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博 士 論 文

多處理器網路的錯誤診斷問題

On the diagnosis problems for multiprocessor systems

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中華民國九十三年十月

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On the diagnosis problems for multiprocessor systems

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Pao-Lien Lai entitled On the diagnosis problems for multiprocessor systems

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多處理器網路的錯誤診斷問題

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中文摘要

在超大型平行計算的處理器(processor)網路中，通常會用一個圖G=(V,E)來代表其內部的連結網路，其中處理器使用點(V)來表示，而處理器之間的連接線則用邊(E)來表示之。因此討論一個圖G的特性，相當於討論這個大型連結網路的特性。錯誤(fault)診斷能力(diagnosability)對網路的容錯能力(fault-tolerant)與可靠度(reliability)是一項極為重要的特性。因為網路架構中處理器或網路連線均有可能發生錯誤，所以很多網路結構的錯誤診斷(fault diagnosis)性質均已被探討。

關於多處理器系統的錯誤自我診斷問題，在文獻中已有幾個不同的模式被提出。Preparata, Metze 與 Chien 三人最早提出一種構想及模式，現在稱為PMC-Model。在此模式下，兩個相連接的處理器可以互相偵測是否錯誤。Maeng與Malek在之後提出一種comparison model稱為MM-model。他們對錯誤診斷的基本構想是由一個處理器向相鄰的兩個處理器送出信號，然後由回收的訊號，比較並判斷是否有錯誤。為了要收集到最多的資料以供錯誤診斷，在MM*-model下,規定任一個處理器都對其所有相鄰的兩個處

理器作偵測及比較。

在本論文中，我們研究一種配對合成網路(matching composition network,MCN)的診斷能力，在 MM*-comparison model 下，配對合成網路的診斷能力，與它的組成成分(Components)之間的診斷能力與連通度(connectivity)有一定關係，我們研究其間的關係，可將文獻中有關各種 cube，如：hypercube，crossed cube，twisted cube 與 Mobius cube 等的診斷能力研究結果有一個統一的解釋。另外，我們也探討了卡迪生乘積(cartesian product)網路在 MM* model 下的的診斷能力，對 Mesh、Tori、k-ary n-cubes 及 generalized hypercubes 等結構網路，也說明他們的診斷能力。

然而，對許多已知的網路結構來說，傳統的錯誤診斷能力經常受限於最小的鄰接點數(minimum degree)，但對網路中任一個處理器而言，他所有的鄰接處理器均是錯誤的機率不高。因此對 PMC-model 下的錯誤診斷，我們介紹“強 t-可診斷系統(strongly t-diagnosable systems)”及“條件式診斷能力(conditional diagnosability)”的觀念。並證明在假設“對網路中任一個處理器，他所有的鄰接處理器不會全是錯誤”的條件下，網路的錯誤診斷能力將會大幅提升，如：超立方體結構網路(hypercube)的條件式錯誤診斷能力在符合此條件下，上升為約傳統錯誤診斷能力的四倍。

關鍵字： 診斷能力， t-可診斷，比較模式，MM* 模式，PMC 模式，配對合成網路，乘積網路，強力 t-可診斷，條件性錯誤集合，條件性診斷能力。

Abstract

Interconnection networks have been an active research area for parallel and distributed computer systems. We usually use a graph $G = (V, E)$ to represent the topology of a network, where vertices represent processors and edges represent links between processors. There are a lot of interconnection network properties proposed in literature. The diagnosability has played an important role in the reliability of an interconnection network. Since processors or links may fail sometimes, diagnosable properties are also concerned in many studies on network topologies.

The classical problem of diagnosability is discussed widely and the diagnosability of many well-known networks have been explored. In this thesis, we consider the diagnosabilities of two families of networks, one is called the Matching Composition Network (MCN); two M-components (which will be defined subsequently) are connected by a perfect matching. The diagnosability of MCN under the comparison model is shown to be one larger than that of the M-component, provided some connectivity constraints are satisfied. Applying our result, the diagnosability of the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n can all be proved to be n, for $n \geq 4$. In particular, we show that the diagnosability of the 4-dimensional Hypercube Q_4 is 4 which is not previously known.

Another family is the product networks, constructed by applying cartesian product operation on factor networks. It would be interesting to combine two known topologies with established properties to obtain a new one that inherits properties from both. We prove that the product network of G_1 and G_2 is (t_1+t_2)-diagnosable, where G_i is either t_i-diagnosable or t_i-connected with regularity t_i for $i = 1, 2$. Furthermore, we use different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks. We show that the product network of G_1, G_2, $\ldots$, and G_k is $(t_1 + t_2 + \ldots + t_k)$-diagnosable, where each G_i is either t_i-diagnosable or t_i-connected with regularity t_i for $1 \leq i \leq k$.

In this thesis, we introduce a new measure of conditional diagnosability by restricting that any faulty set cannot contain all the neighbors of any vertex in the graph. Based on this requirement, the conditional diagnosability of the n-dimensional Hypercube is shown to be $4(n-2)+1$, which is about four times as large as the classical diagnosability. Besides, we propose some useful conditions for verifying if a system is t-diagnosable, and introduce a new concept, called strongly t-diagnosable system, under PMC model. Applying these concepts and conditions, we investigate some t-diagnosable networks which are also strongly t-diagnosable.

Keywords: diagnosability, t-diagnosable, comparison model, MM* model, PMC model, Matching Composition Network, product networks, strongly t-diagnosable, conditional faulty-set, conditional diagnosability.

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Chapter 1

Introduction

With the rapid development of technology, the need for high-speed parallel processing systems has been continuously increasing. The reliability of the processors in parallel computing systems is therefore becoming an important issue. In order to maintain the reliability of a system, whenever a processor (node or vertex) is found faulty it should be replaced by a fault-free processor. The process of identifying all the faulty nodes is called the *diagnosis of the system*. System-level diagnosis appears to be a alternative to circuit-level testing in a complex multiprocessor system.

Somani [59] and T. Chen [7] had some surveys on the diagnosis problems and illustrated many related issues for multi-processor system. Three basic classes of problems have been identified in system-level diagnosis: (1) the characterization problem, (2) the diagnosability problem, and (3) the diagnosis problem. In this thesis, we focus on the first two problems and have some results which will be presented in chapters 3, 4, and 5.

Many terms for system-level diagnosis have been defined and various models have been proposed in literature [4, 31, 50, 55]. If all allowable fault sets can be diagnosed correctly and completely based on a single syndrome (which will be defined in chapter 2), then the diagnosis is referred to as *one-step diagnosis* or *diagnosis without repairs*. A system is called *sequentially t-diagnosable*, if at least one faulty unit can be identified provided the number of faulty nodes does not exceed t. A system is said to be *t-diagnosable* if, for every syndrome, there is a unique set of faulty nodes that could produce the syndrome as long as the number of faulty nodes does not exceed t. The maximum number of faulty nodes that the system can guarantee to identify is called the *diagnosability* of the system.

Another way of diagnosis is to allow a certain number of processors to be incorrectly diagnosed. Friedman [30] introduced the notation of t/s-diagnosability. A system is t/s-diagnosable if, given any syndrome, the faulty units can be isolated to within a set of, at most, s units provided that the number of faulty units does not exceed t. Chwa and Hakimi [8] characterized t/t-diagnosable systems, a special case of t/s-diagnosable systems.

In this thesis, we consider the diagnosabilities of some multiprocessor networks under PMC model [55] and under the comparison model [47, 48]. The diagnosability of some well-known interconnection networks under the comparison model has been investigated. For example, Wang [65, 66] showed that the diagnosability of an n-dimensional hypercube Q_n is n for $n \geq 5$, and the diagnosability of an n-dimensional enhanced hypercube is $n+1$ for $n \geq 6$. The diagnosability of an n-dimensional crossed cube is proved to be n, $n \geq 4$, in literature [6, 29]. Araki [64] proposed that the k-ary r-dimensional butterfly network $BF(k,r)$ is $2k$-diagnosable for $k \geq 2$ and $r \geq 5$. The diagnosability of the Hypercubes, the Crossed cubes, and the Möbius cubes under the PMC diagnostic model were also studied in works [3, 6, 27, 28, 40]. Besides, G.Y. Chang et al. [6] studied the diagnosabilities of regular networks, such as cube-connected cycles, tori, and star graphs.

In chapter 3, we study the diagnosability of a family of interconnection networks, called the Matching Composition Networks (*MCN*), which can be recursively constructed. MCN includes many well-known interconnection networks as special cases, such as the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n. Basically, MCN and these mentioned cubes are all constructed from two graphs G_1 and G_2 with the same number of nodes, by adding a perfect matching between the nodes of G_1 and G_2. We shall call these two graphs G_1 and G_2 as the *M-components* of MCN.

One of our results under the comparison model is illustrated as follows. Suppose that the number of nodes in each component is at least $t+2$, the order (which will be defined in chapter 2) of each node in G_i is t, and the connectivity of G_i is also t, $i = 1, 2$. We prove that the diagnosability of MCN constructed from G_1 and G_2 is $t+1$ under the comparison model, for $t \geq 2$, in chapter 3 and in literature [46]. In other words, the diagnosability of MCN is increased by one as compared with those of the M-components. Using our result, it is straightforward to see that the diagnosability of the Hypercube Q_n, the Crossed cube

CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n are n for $n \geq 4$. Some of these particular applications are previously known results [29, 66], using rather lengthy proofs. Our approach unifies these special cases and our proof is much simpler. We would like to point out that the diagnosability of the 4-dimensional Hypercube Q_4 is 4, which is not previously known [29, 66].

