3 Lyapunov Exponents for Coupled Chaotic Systems

In Section 3.1, we will discuss Lyapunov exponents of the difference system and of the coupled system corresponding to two identical chaotic attractors. In Section 3.2, we present some analytical results for variations of Lyapunov exponent for certain coupled system which undergoes synchronization.

3.1 Lyapunov Exponents for Coupled Chaotic System and Difference System

Consider a system of ordinary differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n, \tag{3.1}$$

where $f \in \mathbb{C}^1$ is such that the system has a chaotic attractor. We couple two such identical systems in a unidirectional manner:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= f(\mathbf{x}_1), & \mathbf{x}_1(0) &= \mathbf{x}_1^{(0)} \in \mathbb{R}^n \\ \dot{\mathbf{x}}_2 &= f(\mathbf{x}_2) + kD(\mathbf{x}_1 - \mathbf{x}_2), & \mathbf{x}_2(0) &= \mathbf{x}_2^{(0)} \in \mathbb{R}^n, \end{aligned} \tag{3.2}$$

where D is a $n \times n$ coupling matrix, k is the coupling parameter.

We note that the $\mathbf{x}_1$-component is not influenced by the $\mathbf{x}_2$-component, while $\mathbf{x}_2$-component affects the $\mathbf{x}_1$-component. Therefore, we sometimes call the $\mathbf{x}_1$-subsystem **driving (or master)** one, and the $\mathbf{x}_2$-subsystem **response (or slave)** one.

Then, its variational system along a solution $(\mathbf{x}_1(t;\mathbf{x}_1^{(0)}), \mathbf{x}_2(t,k;\mathbf{x}_2^{(0)}))$ is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) & 0 \\ kD & A_2(t,k) - kD \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^{(0)} \\ \mathbf{y}_2^{(0)} \end{pmatrix}, \tag{3.3}$$

where

$$A_1(t) = D_x f(\mathbf{x}_1(t;\mathbf{x}_1^{(0)})) \in C([0,\infty)), \qquad ||A_1(t)|| \le M,$$

$$A_2(t,k) = D_x f(\mathbf{x}_2(t,k;\mathbf{x}_0^{(2)})) \in C([0,\infty) \times [0,\infty)), \qquad ||A_2(t,k)|| \le M,$$

and $(\mathbf{x}_1(t;\mathbf{x}_1^{(0)}), \mathbf{x}_2(t,k;\mathbf{x}_2^{(0)}))$ is the solution of this coupled system, which passes through $(\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)})$ at $t = 0$. We hope to understand the dynamical behavior of the whole coupled system in the aspect of its Lyapunov exponents.

When we study the behavior of amplitude synchronization for coupled system, we may calculate the Lyapunov exponent of the difference system, which will tell us whether if the variable difference will damp out or not and hence whether if the synchronization state is stable or not. The difference system corresponding to two identical oscillators coupled in a unidirectionally manner is as follows.

$$\dot{\mathbf{z}} = f(\mathbf{x}_1) - f(\mathbf{x}_2) - kD\mathbf{z} = (A_2(t,k) - kD)\mathbf{z} + h(\mathbf{z},k), \tag{3.4}$$

where $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2 \in \mathbb{R}^n$, $h \in C(\mathbb{R}^n \times [0,\infty))$. Sometimes, (3.4) can be written as

$$\dot{\mathbf{z}} = (A_3(t,k) - kD)\mathbf{z}.$$

as is the case for the coupled Rossler-Rossler system and coupled Lorenz-Lorenz system. where

$$A_3(t,k) \in C([0,\infty) \times [0,\infty)), \qquad ||A_3(t,k)|| \le M.$$

In fact, $h(\mathbf{z},k)$ contains the higher order terms of Taylor expansion of $f(\mathbf{x}_1) - f(\mathbf{x}_2) = f(\mathbf{z}+\mathbf{x}_2) - f(\mathbf{x}_2)$ about $\mathbf{x}_2$. As to $A_3(t,k)$, we may give a example of the difference system of Rossler-Rosseler system. From this example, it helps to understand what $A_3(t,k)$ is and the general form of the difference system. The Rossler system is system (3.1) with $f = (f_1, f_2, f_3)$ and $f_1(\mathbf{x}) = -x_2 - x_3$, $f_2(\mathbf{x}) = x_1 + 0.15x_2$, $f_3(\mathbf{x}) = 0.2 + x_3(x_1 - 10)$, where $\mathbf{x} = (x_1, x_2, x_3)$. Consider the Rossler-Rossler system coupled as in (3.2). Denote $\mathbf{x}_i = (x_{i1}, x_{i2}, x_{i3}), i = 1, 2$. Then, its linear variational system can be written as follows:

