\ \hline 1,2 & \end{array} \rightarrow \begin{array}{c|c} 5,6 & \\ \hline & [1,4] \end{array} \Rightarrow \begin{array}{c|c} 1,5,6 & \\ \hline & 2,3,4 \end{array}$$
$$\rightarrow \begin{array}{c|c} 1,5,6 & 7 \\ \hline & 2,3,4 \end{array}$$

$$n=5: \rightarrow \begin{array}{c|c} 1,5,6 & 7,8 \\ \hline & 2,3,4 \end{array} \rightarrow \begin{array}{c|c} 1 & 7,8 \\ \hline [3,6] & 2 \end{array} \rightarrow \begin{array}{c|c} & 7,8 \\ \hline [3,6] & \end{array}$$
$$\rightarrow \begin{array}{c|c} 9 & 1,2,7,8 \\ \hline [3,6] & \end{array}$$

$$n=6: \rightarrow \begin{array}{c|c} 9,10 & 1,2,7,8 \\ \hline [3,6] & \end{array} \rightarrow \begin{array}{c|c} 9,10 & 1,2 \\ \hline 3,4 & [5,8] \end{array} \rightarrow \begin{array}{c|c} [1,4],9,10 & \\ \hline & [5,8] \end{array}$$
$$\Rightarrow \begin{array}{c|c} [2,4],9,10 & \\ \hline & 1,[5,8] \end{array} \rightarrow \begin{array}{c|c} [2,4],9,10 & 11 \\ \hline & 1,[5,8] \end{array}$$

Corollary 2.2.9. *For 3-stage Clos network $C(n,m,r)$, let s be the state where X, Y, and Z are in cells (i_1,j_2), (i_2,j_1), and (i_1,j_1), respectively,*

$$\begin{array}{c|c|c|c} & & & \\ \hline X & & Z & \\ \hline & & Y & \\ \hline & & & \end{array}$$

where, $\min\{Z\} > k$, $X \cap Y = \varnothing$, $X \cup Y = [1,k]$, $k \leq 2(n-|Z|)$.
For each $\alpha \leq k$ and $\max\{\alpha, k-\alpha\} \leq (n-|Z|)$, let $f_\alpha(s)$ be the state which has $f_\alpha(X)$ in cell (i_1,j_2), $|f_\alpha(X)| = \alpha$, and $f_\alpha(Y)$ in cell (i_2,j_1), such that $f_\alpha(X) \cap f_\alpha(Y) = \varnothing$, $f_\alpha(X) \cup f_\alpha(Y) = [1,k]$. Then $f_\alpha(s)$ can be reached from s under MI.

2.3 Wide-sense nonblocking in asymmetric case

Without loss of generality, we assume $\lfloor \frac{n_1}{n_2} \rfloor = k \geq 1$ throughout this section. If $n_1 \geq r_2 n_2$, then $m = r_2 n_2$ is necessary and sufficient for $C(n_1, r_1, m, n_2, r_2)$ to be either SNB or WSNB. Therefore we assume $r_2 > \frac{n_1}{n_2}$, or $r_2 \geq \lceil \frac{n_1+1}{n_2} \rceil$.

Theorem 2.3.1. *$C(n_1, r_1, m, n_2, r_2)$ for $r_2 \geq 2$ is WSNB under CS and CD if and only if $m \geq n_1 + n_2 - 1$.*

Proof. The "if" part is trivial since $C(n_1, r_1, n_1 + n_2 - 1, n_2, r_2)$ is SNB. To prove the "only if" part, we show that if $m = n_1 + n_2 - 2$, then there exists a blocking state. Clearly, we can reach the state

$[1, n_2]$	$[n_2+1, 2n_2]$	$\ldots$	$[kn_2+1, n_1-1]$	
				$[n_1, n_1+n_2-2]$

(if $[kn_2 + 1, n_1 - 1]$ is an empty set, then the corresponding column does not exist). Since row 1 has only $n_1 - 1$ entries and the last column has only $n_2 - 1$ entries, one new connection can be requested in the cell $(1, \lceil \frac{n_1}{n_2} \rceil + 1)$, but no middle switch is available. □

The MI case is as following. We first prove a lemma.

Lemma 2.3.2. *$C(n_1, r_1, m, n_2, r_2)$ with $n_1 = n_2 + 1$, $min\{r_1, r_2\} \geq 2$, is not WSNB under MI if $m < 2n_2$.*

Proof. We prove, by induction on n_2, the existence of a state which must use $2n_2$ middle switches.

(i) $n_2 = 2$,

$$\begin{array}{c|c} & \\ \hline [1,2] & 3 \end{array} \rightarrow \begin{array}{c|c} [1,2] & \\ \hline & 3 \end{array} \rightarrow \begin{array}{c|c} [1,2] & 4 \\ \hline & 3 \end{array}$$

(ii) suppose that for $n_2 = n$ the statement is true.

(iii) $n_2 = n + 1$, since for $n_2 = n$ the statement is true, we can reach a state s,

$$\begin{array}{c|c} X & 2n \\ \hline & Y \end{array}, \; |X| = n, |Y| = n-1, \; X \cap Y = \varnothing, \text{ and } X \cup Y = [1, 2n-1].$$

Add $2n+1$ to cell $(1,2)$, since $n_2 = n+1$, and delete the four numbers $[2n-4, 2n-1]$ from cell $(1,1)$ and $(2,2)$. By noting that Corollary 2.2.9 also applies to the asymmetric 3-stage clos network,we can get a state s_1,

$$\begin{array}{c|c} X_1 & [2n, 2n+1] \\ \hline & Y_1 \end{array}, \; |X_1| = n-3, \; |Y_1| = n-2, \; X_1 \cap Y_1 = \varnothing, \text{ and } X_1 \cup Y_1 = [1, 2n-5].$$

Then, we add $[2n-4, 2n-1]$ to cell $(2,1)$, and delete the four numbers $[2n-8, 2n-5]$ from cell $(1,1)$ and $(2,2)$. By Corollary 2.2.9 again, we can reach a state s_2,

$$\begin{array}{c|c} X_2 & [2n, 2n+1] \\ \hline [2n-4, 2n-1] & Y_2 \end{array}, \; |X_2| = n-4, |Y_2| = n-5, \; X_2 \cap Y_2 = \varnothing, \text{ and } X_2 \cup Y_2 = [1, 2n-9].$$

Repeat the above steps, without loss of generality, we reach a state s',

$$\begin{array}{c|c} & X' \\ \hline Y' & \end{array}, \; |X'| = n, |Y'| = n+1, \; X' \cap Y' = \varnothing, \text{ and } X' \cup Y' = [1, 2n+1].$$

Finally, we add $2n+2$ to cell $(2,2)$.

