Remark 3.8. *In the derivation of the Theorem 3.5, it needs the condition of integral separateness. However, for the linear system, if the integral separateness is not satisfied, then the result of Theorem 3.5 can be obtained. Let us give a example as follows,*

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \qquad A(t) = \begin{pmatrix} 1 + \frac{\pi}{2}\sin\pi\sqrt{t} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^2.$$

It has two Lyapunov exponents 1 and 0. This system does not have integral separateness, but the result of Theorem 3.5 holds for its coupled system.

Remark 3.9. *In numerical experience, we vary the coupling parameter k in order to receive amplitude synchronization which arises as the largest Lyapunov exponent of the difference system becomes negative by varying k. Hence, by the definition of Lyapunov exponent, we can $e_i(t) \approx \exp\lambda_j t$, for large t, where e_i is the error vector of the difference system, $i, j = 1, \cdots n$, $\lambda_j < 0$ by varying k. From this fact, we perhaps can understand the perturbation term of the coefficient matrix of the system (3.13) and (3.19) that it converges to 0 with a negative exponential rate. It may imply that the Lyapunov exponents of the original system and its perturbed system are the same without Theorem 3.2 and Theorem 3.3.*

4 Numerical Illustrations

In the first two section, we present some results of numerical experiments which is related to the discussion in the preceding section, where the coupled system consists of two Rossler system or two Lorenz ones.

In the third part, we mainly concern ourself with the numerical results about Lyapunov exponents of two different coupled systems. Namely, the coupled system consists of one Rossler and one Lorenz equations.

In the fourth part, we summarize our numerical results and compare variations of Lyapunov exponents for the Rossler and Lorenz systems coupled in different manners. Some parts of the results in the tables need further numerical evidence.

The following numerical experience mainly uses the matlab program to execute, and the algorithm of calculating Lyapunov exponent is adopted from [3], other related calculating algorithms are in the reference [1], [2], [12].

4.1 Lyapunov Exponents for two Coupled Rossler Systems

In Fig.1-Fig.6, we consider the coupled Rossler-Rossler system

$$\begin{aligned}
\dot{x}_1 &= -(y_1 + z_1) + k_1(x_2 - x_1) \\
\dot{y}_1 &= x_1 + 0.15y_1 \\
\dot{z}_1 &= 0.2 + z_1(x_1 - 10) \\
\dot{x}_2 &= -(y_2 + z_2) + k_2(x_1 - x_2) \\
\dot{y}_2 &= x_2 + 0.15y_2 \\
\dot{z}_2 &= 0.2 + z_2(x_2 - 10)
\end{aligned} \tag{4.1}$$

where the coupling is one-component, and $k_{1,2}$ are the coupling parameters between the two oscillators. It is set as $k_1 = 0$ if the coupling scheme is uni-directional, and as $k_1 \neq 0, k_2 \neq 0$ if the coupling scheme is bi-directional .

Herein initial condition $= (1, 1, 1, 1, 1, 0.5)$, time$=10^4$, absolute error$=10^{-8}$, relative error$=10^{-8}$, a hundred values of k is taken within the indicated interval.

Fig.1: variations of the originally positive and null Lyapunov exponents for $k_1 = 0$ and $k_2 \in [0, 0.5]$.
Fig.2: variations of the originally negative Lyapunov exponent for $k_1 = 0$ and $k_2 \in [0, 0.5]$.
Fig.3: variations of the originally positive Lyapunov exponent for $k_1 = 0$ and $k_2 \in [5.5, 10.5]$.
Fig.4: variations of the originally positive and null Lyapunov exponent for $k_1 = k_2 \in [0, 0.5]$.
Fig.5: variations of the originally negative Lyapunov exponent for $k_1 = k_2 \in [0, 0.5]$.
Fig.6: variations of the originally positive Lyapunov exponent for $k_1 = k_2 \in [2.5,\ 7.5]$.