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) & 0 \\ kD & A_2(t,k) - kD \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix},$$

where $\mathbf{y}_i = (y_{i1}, y_{i2}, y_{i3}), i = 1, 2,$

$$A_1(t) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.15 & 0 \\ x_{13}(t) & 0 & x_{11}(t) - 10 \end{pmatrix},$$

and

$$A_2(t,k) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.15 & 0 \\ x_{23}(t,k) & 0 & x_{21}(t,k) - 10 \end{pmatrix}.$$

Let $\mathbf{z} = (z_1, z_2, z_3) = \mathbf{x}_1 - \mathbf{x}_2$, then its difference system can be written as follows:

$$\dot{\mathbf{z}} = (A_3(t,k) - kD)\mathbf{z} = (A_2(t,k) - kD)\mathbf{z} + h(\mathbf{z},k),$$

where

$$A_3(t,k) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.15 & 0 \\ x_{23}(t,k) & 0 & x_{11}(t) - 10 \end{pmatrix},$$

and $h(\mathbf{z}, k) = (0, 0, z_1 z_3)$. From this instance, we observe that $A_3(t, k)$ is a time dependent matrix and some of its coefficients are different from $A_2(t, k)$.

Therefore, the Lyapunov exponents of response system are not the same as the ones of the difference system, as to be seen in the numerical computations in Fig.26 and Fig.27. However, we will see that those Lyapunov exponents seem to be the same as coupling parameter k exceeds some value at which the positive Lyapunov exponent of the difference system becomes negative. The reason is because that amplitude synchronization arises. In physical viewpoint, the term $h(\mathbf{z}, k)$ of the difference system can die out asymptomatically, and the coefficient matrix $A_2(t, k) - kD$ of variational system of the response system and the coefficient matrix $A_3(t, k) - kD$ of the difference system are nearly the same for large t. We believe this also holds for coupled Lorenz-Lorenz system, although it remains to be checked more carefully.

There are a few facts that we are assure of in computing Lyapunov exponents. Namely,

(i) The Lyapunov exponents of the driving system are the same as the ones of the original system (3.1), as long as the coupling is unidirectional. It is clear that under a Lyapunov transformation the linear variational system can be transformed to the triangular one. By means of the property that Lyapunov transformation does not change Lyapunov exponent of the original system, we arrive at this assertion.

(ii) In physics, the coefficient matrix of the variational equation of the response system is the same as the one of the difference system for large time t as coupling parameter k exceeds some value at which the positive Lyapunov exponent of the difference system becomes negative.

On the other hand, we also consider the identical system coupled in a bidirectionally manner:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= f(\mathbf{x}_1) + kD(\mathbf{x}_2 - \mathbf{x}_1), \qquad \mathbf{x}_1(0) = \mathbf{x}_1^{(0)} \in \mathbb{R}^n \\ \dot{\mathbf{x}}_2 &= f(\mathbf{x}_2) + kD(\mathbf{x}_1 - \mathbf{x}_2), \qquad \mathbf{x}_2(0) = \mathbf{x}_2^{(0)} \in \mathbb{R}^n, \end{aligned} \tag{3.5}$$

where D is a $n \times n$ coupling matrix, k is the coupling parameter.

Then, its variational system along a solution $(\mathbf{x}_1(t, k; \mathbf{x}_1^{(0)}), \mathbf{x}_2(t, k; \mathbf{x}_2^{(0)}))$ is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t,k) - kD & kD \\ kD & A_2(t,k) - kD \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^{(0)} \\ \mathbf{y}_2^{(0)} \end{pmatrix}, \tag{3.6}$$

where

$$A_1(t,k) = D_x f(\mathbf{x}_1(t,k;\mathbf{x}_1^{(0)})) \in C([0,\infty) \times [0,\infty)), \qquad ||A_1(t,k)|| \leq M,$$

$$A_2(t,k) = D_x f(\mathbf{x}_2(t,k;\mathbf{x}_0^{(2)})) \in C([0,\infty) \times [0,\infty)), \qquad ||A_2(t,k)|| \leq M.$$

$(\mathbf{x}_1(t,k;\mathbf{x}_1^{(0)}), \mathbf{x}_2(t,k;\mathbf{x}_2^{(0)}))$ is the solution of this coupled system, which passes through $(\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)})$ at $t = 0$.