□

Corollary 2.3.3. *$C(n_1, r_1, m, n_2, r_2)$ with $n_2 < n_1 < 2n_2$, $r_1 \geq 2, r_2 = 2$, is not WSNB under MI if $m < 2n_2$.*

Theorem 2.3.4. *$C(n_1, r_1, m, n_2, r_2)$ with $n_1 > n_2$, $\min\{r_1, r_2\} \geq 2$, is WSNB under MI if and only if $m \geq \min\{n_1 + n_2 - 1, r_2 n_2\}$.*

Proof. The "if" part is trivial. To prove the "only if" part, it suffices to show for $r_1 = 2$.

Case 1. $n_1 \leq (r_2 - 1)n_2$, assume $n_1 = pn_2 + q, 0 \leq q < n_2$. Clearly we can reach the state

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\dots$	$[x, n_1 - n_2]$		

,where $x = (p-2)n_2 + 1$, if $q = 0$; $x = (p-1)n_2 + 1$,if $q \neq 0$.

We can also move $[1, n_1 - n_2]$ from first row to second row by moving cell by cell in the order from left to right.

Our focus is actually on the last two columns, i.e., the 2×2 submatrix M. The Set $[1, n_1 - n_2]$ in the first $p - 1$ or p columns serves the sole purpose that all entries in M are larger than $n_1 - n_2$. This is achieved by moving the set$[1, n_1 - n_2]$ to the row where entries are to be added in M. The entries are added according to the proof of Lemma 2.2.7. Hence, eventually, we reach the state

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\dots$	$[x, n_1 - n_2]$	$\dots$	X	
						Y

, $|X| = |Y| = n_2 - 1$, $X \cap Y = \varnothing$, and $X \cup Y = [n_1 - n_2 + 1, n_1 + n_2 - 2]$.

Finally, add $n_1 + n_2 - 1$ to cell $(1, r_2)$.

Case 2. $(r_2 - 1)n_2 < n_1 < r_2 n_2$, which implies $(n_1 + n_2 - 1) \geq r_2 n_2$. Clearly, we can reach the state

$[1, n_2]$	$\dots$	$[(r_2 - 3)n_2 + 1, (r_2 - 2)n_2]$		

Similar to Case 1, we can reach the state

$[1, n_2]$	$\dots$	$[(r_2 - 3)n_2 + 1, (r_2 - 2)n_2]$	X	
				Y ,

$|X| = n_2, |Y| = n_2 - 1$, $X \cap Y = \varnothing$, and $X \cup Y = [(r_2 - 2)n_2 + 1, r_2 n_2 - 1]$. Finally, add $r_2 n_2$ to cell $(1, r_2)$.

Case 3. $r_2 n_2 \le n_1$. This is a trivial case with $m = r_2 n_2$.

□

Finally, we study the packing and STU strategies. Let X_{ij} denote the set of connections from I_i to O_j. We first prove

Lemma 2.3.5. *Suppose $n_1 \ge n_2$. Then $|X_{11} \cup X_{22}| \le n_2, |X_{12} \cup X_{21}| \le n_2$.*

Proof. Suppose not, say, $|X_{11} \cup X_{22}| = n_2 + 1$. Let y denote the $(n_2 + 1)$st middle switch added to cell $(1, 1)$ or cell $(2, 2)$. Without loss of generality, assume y is added to cell $(2, 2)$. Then $X_{11}/X_{22} = \varnothing$ since otherwise, the (I_2, O_2) connection should be routed through a middle crossbar in X_{11}/X_{22} by the packing strategy. Therefore

$$X_{11} \cup X_{22} = X_{22},$$

and

$$|X_{11} \cup X_{22}| \le n_2 - 1$$

since cell $(2, 2)$ can have at most n_2 connections, including y, contradicting the assumption that y is the $(n_2 + 1)$st middle switch in $X_{11} \cup X_{22}$.

Similarly, we can prove $|X_{12} \cup X_{21}| \le n_2$. □

Theorem 2.3.6. *Suppose $n_1 \ge n_2$. Then $C(n_1, 2, m, n_2, 2)$ is wide-sense nonblocking under the packing or the STU strategy if and only if $m \ge \min\{2n_2, n_2 + \lfloor n_1/2 \rfloor\}$.*

Proof. Suppose $n_1 \ge 2n_2$. Consider $2n_2$ connections for an input switch. They must be routed through $2n_2$ distinct middle switches. On the other hand, there are at most $2n_2$ connections, hence $2n_2$ middle switchs suffice. Next suppose $n_1 \le 2n_2$.

Necessity.

$$\frac{[1,n_2]\ \big|\ \qquad}{\qquad\ \big|\ [1,n_2]} \to \frac{[1,n_2-\lfloor n_1/2\rfloor]\ \big|\ \qquad}{\qquad\ \big|\ [n_2-\lfloor n_1/2\rfloor]+1,n_2]}$$

$$\to \frac{[1,n_2-\lfloor n_1/2\rfloor]\ \big|\ \qquad}{[n_2+1,n_2+\lfloor n_1/2\rfloor]\ \big|\ [n_2-\lfloor n_1/2\rfloor]+1,n_2]}$$

The last state has $n_2-\lfloor n_1/2\rfloor+\lfloor n_1/2\rfloor+\lfloor n_1/2\rfloor = n_2+\lfloor n_1/2\rfloor$ elements.

Sufficiency. Suppose to the contrary that there exists a state such that a new request under the packing strategy will force the use of an idle middle crossbar y which will be the $(n_2+\lfloor n_1/2\rfloor+1)$st middle crossbar in use. Without loss of generality, assume y is in cell $(2,2)$. Then by an argument analogous to the one used in proving Lemma 2.3.5, $X_{11}\subseteq X_{22}$ in that state. Therefore

$$X_{11}\cup X_{12}\cup X_{21}\cup X_{22} = X_{12}\cup X_{21}\cup X_{22}.$$

Further

$$\begin{aligned}
&|X_{12}\cup X_{21}| \le n_2 \qquad \text{(by Lemma 2.3.5)}\\
&|X_{12}\cup X_{22}| \le n_2,\\
&|X_{21}\cup X_{22}| \le n_1.
\end{aligned}$$

Hence

$$\begin{aligned}
&|X_{12}\cup X_{21}\cup X_{22}| \le (n_2+n_2+n_1)/2, \text{ or}\\
&|X_{12}\cup X_{21}\cup X_{22}| \le n_2+\lfloor n_1/2\rfloor.
\end{aligned}$$

□

Note that the proof of sufficiency is simpler than Beneš original proof for the symmetric network.

Theorem 2.3.7. *Suppose* $n_1\ge n_2$ *and* $\max\{r_1,r_2\}\ge 3$. *Then* $C(n_1,r_1,m,n_2,r_2)$ *is wide-sense nonblocking if and only if* $m\ge\min\{r_2n_2,n_1+n_2-1\}$.

Proof. The "if" part is trivial since the condition already guarantees strict nonblockingness by an extension of Clos result [6] to the asymmetric case. We now prove the "only if" part. If $n_1 \geq r_2 n_2$, then trivially, $m \geq r_2 n_2$ is necessary. Therefore we assume $n_1 < r_2 n_2$.

