Primary answers to these questions can be obtained through many numerical computations. We can first use the setting below, as in [7], in which a Rossler (resp. Lorenz) is unidirectionally coupled to a Rossler (resp. Lorenz), or consider a coupled Rossler-Lorenz system and compute the Lyapunov spectrum of the whole system as a function of the coupling parameter. We can also set up a mutually coupled scheme and do the same computations. We hope to gain further understanding after accumulating enough numerical data. This can serve as a preliminary work for the scaling of Lyapunov exponent in coupled chaotic system with different coupling scheme and study connection between Lyapunov exponent and dissipation of the coherence behavior of coupled system.

In Section 2, we will present several definitions of Lyapunov exponent, and discuss the relation among these definitions. In addition, we summarize some theorems about Lyapunov exponent.

In Section 3, we will discuss The subtracted system and the coupled systems as well as their relationships between them. We then investigate the relation between Lyapunov exponents of the whole coupled system along with the ones of the associated difference system. In the second part of this section, some analytic works about variations of Lyapunov exponent with special coupling way will be addressed.

In Section 4, we present some results of numerical experiment of chaotic coupled system with different coupling scheme and provide some numerical illustrations. Section 5 will be devoted to discussion and conclusion.

2 Lyapunov Exponents in Continuous-time Systems

In the first part of this section, we will present the definitions of Lyapunov exponent, and discuss the relations among these definitions. In the second part, we summarize some theorems about Lyapunov exponent.

Before we begin introducing the definitions of Lyapunov exponent formally, we first provide the notion of characteristic exponent of functions, which was first developed by Lyapunov in 1892. The following definition can be found in [Adrianova].

Definition 2.1. *Let $f(t)$ be a complex-valued function defined on the interval $[t_0, \infty)$.*

$$\chi[f] = \limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$$

is called the **characteristic exponent** *of the function $f(t)$.*

The characteristic exponent tells us about the growth of the absolute value of a function. Clearly, if the characteristic exponent for a function f is the number α, then for a large time t, the growth of the absolute value of f is almost $\exp(\alpha t)$.

We shall mainly deal with finite characteristic exponents with the exception of $\chi[f] = \pm\infty$. From Definition 2.1, some basic properties are listed as follows:

(a) $\chi[f] = \chi[|f|]$,

(b) $\chi[cf] = \chi[f]$,

(c) if $|f(t)| \leq |g(t)|$ for $t \geq a$, then $\chi[f] \leq \chi[g]$.

We summarize further properties of characteristic exponent in the following paragraph.

Theorem 2.2 (Adrianova). *The characteristic exponent of the sum of a finite number of the functions $f_k(t), k = 1, 2 \ldots n$, does not exceed the greatest one of the characteristic exponents of these functions, i.e.,*

$$\chi[\sum_{k=1}^{n} f_k(t)] \leq \max_{1\leq k\leq n} \chi[f_k(t)].$$

The equality holds if only one of these functions has the greatest exponent. On the other hand, the characteristic exponent of the product of a finite number of functions does not exceed the sum of the characteristic exponents of factors, i.e.,

$$\chi[\prod_{k=1}^{n} f_k(t)] \leq \sum_{k=1}^{n} \chi[f_k(t)],$$

and the equality holds if one function has a finite limit, i.e.,

$$\lim_{t\longrightarrow\infty} \frac{1}{t} \ln|f_k(t)| = \alpha \neq \pm\infty \qquad \text{for some } k.$$

Now, let us consider the characteristic exponent of an integral. Following Lyapunov, in considering characteristic exponents of integral, we set

$$F(t) = \int_a^t f(s)ds, \qquad \text{where} \quad a = \begin{cases} t_0 & \text{for} \quad \chi[f] \geq 0, \\ \infty & \text{for} \quad \chi[f] < 0. \end{cases}$$

The integral defined in this way is called the **Lyapunov integral**. With this definition, we have the following theorem about the characteristic exponent of integral and integrand.