Its difference system is similar to one of the unidirectional coupled system. The difference system can be written as follow:

$$\dot{\mathbf{z}} = f(\mathbf{x}_1) - f(\mathbf{x}_2) - 2kD\mathbf{z} = (A_2(t,k) - 2kD)\mathbf{z} + h(\mathbf{z},k), \tag{3.7}$$

where $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2 \in \mathbb{R}^n$, $h \in C(\mathbb{R}^n \times [0,\infty))$. Similar to the unidirectional scheme, (3.7) sometimes can be written as

$$\dot{\mathbf{z}} = (A_3(t,k) - 2kD)\mathbf{z}.$$

As is the case for the coupled Rossler-Rossler system and coupled Lorenz-Lorenz system, where

$$A_3(t,k) \in C([0,\infty) \times [0,\infty)), \qquad ||A_3(t,k)|| \leq M.$$

We note that $\mathbf{x}_1$ and $\mathbf{x}_2$ subsystem of the coupled system, which interacts each other, so it can not be solved individually, as the situation in the unidirectional coupled system. To obtain Lyapunov exponent, we must solve the whole system. At the same time, we seek for the relation between Lyapunov exponents of the whole system and the ones of the difference system. Restart, are some Lyapunov exponents of the whole system the same as the ones of the difference system, as the case in the unidirectional-coupled system? Under the special circumstances $\mathbf{x}_1^0 = \mathbf{x}_2^0$, we know some Lyapunov exponents of bidirectional-coupled system are the same as the ones of the difference system. We will see this fact in the theorem in the later subsection. As to $\mathbf{x}_1^0 \neq \mathbf{x}_2^0$, from the numerical observation, we can find the Lyapunov exponents of the coupled system and the ones of the difference system seem to be the same as coupling parameter k exceeds some value. Similarly, we only check coupled Rossler-Rossler system; for coupled coupled Lorenz-Lorenz, further investigations are needed.

Let us consider the question when do two coupled chaotic systems become synchronized ?

In [9], [10], [11], identical synchronization (or amplitude synchronization) arises as the largest Lyapunov exponent of of the difference system becomes negative. On the other hand, [7] declares amplitude synchronization takes place as one original positive Lyapunov exponent of coupled chaotic system becomes negative through some coupling parameter. From our numerical observation, we have observed that the above two significant coupling parameters of amplitude synchronization seem to be the same. There is still the question that why [7] adopts the second view of amplitude synchronization. The possible reason is that before calculating Lyapunov exponents of the difference system, we need to solve the whole coupled system. To obtain the difference system and its associated linear variational equation. However, if we adopt the viewpoint in [7], than we only solve the coupled system and its linear variational one. Herein, we use the viewpoint in [7] as the criterion for amplitude synchronization to take place.

3.2 Variations of Lyapunov Exponents for Synchronized Chaotic Systems

In the first subsection, we discuss the variations of Lyapunov exponents of the unidirectionally coupled system, and in the second subsection, we are concerned with the variations of Lyapunov exponent of the bidirectionally coupled system. In each subsection, we only adopt the full component coupling scheme, which is

$$K = kI_n,$$

where k is the coupling parameter. This special coupling scheme is used in control theory, and we can always synchronize the coupled system with the right choice of coupling parameter k.

As to the other coupling scheme, the one component coupling scheme is described as follows:

$$K = \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}.$$

We only consider the coupling term x_1 (or x_2) component. Several studies on the synchronization of two chaotic systems apoted this coupling scheme. Nevertheless, we only discuss variations of Lyapunov exponents of the coupled system with the full component coupling scheme in numerical results.