Case (i) $r_2 = 2, r_1 \geq 3$.

$$\begin{array}{c|c} [1,n_2] & \\ \hline & n_2 \\ \hline & \end{array} \rightarrow \begin{array}{c|c} [1,n_2-1] & \\ \hline n_2+1 & n_2 \\ \hline & \end{array} \rightarrow \begin{array}{c|c} [1,n_2-1] & \\ \hline n_2+1 & n_2 \\ \hline & n_2+1 \end{array}$$

$$\rightarrow \begin{array}{c|c} [1,n_2-1], n_2+1 & \\ \hline & n_2 \\ \hline & n_2+1 \end{array} \rightarrow \begin{array}{c|c} [1,n_2-1], n_2+1 & \\ \hline & [n_2, n_2+1] \\ \hline & \end{array}$$

$$\rightarrow \begin{array}{c|c} [1,n_2-1] & \\ \hline n_2+2 & [n_2, n_2+1] \\ \hline & \end{array}$$

Repeat such an operation, eventually we obtain

$$\begin{array}{c|c} [1,n_2-1] & \\ \hline & [n_2, n_2+\lfloor n_1/2 \rfloor - 1] \\ \hline & \end{array}$$

$$\rightarrow \begin{array}{c|c} [1,n_2-1] & \\ \hline n_2+\lfloor n_1/2 \rfloor & [n_2, n_2+\lfloor n_1/2 \rfloor - 1] \\ \hline & \end{array}$$

Case (ii) $r_2 \geq 3, r_1 = 2$.

Subcase (1): $n_1 > (r_2 - 1)n_2$.

First,

$$\begin{array}{c|c|c|c|c|} [1,n_2] & [n_2+1, 2n_2] & \cdots & [(r_2-3)n_2+1, (r_2-2)n_2-1] & [(r_2-2)n_2, (r_2-1)n_2-2] \\ \hline & & & & \end{array}$$

Now, consider the last three columns. Define $A = [(r_2-3)n_2+1, (r_2-$

$2)n_2-1]$ and $B=[(r_2-2)n_2,(r_2-1)n_2-1]$. Then

$$\begin{array}{c|c|c} A & B,(r_2-1)n_2-1 & \\ \hline & & \end{array}$$

$$\to\begin{array}{c|c|c} A & B,(r_2-1)n_2-1 & \\ \hline & & (r_2-1)n_2-1 \end{array}$$

$$\to\begin{array}{c|c|c} A & B & (r_2-1)n_2 \\ \hline & & (r_2-1)n_2-1 \end{array}$$

$$\to\begin{array}{c|c|c} A & B,(r_2-1)n_2 & \\ \hline (r_2-1)n_2 & & (r_2-1)n_2-1 \end{array}$$

$$\to\begin{array}{c|c|c} A & B & \\ \hline (r_2-1)n_2 & & (r_2-1)n_2-1 \end{array}$$

$$\to\begin{array}{c|c|c} A & B,(r_2-1)n_2 & \\ \hline & & (r_2-1)n_2-1 \end{array}$$

$$\to\begin{array}{c|c|c} A & B & \\ \hline & & (r_2-1)n_2-1,(r_2-1)n_2 \end{array}\to\cdots$$

$$\to\begin{array}{c|c|c} A & B & \\ \hline & & [(r_2-1)n_2-1,r_2n_2-3] \end{array}$$

define $C=[(r_2-1)n_2-1,r_2n_2-3]$

$$\to\begin{array}{c|c|c} A & B & r_2n_2-2 \\ \hline & & C \end{array}\to\begin{array}{c|c|c} A & B & r_2n_2-2 \\ \hline & r_2n_2-2 & C \end{array}$$

$$\to\begin{array}{c|c|c} A,r_2n_2-2 & B & \\ \hline & r_2n_2-2 & C \end{array}$$

$$\to\begin{array}{c|c|c} A,r_2n_2-2 & B & r_2n_2-1 \\ \hline & & C \end{array}\to\begin{array}{c|c|c} A,r_2n_2-2 & B & \\ \hline & r_2n_2-1 & C \end{array}$$

$$\to\begin{array}{c|c|c} A & B & \\ \hline & r_2n_2-1 & C,r_2n_2-2 \end{array}\to\begin{array}{c|c|c} A,r_2n_2-1 & B & \\ \hline & & C,r_2n_2-2 \end{array}$$

$$\to\begin{array}{c|c|c} A,r_2n_2-1 & B,r_2n_2-2 & \\ \hline & & C \end{array}$$

$$\to\begin{array}{c|c|c|c|c|c} [1,n_2] & [n_2+1,2n_2] & \cdots & A,r_2n_2-1 & B,r_2n_2-2 & r_2n_2 \\ \hline & & & & & C \end{array}$$

Subcase (2): $n_1\le(r_2-1)n_2$.
First,

$$\begin{array}{c|c|c|c|c|}
[1,n_2] & [n_2+1,2n_2] & \cdots & [n_1-2n_2+1,n_1-n_2-1] & [n_1-n_2,n_1-2] \\ \hline
 & & & &
\end{array}$$

Also, consider the last three columns, define $A=[n_1-2n_2+1,n_1-n_2-1]$ and $B=[n_1-n_2,n_1-2]$.

Similar, we can get the following state

$$\begin{array}{c|c|c}
A & B & \\ \hline
 & & [(n_1-1,n_1+n_2-3]
\end{array}$$

define $C=[(n_1-1,n_1+n_2-3]$

$$\to \begin{array}{c|c|c}
A & B & n_1+n_2-2 \\ \hline
 & & C
\end{array}
\to \begin{array}{c|c|c}
A & B & n_1+n_2-2 \\ \hline
 & n_1+n_2-2 & C
\end{array}$$

$$\to \begin{array}{c|c|c}
A & B,n_1+n_2-2 & \\ \hline
 & & C
\end{array}$$

$$\to \begin{array}{c|c|c|c|c|c}
[1,n_2] & [n_2+1,2n_2] & \cdots & A & B,n_1+n_2-2 & n_1+n_2-1 \\ \hline
 & & & & & C
\end{array}$$

Case (iii) $r_1\geq 3, r_2\geq 3$.

Let $p=\lfloor n_1/n_2\rfloor$. Then $n_2\leq n_1<r_2n_2$ implies $1\leq p<r_2$. It suffices to prove the state

$$s=\begin{array}{c|c|c|c|c}
[1,n_2] & [n_2+1,2n_2] & \cdots & [(p-2)n_2+1,(p-1)n_2] & \\ \hline
 & & & & \\ \hline
 & & & &
\end{array}$$

can be reached since we can use Case (i) for the remaining $r_1\times(r_2-p+1)$ array with $n_2'\leq n_1<2n_2'$. Hence the total number of middle crossbars used is $(p-1)n_2+(n_1'+n_2-1)=n_1+n_2-1$.

$$\begin{array}{c|c}
[1,n_2] & \\ \hline
 &
\end{array}$$

$$\to \begin{array}{c|c|c}
[1,n_2] & [n_2+1,2n_2] & \\ \hline
 & &
\end{array}$$

$$\to \begin{array}{c|c|c|c}
[1,n_2] & [n_2+1,2n_2] & [2n_2+1,3n_2] & \\ \hline
 & & &
\end{array}$$

$$\to\cdots\to s$$

□

Chapter 3

Multi-$\log_d N$ networks

The multi-$\log_d N$ network, first proposed by Lea [12], can be obtained by substituting each middle crossbar of a 3-stage Clos stage with a $\log_d N$ network. In this chapter, we study the five WSNB strategies in multi-$\log_d N$ networks and show that the cost is still the same as SNB.