In chapter 4, we investigate the diagnosabilities of Cartesian Product Networks under the comparison model. A product networks is obtained by applying the graph Cartesian product operation on factor networks. It would be interesting to be able to combine two known topologies with established properties to obtain a new one that inherits properties from both. We use the Cartesian product as a tool to achieve this combining. The product networks constitute an important classes for the interconnection networks. Motivated by this observation, the diagnosability of the product networks under the comparison diagnosis model is also studied in this thesis.

Though various properties of the product networks (e.g. connectivity, diameter, shortest path routing, and embedding, etc.) have been investigated by many researchers [15, 16, 24, 33, 34, 53, 54, 56, 71], we study a different topological property from the previous works. The diagnosability of hypercubes and enhanced hypercubes were studied in literature [40, 65, 66] and the diagnosability of crossed cubes was presented in literature [29]. The diagnosability of the product networks under the PMC model was investigated in work [2]. In this thesis and literature [10], we study the diagnosability of the product network of G_1 and G_2, where G_i is t_i-diagnosable or t_i-connected for $i = 1, 2$. Moreover, we use different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks.

Under PMC model, the main studies are present as follows. Reviewing the previous papers [3, 6, 11, 27, 28, 35, 40, 42, 57], the Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are all n-connected and n-diagnosable. In advance, we observe that they are almost $(n + 1)$-connected and $(n + 1)$-diagnosable except the case that all the neighbors of some vertex are faulty simultaneously. Therefore, in chapter 5 and literature [45], we introduce the concept of strongly t-diagnosable system and, furthermore, propose some conditions to assure which networks are strongly t-diagnosable.

For the classical diagnosability, only processors with direct connections are allowed to test one another. For a system, if all the adjacent neighbors of a processor v are faulty simultaneously, it is impossible to determine whether processor v is fault-free or faulty. Hence, for most practical systems that are sparsely connected, only a small number of faulty processors can be recognized with the classical diagnosability theory. So it is an interesting problem to study how the diagnosability varies with some reasonable restrictions. Thulasiraman et al. [20] investigated the fault diagnosis with local constraints. In chapter 5 and literature [45], we propose to study a certain conditional diagnosability and show that the conditional diagnosability of the Hypercube, Q_n, is $4(n-2)+1$.

The diagnosis problem is to determine the working status of each individual processors in a system which has been designed to have a certain level of diagnosability. A diagnosis algorithm can be implemented in two basic ways: centralized algorithm and distributed algorithm. Most of the diagnosis algorithms suggested in earlier works were centralized [9, 18, 19, 32, 61, 63, 70], while testing and diagnosis were done simultaneously. Therefore, a host processor is required in order to collect testing results and to diagnosis the system according to the syndrome. Centralized diagnosis algorithm puts a heavy communication load on the system. Moreover, the supervisory processor is a major performance and reliability bottleneck. However, in either distributed systems or high-speed communication networks, there is no host and all processors run independently. For these reasons, a distributed diagnosis approach in multi-processor systems is preferred rather than a centralized one.

As against a central diagnosis algorithm, a distributed diagnosis algorithm is executed on many or all the processors in the system simultaneously. In a distributed diagnosis algorithm, each processor in the system determines the status of every other processor based on the information it collects through distribution of test results information. Local distributed diagnosis algorithms have been introduced for regular interconnected systems, in which each processor of the system takes part in the diagnosis process, using only the information that is available in its local domain. The problem of testing and diagnosis with distributed algorithms has been investigated by many researchers [5, 38, 41, 60, 67]. Of course, it is still desirable that more works on diagnosis problem are done for practical systems.

1.1 Basic Terms and Notations

A multiprocessor system is modelled as an undirected graph $G = (V, E)$ whose vertices represent processors and edges represent communication links. Throughout this thesis, we will focus on undirected graph without loops (simply abbreviated as graph).

The degree of vertex v in a graph G, written as $d_G(v)$ or $deg(v)$, is the number of edges incident to v. The maximum degree is denoted by $\triangle(G)$, the minimum degree is $\delta(G)$, and G is regular if $\triangle(G) = \delta(G)$. It is k-regular if the common degree is k. The neighborhood of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v. A vertex (node) *cover* of G is a subset $\chi \subseteq V$ such that every edge of E has at least one end vertex in χ. A vertex cover with the minimum cardinality is called a *minimum vertex cover*. The connectivity $\kappa(G)$ of a graph $G(V, E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. A graph G is k-connected if its connectivity is at least k.

The Hypercube structure [57] is a well-known and commercially available interconnection model for multiprocessor system. The fault-tolerant computing for the Hypercube structure has been the interest of many researchers. A Hypercube of dimension n, denoted by Q_n, is an undirected graph consisting of 2^n vertices and $n2^{n-1}$ edges. The Hypercube Q_1 is a complete graph K_2 with two vertices $\{0, 1\}$. For $n \geq 2$, Q_n is constructed from two copies of Q_{n-1} by adding a perfect matching between them. Each vertex u of Q_n can be distinctly labelled by a binary n-bit string, $u_{n-1}u_{n-2} \ldots u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position.

There are several variations of the Hypercube, for example; the Crossed cube [21], the Twisted cube [37], and the Möbius cube [13]. For each of these cubes, an n-dimensional cube can be constructed from two copies of $(n - 1)$-dimensional subcubes by adding a perfect matching between the two subcubes. The main difference is that each of these cubes has various perfect matching between its subcubes. An n-dimensional cube has (i) 2^n vertices, (ii) connectivity n, and (iii) each vertex has the same degree n. We use *cube family* to call all such cubes, which are constructed recursively by joining two subcubes with a perfect matching.

Chapter 2

Diagnosis Models

For the purpose of self-diagnosis of a given system, several different models have been proposed in literature [47, 48, 55]. Preparata, Metze, and Chien [55] first introduced a model, so called PMC-model, for system level diagnosis in multiprocessor systems. In this model, it is assumed that a processor can test the faulty or fault-free status of another processor. The *comparison model*, called *MM model*, proposed by Maeng and Malek [47, 48], is considered to be another practical approach for fault diagnosis in multiprocessor systems.

2.1 PMC model

A multiprocessor system is modelled as an undirected graph $G = (V, E)$ whose vertices represent processors and edges represent communication links. Adjacent processors are capable of performing tests on each other. For adjacent vertices $u, v \in V$, the ordered pair (u, v) represents the test performed by u on v. In this situation, u is called the *tester* and v is called the *tested* vertex. The outcome of a test (u, v) is 1 (respectively 0) if u evaluates v as faulty (respectively fault-free).

A *test assignment* for a system $G = (V, E)$ is a collection of tests (u, v) for some adjacent pairs of vertices. Throughout this thesis, it can be modelled as a directed graph $T = (V, L)$, where $(u, v) \in L$ implies that u tests v in G. The collection of all test results for a test assignment T is called a *syndrome*. Formally, a syndrome is a function $\sigma : L \rightarrow \{0, 1\}$.

2.2 The Comparison Model

In this approach, the diagnosis is carried out by sending the same testing task to a pair $\{u, v\}$ of processors and comparing their responses. The comparison is performed by a third processor w that has direct communication links to both processors u and v. The third processor w is called a *comparator* of u and v.

If the comparator is fault-free, a disagreement between the two responses is an indication of the existence of a faulty processor. To gain as much knowledge as possible about the faulty status of the system, it was assumed that a comparison is performed by each processor for each pair of distinct neighbors with which it can communicate directly. This special case of MM-model is referred to as the *MM*-model*. Sengupta and Dahbura [58] studied the MM-model and the MM*-model, gave a characterization of diagnosable systems under the comparison approach, and proposed a polynomial time algorithm to determine faulty processors under MM*-model. In this thesis, we study the diagnosabilities of MCN (which will be defined subsequently) and Cartesian Product Networks under MM*-model.

In the study of multiprocessor systems, the topology of networks is usually represented by a graph $G = (V, E)$, where each node $v \in V$ represents a processor and each edge $(u, v) \in E$ represents a communication link. The diagnosis by comparison approach can be modelled by a labelled multigraph, called *comparison graph*, $T = (V, L)$, where V is the set of all processors and L is the set of labelled edges. A labelled edge $(u, v)_w \in L$, with w being a label on the edge, connects u and v, which implies that processors u and v are being compared by w. Under the MM-model, processor w is a comparator for processors u and v only if $(w, u) \in E$ and $(w, v) \in E$. The MM*-model is a special case of the MM model, it is assumed that each processor w such that $(w, u) \in E$ and $(w, v) \in E$ is a comparator for the pair of processors u and v. The comparison graph $T = (V, L)$ of a given system can be a multigraph, for the same pair of nodes may be compared by several different comparators.

For $(u, v)_w \in L$, the output of comparator w of u and v is denoted by $r((u, v)_w)$, a disagreement of the outputs is denoted by the comparison results $r((u, v)_w) = 1$, whereas an agreement is denoted by $r((u, v)_w) = 0$.

Therefore, if the comparator w is fault-free and $r((u,v)_w) = 0$, then u and v are both fault-free. If $r((u,v)_w) = 1$, then at least one of u, v, and w must be faulty. The set of all comparison results of a multicomputer system that are analyzed together to determine the faulty processors is called a *syndrome* of the system.

2.3 Common Terms and Notations

For a given syndrome σ, a subset of nodes $F \subseteq V$ is said to be *consistent* with σ, if syndrome σ can be produced from the situation that all nodes in F are faulty and all nodes in $V-F$ are fault-free. Because a faulty comparator (or tester) can lead to unreliable result, a given set F of faulty nodes may produce different syndromes. Let $\sigma(F)$ represent the set of all syndromes which could be produced if F is the set of faulty vertices.