Theorem 2.3 (Adrianova). *Let $F(t)$ be the Lyapunov integral of $f(t)$. The characteristic exponent of an integral does not exceed the characteristic exponent of the integrand, i.e.,*

$$\chi[F] \leq \chi[f].$$

Naturally, we are bound to ask: why do we need a condition on the lower integral limit in the integral ? Let us consider a example, $f(t) = \exp(\alpha t)$, then

$$F(t) = \int_{t_0}^{t} \exp(\alpha s) ds.$$

For $\alpha > 0$ or $\alpha = 0$, $\chi[F] \leq \chi[f]$ holds. But for $\alpha < 0$, $\chi[F] > \chi[f]$. Henceforth, we need to employ the notion of Lyapunov integral in our studies.

Remark 2.4. *When $\chi[f] = \alpha < 0$, we can rewrite the Lyapunov integral in the form*

$$\int_{\infty}^{t} f(s) ds = \int_{t_0}^{t} f(s) ds - \int_{t_0}^{\infty} f(s) ds,$$

where the second integral on the right-hand side of the equality is convergent.

So far we have only discussed the characteristic exponent of a scalar function. Now we shall present some definitions and theorems of the characteristic exponent of the matrix function. The following definition can be found in [Adrianova].

Definition 2.5. *Let $F(t)$ be a $m \times n$ matrix function of t, $m \leq n$. The number defined as*

$$\chi[F] = \max_{i,j} \chi[f_{ij}]$$

is called the **characteristic exponent** *of the matrix $F(t)$.*

Theorem 2.6 (Adrianova). *The characteristic exponent of a finite size matrix $F(t)$ coincides with the characteristic exponent of its norm, i.e.,*

$$\chi[F] = \chi[\|F\|].$$

The result always holds for all norms.

Theorem 2.7 (Adrianova). *The characteristic exponent of the sum of a finite number of matrices does not exceed the greatest one of characteristic exponents of these matrices, i.e.,*

$$\chi[\sum_{k=1}^{n} F_k(t)] \leq \max_{1\leq k\leq n} \chi[F_k(t)].$$

The equality holds if only one matrix has the greatest characteristic exponent. The characteristic exponent of the product of a finite number of matrices does not exceed the sum of the characteristic exponents of these matrices, i.e.,

$$\chi[\prod_{k=1}^{n} F_k(t)] \leq \sum_{k=1}^{n} \chi[F_k(t)].$$

The equality holds if one matrix function has a finite limit.

Obviously, Theorem 2.7 is similar to Theorem 2.2. In addition, for a vector-valued function, the above theorems are also valid.

Lyapunov exponents provide a qualitative and quantitative characterization of dynamical behavior. It is related to the exponentially fast divergence or convergence of nearby orbits in phase space. In fact, they are a generalization of the eigenvalues for linear systems and of characteristic multipliers for periodic linear systems. In the case of linear homogeneous systems with constant coefficients, Lyapunov exponents are the real parts of the eigenvalues of the matrix of coefficients, and in the case of periodic coefficients they are the real parts of the logarithms of the characteristic multipliers divided by the period. They can be used to determine the stability of quasi-periodic and chaotic behavior as well as that of equilibrium points and periodic solutions.

Consider an autonomous continuous dynamical system of the following form:

$$\dot{x}(t) = f(x(t)), \qquad t \geq 0; \qquad x(0) = x_0, \quad x(t) \in \mathbb{R}^n, \tag{2.1}$$

where f is continuously differentiable. Let $\phi_t(x_0)$ be the solution of the system (2.1), which passes through x_0 at $t = t_0$.

Consider the linear variational equation:

$$\dot{Y}(t) = D_x f(\phi_t(x_0))Y =: A(t)Y(t), \quad Y(t) \in \mathbb{R}^{n\times n}, \quad t \geq t_0, \quad Y(t_0) = I, \tag{2.2}$$

where $A(t) = \dfrac{\partial f}{\partial x}(\phi_t(x_0)) \in \mathbb{R}^{n \times n}$. Let $\Phi(t)$ be the fundamental matrix solution (FMS) of the variational equation (2.2). Since the initial condition is the identity matrix, it follows that a perturbation δx_0 of x_0 evolves according to (2.2) is

$$\delta x(t) = \Phi(t)\delta x_0,$$

where δx_0 is the initial **infinitesimal** perturbation at x_0.

The perturbation δx may be interpreted in two ways: as an infinitesimal perturbation of the solution for the original system (2.1) or as a vector-valued solution of the linear variational equation (2.2).