3.1 Strictly nonblocking

For strictly nonblocking, Shyy and Lea [13] proved the following theorem for $d = 2$ and Hwang [8] extended it to the d-nary version.

Theorem 3.1.1. *Multi-*$\log_d N$ *network with* p *copies is strictly nonblocking if* $p \geq p(n)$*, where*

$$p(n) = \begin{cases} (d+1) \times d^{\frac{n}{2}-1} - 1 & \textit{for } n \textit{ even,} \\ 2 \times d^{\frac{n-1}{2}} - 1 & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. We consider the graph model of a copy (baseline network). For the note in the j-th link stage is the path of a request (i, o), $j = 1, \ldots, n$, there are at most $k(j)$ requests can intersect it and doesn't intersect the other node on the path, where

$$k(j) = \begin{cases} d^j - d^{j-1} & \text{for } j < n/2, \\ d^{n-j} - d^{n-j+1} & \text{otherwise.} \end{cases}$$

Therefore, there are at most

$$\sum_{j=0}^{n}(k(j)-1) = \begin{cases} 2\displaystyle\sum_{j=1}^{n/2-1}(d^j - d^{j-1}) + (d^{n/2} - d^{n/2-1}) & \text{for } n \text{ even,} \\ 2\displaystyle\sum_{j=1}^{(n-1)/2}(d^j - d^{j-1}) & \text{for } n \text{ odd.} \end{cases}$$

$$= \begin{cases} (d+1)\times d^{\frac{n}{2}-1} - 2 & \text{for } n \text{ even,} \\ 2\times d^{\frac{n-1}{2}} - 2 & \text{for } n \text{ odd.} \end{cases}$$

If these requests are route in distinct copies, then (i, o) must route in another copy. Hence, the theorem holds. □

Any request has a unique path in a $\log_d N$ network. Hence two intersecting paths must be routed through different copies of $\log_d N$ network.

Theorem 3.1.1 was stated in [8] only as a sufficient condition. Chang, Guo, and Hwang [5] proved that it is also necessary.

Theorem 3.1.2. *Multi-*$\log_d N$ *network with* p *copies is strictly nonblocking only if* $p \geq p(n)$.

Proof. For any request $\gamma = (x, y)$, assume that the path of γ consists of links $L_0, L_1, \ldots, L_n$. For n odd, let $I_1(O_2)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{\frac{n-1}{2}}$, then $|I_1| = d^{\frac{n-1}{2}} - 1$ and $|O_2| = d^{\frac{n+1}{2}} - 1$. Let $O_1(I_2)$ be the set of outputs(inputs), except $y(x)$, which can reach $L_{\frac{n+1}{2}}$. Then $|O_1| = d^{\frac{n-1}{2}} - 1$ and $|I_2| = d^{\frac{n+1}{2}} - 1$. Note that γ cannot be routed through the same copy with any request from I_1 to O_2 or I_2 to O_1. Suppose $p = p(n) - 1$ while $|I_1|$ requests from I_1 to $O_2 \setminus O_1$ and $|O_1|$ requests from O_1 to $I_2 \setminus I_1$ have already been connected in different copies. In this case, they can occupy $|I_1| + |O_1| = p(n) - 1 = p$ copies, with no copy left for γ. For n even, let $I_1(O_2)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{\frac{n}{2}-1}$, then $|I_1| = d^{\frac{n}{2}-1} - 1$ and $|O_2| = d^{\frac{n}{2}+1} - 1$. Let $O_1(I_2)$ be the set of outputs(inputs), except $y(x)$, which can reach $L_{\frac{n}{2}+1}$. Then $|O_1| = d^{\frac{n}{2}-1} - 1$ and $|I_2| = d^{\frac{n}{2}+1} - 1$. Let $I_3(O_3)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{\frac{n}{2}}$. Then $|I_3| = |O_3| = d^{\frac{n}{2}}$. Note that γ cannot be routed through the same copy with any request from I_1 to O_2, I_2 to O_1, or I_3 to O_3. Suppose $p = p(n) - 1$ while $|I_1|$ requests from I_1 to $O_2 \setminus O_3$, $|O_1|$ requests from O_1 to $I_2 \setminus I_3$, and $|I_3 \setminus I_1|$ requests from $I_3 \setminus I_1$ to $O_3 \setminus O_1$ have already been connected in different copies. In this case, they can occupy $|I_1| + |O_1| + |I_3 \setminus I_1| = |I_1| + |O_1| + |O_3 \setminus O_1| = p(n) - 1 = p$ copies, with no copy left for γ. Hence p must be greater than or equal to $p(n)$. □

We call such a set of $p(n) - 1$ requests blocking γ the maximal blocking configuration (MBC), denote by $M(n, \gamma)$.

Note that if a network is SNB, then it is also WSNB. i.e. multi-$\log_d N$ is WSNB if $p \geq p(n)$. Therefore, we only need to prove necessity in the following proofs. In all these proofs, we assume that the network carries no traffic at the beginning.

3.2 Wide-sense nonblocking

We consider strategy CD first.

Theorem 3.2.1. *Multi-*$\log_d N$ *network with* p *copies is WSNB under CD if and only if* $p \geq p(n)$.

Proof. Suppose $p < p(n)$. Consider a sequence of $p + 1$ requests with p requests from $M(n, \gamma)$ followed by the request γ. By the property of strategy CD, these p requests will be routed in p copies. Then we cannot route γ any more. Hence p must be greater than or equal to $p(n)$. □

For strategy CS,

Theorem 3.2.2. *Multi-*$\log_d N$ *network with* p *copies is WSNB under CS if and only if* $p \geq p(n)$.

Proof. Suppose $p < p(n)$. For a request γ and any p requests of $M(n, \gamma)$, say $\gamma_1, \gamma_2, \ldots, \gamma_p$, route γ_1 in copy 1, then route γ in copy 2(because γ_1 blocks γ

in copy 1). Then disconnect γ and route γ_2 in copy 2. Then route γ in copy 3. Again disconnect it and route γ_3 in copy 3. Doing this iteratively until γ_p is routed in copy p. Then γ cannot be routed any more. Hence p must be greater than or equal to $p(n)$. $\square$

For strategies P or STU, we introduce a lemma.