3.2.1 The unidirectional coupling

Before we present the main result, we first introduce the notion of stability of Lyapunov exponent and related theorem. Consider the system

$$\dot{\mathbf{x}} = C(t)\mathbf{x}, \tag{3.8}$$

where

$$C \in C(\mathbb{R}^+), \qquad \sup_{t\in\mathbb{R}^+} ||C(t)|| \le M, \qquad \mathbf{x} \in \mathbb{R}^n,$$

with Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the perturbed system

$$\dot{\mathbf{y}} = [C(t) + E(t)]\mathbf{y},$$

$$E \in C(\mathbb{R}^+), \qquad \sup_{t\in\mathbb{R}^+} ||E(t)|| \le \delta,$$

with Lyapunov exponents $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$.

Definition 3.1 (Adrianova). *The Lyapunov exponents of the system (3.8) are said to be stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that the inequality $\sup_{t\in\mathbb{R}^+} ||E(t)|| < \delta$ implies the inequality*

$$|\lambda_i - \lambda'_i| < \epsilon, \qquad i = 1, \ldots, n.$$

Theorem 3.2 (Adrianova). *If the Lyapunov exponent of the system (3.8) is stable and*

$$||E(t)|| \longrightarrow 0 \qquad as \qquad t \longrightarrow \infty,$$

then

$$\lambda_i = \lambda'_i, \qquad i = 1, \ldots, n.$$

Theorem 3.3 (Adrianova). *(Necessary and sufficient conditions for stability of Lyapunov exponent.)*
If the system (3.8) has different Lyapunov exponent., then they are stable if and only if the system (3.8) has integral separateness property, i.e. it has solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ *such that the inequalty*

$$\frac{||\mathbf{x}_i(t)||}{||\mathbf{x}_i(s)||} \cdot \frac{||\mathbf{x}_{i+1}(s)||}{||\mathbf{x}_{i+1}(t)||} \geq de^{a(t-s)}, \qquad i = 1, \ldots, n-1,$$

holds for some constants $a > 0$, $d > 0$ *and for all* $t \geq s$.

Remark 3.4.

(i) Integral separateness is invariant under Lyapunov transformation.
(ii) We can confirm the integral separateness of the original system by means of checking whether diagonal elements $b_{i,i}$ *of the triangular system have integral separateness. Herein, the triangular system is the system which be reduced by Lyapunov transformation. In other words, we only check whether if the following inequality are satisfied, i.e.*

$$\int_s^t [b_{i,i}(\tau) - b_{i+1,i+1}(\tau)]d\tau \geq a(t-s) - d, \quad t \geq s, \quad a > 0 \quad d \in \mathbb{R} \quad i = 1, \ldots, n-1.$$

(iii) Diagonal elements $b_{i,i}$ *of the triangular system are integrally separated if and only if for sufficiently large* H *such that the following inequality holds*

$$\frac{1}{H} \int_t^{t+H} [b_{i,i}(\tau) - b_{i+1,i+1}(\tau)]d\tau \geq a > 0, \quad t \geq 0.$$

Now, let us recall a dynamical system (3.1)

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x} \in \mathbb{R}^n,$$

where the f is assumed to be continuously differentiable, and has a chaotic attractor. Let $\mathbf{x}_1(t; \mathbf{x}_0)$ be the solution of the system (3.1) passing through $\mathbf{x}_0$ at $t = 0$.

Consider a unidirectionally coupled system of the following forms:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= f(\mathbf{x}_1), & \mathbf{x}_1(0) &= \mathbf{x}_1^0 \\ \dot{\mathbf{x}}_2 &= f(\mathbf{x}_2) + K(\mathbf{x}_1 - \mathbf{x}_2), & \mathbf{x}_2(0) &= \mathbf{x}_2^0, \end{aligned} \tag{3.9}$$

Theorem 3.5. *If the linear variational system of (3.1) has the integral separateness property, then the Lyapunov exponents of the unidirectionally coupled system (3.9) are*

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 - k, \lambda_1 - k, \ldots, \lambda_n - k, \quad \text{for } k \geq k^*,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Lyapunov exponents of the original system (3.1). k^* is some coupling parameter value.

Proof. We divide the discussions into two cases.

When $\mathbf{x_1^0} = \mathbf{x_2^0}$, then the variational equation of (3.9) is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) & 0 \\ K & A_1(t) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}, \tag{3.10}$$

where $\mathbf{y}_i \in \mathbb{R}^n, i = 1, \cdots, n$,

$$A_1(t) = D_x f(\mathbf{x}_1(t; x_1^0)) \in C([0, \infty)), \qquad ||A_1(t)|| \leq M,$$

and $(\mathbf{x}(t; \mathbf{x}^0), \mathbf{x}(t; \mathbf{x}^0))$ is the solution of the system (3.9) passing through $(\mathbf{x}^0, \mathbf{x}^0)$ at $t = 0$.