Now, we will present the definition of Lyapunov exponent as follow:

Definition 2.8. *Consider the system (2.1), and n linearly independent initial perturbation vectors* $\{\delta_i x_0 \mid i = 1, \cdots, n\}$ *at* x_0*. Let* $\Phi(t)$ *be the fundamental matrix solution of the variational equation (2.2), then the number defined as*

$$\lambda_i(x_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\delta_i x(t)\| = \limsup_{t\to\infty} \frac{1}{t} \ln \|(\Phi(t)\delta x_0)_i\|$$

is called the **Lyapunov exponent** *of the system (2.1) at* x_0*, where* $\delta_i x(t)$ *is regarded as a perturbation function of the solution for the original system (2.1),* $i = 1 \ldots n$*.*

In fact, Lyapunov exponents are characteristic exponents of the perturbation functions of the system (2.1).

Another aspect of this notion is on how $\Phi(t)$ changes lengths of all perturbation vectors. We can consider the square of the length of the image of a perturbation vector δx_0 under $\Phi(t)$ by

$$\begin{aligned} ||\Phi(t)\delta x_0||^2 &= (\Phi(t)\delta x_0)^T \Phi(t)\delta x_0 \\ &= \delta x_0^T [(\Phi(t))^T \Phi(t)]\delta x_0. \end{aligned}$$

It is possible to take roots: $[(\Phi(t))^T \Phi(t)]^{1/2}$ measures how much lengths are changed by $\Phi(t)$, and $[(\Phi(t))^T \Phi(t)]^{1/2t}$ measures the average amount vectors are stretched. Therefore, we can also rewrite the definition 2.8 as follow:

Definition 2.9 (Eckmann and Ruelle). *Let $\Phi(t)$ be the FMS of the linear variational equation (2.2), then the following symmetric positive-definite matrix exists:*

$$\Lambda = \limsup_{t \longrightarrow \infty} (\Phi(t)^T \Phi(t))^{1/2t},$$

and the logarithms of their eigenvalues are called the **Lyapunov exponents** *of (2.2).*

From definition 2.8 and definition 2.9, we know there are n Lyapunov exponents for (2.2) at x_0. All these exponents are probably different. Next, we shall present another definition that which requires the FMS, $\Phi(t)$, be the normal FMS.

Let us turn again to the linear homogeneous system (2.2). Suppose its spectrum contains p elements, $p \leq n$, i.e., let

$$\infty > \lambda_1 > \lambda_2 > \cdots > \lambda_p > -\infty$$

be the set of distinct characteristic exponents of solutions for the linear system (2.2).

Consider the FMS $\Phi(t)$ of the linear system, which has form as $\Phi(t) = \{\phi_1(t), \ldots, \phi_n(t)\}$ ordered in such a way that $\chi[\phi_k] \leq \chi[\phi_{k+1}]$, $k = 1, \ldots n-1$. Of course, some of these characteristic exponents could coincide. Let this FMS contain r_k solutions with the characteristic exponent λ_k, $k = 1, \ldots, p$. We note that some r_k may be zero.

We call the number

$$\sigma_\Phi = \sum_{k=1}^{p} r_k \lambda_k,$$

the **sum of characteristic exponents** of the FMS $\Phi(t)$.

Definition 2.10 (Adrianova). *A FMS $\Phi(t)$ of the linear homogeneous system is called* **normal** *if $\sigma_\Phi \leq \sigma_\Psi$, for all possible FMS, $\Psi(t)$, of the linear homogeneous system.*

In fact, a FMS $\Phi(t) = \{\phi_1(t), \ldots, \phi_n(t)\}$ is normal if and only if it has the property of **incompressibility**, i.e.

$$\chi[\sum_{i=1}^{n} c_i \phi_i(t)] = \max_{1 \leq i \leq n} \chi[\phi_i(t)],$$

where $|c_i| \neq 0$.

By the normal FMS of the linear homogenous system, the definition of Lyapunov exponent can be written as follow:

Definition 2.11. *Consider the linear variational equation (2.2) of the original system (2.1) and its normal FMS* $\Phi(t)$*, then the number defined as*

$$\lambda_i(x_0) = \limsup_{t \to \infty} \frac{1}{t} \ln \|\Phi(t)e_i\|$$

is called the **Lyapunov exponent** *of the system (2.1) at* x_0*, where* e_i *is the standard basis of* $\mathbb{R}^n$*,* $i = 1 \ldots n$*.*

However, we would like to know how does a FMS be transformed to a normal FMS. With the theorem below, Lyapunov showed how to construct a normal FMS.