Lemma 3.2.3. *For any request γ and $M(n, \gamma)$, there exists a request γ' which does not block γ or any request in $M(n, \gamma)$ in the $\log_d N$ network.*

Proof. Use the graph model of the baseline network as an example. Without loss of generality, let $\gamma = (0, 0)$. For all requests (i, j) in $M(n, \gamma)$, we obtain $i < \frac{N}{d}$ and $j < \frac{N}{d}$. Hence $\gamma' = (N-1, N-1)$ will satisfy our claim. $\square$

Theorem 3.2.4. *Multi-$\log_d N$ network with p copies is WSNB under P or STU if and only if $p \geq p(n)$.*

Proof. Suppose to the contrary, $p < p(n)$. For any request γ and any p requests of $M(n, \gamma)$, say $\gamma_1, \gamma_2, \ldots, \gamma_p$, we route γ_1 in copy 1 first. Then route γ in copy 2 and route γ' in copy 2 (because copy 1 are as busy as copy 2, we can choose copy 2). Now, we disconnect γ and route γ_2 in copy 2. Then disconnect γ'. Similarly, we route γ in copy 3 and γ' in copy 3, then disconnect γ and route γ_3 in copy 2. Finally, we route γ_p in copy p. Then γ cannot be routed any more. Hence p must be greater than or equal to $p(n)$. $\square$

MI is more complicated. It's amazing that we construct a relation between 3-stage Clos network and multi-$\log_d N$ network, and then we can apply the same process in Theorem 2.2.8 to multi-$\log_d N$ network.

Theorem 3.2.5. *Multi-$\log_d N$ network with p copies is WSNB under MI if and only if $p \geq p(n)$.*

Proof. We discuss two cases:

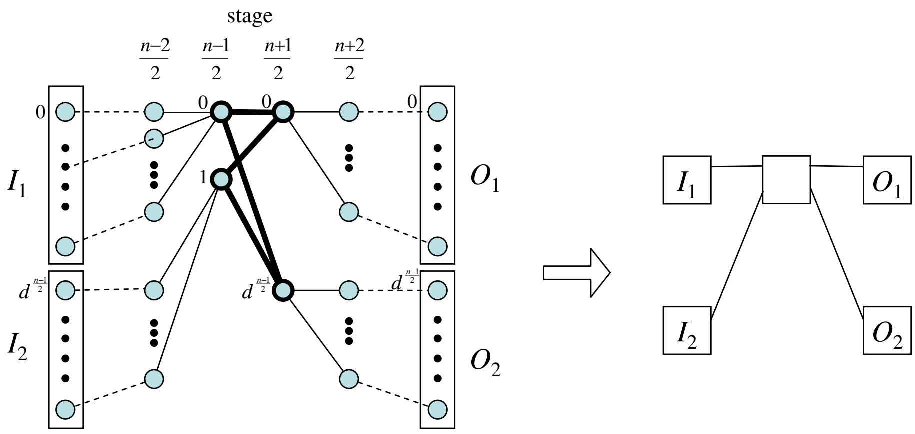


Figure 3.1: The left figure is an induced graph of the graph model of a multi-$\log_d N$ network, for n odd. And the right figure is its correspondence to a 3-stage Clos network.

(i) n is odd. Select two subset I_1 and I_2 of inputs and two subset O_1 and O_2 of outputs. Set $I_1 = O_1 = \{0, 1, 2, \ldots, d^{\frac{n-1}{2}} - 1\}$, $I_2 = O_2 = \{d^{\frac{n-1}{2}}, \ldots, 2 \times d^{\frac{n-1}{2}} - 1\}$. See Figure 3.1. By the configuration of baseline network, every request from I_1 to $O_1 \cup O_2$ must intersect node 0 in stage $\frac{n-1}{2}$ and every request from I_2 to $O_1 \cup O_2$ must intersect node 1 in stage $\frac{n-1}{2}$. Therefore, for $i = 1$ or 2, all requests from I_i to $O_1 \cup O_2$ must use different copies. Similarly, every request from $I_1 \cup I_2$ to O_1 must intersect node 0 in stage $\frac{n+1}{2}$ and every request from $I_1 \cup I_2$ to O_2 must intersect node $d^{\frac{n-1}{2}}$ in stage $\frac{n+1}{2}$. Therefore, for $i = 1$ or 2, all requests from $I_1 \cup I_2$ to O_i must use different copies. Now, we match this to a 3-stage Clos network $C(d^{\frac{n-1}{2}}, 1, 2)$, where I_i is the i-th input switch, O_i is the i-th output switch, for $i = 1$ or 2, and the complete bipartite graph induced by nodes 0 and 1 of stage $\frac{n-1}{2}$ and nodes 0 and $d^{\frac{n-1}{2}}$ of stage $\frac{n+1}{2}$ is the middle switch. Then a request (i, j) in $C(d^{\frac{n-1}{2}}, p, 2)$ routed through the k-th middle switch under MI corresponds to a request (i, j) in the multi-$\log_d N$ using copy

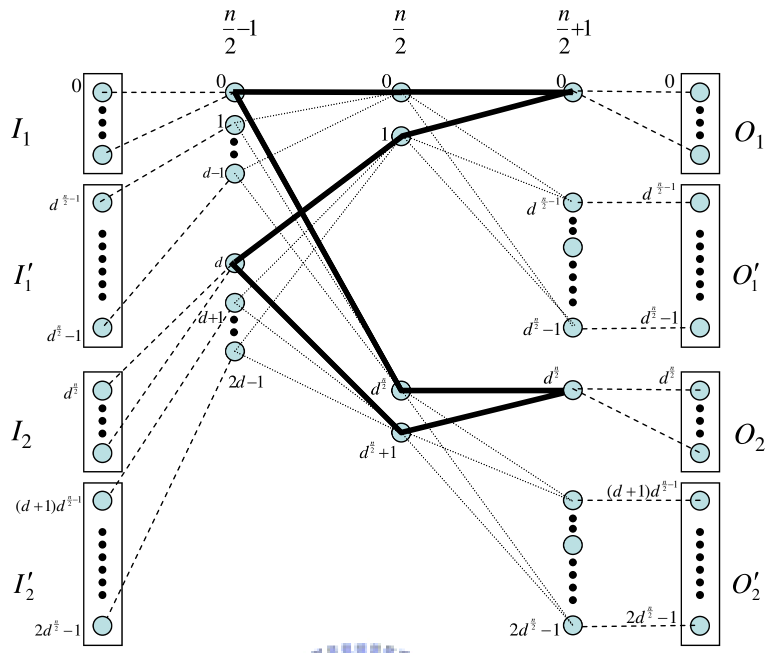


Figure 3.2: This is an induced graph of the graph model of a multi-$\log_d N$ network, for n is even.