Then, the FMS of (3.10) is

$$\Phi(t) = \begin{pmatrix} \Phi_1(t) & 0 \\ \Phi_{21}(t) & e^{-kt}\Phi_1(t) \end{pmatrix},$$

where $\Phi_1(t)$ is a FMS of the system $\dot{\mathbf{y}}_1 = A_1(t)\mathbf{y}_1$, hence we can find a Lyapunov transformation

$$Q = \begin{pmatrix} 0 & Q_1 \\ Q_1 & 0 \end{pmatrix},$$

so that the system (3.10) is transformed to the triangular system (3.11).

$$\begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{pmatrix} = \begin{pmatrix} B_1(t) - K & K \\ 0 & B_1(t) \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{z}_1(0) \\ \mathbf{z}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1^0 \\ \mathbf{z}_2^0 \end{pmatrix}, \tag{3.11}$$

where $\mathbf{Y} = Q\mathbf{Z}$, $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2)$ and Q_1 is the $n \times n$ unitary matrix of QR decomposition of FMS Φ_1 in Theorem 2.15, and

$$\dot{\mathbf{y}}_1 = A_1(t)\mathbf{y}_1 \Longrightarrow \dot{\mathbf{z}}_1 = B_1(t)\mathbf{z}_1, \quad B_1(t) = Q_1^{-1}(t)A_1(t)Q_1(t) - Q_1^{-1}(t)\dot{Q}_1(t),$$

with $B_1(t)$ also a $n \times n$ upper triangular matrix.

Hence, by Theorem 2.17, we obtain the result that the Lyapunov exponents of the system (3.9) are

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Lyapunov exponents of the original system (3.1). In this case k^* is 0.

When $\mathbf{x_1^0} \neq \mathbf{x_2^0}$, then the variational equation of the system (3.9) arises is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) & 0 \\ K & A_2(t,k) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}, \tag{3.12}$$

where

$$A_1(t) = D_x f(\mathbf{x}_1(t; \mathbf{x}_0^{(1)})) \in C([0, \infty)),$$

$$A_2(t,k) = D_x f(\mathbf{x}_2(t, k; \mathbf{x}_0^{(2)})) \in C([0, \infty) \times [0, \infty)),$$

$$||A_1(t)|| \leq M, \qquad ||A_2(t,k)|| \leq M,$$

and $(\mathbf{x}_1(t; \mathbf{x}_1^0), \mathbf{x}_2(t, k; \mathbf{x}_2^0))$ is the solution of the system (3.9) passing through $(\mathbf{x}_1^0, \mathbf{x}_2^0)$ at $t = 0$.

Since the system (3.9) has amplitude synchronization for k large enough, it follows that

$$\lim_{t \longrightarrow \infty} (\mathbf{x}_1(t; \mathbf{x}_1^0) - \mathbf{x}_2(t, k; \mathbf{x}_2^0)) = 0, \qquad \text{for } k \geq k^*.$$

For this $k^* = k_{as}$, at which amplitude synchronization takes place.

Write

$$\mathbf{x}_1(t; \mathbf{x}_1^0) + \mathbf{e}(t,k) = \mathbf{x}_2(t, k; \mathbf{x}_0^{(2)}),$$

where $\mathbf{e}(t,k) \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$. Then, the variational equation (3.12) can be rewritten as follows

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) & 0 \\ K & A_1(t) + E(t,k) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}, \tag{3.13}$$

where $||E(t,k)|| \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$.

Once again, we use the same Lyapunov transformation Q in $\mathbf{x}_1^0 = \mathbf{x}_2^0$ case. Then, the system (3.13) can be rewritten the system (3.14)

$$\begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{pmatrix} = \begin{pmatrix} B_1(t) - K + E_1(t,k) & K \\ 0 & B_1(t) \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{z}_1(0) \\ \mathbf{z}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1^0 \\ \mathbf{z}_2^0 \end{pmatrix}, \quad (3.14)$$

where $\mathbf{Y} = Q\mathbf{Z}$, $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2)$, $||E_1(t,k)|| \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$.