Remark 4.1. *In Fig.3 and Fig.6, the originally positive Lyapunov exponent becomes negative and then positive again as the coupling parameter increases. If we keep increasing the coupling parameter, then this positive Lyapunov exponent still increases asymptotically. This phenomena only occurs in one-component coupling, not in full-component coupling.*

In Fig.7-Fig.10, we consider the coupled Rossler-Rossler system

$$\begin{aligned}
\dot{x}_1 &= -(y_1 + z_1) + k_1(x_2 - x_1) \\
\dot{y}_1 &= x_1 + 0.15y_1 + k_1(y_2 - y_1) \\
\dot{z}_1 &= 0.2 + z_1(x_1 - 10) + k_1(z_2 - z_1) \\
\dot{x}_2 &= -(y_2 + z_2) + k_2(x_1 - x_2) \\
\dot{y}_2 &= x_2 + 0.15y_2 + k_2(y_1 - y_2) \\
\dot{z}_2 &= 0.2 + z_2(x_2 - 10) + k_2(z_1 - z_2),
\end{aligned} \tag{4.2}$$

where the coupling is of full-component, and $k_{1,2}$ are the coupling parameters between the two oscillators. It is set as $k_1 = 0$ if the coupling scheme is unidirectional, and as $k_1 \neq 0, k_2 \neq 0$ if the coupling scheme is bi-directional.

Herein initial condition $(1, 1, 1, 1, 1, 0.5)$, time$=10^4$, absolute error$=10^{-8}$, relative error$=10^{-8}$, a hundred values of k is taken within the indicated interval.

Fig.7: variations of the originally positive and null Lyapunov exponents for $k_1 = 0$ and $k_2 \in [0, 0.4]$.

Fig.8: variations of the originally negative Lyapunov exponent for $k_1 = 0$ and $k_2 \in [0, 0.4]$.

Fig.9: variations of the originally positive and null Lyapunov exponent for $k_1 = k_2 \in [0, 0.35]$.

Fig.10: variations of the originally negative Lyapunov exponent for $k_1 = k_2 \in [0, 0.5]$.

Remark 4.2. *In Fig.7-Fig.10, the originally positive, null, and negative Lyapunov exponents of the coupled system (4-2) seem to have pairwise linear difference (differ by k for uni-direction case and by* $2k$ *for bi-direction case), as the parameter is large enough.*

In Fig.11, Fig.12, we consider the coupled Rossler-Rossler along with the associated difference system

$$
\begin{aligned}
\dot{x_1} &= -(y_1 + z_1) \\
\dot{y_1} &= x_1 + 0.15y_1 \\
\dot{z_1} &= 0.2 + z_1(x_1 - 10) \\
\dot{x_2} &= -(y_1 + z_1)) + k(x_1 - x_2) \\
\dot{y_2} &= x_1 + 0.15y_1 + k(y_1 - y_2) \\
\dot{z_2} &= 0.2 + z_1(x_1 - 10) + k(z_1 - z_2) \\
\dot{e_1} &= -ke_1 - e_2 - e_3 \\
\dot{e_2} &= e_1 + (0.15 - k)e_2 \\
\dot{e_3} &= z_2e_1 + (x_1 - 10 - k)e_3,
\end{aligned}
\tag{4.3}
$$

where k is the coupling parameter. $e_1 = x_1 - x_2$, $e_2 = y_1 - y_2$, $e_3 = z_1 - z_2$.

With initial condition $(1, 1, 1, 1, 1, 0.5, 0, 0, 0.5)$, time$=10^4$, absolute error$=10^{-8}$, relative error$=10^{-8}$, one hundred and fifty values of k is taken within $[0.0025, 0.13]$.

Fig.11: variations of the positive, null Lyapunov exponents for $k \geq 0$.

Fig.12: variations of the negative Lyapunov exponent for $k \geq 0$.