k. Therefore, by Theorem 2.2.8, the network is not WSNB if

$$p < 2 \cdot (d^{\frac{n-1}{2}}) - 1 = 2 \times d^{\frac{n-1}{2}} - 1 = p(n).$$

(ii) n is even. Select four subset I_1, I'_1, I_2 and I'_2 of inputs and four subset O_1, O'_1, O_2, and O'_2 of outputs. Set $I_1 = O_1 = \{0, 1, 2, \ldots, d^{\frac{n}{2}-1} - 1\}$, $I'_1 = O'_1 = \{d^{\frac{n}{2}-1}, \ldots, d^{\frac{n}{2}} - 1\}$, $I_2 = O_2 = \{d^{\frac{n}{2}}, \ldots, (d+1)d^{\frac{n}{2}-1} - 1\}$, and $I'_2 = O'_2 = \{(d+1)d^{\frac{n}{2}-1}, \ldots, 2 \times d^{\frac{n}{2}} - 1\}$. See Figure 3.2. Then every request from I_1 to $O_1 \cup O_2$ must intersect node 0 in stage $\frac{n}{2} - 1$, every request from I_2 to $O_1 \cup O_2$ must intersect node d in stage $\frac{n}{2} - 1$, every request from $I_1 \cup I_2$ to O_1 must intersect node 0 in stage $\frac{n}{2} + 1$, and every request from $I_1 \cup I_2$ to O_2 must intersect node $d^{\frac{n}{2}}$ in stage $\frac{n}{2} + 1$. Similar to case (i), we can treat I_1, I_2, O_1, O_2 as the inputs and outputs of $C(d^{\frac{n}{2}-1}, 1, 2)$, and the subgraph sketch in bold line in Figure

3.2 is the middle switch. Therefore, by Theorem 2.2.8, the network is not WSNB if

$$p < 2 \cdot (d^{\frac{n}{2}-1}) - 1. \tag{3.2.1}$$

Besides, we observe that, for $i, j = 1, 2$, every request from I_i to O_j must block every request from I'_i to O'_j in the same node in the stage $\frac{n}{2}$. Therefore, if we connect all $(d-1)d^{\frac{n}{2}-1}$ requests in I'_i to O'_j in copy 0 to copy $(d-1)d^{\frac{n}{2}-1} - 1$ before every time we connect a request γ from I_i to O_j and disconnect them after γ connected, then we can force the copy chosen to route γ begin at least $(d-1)d^{\frac{n}{2}-1}$-th copy. Hence (3.2.1) can be enlarged to

$$p < 2 \times (d^{\frac{n}{2}-1}) - 1 + (d-1)d^{\frac{n}{2}-1} = p(n).$$

□

Note that, in Theorem 3.2.5, it doesn't need to consider all inputs and outputs, because $I_1 \cup I_2$ and $O_1 \cup O_2$ are enough to force $p \geq p(n)$ which is the bound of SNB.

Example 2. Here is an example that multi-$\log_2 32$ network can be routed through 7-th copy under MI. A triple (i, o, k) means the request (i, o) routes through k-th copy and $(i, o, k)^-$ means the request (i, o) routed through k-th copy disconnected.

$$\begin{aligned} &(0,0,1) \to (1,1,2) \to (0,0,1)^- \to (4,4,1) \to (0,5,3) \to (2,6,4) \\ \to &(1,1,2)^- \to (4,4,1)^- \to (4,0,1) \to (5,1,2) \to (1,2,5) \to (3,3,6) \\ \to &(4,0,1)^- \to (5,1,2)^- \to (0,5,3)^- \to (2,6,4)^- \to (4,4,1) \to (5,5,2) \\ \to &(6,6,3) \to (7,7,4) \to (4,4,1)^- \to (0,0,1) \to (2,4,7). \end{aligned}$$

Example 3. Here is an example that multi-$\log_2 64$ network can be routed through 11-th copy under MI. Note that every step following start from 5-th copy because the 4 requests from I'_i to O'_j through 1-st to 4-th copy.

$$(0,0,5) \to (1,1,6) \to (0,0,5)^- \to (8,8,5) \to (0,9,7) \to (2,10,8)$$
$$\to(1,1,6)^- \to (8,8,5)^- \to (8,0,5) \to (9,1,6) \to (1,2,9) \to (3,3,10$$
$$\to(8,0,5)^- \to (9,1,6)^- \to (0,9,7)^- \to (2,10,8)^- \to (8,8,5) \to (9,9,6)$$
$$\to(10,10,7) \to (11,11,8) \to (8,8,5)^- \to (0,0,5) \to (2,8,11).$$

3.3 Generalizations

In this chapter, we extend our results to a class of networks including the 3-stage Clos networks, the multi-$\log_d N$ and the $\log_d(N,k,m)$ networks as special cases.

A vertical-copy network V consists of an input stage of r_1 $n_1 \times m$ crossbars, an output stage of r_2 $m \times n_2$ crossbars and a middle stage of m copies of a network ν with r_1 inputs and r_2 outputs. There exists exactly one link between each input(output) crossbar and each copy of ν. When ν is the $r_1 \times r_2$ crossbar, V is a 3-stage Clos network. When $n_1 = n_2 = 1$ and ν is the $\log_d N$ network, V is a multi-$\log_d N$ network. When $n_1 = n_2 = 1$ and ν is the k-extra-stage $\log_d N$ network, then V is the $\log_d(N,k,m)$ network. In particular, if $k = n-1$, then V is the Cantor network.

Suppose that the necessary and sufficient condition for ν to be SNB is known. Consider $p = p(n) - 1$. For any request γ, there must be a state s such that γ is blocked in each of the $p(n)-1$ copies $\nu_1, \nu_2, \ldots, \nu_{p(n)-1}$. Let R_i be the set of all requests routing through ν_i in s and $M(\nu,\gamma) = \{R_i \mid i = 1,2,\ldots,p(n)-1\}$. i.e., V is SNB if and only if the number of copies is larger than $|M(\nu,\gamma)|$. Let "Route R_i in ν_j" mean "Route all requests in R_i in ν_j consecutively".

Theorem 3.3.1. *A vertical-copy network V is WSNB under the CS routing if and only if V is SNB.*

Proof. Suppose there are $p < p(n)$ copies $\nu_1, \nu_2, \ldots, \nu_p$ in V. For a request γ, we route R_1 in ν_1, then route γ in ν_2(γ is blocked in ν_1). Then disconnect γ

and route R_2 in ν_2. Then route γ in ν_3. Again disconnect it and route R_3 in ν_3. Doing this iteratively until R_p is routed in ν_p. Then γ cannot be routed in any copy. Hence p must be greater than or equal to $p(n)$. □

For CD, we use another argument.

Theorem 3.3.2. *A vertical-copy network V is WSNB under the CD routing if and only if V is SNB.*

Proof. First, we claim every request γ can be routed in ν_k for a given k. Route γ in ν_i. If $i \neq k$, then disconnect γ and route it again in ν_{i+1}. Similarly, if $i+1 \neq k$, then disconnect γ and route it again in ν_{i+2} until γ is routed in ν_k. Note that if $i = p$, then we let $i+1$ be 1. Therefore, if $p < p(n)$, then we can route R_i in ν_i for $i = 1$ to p as we want. Then γ cannot be routed in any copy. Hence p must be greater than or equal to $p(n)$. □

For STU, if there exists a request γ_i' which does not block $\{\gamma\} \cup R_i$ for all i, theorem 3.2.4 remains true if $M(n, \gamma)$ is replaced by $M(\nu, \gamma)$ and γ_i is replaced by R_i. But we use a different argument for P.