The set of all faulty processors in the system is called a *faulty-set*. This can be any subset of V. Two distinct sets $F_1, F_2 \subset V$ are said to be *indistinguishable* if and only if $\sigma(F_1) \bigcap \sigma(F_2) \neq \emptyset$; otherwise, F_1, F_2 are said to be *distinguishable*. Besides, we say (F_1, F_2) is an *indistinguishable-pair* if $\sigma(F_1) \bigcap \sigma(F_2) \neq \emptyset$, else, (F_1, F_2) is a *distinguishable-pair*. A system is said to be *t-diagnosable* if for every syndrome, there is a unique set of faulty nodes that could produce the syndrome, provided the number of faulty nodes does not exceed t.

Let $G = (V, E)$. For a set $S \subset V$, the notation $G - S$ represents the graph obtained by removing the vertices in S from G and deleting those edges with at least one end vertex in S simultaneously. If $G - S$ is disconnected, then S is called a vertex *cut* or a *separating set*.

Definition 1 *[68] The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise it is nontrivial.*

Let G_1, G_2 be two subgraphs of G, if there are ambiguities, we shall write the vertex set of G_1 as V_{G_1} or $V(G_1)$. The neighborhood set of the vertex set V_{G_1} is defined as $N(V_{G_1}) = \{y \in V(G) |$ there exists a vertex $x \in V_{G_1}$ such that $(x,y) \in E(G)\} - V_{G_1}$. The restricted neighborhood set of V_{G_1} in G_2 (V_{G_2}) is defined as $N(V_{G_1}, G_2)$ ($N(V_{G_1}, V_{G_2})$ respectively) $= \{y \in V(G_2) |$ there exists a vertex $x \in V_{G_1}$ such that $(x,y) \in E(G)\} - V_{G_1}$.

We use $|X|$ to denote the cardinality of set X. For $v \in V$, we use $d_{in}(v)$ to denote the number of edges directed toward v in G. The restricted degree of a vertex v in a subgraph G_1 is defined as $deg_{G_1}(v) = |N(\{v\}, G_1)|$.

2.4 Preliminaries for PMC model

For PMC model, some known results about the definition of t-diagnosable system and related concepts are listed as follows. Some of these previous results are on directed graphs and others are on undirected graphs.

Definition 2 *[55] A system of n units is t-diagnosable if all faulty units can be identified without replacement provided that the number of faults presented does not exceed t.*

Let $F_1, F_2 \subset V$ be two distinct sets and let the symmetric difference $F_1 \triangle F_2 = (F_1 - F_2) \bigcup (F_2 - F_1)$. DahBura and Masson [14] proposed a polynomial time algorithm to check whether a system is t-diagnosable.

Lemma 1 *[14] A system $G(V, E)$ is t-diagnosable under PMC model if and only if for each pair $F_1, F_2 \subset V$ with $|F_1|, |F_2| \leq t$ and $F_1 \neq F_2$, there is at least one test from $V - (F_1 \bigcup F_2)$ to $F_1 \triangle F_2$.*

The following two results related to t-diagnosable systems are due to Hakimi et al. [35], and Preparata et al. [55], respectively.

Lemma 2 *[55] Let $G(V, E)$ be the graph representation of a system G, with V representing the processors and E the interconnection among them. Let $|V| = n$. The following two conditions are necessary for G to be t-diagnosable under PMC model:*

(i) $n \geq 2t + 1$, *and*

(ii) each processor is tested by at least t other processors.

Lemma 3 *[35] The following two conditions are sufficient for a system G of n processors to be t-diagnosable under PMC model:*

(i) $n \geq 2t+1$, *and*

(ii) $\kappa(G) \geq t$.

For a directed graph G and vertex $v \in V(G)$, let $\Gamma(v) = \{v_i | (v, v_i) \in E\}$ and $\Gamma(X) = \bigcup_{v \in X} \Gamma(v) - X$, $X \subset V$. Hakimi and Amin presented a necessary and sufficient condition for a system G to be t-diagnosable as follows:

Theorem 1 *[35] Let $G(V, E)$ be the directed graph of a system G with n units. Then G is t-diagnosable under PMC model if and only if: (i) $n \geq 2t+1$, (ii) $d_{in}(v) \geq t$ for all $v \in V$, and (iii) for each integer p with $0 \leq p \leq t-1$, and each $X \subset V$ with $|X| = n - 2t + p$, $|\Gamma(X)| > p$.*

In this thesis, we propose some new viewpoints on diagnosis, and we will focus on undirected graph (simply abbreviated as graph). Let $G(V, E)$ be an undirected graph of a system G. The following lemma follows directly from Lemma 1.

Lemma 4 *For any two distinct sets F_1, $F_2 \subset V$, (F_1, F_2) is a distinguishable-pair under PMC model if and only if $\exists u \in X$ and $\exists v \in F_1 \triangle F_2$ such that $(u, v) \in E$ (See Fig. 2.1).*

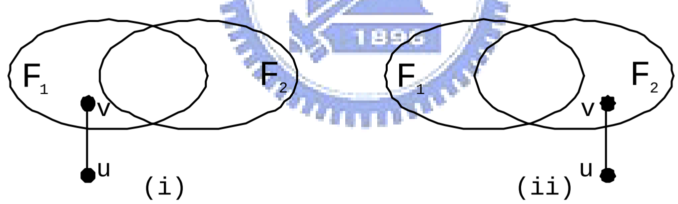

Figure 2.1: Illustration for a distinguishable pair (F_1, F_2).

It follows from Definition 2 that the following lemma holds.

Lemma 5 *A system is t-diagnosable under PMC model if and only if for each distinct pair of sets F_1, $F_2 \subset V$ with $|F_1| \leq t$ and $|F_2| \leq t$, F_1 and F_2 are distinguishable.*

An equivalent way of stating the above lemma is the following:

Lemma 6 *A system is t-diagnosable under PMC model if and only if for each indistinguishable pair of sets* F_1, $F_2 \subset V$, *it implies that* $|F_1| > t$ *or* $|F_2| > t$.

By Lemma 2, a similar result for undirected graph is stated as follows.

Corollary 1 *[55] Let* $G(V, E)$ *be an undirected graph. The following two conditions are necessary for* G *to be* t*-diagnosable under PMC model:*

(i) $n \geq 2t + 1$, *and*

(ii) $\delta(G) \geq t$.

For our discussion later, an alternative characterization of t-diagnosable system is given.

Theorem 2 *Let* $G(V, E)$ *be the graph of a system* G. *Then* G *is* t*-diagnosable under PMC model if and only if for each vertex set* $S \subset V$ *with* $|S| = p$, $0 \leq p \leq t - 1$, *every component* C *of* $G - S$ *satisfies* $|V_C| \geq 2(t - p) + 1$.

Proof: To see that $|V_C| \geq 2(t - p) + 1$ is necessary, by contradiction. Then there exists a set of vertex $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, such that one of the components $G - S$ has strictly less than $2(t-p)+1$ vertices. Let C be such a component with $|V_C| \leq 2(t-p)$. We then arbitrarily partition V_C into two disjoint subsets, $V_C = A_1 \bigcup A_2$ with $|A_1| \leq t-p$ and $|A_2| \leq t - p$. Let $F_1 = A_1 \bigcup S$ and $F_2 = A_2 \bigcup S$. Then $|F_1| \leq t$ and $|F_2| \leq t$. It is clear that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are indistinguishable. This contradicts with the assumption that G is t-diagnosable.

To prove the sufficiency, suppose on the contrary that G is not t-diagnosable, i.e, there exists an indistinguishable pair (F_1, F_2) with $|F_i| \leq t$, $i = i, 2$. By Lemma 4, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. Let $S = F_1 \bigcap F_2$. Thus, in $G - S$, $F_1 \triangle F_2$ is disconnected from other parts. We observe that $|F_1 \triangle F_2| = 2(t - p)$, where $|S| = p$ and $0 \leq p \leq t-1$. Therefore, there is at least one component C of $G-S$ with $|V_C| \leq 2(t-p)$, which is a contradiction. This completes the proof of the theorem. □

2.5 Preliminaries for the comparison model

Next, we discuss the diagnosability under the comparison model. Given a graph G, let T be the comparison graph of G. For a node $v \in V(G)$, we define X_v to be the set of nodes $\{u \mid (v,u) \in E(G)\} \bigcup \{u \mid (v,u)_w \in E(T) \text{ for some } w\}$ and Y_v to be the set of edges $\{(u,w) \mid u,w \in X_v \text{ and } (v,u)_w \in E(T)\}$. In [58], the *order graph* of node v is defined as $G(v) = (X_v, Y_v)$ and the *order* of the node v, denoted by $order_G(v)$, is defined to be the cardinality of a minimum vertex cover of $G(v)$. Let $U \subset V(G)$, we use $\Gamma(G,U)$ to denote the set $\{v \mid (u,v)_w \in E(T) \text{ and } w,u \in U, v \in \bar{U}\}$. We observe that $\Gamma(G,U) = N(\bar{U},U)$ if $G[U]$ is connected and $|U| > 1$. This observation can be extended to the following lemma.

Lemma 7 *Let U be a subset of $V(G)$ and $G[U_i]$, $1 \le i \le k$, be the connected components of the subgraph $G[U]$ such that $U = \bigcup_{i=1}^{k} U_i$. Then $\Gamma(G,U) = \bigcup_{i=1}^{k} \{N(\bar{U},U_i) \mid |U_i| > 1\}$.*

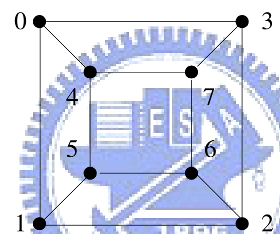

Figure 2.2: An example for $\Gamma(G,U)$ of Q_3.

In Fig. 2.2, taking Q_3 as an example, we have $\Gamma(G,U)$ =$\{4,5,6,7\}$, where U =$\{0,1,2,3\}$.

The next lemma follows directly from the definition of connectivity of G.