Remark 4.3. *From Fig.11 and Fig.12, we can observe three pairs of Lyapunov exponents. They seem to be identical when k passes about* 0.09. *There are three Lyapunov exponents* *of the response system and three Lyapunov exponents of the difference system among three pairs.*

4.2 Lyapunov Exponents for two Coupled Lorenz Systems

In Fig.13-Fig.20, we consider the coupled Lorenz-Lorenz system

$$\begin{array}{l} \dot{x}_1 = 10(y_1 - x_1) + k_1(x_2 - x_1) \\ \dot{y}_1 = 28x_1 - y_1 - x_1 z_1 \\ \dot{z}_1 = -\frac{8}{3}z_1 + x_1 y_1 \\ \dot{x}_2 = 10(y_2 - x_2) + k_2(x_1 - x_2) \\ \dot{y}_2 = 28x_2 - y_2 - x_2 z_2 \\ \dot{z}_2 = -\frac{8}{3}z_2 + x_2 y_2, \end{array} \tag{4.4}$$

where the coupling is of one-component, and $k_{1,2}$ are the coupling parameters between the two oscillators. It is set as $k_1 = 0$ if the coupling scheme is uni-directional, and as $k_1 \neq 0, k_2 \neq 0$ if the coupling scheme is bi-directional .

Herein initial condition $(1,1,1,1,1,0.5)$, time=10^4, absolute error=10^{-8}, relative error=10^{-8}, a two hundred values of k is taken within $[0,2]$ in Fig.13-Fig.15, one hundred values of k is taker within $[6.5,12]$ in Fig.16, one hundred and forty values of $k = k_1 = k_2$ is taken within $[0,1.4]$ in Fig.17-Fig.19, one hundred values of k is taker within $[3,7]$ in Fig.20,.

Fig.13: variations of the originally positive Lyapunov exponents for $k_1 = 0$ and $k_2 \in [0,2]$.
Fig.14: variations of the originally null Lyapunov exponent for $k_1 = 0$ and $k_2 \in [0,2]$.
Fig.15: variations of the originally negative Lyapunov exponent for $k_1 = 0$ and $k_2 \in [0,2]$.
Fig.16: variations of the originally positive, null Lyapunov exponents for $k_1 = 0$ and $k_2 \in [6.5,12]$.
Fig.17: variations of the originally positive Lyapunov exponent for $k_1 = k_2 \in [0,1.4]$.
Fig.18: variations of the originally null Lyapunov exponent for $k_1 = k_2 \in [0,1.4]$.
Fig.19: variations of the originally negative Lyapunov exponent for $k_1 = k_2 \in [0,1.4]$.
Fig.20: variations of the originally positive, null Lyapunov exponents for $k_1 = k_2 \in [3,7]$.

Remark 4.4. *In Fig.16 and Fig.20, the originally positive Lyapunov exponent asymptotically becomes negative as the coupling parameter increases. If we keep increasing the coupling parameter, then this positive Lyapunov exponent still negative. The increasing behavior of Lyapunov exponents in Fig.3 and Fig.6 does not take place in Fig.16 and Fig.20.*

In Fig.21-Fig.29, we consider the coupled Lorenz-Lorenz system

$$\begin{aligned}
\dot{x}_1 &= 10(y_1 - x_1) + k_1(x_2 - x_1) \\
\dot{y}_1 &= 28x_1 - y_1 - x_1 z_1 + k_1(y_2 - y_1) \\
\dot{z}_1 &= -\tfrac{8}{3}z_1 + x_1 y_1 + k_1(z_2 - z_1) \\
\dot{x}_2 &= 10(y_2 - x_2) + k_2(x_1 - x_2) \\
\dot{y}_2 &= 28x_2 - y_2 - x_2 z_2 + k_2(y_1 - y_2) \\
\dot{z}_2 &= -\tfrac{8}{3}z_2 + x_2 y_2 + k_2(z_1 - z_2),
\end{aligned} \tag{4.5}$$

where the coupling is of full component, $k_{1,2}$ are the coupling parameters between the two oscillators. It is set as $k_1 = 0$ if the coupling scheme is unidirectional, and as $k_1 \neq 0, k_2 \neq 0$ if the coupling scheme is bi-directional.