Theorem 3.3.3. *Suppose there exists a request γ_i' which does not block $\{\gamma\} \cup R_i$ for all i. A vertical-copy network V is WSNB under the P routing if and only if V is SNB.*

Proof. It suffices to prove the "only if" part. Suppose there are only $p = p(n) - 1$ copies $\nu_1, \nu_2, \ldots, \nu_p$ in V. For the request $\gamma = (0, 0)$, without loss of generality, suppose $R_i = \{\gamma_{i,j} \mid j = 1, \ldots, \lambda_i\}$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Let $|\nu_i|$ denote the number of connections in ν_i. For a given k, let $s(k, \mathbb{B})$ be a state satisfying the following conditions:

(i) $|\nu_k| < \lambda_k$,

(ii) Connections in ν_i are those from R_i,

(iii) $|\nu_i| = |\nu_k| + 1$ or $|\nu_i| = \lambda_i$ if $i \in \mathbb{B}$, where $\mathbb{B}$ denotes the set of i such that $|\nu_i| > |\nu_k|$.

Let $S(k)$ denote the state that ν_i contains R_i for all $1 \le i \le k$. We make two claims:

Claim A. We can add another connection δ of R_k in ν_k in state $s(k, \mathbb{B})$.

Claim B. $S(k)$ can be realized.

We prove both claims by induction on k. For $k = 1$, then $\mathbb{B} = \varnothing$. Clearly, we can add δ to ν_1, and keep on adding other connections until ν_i contains R_i. So consider general $k > 1$. From $s(k, \mathbb{B})$ we can obtain the state $s^*(k, \mathbb{B})$, which differs from $s(k, \mathbb{B})$ by having ν_i containing R_i for all $1 \le i \le k-1$, by applying induction to claim B(with $k = k-1$). In state $s^*(k, \mathbb{B})$, γ must be routed in ν_k. Now delete all connections in $s^*(k, \mathbb{B}) \setminus s(k, \mathbb{B})$ so that $|\nu_k| \ge |\nu_i|$ for all i. Then γ'_k can be routed in ν_k. Delete γ and route δ in ν_k. Delete γ'_k and Claim A is proved. Also, we can keep on adding all remaining connections of R_k to ν_k to prove Claim B.

Setting $k = p$ in Claim B, then γ cannot be routed in any of the p copies. Hence at least $p(n)$ copies are needed.

$\square$

Example 4. For simplicity, we will represent a state by its $|\nu|$-sequence. To help clarify the state, let $|\nu_i|^*$ denote the fact that γ is in the ν_i, $|\nu_i|'$ the fact that γ' is and $|\nu_i|''$ the fact that both are. Suppose $p = 3$ and we want to reach the state $S(3) = (\lambda_1, \lambda_2, \lambda_3) = (2, 3, 4)$. The the $|\nu|$-sequence of our construction in Theorem 3.3.3 would be:

$(0,0,0)$	$\Rightarrow(1,0,0)$	$\Rightarrow(2,0,0)$	$\Rightarrow(2,1^*,0)$	$\Rightarrow(1,1^*,0)$	$\Rightarrow(1,2'',0)$
$\Rightarrow(1,1',0)$	$\Rightarrow(1,2',0)$	$\Rightarrow(1,1,0)$	$\Rightarrow(2,1,0)$	$\Rightarrow(2,2^*,0)$	$\Rightarrow(2,3'',0)$
$\Rightarrow(2,2',0)$	$\Rightarrow(2,3',0)$	$\Rightarrow(2,2,0)$	$\Rightarrow(2,3,0)$	$\Rightarrow(2,3,1^*)$	$\Rightarrow(1,1,1^*)$
$\Rightarrow(1,1,2'')$	$\Rightarrow(1,1,1')$	$\Rightarrow(1,1,2')$	$\Rightarrow(1,1,1)$	$\Rightarrow(2,1,1)$	$\Rightarrow(2,2^*,1)$
$\Rightarrow(2,3'',1)$	$\Rightarrow(2,2',1)$	$\Rightarrow(2,3',1)$	$\Rightarrow(2,2,1)$	$\Rightarrow(2,3,1)$	$\Rightarrow(2,3,2^*)$
$\Rightarrow(2,2,2^*)$	$\Rightarrow(2,2,3'')$	$\Rightarrow(2,2,2')$	$\Rightarrow(2,2,3')$	$\Rightarrow(2,2,2)$	$\Rightarrow(2,3,2)$
$\Rightarrow(2,3,3^*)$	$\Rightarrow(2,3,4'')$	$\Rightarrow(2,3,3')$	$\Rightarrow(2,3,4')$	$\Rightarrow(2,3,3)$	$\Rightarrow(2,3,4).$

Therefore, we obtain the state $S(3)$.

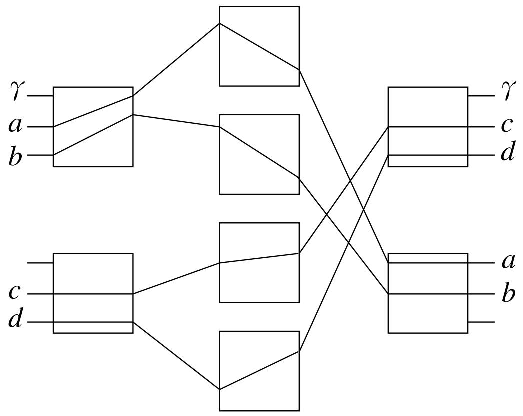


Figure 3.3: γ and $M(V,\gamma) = \{a, b, c, d\}$ in $C(3, 4, 2)$

Corollary 3.3.4. $\log_d(N, k, m)$ *is WSNB under any of CS, CD, STU, and P if and only if it is SNB, i.e.,* [8],

$$m > \begin{cases} k + 3 \cdot 2^{\frac{n-k}{2}-1} - 2, & \text{for } n-k \text{ even,} \\ k + 2^{\frac{n-k+1}{2}} - 2, & \text{for } n-k \text{ odd.} \end{cases}$$

Proof. Note that $\log_d(N, k, m)$ is a vertical copy network. Then the results for CS and CD follow from Theorem 3.3.1 and 3.3.2. For P and STU, it is easily verified that $\gamma_i' = (N-1, N-1)$ doesn't block any request in $\{\gamma\} \cup R_i$ for all i Then the results follow from Theorems 3.3.3. □

That packing is a good routing strategy has been a folklore for a long time and documented in literature [1]. One motivation for that folklore is that $C(n, m, 2)$ is WSNB under P if and only if $m \geq \lfloor \frac{3n}{2} \rfloor$ [1], while it is SNB if and only if $m \geq 2n - 1$. The seemingly discrepancy between the $m \geq \lfloor \frac{3n}{2} \rfloor$ result and Theorem 3.3.3 is explained by the fact that γ' does not exist in $C(n, m, 2)$ since $M(V, \gamma)$ occupies both input switches(see Figure 3.3).