Lemma 8 *[27] Let G be a connected graph and U be a subset of $V(G)$. Then $|N(\bar{U},U)| \ge \kappa(G)$ if $|\bar{U}| \ge \kappa(G)$, and $N(\bar{U},U) = \bar{U}$ if $|\bar{U}| < \kappa(G)$.*

Five lemmas and theorems presented by Sengupta and Dahbura [58] must be applied to characterize whether a system to be t-diagnosable. The results of these theorems are as follows.

First one is a necessary and sufficient condition for ensuring distinguishability.

Theorem 3 *[58] For any S_1,S_2 where S_1, $S_2 \subset V$ and $S_1 \neq S_2$, (S_1, S_2) is a distinguishable pair under the comparison model if and only if at least one of the following conditions is satisfied: (See Fig. 2.3)*

(i) $\exists i, k \in V - S_1 - S_2$ *and* $\exists j \in (S_1 - S_2) \bigcup (S_2 - S_1)$ *such that* $(i, j)_k \in C$*,*

(ii) $\exists i, j \in S_1 - S_2$ *and* $\exists k \in V - S_1 - S_2$ *such that* $(i, j)_k \in C$*, or*

(iii) $\exists i, j \in S_2 - S_1$ *and* $\exists k \in V - S_1 - S_2$ *such that* $(i, j)_k \in C$*.*

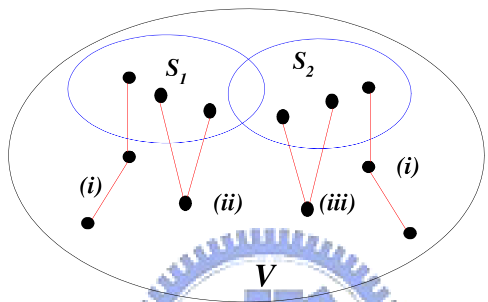

Figure 2.3: Description of distinguishability.

Two necessary conditions for checking a system to be t-diagnosable are as follows.

Lemma 9 *[58] If a system with N nodes is t-diagnosable, then $N \geq 2t + 1$.*

Lemma 10 *[58] If, in a system, each node has order at least t, then for each $S_1, S_2 \subset V$ such that $|S_1 \bigcup S_2| \leq t$, (S_1, S_2) is a distinguishable pair.*

Another necessary and sufficient condition for ensuring distinguishability is the following theorem.

Theorem 4 *[58] A system is t-diagnosable under the comparison model if and only if each node has order at least t and for each distinct pair of sets S_1, $S_2 \subset V$ such that $|S_1| = |S_2| = t$, at least one of the conditions of Theorem 3 is satisfied.*

The following theorem is a sufficient condition for verifying a system to be t-diagnosable.

Theorem 5 *[58] A system G with N nodes is t-diagnosable under the comparison model if*

(1) $N \geq 2t + 1$;

(2) $order_G(v) \geq t$ *for every node* v *in* G;

(3) $|\Gamma(G, U)| > p$ *for each* $U \subset V(G)$ *such that* $|U| = N - 2t + p$ *and* $0 \leq p \leq t - 1$.

According to the Theorems 3, 4, and 5, we observe that condition (3) of Theorem 5 restricts G satisfying the first condition of Theorem 3 and ignores conditions 2 and 3. Hence, we present a hybrid theorem to test whether a system is t-diagnosable.

Theorem 6 *A system G with N nodes is t-diagnosable under the comparison model if*

(1) $N \geq 2t + 1$;

(2) $order_G(v) \geq t$ *for every node* v *in* G;

(3) *for any two distinct subsets* $S_1, S_2 \subset V(G)$ *such that* $|S_1| = |S_2| = t$

either (a) $|\Gamma(G, U)| > p$, *where* $U = V(G) - (S_1 \bigcup S_2)$, *and* $|S_1 \bigcap S_2| = p$;

or (b) *The pair* (S_1, S_2) *satisfies condition* (ii) *or* (iii) *of Theorem 3.*

Proof: Conditions (1) and (2) are the same as conditions (1) and (2) of Theorem 5. Consider condition $(3.a)$. S_1 and S_2 are two distinct subsets of $V(G)$ with $|S_1| = |S_2| = t$, $U = V(G) - (S_1 \bigcup S_2)$, and $|S_1 \bigcap S_2| = p$. Then $0 \leq p \leq t - 1$ and $|U| = N - 2t + p$. If $|\Gamma(G, U)| > p$, it implies that the pair (S_1, S_2) satisfies condition (i) of Theorem 3. Combining conditions $(3.a)$ and $(3.b)$, by Theorems 3 and 4, this theorem follows. $\square$

Chapter 3

Diagnosability of the Matching Composition Networks

Now we define the Matching Composition Network (MCN) as follows. Let G_1 and G_2 be two graphs with the same number of nodes. Let M be an arbitrary *perfect matching* between the nodes of G_1 and G_2; i.e., M is a set of edges connecting the nodes of G_1 and G_2 in a one to one fashion, the resulting composition graph is called a *Matching Composition Network* (*MCN*). For convenience, G_1 and G_2 are called the *M-components* of the MCN. Formally, we use the notation $G_1 \bigoplus_M G_2$ to denote a MCN, which has node set $V(G_1 \bigoplus_M G_2) = V(G_1) \bigcup V(G_2)$ and edge set $E(G_1 \bigoplus_M G_2) = E(G_1) \cup E(G_2) \cup M$. See Fig. 3.1.

3.1 Diagnosability of the Matching Composition Networks under PMC Model

Theorem 7 *Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two t-diagnosable systems with the same number of vertices, where $t \geq 1$. Then MCN $G = G_1 \bigoplus_M G_2$ is $(t+1)$-diagnosable.*

Proof: We shall use Theorem 2 to prove this theorem. Let $G = G(V, E) = G_1 \bigoplus_M G_2$ and $S \subset V$ with $|S| = p$, $0 \leq p \leq t$. Let $S_1 = S \bigcap V_1$ and $S_2 = S \bigcap V_2$ with $|S_1| = p_1$ and $|S_2| = p_2$. In the following proof, we consider two cases: (1) $S_1 = \emptyset$ or $S_2 = \emptyset$, and (2) $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. We shall prove that: $|V_C| \geq 2((t+1) - p) + 1$ for every component C of $G - S$ as $0 \leq p \leq t$.

Case 1: $S_1 = \emptyset$ or $S_2 = \emptyset$.

Without loss of generality, assume $S_1 = \emptyset$ and $S_2 = S$. We know that each vertex of V_2 has an adjacent neighbor in V_1, so, $G - S$ is connected. The only component C of $G - S$ is $G - S$ itself. Hence, $|V_C| = |V - S| = |V_1| + |V_2| - p$. G_i is t-diagnosable, $i = 1, 2$, by Lemma 2, $|V_i| \geq 2t + 1$. So $|V_C| \geq 2(2t + 1) - p \geq 2((t + 1) - p) + 1$ for $t \geq 1$.

Case 2: $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

$S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, it implies $1 \leq p_1 \leq t-1$ and $1 \leq p_2 \leq t-1$. Let C_1 be a component of $G_1 - S_1$. G_1 is t-diagnosable, by Theorem 2, $|V_{C_1}| \geq 2(t - p_1) + 1$. We claim that $2(t-p_1)+1 \geq p_2+1$. Since $p = p_1+p_2$, $2(t-p_1)+1 = 2(t-(p-p_2))+1 = 2p_2+2(t-p)+1$. Notice that $p \leq t$. Hence, $|V_{C_1}| \geq 2(t - p_1) + 1 \geq p_2 + 1$. That is, V_{C_1} has at least one adjacent neighbor $v \in V_2$ and $v \notin S_2$. G_2 is t-diagnosable, by Theorem 2, every component of $G_2 - S_2$ has at least $2(t - p_2) + 1$ vertices. Let C_2 be the component of $G_2 - S_2$ such that $v \in V_{C_2}$ and let C be the component of $G - S$ such that $V_{C_1} \bigcup V_{C_2} \subset V_C$. Then $|V_C| \geq |V_{C_1}| + |V_{C_2}| \geq (2(t - p_1) + 1) + (2(t - p_2) + 1) = 2(2t - p + 1) \geq 2((t + 1) - p) + 1$ as $t \geq 1$.

So every component of $G - S$ has at least $2((t + 1) - p) + 1$ vertices in this subcase. Consequently, the lemma follows. □

3.2 Diagnosability of the Matching Composition Networks under the Comparison Model

What we have in mind is the following: Let G_1 and G_2 be two t-connected networks with the same number of nodes and $order_{G_i}(v) \geq t$ for every node v in G_i, where $i = 1, 2$, and let M be an arbitrary perfect matching between the nodes of G_1 and G_2. Then the degree of any node v in $G(G_1 \bigoplus_M G_2)$ as compared with that of node v in G_i, $i = 1, 2$, is increased by one. We expect that the diagnosability of $G(G_1 \bigoplus_M G_2)$ is also increased to $t + 1$. For example, the Hypercube Q_{n+1} is constructed from two copies of Q_n by adding a perfect matching between the two and the diagnosability is increased from n to $n + 1$ for $n \geq 5$. Other examples such as the Twisted cube TQ_{n+1}, the Crossed cube CQ_{n+1}, and the Möbius cube MQ_{n+1} are all constructed recursively using the same method as above.

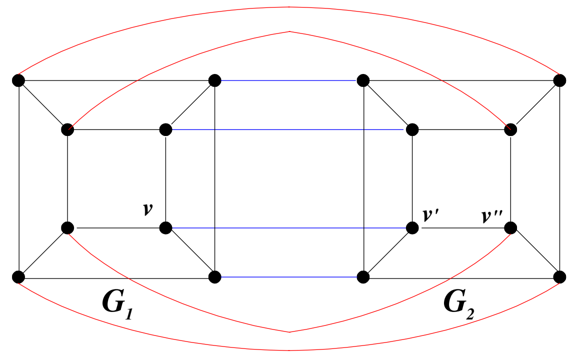

Figure 3.1: An example of MCN $G(G_1 \bigoplus_M G_2)$.