Definition 2.12 (Adrianova). *Consider a FMS* $\Psi(t) = [\psi_1(t), \ldots, \psi_n(t)]$*, such that the characteristic exponent of the columns of* $\Psi(t)$ *are ordered as* $\chi[\psi_1] \leq \cdots \leq \chi[\psi_n]$*. There exists a nonsingular triangular matrix* C *of the form*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ c_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_{n1} & \cdots & c_{n,n-1} & 1 \end{pmatrix},$$

such that

$$\Phi(t) = \Psi(t)C = \{\phi_1(t), \ldots, \phi_n(t)\}$$

is a normal FMS. Similarly, if the characteristic exponent of the columns of $\Psi(t)$ *are ordered as* $\chi[\psi_1] \geq \cdots \geq \chi[\psi_n]$*, then there exists a unit upper triangular matrix* C *such that* $\Phi(t) = \Psi(t)C$ *is a normal FMS. A unit triangular matrix is a triangular matrix whose diagonal elements are 1.*

In some references, Lyapunov exponent is also defined as follow:

Definition 2.13 (Parker and Chua). *Let* $\Phi(t)$ *is the FMS of the linear variational equation (2.2), let* $m_1(t), \ldots, m_n(t)$ *be the eigenvalues of FMS* $\Phi(t)$*, then the* **Lyapunov exponent** *of* x_0 *are defined as*

$$\lambda(x_0) = \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln |m_i(t)|, \qquad i = 1, \ldots, n.$$

There are similar definitions for nonautonomous continuous dynamical system. Basically, Definition 2.8, Definition 2.9, and Definition 2.11 are equivalent, but Lyapunov exponents calculated by Definition 2.13 are different from the above three definitions. We can give a example as follow:

Consider the linear homogenous system

$$\dot{x} = Ax, \qquad x(0) = x_0, \qquad x \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ e^t & 2 \end{pmatrix}.$$

We can also solve the FMS $\Phi(t)$ of this system:

$$\Phi(t) = \begin{pmatrix} e^t & 0 \\ te^{2t} & e^{2t} \end{pmatrix}.$$

By the above Theorem 2.7 and Definition 2.5, we can calculate the Lyapunov exponent by using Definitions 2.8, 2.9, 2.11, and they are $\lambda_1(x_0) = 2, \lambda_2(x_0) = 2$. However, calculating the Lyapunov exponent by using Definition 2.13 gives $\lambda_1(x_0) = 1, \lambda_2(x_0) = 2$. Nevertheless, four definitions may be the same under some conditions, and we will discuss it later.

Next part, we present some properties of Lyapunov exponents.

Definition 2.14 (Adrianova). *L is a* **Lyapunov transformation** *or* **Lyapunov matrix** *if*

(i) $L \in C^1[t_0, \infty)$.

(ii) $L(t), L^{-1}(t), \dot{L}(t)$ *are bound for* $t \geq t_0$.

Let $x = L(t)y$ be a Lyapunov transformation. It follows that

$$\dot{x} = A(t)x \Longrightarrow \dot{y} = B(t)y, \qquad B(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t).$$

¿From the second condition of Definition 2.14, it can be seen that Lyapunov transformations do not change characteristic exponents.

Consider a linear system

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad A \in C([t_0, \infty)). \tag{2.3}$$

Theorem 2.15 (Adrianova). *By means of a unitary transformation any linear system (2.3) can be reduced to a system with an upper triangular matrix whose diagonal coefficients are real. If coefficient matrix $A(t)$ of the system (2.3) is bounded, then the coefficient matrix of the triangular system are also bounded, and the unitary transformation is a Lyapunov transformation.*

Remark 2.16. *In fact, via an unitary transformation Q, any linear system (2.3) can be reduced to a system whose coefficients matrix is an upper triangular matrix, and the unitary matrix Q can transform FMS $\Phi(t)$ to a upper matrix R.*