Because n Lyapunov exponents of the $\mathbf{z}_2$-subsystem of the system (3.14) and the ones of the triangular system of linear variation system of the system (3.1) are the same. Hence, the other Lyapunov exponents of the system (3.14) can determine by finding Lyaupunov exponents of $\dot{\mathbf{z}}_1 = (B_1(t) + E_1(t,k) - K)\mathbf{z}_1$. Since there is the perturbation matrix $E_1(t,k)$ in the coefficient matrix of this system, we can use Theorems 3.2 and 3.3 to obtain that the Lyapunov exponents for $\dot{\mathbf{z}}_1 = (B_1(t) - K)\mathbf{z}_1$ and this system with a perturbation system $\dot{\mathbf{z}}_1 = (B_1(t) + E_1(t,k) - K)\mathbf{z}_1$ are the same.

Therefore, the Lyapunov exponents of the whole coupled system can be obtained by finding the ones of $\dot{\mathbf{z}}_1 = (B_1(t) - K)\mathbf{z}_1$ and $\dot{\mathbf{z}}_2 = B_1(t)\mathbf{z}_2$. Since the above two systems are also triangular systems, we may use Theorem 2.17 to conclude that

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 - k, \lambda_1 - k, \ldots, \lambda_n - k, \quad \text{for } k \geq k^*,$$

where

$$\lambda_i = \limsup_{t \longrightarrow \infty} \frac{1}{t} \int_0^t b_{i,i}(s) ds, \qquad i = 1, \cdots, n,$$

$b_{i,i}$ is the diagonal element of the upper triangular coefficient matrix $B_1(t)$. Then, λ_i, $i = 1, \cdots, n$ are also the Lyapunov exponents of the original original system (3.1) by the property that Lyapunov transformation does not change the Lyapunov exponent. Hence, we have completed the proof.

3.2.2 The bidirectional coupling

In this subsection, We only show that the proposition is analogous to $\mathbf{x}_1^0 = \mathbf{x}_2^0$ case in Theorem 3.5. As to $\mathbf{x}_1^0 \neq \mathbf{x}_2^0$, although we also observe such a result similar to Theorem 3.5 from numerical observation. But we are not able to verify it theoretically.

Consider a bidirectional coupled system of the following forms:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= f(\mathbf{x}_1) + K(\mathbf{x}_2 - \mathbf{x}_1), \qquad \mathbf{x}_1(0) = \mathbf{x}_1^0 \\ \dot{\mathbf{x}}_2 &= f(\mathbf{x}_2) + K(\mathbf{x}_1 - \mathbf{x}_2), \qquad \mathbf{x}_2(0) = \mathbf{x}_2^0, \end{aligned} \quad (3.15)$$

where $\mathbf{x}_i \in \mathbb{R}^n, i = 1, \cdots, n$.

Proposition 3.6. *If the initial condition of the system (3.15) satisfies* $\mathbf{x}_1^0 = \mathbf{x}_2^0$*, then the Lyapunov exponents of the bidirectional coupled system (3.15) are*

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 - 2k, \lambda_1 - 2k, \ldots, \lambda_n - 2k, \quad for\ k \geq 0,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are Lyapunov exponents of the original system (3.1).*

Proof.

When $\mathbf{x_1^0} = \mathbf{x_2^0}$, the variational equation of the system (3.15) is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t) - K & K \\ K & A_1(t) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}, \tag{3.16}$$

where $\mathbf{y}_i \in \mathbb{R}^n, i = 1, 2,$

$$A_1(t) = D_x f(\mathbf{x}_1(t; \mathbf{x}_1^0)) \in C([0, \infty)), \qquad ||A_1(t)|| \leq M,$$

and $(\mathbf{x}(t; \mathbf{x}^0), \mathbf{x}(t; \mathbf{x}^0))$ is the solution of the system (3.15) passing through $(\mathbf{x}_0, \mathbf{x}_0)$ at $t = 0$. In fact, $\mathbf{x}(t; \mathbf{x}_0)$ is also the solution of the original system (3.1) passing through $\mathbf{x}_0$ at $t = 0$. The FMS of the system (3.16) is

$$\Phi(t) = \begin{pmatrix} (\frac{1}{2}e^{-2kt})\Phi_1(t) & \Phi_1(t) \\ (-\frac{1}{2}e^{-2kt})\Phi_1(t) & \Phi_1(t) \end{pmatrix},$$

where $\Phi_1(t)$ is the FMS of the system $\dot{\mathbf{y}}_1 = A_1(t)\mathbf{y}_1$. Hence we can find a Lyapunov transformation