Theorem 8 *Let G_1 and G_2 be two networks with the same number of nodes, and t be a positive integer. Suppose that $order_{G_i}(v) \geq t$ for every node v in G_i, where $i = 1, 2$. Then $order_{G_1 \bigoplus_M G_2}(v) \geq t + 1$ for node v in $G_1 \bigoplus_M G_2$.*

Proof: See Fig. 3.1. Let v be a node of $G = G_1 \bigoplus_M G_2$. Without loss of generality, we assume that $v \in V(G_1)$, $v' \in V(G_2)$, and $(v, v') \in M$. Of course, node v' is connected to at least one other node v'' in $V(G_2)$. Let $G_1(v)$ and $G(v)$ be the order graph of v in graph G_1 and G, respectively. We observe that $G_1(v)$ is a proper subgraph of G, both v' and v'' are in the latter, none of them in the former, and (v', v'') is an edge in $G(v)$. Therefore, every vertex cover of the order graph $G(v)$ contains a vertex cover of the order graph $G_1(v)$. Besides, any vertex cover of $G(v)$ has to include at least one of v' and v''. Thus, $order_{G_1 \bigoplus_M G_2}(v) \geq order_{G_i}(v) + 1$ for any node v in G_i, $i = 1, 2$. This completes the proof. □

We need the following lemma later in Theorem 9.

Lemma 11 *Let G be a t-connected network, $|V(G)| \geq t + 2$ and $order_G(v) \geq t$ for every node v in G, where $t \geq 2$. Suppose that U is a subset of nodes of $V(G)$ with $|\bar{U}| \leq t$. Then $\Gamma(G, U) = \bar{U}$.*

Proof: By assumption $|\bar{U}| \leq t$ and $\kappa(G) \geq t$, we prove the lemma by two cases; the first for $|\bar{U}| < \kappa(G)$ and the second for $|\bar{U}| = \kappa(G)$.

If $|\bar{U}| < \kappa(G)$, the induced graph $G[U]$ is connected. By Lemma 7, $\Gamma(G, U) = N(\bar{U}, U)$. By Lemma 8, $N(\bar{U}, U) = \bar{U}$. This case holds.

Suppose that $|\bar{U}| = \kappa(G)$. We observe that, adding any node v of $\bar{U}$ to U, the induced subgraph $G[U \bigcup \{v\}]$ forms a connected graph. It implies that every node v of $\bar{U}$ is adjacent to every connected components of $G[U]$. We claim that the subgraph induced by U contains a connected component A with cardinality at least two (See Fig. 3.2(a)). Then, the connected component A is adjacent to all nodes in $\bar{U}$ and, so $\Gamma(G, U) = \bar{U}$.

Now, we prove the claim. Suppose on the contrary that every connected component of the subgraph induced by U is an isolated node. Let v be an arbitrary node in $\bar{U}$, and let $G(v) = (X_v, Y_v)$ be the order graph of v in G. Then $\bar{U} - \{v\}$ is a vertex cover of $G(v)$, because every connected component of $G[U]$ is an isolated node. Since $|\bar{U}| \leq t$, we have $|\bar{U} - \{v\}| \leq t - 1$. Therefore, even if the induced graph $G[\bar{U} - \{v\}]$ is a complete graph (See Fig. 3.2(b)), the cardinality of a minimum vertex cover of the order graph $G(v)$ is at most $t - 1$. However, this contradicts to the hypothesis of $order_G(v) \geq t$ for every node v in G. So $G[U]$ has a connected component A with cardinality at least two. This proves the claim, and the lemma follows. □

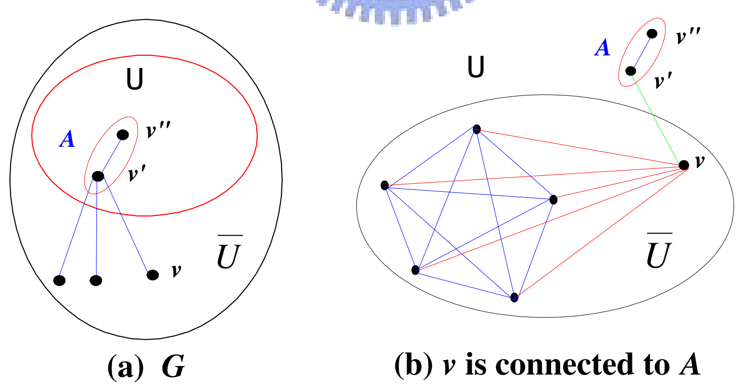

Figure 3.2: An example of the $\Gamma(G, U)$ when $|U| = t$.

We are now ready to state and prove the following theorem about the diagnosability of Matching Composition Network under the comparison model. As an illustration, the conditions of the following theorem are applicable to some well-known interconnection networks, such as Q_n, CQ_n, TQ_n, and MQ_n for $n = t \geq 3$.

Theorem 9 *For $t \geq 2$, let G_1 and G_2 be two graphs with the same number of nodes N, where $N \geq t+2$. Suppose that $order_{G_i}(v) \geq t$ for every node v in G_i and the connectivity $\kappa(G_i) \geq t$, where $i = 1, 2$. Then MCN $G_1 \bigoplus_M G_2$ is $(t+1)$-diagnosable under the comparison model.*

Proof: Since $|V(G_1)| = |V(G_2)| = N$, $2N \geq 2(t+2) > 2(t+1)+1$. By Theorem 8, $order_{G_1 \bigoplus_M G_2}(v) \geq t+1$ for any node v in $G_1 \bigoplus_M G_2$. It remains to prove that $G_1 \bigoplus_M G_2$ satisfies condition 3 of Theorem 6.

Let F_1 and F_2 be two distinct subsets of $V(G)$ with the same number $t+1$ of nodes, and let $|F_1 \bigcap F_2| = p$, then $0 \leq p \leq t$. In order to prove this theorem, we will prove that F_1 and F_2 are distinguishable, i.e., this pair (F_1, F_2) satisfies either condition $(3.a)$ or $(3.b)$ of Theorem 6.

Let $G = G_1 \bigoplus_M G_2$ and $U = V(G) - (F_1 \bigcup F_2)$, then $|U| = 2N - 2(t+1) + p$. Let $U = U_1 \bigcup U_2$ with $U_i = U \bigcap V(G_i)$ and $\bar{U}_i = V(G_i) - U_i$, $i = 1, 2$. Without loss of generality, we assume that $|U_1| \geq |U_2|$. Let $|\bar{U}_1| = n_1$, $|\bar{U}_2| = n_2$, $n_1 + n_2 = 2(t+1) - p$, and $n_1 \leq n_2$. Since $0 \leq n_1 \leq \frac{2(t+1)-p}{2}$, the maximum value of n_1 is equal to $t+1$ when $p = 0$ and $n_2 = t+1$. According to different values of n_1 and n_2, we divide the proof into two cases. The first case $n_2 \leq t$ which implies $n_1 \leq t$. The second case $n_2 > t$, and this case is further divided into three subcases $n_1 < t$, $n_1 = t$, and $n_1 > t$.

Case 1: $n_1 \leq t$ and $n_2 \leq t$.

By Lemma 11, we have $|\Gamma(G, U)| \geq |\Gamma(G_1, U_1)| + |\Gamma(G_2, U_2)| = |\bar{U}_1| + |\bar{U}_2| = n_1 + n_2 = 2(t+1) - p$. We know that $0 < p \leq t$, $|\Gamma(G, U)| \geq 2(t+1) - p > p$, and condition $(3.a)$ of Theorem 6 is satisfied.

Case 2: $n_2 > t$.

We discuss the case according to the following three subcases, (2a) $n_1 < t$, (2b) $n_1 = t$, and (2c) $n_1 > t$.

Subcase 2a: $n_1 < t$.

Since $\kappa(G_1) \geq t$ and $|\bar{U}_1| = n_1 < t$, $G[U_1]$ is connected. By Lemmas 7 and 8, $\Gamma(G_1, U_1) = N(\bar{U}_1, U_1) = n_1$. There are n_1 and n_2 nodes in $\bar{U}_1$ and $\bar{U}_2$, respectively, and $n_2 = 2t + 2 - p - n_1$ (See Fig. 3.3). If all the nodes in $\bar{U}_1$ are adjacent to some n_1 nodes in $\bar{U}_2$, there are still at least $n_2 - n_1 = 2t + 2 - p - 2n_1$ nodes in $\bar{U}_2$ such that each of them is adjacent to some node in U_1 under the matching M. So, $|\Gamma(G, U)| \geq |\Gamma(G_1, U_1)| + (n_2 - n_1) = n_1 + (n_2 - n_1) = n_2$. Because $n_2 > t \geq p$, the proof of this subcase is complete.

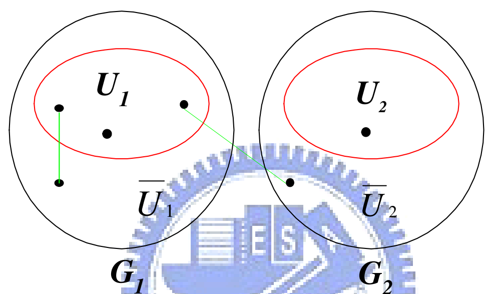

Figure 3.3: Illustration in Theorem 9 for an example of subcase 2a.

Subcase 2b: $n_1 = t$.