Consider the following lower triangular coefficient matrix system,

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad A \in C[t_0, \infty), \quad \sup \|A(t)\| \leq M, \tag{2.4}$$

where

$$A(t) = \begin{pmatrix} a_{1,1}(t) & 0 & \cdots & 0 \\ a_{2,1}(t) & a_{2,2}(t) & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}.$$

Theorem 2.17. *The Lyapunov exponents of the triangular system (2.4) with real diagonal have the form*

$$\lambda_k = \chi[\phi_{k,k}(t)] = \limsup_{t \longrightarrow \infty} \frac{1}{t} \int_{t_0}^{t} a_{k,k}(s)ds, \qquad k = 1, \ldots, n.$$

Proof. We introduce the notation

$$A_k = \exp \int_{t_0}^{t} a_{k,k}(s)ds.$$

Consider the FMS

$$\Phi(t) = \{\phi_1(t), \ldots, \phi_n(t)\},$$

where

$$\phi_k(t) = \begin{pmatrix} \phi_{k,1}(t) \\ \phi_{k,2}(t) \\ \vdots \\ \phi_{k,n}(t) \end{pmatrix},$$

and

$$\begin{array}{ll} \phi_{k,j}(t) = 0 & \text{for} \quad j < k, \\ \phi_{k,k}(t) = A_k & \text{for} \quad j = k, \\ \phi_{k,j}(t) = A_j \displaystyle\int_{\alpha_j}^{t} A_j^{-1} \sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du = A_j \int_{\alpha_j}^{t} B(u)du & \text{for} \quad j > k, \end{array}$$

where

$$\alpha_j = \begin{cases} t_0 & \text{if} \quad \chi[B] \geq 0, \\ \infty & \text{if} \quad \chi[B] < 0. \end{cases}$$

Now, let us show that $\chi[\phi_k(t)] = \lambda_k$,

$$\begin{aligned} \chi[\phi_{k,k}] &= \lambda_k, \\ \chi[\phi_{k,k+1}] &= \chi[A_{k+1} \int_{\alpha_{k+1}}^{t} A_{k+1}^{-1} a_{k+1,k}(u)\phi_{k,k}(u)du] \\ &\leq \chi[A_{k+1}] + \chi[\int_{\alpha_{k+1}}^{t} A_{k+1}^{-1} a_{k+1,k}(u)\phi_{k,k}(u)du] \\ &\leq \lambda_{k+1} + \chi[A_{k+1}^{-1} a_{k+1,k}(u)\phi_{k,k}(u)du] \\ &\leq \lambda_{k+1} + \chi[A_{k+1}^{-1}] + \chi[a_{k+1,k}(t)] + \chi[\phi_{k,k}] \\ &\leq \lambda_{k+1} - \lambda_{k+1} + 0 + \lambda_k \\ &\leq \lambda_k. \end{aligned}$$

In the above derivations, we have used Theorems 2.2 and 2.3. Next, it can be shown by induction that

$$\chi[\phi_{k,i}] \leq \lambda_k, \qquad \text{for} \quad i = k, \ldots, j-1.$$

Thus, we get

$$\begin{aligned} \chi[\phi_{k,j}] &= \chi[A_j \int_{\alpha_j}^{t} A_j^{-1} \sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du] \\ &\leq \lambda_j - \lambda_j + \max_s\{\chi[a_{j,s}]\} + \max_s \chi[\phi_{k,s}] \\ &\leq \lambda_k. \end{aligned}$$

By Definition 2.5, we obtain

$$\chi[\phi_k] = \lambda_k.$$

Let us show that $\phi_k(t)$ is a solution of system (2.4). We will substitute $\phi_k(t)$ into the system (2.4), and we will check that the equality holds. Since it is obvious for $j = k$ and $j < k$, we will only check the case $j > k$. Firstly, we look the left-hand side of the equality,

$$\frac{d}{dt}\phi_{k,j}(t) = \frac{d}{dt}A_j\int_{\alpha_j}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du + A_j\frac{d}{dt}\int_{\alpha_j}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du. \quad (2.5)$$

When $\chi[B(t)] < 0$, we takes $\alpha_j = \infty$. Hence, the equation (2.5) becomes

$$\frac{d}{dt}\phi_{k,j}(t) = \frac{d}{dt}A_j\int_{\infty}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du + A_j\frac{d}{dt}\int_{\infty}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du.$$