For $r \geq 3$, $C(n, m, r)$ is WSNB under P if and only if it is SNB. Thus the saving of $C(n, m, 2)$ under P seems to be a fluke rather than a testimony of its goodness. In this chapter, again we showed that in the worst-case scenario, P does not help. Instead, MI is the only routing strategy which is still not ruled out to be useful.

Chapter 4

Conclusions and future works

From Chapter 2 and 3, we see that the costs of WSNB under the five routing strategies on the 3-stage Clos network, the multi-$\log_d N$, and some more general vertical-copy networks are same as SNB, except $C(n, m, 2)$ with packing strategy. It seems like WSNB is no better than SNB. But for multicast traffic, Yang-Masson [17] suggested a WSNB algorithm such that the required number of middle crossbars of the 3-stage Clos network is strictly less than SNB. For $\log_2 N$ networks, Tscha and Lee [16] used the window algorithm in the $\log_2(N, 0, m)$ network (also see Kabacinski and Danilewicz [11]) to give another example that the number of copies in WSNB is less than in SNB. Recently, Hwang and Lin [10] further extended their result to $\log_2(N, k, m)$. Note that if the window algorithm is used on one-to-one (1-cast) traffic, then the result is still same as SNB.

For larger r, Tsai, Wang, and Hwang [15] proved that $C(n, m, r)$ is not WSNB under any algorithm. We extended it to the asymmetric version (See Appendix for the proof).

Theorem 4.1. *For* $n_1 r_1 \leq n_2 r_2$ *and* $r_1 \geq (n_2 - 1)\binom{n_1+n_2-2}{n_1-1} + 1$, $C(n_1, r_1, m, n_2, r_2)$ *is WSNB if and only if* $m \geq n_1 + n_2 - 1$. *For* $n_1 r_1 > n_2 r_2$ *and* $r_2 \geq (n_1 - 1)\binom{n_1+n_2-2}{n_2-1} + 1$, $C(n_1, r_1, m, n_2, r_2)$ *is WSNB if and only if* $m \geq n_1 + n_2 - 1$.

Corollary 4.2. *For* $r \geq (n-1)\binom{2n-2}{n-1} + 1$*,* $C(n, m, r)$ *is WSNB if and only if* $m \geq 2n - 1$*.*

Note that further increasing r does not lead to a stronger result since O_1 can be connected to at most n inputs. Moreover, if $m \geq n_1 + n_2 - 1$, then it is SNB.

From these theorems, it seems that it is hopeless to find a good algorithm for WSNB in one-to-one traffic for large r. However, finding a better algorithm for smaller r is still possible. Note that the lower bound of r in Corollary 4.2 is exponential in n. Obtaining a polynomial bound will greatly reduce the range of uncertainty whether WSNB can improve over SNB.

Appendix

Proof of Theorem 4.1. For $n_1 r_1 \le n_2 r_2$, it suffices to prove the "only if" part. Suppose $r_1 = (n_2 - 1)\binom{n_1+n_2-2}{n_1-1} + 1$ and $m = n_1 + n_2 - 2$. Consider the state s that every input crossbar has $n_1 - 1$ inputs connect to arbitrary output crossbar except O_1. Then every input crossbar connects to $n_1 - 1$ middle crossbars. By the pigeonhole principle, there are $\left\lceil \frac{r_1}{\binom{m}{n_1-1}} \right\rceil = n_2$ input crossbars, say X, connected to the same n_2 middle crossbars, say M. Under s, we add n_2 new requests from idle input links in X to O_1. Since X is already routed through M, the new requests cannot use M any more. And since the new requests involve the same output crossbar, they must be routed through distinct middle crossbars. Therefore

$$m \ge |X| + |M| = n_1 + n_2 - 1.$$

For $n_1 r_1 > n_2 r_2$, the argument is similar by exchanging "input" to "output". □

References

[1] V. E. Beneš, Mathematical Theory of Connecting Networks and Telephone Traffic. *Academic Press*, 1965, New York.

[2] V. E. Beneš, Blocking in the NAIU network, AT&T *Bell Labs Tech. Memo.* 1985.

[3] J. C. Bermond, J. M. Fourneau, and A. Jean-Marie, Equivalence of multistage interconnection networks, *Information Processing Letter* 26 (1987) 45–50.

[4] F. H. Chang, J. Y. Guo, F. K. Hwang, C. K. Lin, Wide-sense nonblocking for symmetric or asymmetric 3-stage Clos networks under various routing strategies, *Theoretical Computer Science* 314 (2004) 375–386.

[5] F. H. Chang, J. Y. Guo, F. K. Hwang, Wide-sense nonblocking for multi-$\log_d N$ networks under various routing strategies, *Theoretical Computer Science*, to appear.

[6] C. Clos, A Study of nonblocking switching networks, *Bell System Tech. J.*, 32 (1953) 406-424.

[7] D. Z. Du, P. C. Fishburn, B. Gao and F. K. Hwang, Wide-sense nonblocking for 3-stage Clos networks, in Switching Networks: Recent Advances, Eds:D.Z.Du and H.Q.Ngo, pp. 89-100, Kluwer 2001, Boston.

[8] F. K. Hwang, Choosing the best $\log_k(N, m, P)$ strictly nonblocking networks. *IEEE Transactions on Communications*, Vol. 46, No. 4 (1998) 454–455.

[9] F. K. Hwang, The Mathematical Theory of Nonblocking Switching Networks, *World Scientific*, Singapore, 1st Ed.(1998) and 2nd Ed.(2004).

[10] F. K. Hwang and B. C. Lin, Wide-sense nonblocking multicast $\log_2(N, m, p)$ network. *IEEE Trans. Commun.* 51 (2003) 1730–1735.

[11] W. Kabacinski and G. Danilewicz, Wide-sense and strict sense nonblocking operation of multicast multi-$\log_2 N$ switching networks. *IEEE Trans. Commun.* 50 (2002) 1025–1036.

[12] C.-T. Lea, Multi-$\log_2 N$ networks and their applications in high-speed electronic and photonic switching systems, *IEEE Transactions on Communications*, Vol. 38, No. 10 (1990) 1740–1749.

[13] D.-J. Shyy and C.-T. Lea, $\log_2(N, m, p)$ strictly nonblocking networks. *IEEE Transactions on Communications*, Vol. 39, No. 10 (1991) 1502–1510.

[14] D.G. Smith, Lower bound in the size of a 3-stage wide-sense nonblocking network, *Elec. Lett.* 13 (1977) 215-216.

[15] K.H. Tsai, D.W. Wang and F.K. Hwang, Lower bounds of wide-sense nonblocking Clos networks, *Theor. Comput. Sci.* 261 (2001) 323-328.

[16] Yeonghwan Tscha and Kyoon-Ha Lee, Yet another result on multi-$\log_2 N$ network, *IEEE Trans. Commun.*, 47 (1999) 1425–1431.

[17] Y. Yang and G. M. Masson, Nonblocking broadcast switching networks, *IEEE Trans. Comput.* 9 (1991) 1005–1015.

[18] Y. Yang and W.Wang, Wide-sense nonblocking Clos networks under packing strategy, *IEEE Trans. Comput.* 45 (1999) 265-284.