Herein initial condition $(1, 1, 1, 1, 1, 0.5)$, time=10^4, absolute error=10^{-8}, relative error=10^{-8}, one hundred and forty values of k is taken within $[0, 1.4]$ in Fig.21 and Fig.23, a hundred values of k is taken within $[0, 0.2]$ in Fig.22, a hundred and twenty values of k is taken within $[0, 1]$ in Fig.24 and Fig.26, two hundred value of k is taken within $[0, 0.2]$ in Fig.25. 6×10^6 values of time t is taken within $[0, 10^4]$ in Fig.27-28. 10^7 values of time t is taken within $[0, 10^6]$ in Fig.29.

Fig.21: variations of the originally positive and null Lyapunov exponents for $k_1 = 0$ and $k_2 \geq 0$.
Fig.22: variations of the originally null Lyapunov exponent for $k_1 = 0$ and $k_2 \in [0, 0.2]$.
Fig.23: variations of the originally negative Lyapunov exponent for $k_1 = 0$ and $k_2 \geq 0$.
Fig.24: variations of the originally positive and null Lyapunov exponent for $k_1 = k_2 \geq 0$.
Fig.25: variations of the originally positive and null Lyapunov exponent for $k_1 = k_2 \in [0, 0.2]$.
Fig.26: variations of the originally negative Lyapunov exponent for $k_1 = k_2 \geq 0$.
Fig.27: Behavior of $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$ v.s. time of the system (4.4) for coupling parameter $k_1 = k_2 = 0.26$.
Fig.28: Behavior of $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$ v.s. time of the system (4.4) for coupling parameter $k_1 = k_2 = 0.28$.
Fig.29: Behavior of $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$ v.s. time of the system (4.4) for coupling parameter $k = 0.3$.

Remark 4.5. *In Fig.21, Fig.23, Fig.24, and Fig.26, the originally positive, null, and negative Lyapunov exponents of the coupled system (4-5) seem to have linear difference pairwise as the coupling parameter is large enough. In Fig.21 and Fig.23, As k varies* *from 0 to positive, and it differs with one of the positive Lyapunov exponents by k when k passes about 0.9. In Fig.24 and Fig.26, As k varies from 0 to positive, and it differs with one of the positive Lyapunov exponents by $2k$ when k passes about 0.45.*

Remark 4.6. *In Fig.24 and Fig.26, variations of Lyapunov exponent are rather strange for coupling parameter $k_1 = k_2 \in [0.25, 0.45]$. When we can observe the behavior of $x_1 - x_2$, $y_1 - y_2$, and $z_1 - z_2$ in Fig.27-Fig.29 (the respective coupling parameter is $k_1 = k_2 = 0.26, 0.28, 0.3$), we find that no synchronization takes place for in Fig.27; synchronization arises in Fig.28; no synchronization arises in Fig.29, but the behavior of variable difference preserves some constant for large time t. In fact, we have verified the three above situations take places by varying for coupling parameter $k_1 = k_2 \in [0.25, 0.45]$. Nevertheless, we observe the situation as one which arises in Fig.30 for many coupling parameter that pass through about 0.3 to 0.45.*

Remark 4.7. *From the numerical observation, we can understand that decreasing variation amplitude of Lyapunov exponent of coupled system adopting the bidirectional coupling scheme is larger than one of coupled system adopting the unidirectional coupling scheme.*

4.3 Lyapunov Exponents for Coupled Rossler and Lorenz Systems

In Fig.30-Fig.32, we consider the coupled Rossler-Lorenz system

$$\begin{aligned}
\dot{x_1} &= -(y_1 + z_1) \\
\dot{y_1} &= x_1 + 0.15y_1 \\
\dot{z_1} &= 0.2 + z_1(x_1 - 10) \\
\dot{x_2} &= 10(y_2 - x_2) + k(x_1 - x_2) \\
\dot{y_2} &= 28x_2 - y_2 - x_2z_2 \\
\dot{z_2} &= -\tfrac{8}{3}z_2 + x_2y_2,
\end{aligned} \tag{4.6}$$

where k is the coupling parameter.