By Remark 2.4 and the fundamental theorem of calculus,

$$\begin{aligned}
\frac{d}{dt}\phi_{k,j}(t) &= \frac{d}{dt}A_j\int_{\infty}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du + A_j\frac{d}{dt}[\int_{t_0}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du - c] \\
&= a_{jj}(t)A_j\int_{\infty}^{t} A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(u)\phi_{k,s}(u)du + A_j[A_j^{-1}\sum_{s=k}^{j-1} a_{j,s}(t)\phi_{k,s}(t)] \\
&= a_{j,j}(t)\phi_{k,j} + \sum_{s=k}^{j-1} a_{j,s}(t)\phi_{k,s} \\
&= \sum_{s=1}^{k} a_{j,s}(t)\cdot 0 + \sum_{s=k}^{j} a_{j,s}(t)\phi_{k,s} + \sum_{s=j+1}^{n} 0\cdot\phi_{k,s} \\
&= \text{the } j \text{ component of the right} - \text{hand side of the system (2.4).}
\end{aligned}$$

Therefore, we have already checked that the equality of the system (2.4) holds. Similarly, it holds for the case $\chi[B(t)] \geq 0$.

Next, we show that the FMS $\Phi(t)$ is a normal FMS. We will use the notion of incompressibility, which is the same as the notion of normal. Without loss of generality, we assume that

$$\chi[\Phi(t)] = \chi[\phi_{\gamma_1}(t)] = \cdots = \chi[\phi_{\gamma_s}(t)] = \max_k \chi[\phi_k] = \lambda,$$

where $\gamma_s \in \{1, 2, \cdots, n\}$, $1 \leq s < n$, and $\gamma_1 < \gamma_2 < \cdots < \gamma_s$. Further, by the preceding proof and Definition 2.5, we know that

$$\chi[\Phi(t)] = \chi[\phi_{\gamma_s}(t)] = \chi[\phi_{\gamma_s,\gamma_s}(t)] = \lambda > \chi[\phi_{l,j}(t)],$$

where $l \in \{1, 2, \cdots, n\} \setminus \{\gamma_1, \gamma_2, \cdots, \gamma_s\}$, $j = 1, \cdots, n$. Then,

$$\sum_{k=1}^{n} c_k\phi_k = \begin{pmatrix} c_1\phi_{1,1} \\ c_1\phi_{1,2} + c_2\phi_{2,2} \\ \vdots \\ \sum_{k=1}^{n} c_k\phi_{k,n} \end{pmatrix},$$

where $|c_k| \neq 0$, by Theorem 2.2, we can deduce that

$$\begin{cases} \chi[\sum_{k=1}^{t_1} c_k\phi_{k,t_1}] < \lambda, & 1 \leq t_1 \leq \gamma_1 - 1 \\ \chi[\sum_{k=1}^{\gamma_1} c_k\phi_{k,\gamma_1}] = \lambda, & \\ \chi[\sum_{k=1}^{t_2} c_k\phi_{k,t_2}] \leq \lambda, & \gamma_1 + 1 \leq t_2 \leq n. \end{cases}$$

Once again, by Definition 2.5, we obtain

$$\chi[\sum_{k=1}^{n} c_k\phi_k] = \max_k \chi[\phi_k] = \lambda,$$

Similarly, as

$$\chi[\Phi(t)] = \chi[\phi_k(t)] = \lambda, \qquad \forall\ k = 1, 2, \cdots, n,$$

we can show the same manner. Therefore, we have completed the proof the FMS $\Phi(t)$ is a normal FMS. Hence,

$$\lambda_k = \chi[A_k] = \chi[\phi_k] = \limsup_{t \longrightarrow \infty} \frac{1}{t} \int_{t_0}^{t} a_{k,k}(s)ds, \qquad k = 1, \ldots, n.$$

Remark 2.18. *In fact, we still need to discuss whether definition 2.13 is the same under unitary transformation (here, it also satisfies the condition of Lyapunov transformation) which can transform the original system to a triangular one. Hence, the question is the same as*

$$\limsup_{t \longrightarrow \infty} \frac{1}{t} \ln |m_i(t)| = \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln |p_j(t)| \quad ? \quad \forall i, j = 1, \cdots, n,$$

where $m_i(t)$ is the eigenvalue of the FMS $\Phi_1(t)$ of original linear variational system, and $p_j(t)$ is the eigenvalue of the FMS $\Phi_2(t)$ of linear system that was transformed.