We know that $n_1 + n_2 = 2(t+1) - p$, $0 \leq p \leq t$, $n_2 > t$, and $n_1 = t$, the only two valid values for n_2 are $t + 1$ and $t + 2$. $n_2 = t + 1$ implies $p = 1$, and $n_2 = t + 2$ implies $p = 0$. By Lemma 11, $|\Gamma(G_1, U_1)| = |\bar{U}_1| = t \geq 2 > p$ for $p = 0$ or 1. Then the subcase holds.

Subcase 2c: $n_1 > t$.

Observing that $0 \leq n_1 \leq \frac{2(t+1)-p}{2}$, where $0 \leq p \leq t$ and $n_2 \geq n_1 > t$, so $n_1 = n_2 = t+1$. It also implies $p = 0$. Here, we will prove that the subcase satisfies either condition $(3a)$ or condition $(3b)$ of Theorem 6.

First, if the subgraph induced by U contains a connected component A_1 with cardinality at least two (See Fig. 3.4), then it must be adjacent to some node in $\bar{U}$. Thus, we know that $|\Gamma(G, U)| > 0 = p$, and Condition $(3.a)$ of Theorem 6 is satisfied.

Otherwise, every connected component of U contains a single node only. By Theorem 3, we know that F_1 and F_2 are distinguishable if there exists a path $\langle u_1 \rightarrow u \rightarrow u_2 \rangle$ such that $u \in U$, and $u_1, u_2 \in F_1 - F_2$ or $u_1, u_2 \in F_2 - F_1$. If $p = 0$, it implies $F_1 \bigcap F_2 = \phi$, any node u in $G[U]$ with degree more than two must be connected to at least two nodes in F_1 or F_2 (See Fig. 3.4). By Theorem 8, $order_{G_1 \bigoplus_M G_2}(v) \geq t + 1$ for every node v in $G_1 \bigoplus_M G_2$, therefore $deg(v) \geq t+1$ for every node v in $G_1 \bigoplus_M G_2$. Since $t \geq 2$, condition $(3.b)$ of Theorem 6 is satisfied.

Hence, the subcase holds and the theorem follows. □

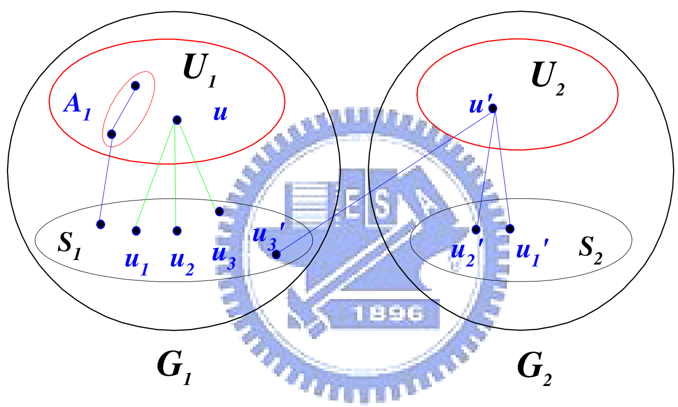

Figure 3.4: Illustration in Theorem 9 for an example of subcase 2c.

By Theorem 5 and Theorem 9, we have the following corollary.

Corollary 2 *Let G_1 and G_2 be two graphs with the same number of nodes N. Suppose that both G_1 and G_2 are t-diagnosable under the comparison model and have connectivity $\kappa(G_1) = \kappa(G_2) \geq t$, where $t \geq 2$. Then MCN $G_1 \bigoplus_M G_2$ is $(t+1)$-diagnosable under the comparison model.*

In [66], D. Wang has proved that the diagnosability of hypercube-structured multiprocessor systems under the comparison model is n when $n \geq 5$. However, the diagnosability of Q_4 is not known to be 4. Using our Theorem 9, we can strengthen the result as follows.

Theorem 10 *The Hypercube Q_n is n-diagnosable for $n \geq 4$.*

Proof: We observe that Q_3 is 3-connected, $order_{Q_3}(v) = 3$ for every node v in Q_3, and the number of nodes of Q_3 is 8, $8 \geq t+2 = 5$ for $t = 3$. It is well-known that Q_4 can be constructed from two copies of Q_3 by adding a perfect matching between these two copies. Therefore, by Theorem 9, Q_4 is 4-diagnosable.

Then, the proof is by induction on n. We have shown that Q_4 is 4-diagnosable. Assume that it is true for $n = m-1$. Considering $n = m$, Q_m is obtained from two copies G_1, G_2 of Q_{m-1} by adding a perfect matching joining corresponding nodes in G_1 and G_2. It is well-known that Q_{m-1} is $(m-1)$-connected. By Corollary 2, Q_m is m-diagnosable. This completes the induction proof. □

However, Q_3 is not 3-diagnosable. In Fig. 3.5, there is a Q_3, let $S_1 = \{0, 5, 7\}$ and $S_2 = \{2, 5, 7\}$. Then, by Theorem 3, S_1 and S_2 are not distinguishable as shown in Fig. 3.5.

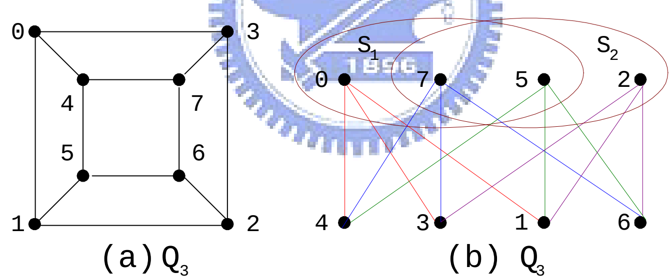

Figure 3.5: An example of an indistinguishable pair for Q_3.

As we observe that most of the related results on diagnosability of multiprocessors systems [29, 66] are based on a sufficient theorem, namely Theorem 5. Not satisfying this sufficient condition, such as in the case of Q_4, does not necessarily imply that the network is not 4-diagnosable. Therefore, we propose a hybrid condition, $3(a)$ and $3(b)$ of Theorem 6, to check the diagnosability of multiprocessor systems under the comparison model. It is more powerful to use. Applying our Theorem 6 and Theorem 9, we show that the diagnosability of Q_4 is indeed 4.

It is known [11, 28, 42] that the Crossed cube CQ_n [21], the Twisted cube TQ_n [37], and the Möbius cube MQ_n [13] are all n-connected. By Theorem 8, we can prove that the order of each node in these two cubes is n. We observe that the two cubes are both constructed recursively using a similar way satisfying the requirements of Theorem 9 and Corollary 2. Therefore, we can prove that CQ_n, TQ_n and MQ_n are all n-diagnosable for $n \geq 4$. Then, we list the following three theorems.

Theorem 11 *[29] The Crossed cube CQ_n is n-diagnosable for $n \geq 4$.*

Theorem 12 *The Twisted cube TQ_n is n-diagnosable for $n \geq 4$.*

Theorem 13 *The Möbius cube MQ_n is n-diagnosable for $n \geq 4$.*

Chapter 4

Diagnosability of Cartesian Product Networks

Many multiprocessor networks are constructed by the cartesian product, such as grids, hypercubes, meshes, and tori. The product networks constitute very important classes for the interconnection networks. The diagnosability of the product networks under the PMC model was investigated in [2]. In this thesis, we study the diagnosability of the product network of G_1 and G_2, where G_i is t_i-diagnosable or t_i-connected for $i = 1, 2$. Furthermore, we use different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks. We show that the product network of $G_1, G_2, \ldots$, and G_k is $(t_1+t_2+\ldots+t_k)$-diagnosable, where each G_i is either t_i-diagnosable or t_i-connected with regularity t_i for $1 \leq i \leq k$.

Definition 3 *The Cartesian product $G = G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$, where the set of nodes V and the set of edges E are given by:*

(i) $V = \{\langle x, y\rangle | x \in V_1$ *and* $y \in V_2\}$, *and*

(ii) for $u = \langle x_u, y_u\rangle$ *and* $v = \langle x_v, y_v\rangle$ *in* V, $(u, v) \in E$ *if and only if* $(x_u, x_v) \in E_1$ *and* $y_u = y_v$, *or* $(y_u, y_v) \in E_2$ *and* $x_u = x_v$.

Let y be a fixed node of G_2. The subgraph G_1^y-component of $G_1 \times G_2$ has node set $V_1^y = \{(x, y) | x \in V_1\}$ and edge set $E_1^y = \{(u, v) | u =< x_u, y >, v =< x_v, y >, (x_u, x_v) \in E_1\}$. Similarly, let x be a fixed node of G_1, the subgraph G_2^x-component of $G1 \times G2$ has node set $V_2^x = \{(x,y)|y \in V_2\}$ and edge set $E_2^x = \{(u,v)|u =< x, y_u >, v =< x, y_v >, (y_u, y_v) \in E_2\}$. It is clear that G_1^y-component (abbreviated as G_1^y) and G_2^x-component (abbreviated as G_2^x) are isomorphic to G_1 and G_2, respectively. (as illustrated in Fig. 4.1). The following lemma lists a set of known results [15, 16, 24, 33, 71] related to the topological properties of the Cartesian product of $G_1 \times G_2$ of two graphs G_1 and G_2.

Lemma 12 *Let $u = \langle x_u, y_u \rangle$ and $v = \langle x_v, y_v \rangle$ be two nodes in $G_1 \times G_2$. The following properties hold:*

(1) *$G_1 \times G_2$ is isomorphic to $G_2 \times G_1$,*

(2) *$|G_1 \times G_2| = |G_1| \cdot |G_2|$, where $|G|$ is the number of nodes in G,*

(3) *$deg_{G_1 \times G_2}(u) = deg_{G_1}(x_u) + deg_{G_2}(y_u)$,*

(4) *$dist_{G_1 \times G_2}(u,v) = dist_{G_1}(x_u, x_v) + dist_{G_2}(y_u, y_v)$, where $dist_G(u,v)$ is the distance between u and v in G,*

(5) *$D(G_1 \times G_2) = D(G_1) + D(G_2)$, where $D(G)$ is the diameter of G,*

(6) *$\kappa(G_1 \times G_2) \geq \kappa(G_1) + \kappa(G_2)$, where $\kappa(G)$ is the connectivity of G.*

4.1 Diagnosability of Cartesian Product Networks Under PMC Model

Araki and Shibata [2] proposed some results for the t-diagnosability and t/t-diagnosability of cartesian product systems under PMC model, listed as follows.