With initial condition $(1, 1, 1, 0.5, 0.5, 0.5)$, time=10^4, absolute error=10^{-8}, relative error=10^{-8}, one hundred and fifty values of k is taken within $[0, 1.5]$.
Fig.30: variations of the originally positive Lyapunov exponents for $k \geq 0$.
Fig.31: variations of the originally null Lyapunov exponent for $k \geq 0$.
Fig.32: variations of the originally negative Lyapunov exponent for $k \geq 0$.

Remark 4.8. *We compute Lyapunov exponents in the coupled system (4-6) with different time, in order to observe the convergence of the originally null Lyapunov exponents for some k.*

For the coupling parameter $k = 0.92$, we compute the originally null Lyapunov exponents, with different computation time as

Time	λ_1	λ_2
10^4	0.00131296	-0.00023921
10^5	0.00030551	-0.00002626
10^6	0.00029458	-0.00000256

For the coupling parameter $k = 0.99$, we compute the originally null Lyapunov exponents with different computation time as

Time	λ_1	λ_2
10^4	0.00195271	-0.00025496
10^5	0.00044135	-0.00002526
10^6	0.00042638	-0.00000354

For the coupling parameter $k = 1.23$, we compute the originally null Lyapunov exponents with different computation time as

Time	λ_1	λ_2
10^4	0.00108621	-0.00001271
10^5	0.00065541	-0.00000419
10^6	0.00082521	-0.00000045

Remark 4.9. *In [6], it declares that generalized synchronization takes place as some original positive Lyapunov exponent becomes negative by varying k beyond some value.*

4.4 The comparison table

$(\text{R-R})_{\text{s}-x}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \searrow$	$(\text{L-L})_{\text{s}-x}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \nearrow\searrow$
$(\text{R-R})_{\text{bi}-x}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \searrow$	$(\text{L-L})_{\text{bi}-x}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \nearrow\searrow$
$(\text{R-R})_{\text{s}-xyz}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \searrow$	$(\text{L-L})_{\text{s}-xyz}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \nearrow\searrow$
$(\text{R-R})_{\text{bi}-xyz}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \searrow$	$(\text{L-L})_{\text{bi}-xyz}$	$\lambda_0^{(1)} \equiv 0$; $\lambda_0^{(2)} \nearrow\searrow ? \searrow$

Table 1: The behavior of the originally null Lyapunov exponents $\lambda_0^{(1)}, \lambda_0^{(2)}$ of the coupled system as the coupling parameter k varies from 0 to positive.

Amplitude	$A_{\text{s}-x} > A_{\text{bi}-x} > A_{\text{s}-xyz} > A_{\text{bi}-xyz}$
coupling parameter range	$R_{\text{s}-x} > R_{\text{bi}-x} > R_{\text{s}-xyz} > R_{\text{bi}-xyz}$

Table 2: The comparison of amplitude and coupling parameter range for $\lambda_0^{(2)}$ as it turns to positive from zero for the coupled (L-L) system case.

(R-R): Coupled Rossler-Rossler system.

(L-L): Coupled Lorenz-Lorenz system.

s$-x$: The coupling scheme: two systems are coupled through x component in single direction.

bi$-x$: The coupling scheme: two systems are coupled through x component bi-directionally.

s$-xyz$: The coupling scheme: two systems are coupled through x, y, and z component in single direction.

bi$-x$: The coupling scheme: two systems are coupled through x, y, and z component bi-directionally.