Intuitively, since this Lyapunov transformation (or Lyapunov matrix) is a unitary matrix, the dynamical behavior of the original system should not be influenced from the fact that the any unitary matrix is the composite of rotations and reflections. From this view, it perhaps tells us the answer to the above question is right.

If this answer is right, then Theorem 2.17 can tell us calculate results using Definition 2.8, 2.9, 2.11, and 2.13 are the same under the coefficient matrix $A(t)$ in the system (2.2) is uniformly bounded. In fact, we mainly discuss the Lyapunov exponent of the dissipative system with attractor. Therefore, the condition the coefficient matrix $A(t)$ in the system (2.2) is uniformly bounded can satisfy.

For the limit existence of Lyapunov exponent, we also present as follows:

Consider a system

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad A \in C[t_0, \infty), \quad \sup \|A(t)\| \leq M. \tag{2.6}$$

Definition 2.19 (Adrianova). *The system (2.6) is* **regular** *if and only if the following two conditions are satisfied simultaneously:*

(i) *The limit*

$$\lim_{t \longrightarrow \infty} \frac{1}{t} \int_{t_0}^{t} tr(A(s))ds = \Lambda$$

exists.

(ii) *For any normal FMS $\Phi(t)$ such that $\sigma_\Phi = \Lambda$.*

Remark 2.20. *A regular system transformed by Lyapunov transformation is still a regular system.*

Theorem 2.21 (Adrianova). *The triangular system (2.4) with real diagonal is regular if and only if its Lyapunov exponent λ_k has finite limit value, i.e,*

$$\lambda_k = \chi[\phi_{kk}(t)] = \lim_{t \longrightarrow \infty} \frac{1}{t} \int_{t_0}^{t} a_{kk}(s)ds, \qquad k = 1, \ldots, n.$$

Remark 2.22. *In fact, by means of remark 2.20 and theorem 2.21, the Lyapunov exponent of a regular system has finite limit value.*

Theorem 2.23 (Oseledec). *The continuous dynamical system (2.1), and the variational equation (2.2), $\Phi(t)$ is its FMS. For the ergodic measure ρ, then almost all $x \in \mathbb{R}^n$, the following limits exist:*

(i)

$$\lim_{t \longrightarrow \infty} (\Phi(t)^T \Phi(t))^{1/2t} = \Lambda_x.$$

(ii)

$$\lim_{t \longrightarrow \infty} ||\Phi(t)u|| = \lambda_k(x), \qquad if\ u\ \in E_k(x) \setminus E_{k+1}(x),$$

where $\lambda_1(x) > \lambda_2(x) > \ldots$ *are the logarithms of the eigenvalue of* $\Lambda(x)$*, and* $E_k(x)$ *is the subspace of* $\mathbb{R}^n$ *corresponing to the eigenvalues* $\leq \exp \lambda_k(x)$.

In fact, regular systems we discussed above are prevalent in a certain measure theoretic sense. We can see Oseledec's work [8]. For example, the work [8] implies that if the variational equation (2.2) comes from linearization of a trajectory of a nonlinear system, and the orbit through the initial condition x_0 generates an ergodic measure, then the Lyapunov exponents exist as limits (and the system (2.2) is regular) and they are the same for almost every x_0 with respect to the ergodic measure.

Next, we shall summary some significant properties of Lyapunov exponent for dynamical system as following:

Proposition 2.22:

(i) Any continuous dynamical system without a fixed point will have at least one zero exponent [5], corresponding to the slowly changing magnitude of a principal axis tangent to the flow.

(ii) The sum of the Lyapunov exponents is the time-averaged divergence of the phase space velocity; hence any dissipative dynamical system will have at least one negative exponent.

(iii) By the definition of sensitive dependence of chaos, any chaotic continuous dynamical will have at least one positive exponent.

(iv) For the ergodic measure, almost every point in the basin of attractor of strange attractor has the same Lyapunov exponent. But, for every point (Not need almost every) in the basin of attractor of nonstrange attractor has the same Lyapunov exponent.