Theorem 14 *[2] Let G_1 and G_2 be digraphs of t_1- and t_2-diagnosable systems, respectively. Then, the system $G = G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable.*

Lemma 13 *[2] Let G be a digraph of a t-diagnosable system. Then $(G \times K_2)$ is a digraph of a $(t+1)$-diagnosable system.*

Theorem 15 *[2] Let G_1 and G_2 be digraphs of t_i/t_i-diagnosable system ($i = 1, 2$). If $\delta_i \geq \lceil \frac{(t_i+1)}{2} \rceil$ for $i = 1, 2$, then, the system $G = G_1 \times G_2$ is $(t_1 + t_2)/(t_1 + t_2)$-diagnosable.*

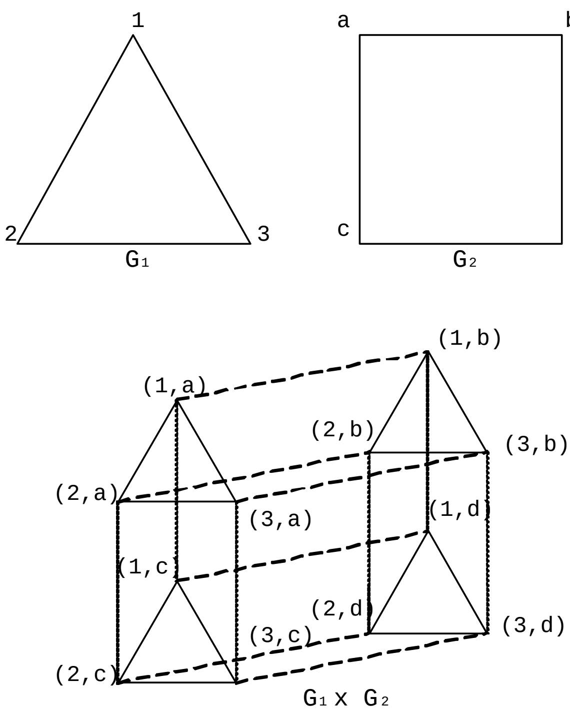

Figure 4.1: An example of product network $G_1 \times G_2$.

Lemma 14 *[2] Let G be a digraph of a t/t-diagnosable system such that $\delta_{in}(G) \geq \lceil \frac{t+1}{2} \rceil$. Then a system represented by $(G \times K_2)$ is $(t+1)/(t+1)$-diagnosable.*

Theorem 16 *[2] Let G_1 and G_2 be digraphs of t_i/t_i-diagnosable system $(i = 1, 2)$. If $\delta_i \geq \lceil \frac{t_i}{2} \rceil + 1$ and $t_i \geq 2 (i = 1, 2)$, then, the system $G = G_1 \times G_2$ is $(t_1+t_2+2)/(t_1+t_2+2)$-diagnosable.*

4.2 Diagnosability of Cartesian Product Networks Under the Comparison Model

In this section, we distinguish the product networks into homogeneous product networks and heterogeneous product networks. By a homogeneous product networks, we mean every factor network of the product has the same properties of being t-diagnosable and t-regular (or being t-connected and t-regular, respectively), while heterogeneous product in the sense that one of the factor networks is t-diagnosable and another is t-connected. Before discussing homogeneous product networks and heterogeneous product networks, we consider the problem that, whether a t-regular and t-connected interconnection network is t-diagnosable.

4.2.1 Diagnosability of t-connected networks

This section considers the problem that, under suitable conditions, a t-regular and t-connected interconnection network is also t-diagnosable. A t-regular and t-connected interconnection network with at least $2t+3$ nodes is first proven also to be t-diagnosable. Moreover, the product network of G_1 and G_2 is shown to be (t_1+t_2) diagnosable, where G_i is t_i connected with regularity t_i for $i = 1, 2$.

Lemma 15 *Let G be a t-regular and t-connected network with $N \geq 2t+1$ nodes and $t > 2$. Then, each node v of G has order t.*

Proof: Let v be a node of G and let $G(v)$ be the order graph of v in G. Let $\chi(v)$ be a node cover of $G(v)$. Assume that node v has order $k < t$. Since G contains $N \geq 2t+1$ nodes and the order of v is $k < t$, there exists at least one node $y \in V, y \neq v, y \notin N(v)$ and $y \notin \chi(v)$. The distance between v and y is at least 2. Each edge of $G(v)$ has at least one endpoint in $\chi(v)$, so all paths from v to y in G must be from v via z, which is a node in $\chi(v)$. Deleting all the nodes of $\chi(v)$ in G ensures that no path exists from v to y. However, exactly k nodes are deleted, contradicting the assumption that G is a t-connected network, so $k \geq t$. $N(v)$ is a node cover of $G(v)$ so the node v must have order $k = t$. □

Given a t-diagnosable system, by Lemma 9 the number of nodes must exceed or be equal to $2t+1$. However, a t-regular and t-connected network with $N = 2t+1$ nodes is not necessarily t-diagnosable. The graph shown in Fig. 4.2 is 4-regular and 4-connected network with $N = 9$ nodes, since any two arbitrarily distinct nodes in Fig. 4.2 are contained in two disjoint cycles. For example, two distinct nodes 4 and 5 are present in cycles $\langle 4, 9, 8, 5 \rangle$ and $\langle 4, 1, 6, 5, 2, 7, 3 \rangle$. This graph can be easily seen to be not 4-diagnosable, since $\{4, 5, 6, 7\}$ and $\{6, 7, 8, 9\}$ constitute an indistinguishable pair. With regard to $N = 2t+2$, the three dimensional crossed cube CQ_3 and the three dimensional hypercube Q_3 are 3-regular, 3-connected networks and each node has order

$t = 3$. However, [29, 40] demonstrated that CQ_3 and Q_3 are not 3-diagnosable under the comparison diagnosis model. The t-regular and t-connected network G with $N \geq 2t + 3$ nodes is thus considered in the following theorem.

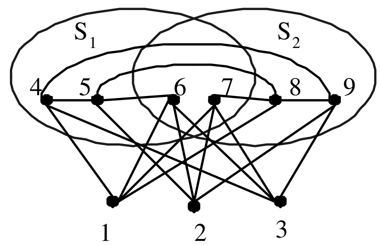

Figure 4.2: An example of 4-connected and 3-diagnosable system.

Theorem 17 *Let $G = (V, E)$ be a t-regular and t-connected network with N nodes and $t > 2$. G is t-diagnosable if $N \geq 2t + 3$.*

Proof: Let S_1 and S_2 be two distinct subsets of V with $|S_1| = |S_2| = t$, $|S_1 \bigcap S_2| = p$ and $0 \leq p \leq t - 1$. By Theorem 4 and Lemma 15, G can be shown to be t-diagnosable by showing that (S_1, S_2) is a distinguishable pair. Let $V'' = S_1 \bigcup S_2$ and $V' = V - V''$. Then $|V''| = 2t - p > t$. Notably, V' may not be connected. The case in which all connected components of the subgraph induced by V' are isolative nodes, is considered first. For $0 \leq p \leq t - 1$, the following cases are considered.

Case 1: $0 \leq p \leq t - 3$.

Since $0 \leq p \leq t - 3$ and G is a t-regular graph, each node of V' has at least two neighbors in $S_1 - S_2$ or $S_2 - S_1$ for $t > 2$. Thus, either condition (2) or condition (3) in Theorem 3 is satisfied.

Case 2: $p = t - 2$.

In this case, $|V''| = t+2, N \geq 2t+3$ and $|V'| = N-(t+2) \geq t+1$. Assume that the pair S_1, S_2 are indistinguishable. Therefore, conditions (2) and (3) in Theorem 3 cannot be satisfied, implying that each node of V' must be connected to $t-2$ nodes in $S_1 \bigcap S_2$, one node in $S_1 - S_2$ and one node in $S_2 - S_1$. Therefore, at most t nodes in V' satisfy this assumption, contradicting the condition $|V'| \geq t+1$. Hence, either condition (2) or condition (3) in Theorem 3 must be satisfied.

Case 3: $p = t-1$.

$|V''| = t+1$ and $|V'| = N-t-1$. The subgraph induced by V' consists of isolative nodes and G is a t-regular graph, so $(N-t-1)t$ edges are adjacent to the nodes of V' and V''. However, G has exactly $\frac{Nt}{2}$ edges. For $N \geq 2t+3$, we have $(N-t-1)t > \frac{Nt}{2}$, which is a contradiction, so $p = t-1$ is impossible.