$\lambda_0^{(1)}$, $\lambda_0^{(2)}$: The originally null Lyapunov exponents of the coupled system.

?: The behavior of the Lyapunov exponent of the coupled system is not affirmative from numerical examination in some coupling parameter interval.

$\equiv$: The indicated value is preserved throughout the computations.

A: The largest amplitude of $\lambda_0^{(2)}$ for the (L-L) coupled system as it turns from zero to positive.

R: The coupling parameter range of $\lambda_0^{(2)}$ for the (L-L) coupled system as it turns from zero to positive.

(R-R)$_{\text{s}-x}$	$\lambda_+^{(1)} \equiv \lambda_+^R$; $\lambda_+^{(2)} \searrow\nearrow$
(R-R)$_{\text{bi}-x}$	$\lambda_+^{(1)} \approx \lambda_+^R,\ as\ k > k_1$; $\lambda_+^{(2)} \searrow\nearrow,\ as\ k > k_1$
(R-R)$_{\text{s}-xyz}$	$\lambda_+^{(1)} \equiv \lambda_+^R$; $\lambda_+^{(2)} \searrow$
(R-R)$_{\text{bi}-xyz}$	$\lambda_+^{(1)} \approx \lambda_+^R,\ as\ k > k_2$; $\lambda_+^{(2)} \searrow,\ as\ k > k_2$
(L-L)$_{\text{s}-x}$	$\lambda_+^{(1)} \equiv \lambda_+^L$; $\lambda_+^{(2)} \searrow$
(L-L)$_{\text{bi}-x}$	$\lambda_+^{(1)} \approx \lambda_+^R,\ as\ k > k_3$; $\lambda_+^{(2)} \searrow,\ as\ k > k_3$
(L-L)$_{\text{s}-xyz}$	$\lambda_+^{(1)} \equiv \lambda_+^L$; $\lambda_+^{(2)} \searrow\nearrow\searrow$
(L-L)$_{\text{bi}-xyz}$	$\lambda_+^{(1)} \approx \lambda_+^R,\ as\ k > k_4$; $\lambda_+^{(2)} \searrow,\ as\ k > k_4$

Table 3: The variations of the originally positive Lyapunov exponents $\lambda_+^{(1)}, \lambda_+^{(2)}$ of the coupled system as the coupling parameter k varies from 0 to positive.

(R-R)$_{\text{s}-x}$	$\lambda_-^{(1)} \equiv \lambda_-^R$; $\lambda_-^{(2)} \searrow$
(R-R)$_{\text{bi}-x}$	$\lambda_-^{(1)} \approx \lambda_-^R,\ as\ k > k_1$; $\lambda_-^{(2)} \searrow,\ as\ k > k_1$
(R-R)$_{\text{s}-xyz}$	$\lambda_-^{(1)} \equiv \lambda_-^R$; $\lambda_-^{(2)} \searrow$
(R-R)$_{\text{bi}-xyz}$	$\lambda_-^{(1)} \approx \lambda_-^R,\ as\ k > k_2$; $\lambda_-^{(2)} \searrow,\ as\ k > k_2$
(L-L)$_{\text{s}-x}$	$\lambda_-^{(1)} \equiv \lambda_-^L$; $\lambda_-^{(2)} \searrow$
(L-L)$_{\text{bi}-x}$	$\lambda_-^{(1)} \approx \lambda_-^R,\ as\ k > k_3$; $\lambda_-^{(2)} \searrow,\ as\ k > k_3$
(L-L)$_{\text{s}-xyz}$	$\lambda_-^{(1)} \equiv \lambda_-^L$; $\lambda_-^{(2)} \searrow\nearrow\searrow$
(L-L)$_{\text{bi}-xyz}$	$\lambda_-^{(1)} \approx \lambda_-^R,\ as\ k > k_4,\ \lambda_-^{(2)} \searrow,\ as\ k > k_4$

Table 4: The variations of the originally negative Lyapunov exponent $\lambda_-^{(1)}, \lambda_-^{(2)}$ of the coupled system v.s. the coupling parameter k.