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_1 & Q_1 \\ Q_1 & -Q_1 \end{pmatrix},$$

such that the system (3.16) is transformed to the triangular system (3.17),

$$\begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{pmatrix} = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_1(t) - 2K \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{z}_1(0) \\ \mathbf{z}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1^0 \\ \mathbf{z}_2^{0)} \end{pmatrix}, \tag{3.17}$$

where $\mathbf{Y} = Q\mathbf{Z}$, $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2)$, and Q_1 is the $n \times n$ unitary matrix of QR decomposition of FMS Φ_1 in Theorem 2.15, which makes that

$$\dot{\mathbf{y}}_1 = A_1(t)\mathbf{y}_1 \Longrightarrow \dot{\mathbf{z}}_1 = B_1(t)\mathbf{z}_1, \quad B_1(t) = Q_1^{-1}(t)A_1(t)Q_1(t) - Q_1^{-1}(t)\dot{Q}_1(t),$$

and $B_1(t)$ is also a $n \times n$ upper triangular matrix.

By Theorem 2.17 and the fact that Lyapunov transformation does not change the Lyapunov exponents, we can obtain the result that Lyapunov exponents of (3.15) are

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1 - 2k, \lambda_2 - 2k, \ldots, \lambda_n - 2k, \quad \text{for } k \ \geq 0,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Lyapunov exponents of the original system (3.1).

For the situation $\mathbf{x_1^0} \neq \mathbf{x_2^0}$, we slightly discuss in later.

Similarly, the variational equation of (3.15) is

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t,k) - K & K \\ K & A_2(t,k) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}, \quad (3.18)$$

where $\mathbf{y}_i \in \mathbb{R}^n, i = 1, 2,$

$$A_1(t,k) = D_x f(\mathbf{x}_1(t,k;\mathbf{x}_1^0)) \in C([0,\infty) \times [0,\infty)),$$

$$A_2(t,k) = D_x f(\mathbf{x}_2(t,k;\mathbf{x}_2^0)) \in C([0,\infty) \times [0,\infty)),$$

$$||A_1(t,k)|| \leq M, \qquad ||A_2(t,k)|| \leq M,$$

and $(\mathbf{x}_1(t,k;\mathbf{x}_1^0), \mathbf{x}_2(t,k;\mathbf{x}_2^0))$ is the solution of the system (3.15) passing through $(\mathbf{x}_1^0, \mathbf{x}_2^0)$ at $t = 0$. Since the system (3.15) has amplitude synchronization for k large enough, then

$$\lim_{t \longrightarrow \infty} (\mathbf{x}_1(t,k;\mathbf{x}_1^0) - \mathbf{x}_2(t,k;\mathbf{x}_2^0)) = 0, \qquad \text{for k} \geq \text{k}^*$$

for this $k^* = k_{as}$, at which amplitude synchronization takes place.

Let

$$\mathbf{x}_1(t,k;\mathbf{x}_1^0) + \mathbf{e}(t,k) = \mathbf{x}_2(t,k;\mathbf{x}_2^0),$$

where $||\mathbf{e}(t,k)|| \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$. Then, the variational system (3.18) can be rewritten as following:

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} A_1(t,k) - K & K \\ K & A_1(t,k) + E(t,k) - K \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \quad (3.19)$$

$$\begin{pmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1^0 \\ \mathbf{y}_2^0 \end{pmatrix}.$$

where $||E(t,k)|| \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$

Similarly, we could consider a Lyapunov transformation $\mathbf{Y} = Q\mathbf{Z}$, where

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_1(t,k) & Q_1(t,k) \\ Q_1(t,k) & -Q_1(t,k) \end{pmatrix},$$