$\lambda_+^{(1)}$, $\lambda_+^{(2)}$: The originally positive Lyapunov exponents of the coupled system.
λ_+^R: The constant and positive Lyapunov exponent of the coupled system (R-R)$_{\text{s}-x}$.
λ_+^L: The constant and positive Lyapunov exponent of the coupled system (L-L)$_{\text{s}-x}$.
k_1, k_2, k_3, k_4: These are some constants, with $k_1 > k_2$, $k_3 > k_4$.
$\approx$: The Lyapunov exponent is near a fixed number.

$\lambda_-^{(1)}$, $\lambda_-^{(2)}$: The originally negative Lyapunov exponents of the coupled system.
λ_-^R: The constant and negative Lyapunov exponent of the coupled system (R-R)$_{\text{s}-x}$.
λ_-^L: The constant and negative Lyapunov exponent of the coupled system (L-L)$_{\text{s}-x}$.

$(\text{R-R})_{\text{s}-x}$	None
$(\text{R-R})_{\text{bi}-x}$	None
$(\text{R-R})_{\text{s}-xyz}$	$\lambda_+^{(1)} - \lambda_+^{(2)} \equiv k,\ \lambda_0^{(1)} - \lambda_0^{(2)} \equiv k,\ \lambda_-^{(1)} - \lambda_-^{(2)} \equiv k,\ as\ k > 2k_2 \approx 2k_{as}$
$(\text{R-R})_{\text{bi}-xyz}$	$\lambda_+^{(1)} - \lambda_+^{(2)} \equiv 2k,\ \lambda_0^{(1)} - \lambda_0^{(2)} \equiv 2k,\ \lambda_-^{(1)} - \lambda_-^{(2)} \equiv 2k,\ as\ k > k_2 \approx k_{as}$
$(\text{L-L})_{\text{s}-x}$	None
$(\text{L-L})_{\text{bi}-x}$	None
$(\text{L-L})_{\text{s}-xyz}$	$\lambda_+^{(1)} - \lambda_+^{(2)} \equiv k,\ \lambda_0^{(1)} - \lambda_0^{(2)} \equiv k,\ \lambda_-^{(1)} - \lambda_-^{(2)} \equiv k,\ as\ k > 2k_4 > 2k_{as}$
$(\text{L-L})_{\text{bi}-xyz}$	$\lambda_+^{(1)} - \lambda_+^{(2)} \equiv 2k,\ \lambda_0^{(1)} - \lambda_0^{(2)} \equiv 2k,\ \lambda_-^{(1)} - \lambda_-^{(2)} \equiv 2k,\ as\ k > k_4(>?)k_{as}$

Table 5: The linear difference between the originally two positive, two null, and two negative Lyapunov exponents of the coupled system as the coupling parameter k varies from 0 to positive.

k_{as}: The coupling parameter value at which the amplitude synchronization takes place.

5 Conclusions

This thesis has studied numerically the variations of Lyapunov exponents of some chaotic systems with several coupling schemes. We have also attempted to explore the behaviors of Lyapunov exponents as the coupled system undergoes synchronization. In the investigations, some analytical results about variations of Lyapunov exponent for system with specific coupling scheme have been obtained. However, because of inherent and unknown characteristic in the considered chaotic system, and the interaction of respective chaotic system through the coupled terms, analytic works about variations of Lyapunov exponent are rather difficult to establish. An alternative way to accomplish this task is to study the relation between Lyapunov exponent and the discrete type chaotic coupled system with different coupling schemes. At the same time, observations from thorough and precise numerical computations, and more mathematics machinery such as ergodic theory should help make the analytic studies more successful. On the other hand, studying the connection between Lyapunov exponent and the dissipation of different coupling scheme is also an interesting project. These investigations are useful in synchronizing the chaotic system effectively in chaotic control.