$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2)$, and $Q_1(t,k)$ is the $n \times n$ unitary matrix of QR decomposition of FMS Φ_1 in theorem 2.15, which makes that

$$\dot{\mathbf{y}}_1 = A_1(t,k)\mathbf{y}_1 \Longrightarrow \dot{\mathbf{z}}_1 = B_1(t,k)\mathbf{z}_1,$$

$$B_1(t,k) = Q_1^{-1}(t,k)A_1(t,k)Q_1(t,k) - Q_1^{-1}(t,k)\dot{Q}_1(t,k),$$

and $B_1(t,k)$ is also a $n \times n$ upper triangular matrix. Q could transform (3.19) to (3.20)

$$\begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{pmatrix} = \begin{pmatrix} B(t,k) + E_1(t,k) & 0 \\ 0 & B(t,k) + E_1(t,k) - 2K \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}, \tag{3.20}$$

$$\begin{pmatrix} \mathbf{z}_1(0) \\ \mathbf{z}_2(0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1^0 \\ \mathbf{z}_2^0 \end{pmatrix},$$

where

$$B(t,k) \in C([0,\infty) \times [0,\infty)), \qquad ||B(t,k)|| \leq M,$$

and $B(t,k)$ is a upper triangular matrix. $||E_1(t,k)|| \longrightarrow 0$, as $t \longrightarrow \infty$, for $k \geq k^*$.

Here, the original coupled system has became a decoupled system. Therefore, the Lyapunov exponents of $\mathbf{z}_1$-subsystem and $\mathbf{z}_2$-one in the system (3.20) can be calculated respectively. Similarly, $\dot{\mathbf{z}}_1 = (B(t,k)+E_1(t,k))\mathbf{z}_1$ can be considered as a perturbed system of $\dot{\mathbf{z}}_1 = B(t,k)\mathbf{z}_1$. Once again, if we add the integrally separateness condition, then we have following results.

Lyapunov exponents of the system (3.15) are

$$\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n, \bar{\lambda}_1 - 2k, \bar{\lambda}_2 - 2k, \ldots, \bar{\lambda}_n - 2k,$$

where $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ are Lyapunov exponents of the system $\dot{\mathbf{z}}_1 = B(t,k)\mathbf{z}_1$ (or the system $\dot{\mathbf{x}}_1 = A_1(t,k)\mathbf{x}_1$).

However, we may ask the question: does the linear system $\dot{\mathbf{z}}_1 = B(t,k)\mathbf{z}_1$ always have the integrally separateness for $k \geq k^*$? The answer needs to be found.

Remark 3.7. *From numerical results of bidirectionally coupled system in the next section,* $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ *seem to be the Lyapunov exponents of the original system (3.1).*

Remark 3.8. *In the derivation of the Theorem 3.5, it needs the condition of integral separateness. However, for the linear system, if the integral separateness is not satisfied, then the result of Theorem 3.5 can be obtained. Let us give a example as follows,*

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \qquad A(t) = \begin{pmatrix} 1 + \frac{\pi}{2}\sin\pi\sqrt{t} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^2.$$

It has two Lyapunov exponents 1 and 0. This system does not have integral separateness, but the result of Theorem 3.5 holds for its coupled system.

Remark 3.9. *In numerical experience, we vary the coupling parameter k in order to receive amplitude synchronization which arises as the largest Lyapunov exponent of the difference system becomes negative by varying k. Hence, by the definition of Lyapunov exponent, we can $e_i(t) \approx \exp \lambda_j t$, for large t, where e_i is the error vector of the difference system, $i, j = 1, \cdots n$, $\lambda_j < 0$ by varying k. From this fact, we perhaps can understand the perturbation term of the coefficient matrix of the system (3.13) and (3.19) that it converges to 0 with a negative exponential rate. It may imply that the Lyapunov exponents of the original system and its perturbed system are the same without Theorem 3.2 and Theorem 3.3.*

4 Numerical Illustrations

In the first two section, we present some results of numerical experiments which is related to the discussion in the preceding section, where the coupled system consists of two Rossler system or two Lorenz ones.

In the third part, we mainly concern ourself with the numerical results about Lyapunov exponents of two different coupled systems. Namely, the coupled system consists of one Rossler and one Lorenz equations.

In the fourth part, we summarize our numerical results and compare variations of Lyapunov exponents for the Rossler and Lorenz systems coupled in different manners. Some parts of the results in the tables need further numerical evidence.

The following numerical experience mainly uses the matlab program to execute, and the algorithm of calculating Lyapunov exponent is adopted from [3], other related calculating algorithms are in the reference [1], [2], [12].