Now consider that the subgraph induced by V' contains a connected component R with cardinality of at least 2. Let $u \in R$ and $v \in (S_1 - S_2) \bigcup (S_2 - S_1)$. G is t-connected, so there exist t disjoint paths from u to v. However, at most p disjoint paths exist from u to v via the nodes of $S_1 \bigcap S_2$. Therefore, there exists at least one path from u to v such that no node of the path belongs to $S_1 \bigcap S_2$. Since u is a node in R, there exists another node w adjacent to u. Hence, the condition (1) in Theorem 3 is satisfied, completing the proof of the theorem. □

Corollary 3 *For $t_1, t_2 > 2$, let G_1 and G_2 be two t_1-connected and t_2-connected networks, with regularity t_1 and t_2, respectively. Let $G = (V, E)$ be the product network of G_1 and G_2. Then, the product network $G = G_1 \times G_2$ is (t_1+t_2)-diagnosable with regularity t_1+t_2.*

Proof: G_1 is t_1-regular and t_1-connected, so at least t_1+1 nodes exist in G_1. Similarly, the number of nodes in G_2 is at least t_2+1. Therefore, G contains at least $(t_1+1)(t_2+1)$ nodes. Moreover, by Lemma 12, the degree of every node in G is t_1+t_2 (regularity t_1+t_2). $\delta(G)$ is used to denote the minimum degree of G. That,[39], $\kappa(G) \leq \delta(G)$ is well known. However , by Lemma 12, $\kappa(G) \geq \kappa(G_1) + \kappa(G_2) = t_1 + t_2$. Since $t_1 + t_2 \leq \kappa(G) \leq \delta(G) = t_1 + t_2$, $\kappa(G) = t_1 + t_2$. Since $(t_1+1)(t_2+1) > 2(t_1+t_2)+3$ for $t_1, t_2 > 2$, Theorem 17 implies that G is (t_1+t_2)-diagnosable. Therefore, the corollary follows. □

Corollary 4 *Let G be a product network of $G_1, G_2, \ldots,$ and G_k. Each G_i is t_i-regular, t_i-connected and $t_i > 2$ for $1 \leq i \leq k$ where $k > 2$. Then, the product network G is*

$(t_1 + t_2 + \ldots + t_k)$-regular and $(t_1 + t_2 + \ldots + t_k)$-diagnosable.

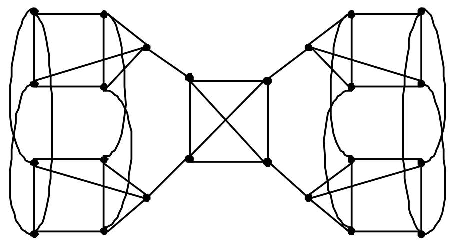

Figure 4.3: An example of 2-connected and 4-diagnosable system.

Theorem 17 indicates that a t-connected network with $N \geq 2t + 3$ nodes is also t-diagnosable. However, a t-diagnosable network is not necessarily a t-connected network (as depicted in Fig. 4.3). The example shown in Fig. 4.3 is 4-regular and 4-diagnosable, but not 4-connected. The t diagnosability and t connectivity are not equivalent terms, but these two concepts are closely related; Theorem 17 provides an example.

4.2.2 Diagnosability of homogeneous product networks

By Corollary 3, the homogeneous product network $G_1 \times G_2$ is (t_1+t_2)-diagnosable, where G_i is t_i-connected and t_i-regular, $t_i > 2$, $i = 1, 2$. The homogeneous product network $G_1 \times G_2$ is also $(t_1 + t_2)$-diagnosable, where G_i is t_i-diagnosable and t_i-regular, $t_i > 2$, $i = 1, 2$. Several lemmas must be proven first.

Lemma 16 *Let $G = (V, E)$ be a t-regular network with $N \geq 2t + 1$ nodes. Suppose each node of G has order t, $t > 2$. If $V' \subset V$ and $|V - V'| \leq t$, then $\Gamma(G, V') = V - V'$.*

Proof: Let v be an arbitrary node in $V - V'$, and let $G(v)$ be the order graph of v in G. The following two cases are considered.

Case 1: $|V - V'| < t$.

For $|V - V'| < t$, the degree of each node is t, so each node in V' has at least one neighbor in V'. Therefore, no isolated node exists in V'. Similarly, every node in $V - V'$ has at least one neighbor in V'. Hence, $\Gamma(G, V') = V - V'$.

Case 1: $|V - V'| = t$.

For $|V - V'| = t$, each node in $V - V'$ has at least one neighbor in V'. $N(v, V')$ is used to denote the neighbor set of v in V'. Assume that no node in $N(v, V')$ is adjacent to any other node in V'. Then, every node in $N(v, V')$ is adjacent only to $V - V'$ (as shown in Fig. 4.4). Thus, $V - V' - \{v\}$ is a node cover of $G(v)$, because every node in $N(v, V')$ is an isolated node in V'. The cardinality of a minimum node cover of the order graph $G(v)$ can be easily determined to be at most $t - 1$. However, this contradicts the hypothesis that each node has order t. Therefore, $N(v, V')$ contains at least one neighbor u of v such that the node u is adjacent to another node w in V'. Hence, $\Gamma(G, V') = V - V'$. □

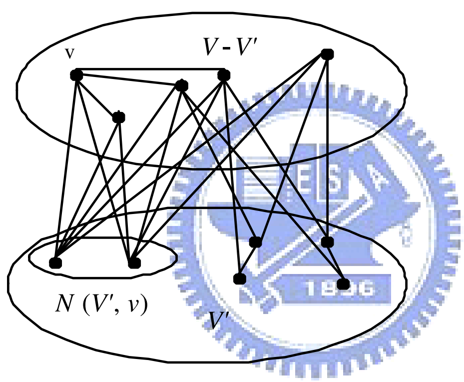

Figure 4.4: Illustration in Lemma 16 for an example of case 2.

Lemma 17 *Let H be a t-regular network, $t > 2$, and let K_2 be the complete network with two nodes. Suppose that the order of each node in H is t. Then, each node of the product network $G = H \times K_2$ has order $t + 1$.*

Proof: Let G^0 and G^1 be two copies of H in G. $M = (V, C)$ represents the comparison scheme of G. Let v be a node of G and let $G(v)$ be the order graph of v in G. Without loss of generality, assume that v is a node in G^0, and that u is a neighbor of v in G^1. There exists at least one node w in G^1 such that $(v, w)_u \in C$. Then, let $G^0(v)$ be the order graph of v in G^0. Since $G^0(v)$ is a proper subgraph of $G(v)$, every node cover of $G(v)$ must contain a node cover of $G^0(v)$. However, (w, u) is an edge in $G(v)$ rather than in $G^0(v)$. Therefore, a node cover of $G(v)$ must include at least either u or w. The order of v in G therefore exceeds that of v in G^0 by one. Thus, the lemma is proven. □

Theorem 18 *For $t > 2$, let H be a t-regular and t-diagnosable network with N nodes. Then the product network $G = H \times K_2$ is $(t+1)$-diagnosable.*

Proof: Let $G^0 = (V^0, E^0)$ and $G^1 = (V^1, E^1)$ be two copies of H in $G = (V, E)$. Let S_1 and S_2 be two distinct subsets of V and let $V'' = S_1 \bigcup S_2$ with $|S_1| = |S_2| = t + 1$, $|S_1 \bigcap S_2| = p$ and $0 \leq p \leq t$. Then, let $V' = V - V''$ with $|V'| = 2N - 2(t+1) + p$. Since G has $2N$ nodes, $2N \geq 2(2t+1) > 2(t+1) + 1$. Lemma 17 implies that each node of G has order $t + 1$. Hence, the theorem is proven if one of the conditions of Theorem 3 is satisfied. Now, let $V^{0'} = V' \bigcap V^0$ and $V^{1'} = V' \bigcap V^1$. G^0 and G^1 are isomorphic to H, so without loss of generality, assume that $|V^{0'}| \geq |V^{1'}|$. Let $|V^0 - V^{0'}| = k$ and $|V^1 - V^{1'}| = 2(t+1) - p - k$. Since $|V^{0'}| \geq |V^{1'}|$, $k \leq 2(t+1) - p - k$. Thus, the proof is divided into the following cases.

Case 1: $2(t+1) - p - k \leq t$ and $k \leq t$.

From Lemma 16, $|\Gamma(G, V')| \geq |\Gamma(G^0, V^{0'})| + |\Gamma(G^1, V^{1'})| = k + 2(t+1) - p - k = 2(t+1) - p$. Since $p \leq t$, $|\Gamma(G, V')| \geq 2(t+1) - p > p$. By Theorem 5, this case holds.

Subcase 2.1: $2(t+1) - p - k > t$ and $k < t$.

From Lemma 16, $|\Gamma(G^0, V^{0'})| = k$. Since $V^0 - V^{0'}$ contains $k < t$ nodes so each node in $V^{0'}$ has at least one neighbor in $V^{0'}$. Therefore, no isolated node is present in $V^{0'}$. Notably, at least $2(t+1) - p - 2k$ nodes in $V^1 - V^{1'}$ are adjacent to some $2(t+1) - p - 2k$ nodes in $V^{0'}$. Thus, $|\Gamma(G, V')| \geq |\Gamma(G^0, V^{0'})| + N(V^1 - V^{1'}, V^{0'}) \geq k + 2(t+1) - p - 2k = 2(t+1) - p - k$. Since $2(t+1) - p - k > t \geq p$, by Theorem 5, the case holds.

Subcase 2.2: $2(t+1) - p - k > t$ and $k = t$.

Since $2(t+1) - p - k > t$ and $k = t$,$(t+2) - p > t$, implying $p < 2$. From Lemma 16, $|\Gamma(G, V')| \geq |\Gamma(G^0, V^{0'})| = t > 2 > p$. Then, the case follows.

Subcase 2.3: $2(t+1) - p - k > t$ and $k > t$.

Since $2(t+1) - p - k > t$ and $k > t$, the number of nodes in $V - V'$ is $2(t+1)$, indicating $p = 0$. Condition (1) in Theorem 3 is first supposed to be satisfied in G^0. Then, the subgraph induced by $V^{0'}$ includes at least one connected component R with a cardinality of at least 2. Given $|V^0 - V^{0'}| = t+1$, Theorem 5 implies $|\Gamma(G^0, V^{0'})| \geq t > 2$ since G^0 is t-diagnosable. Therefore, $|\Gamma(G, V')| \geq |\Gamma(G^0, V^{0'})| > 2 > p$. This result implies that condition (1) in Theorem 3 is also satisfied in G.

Next, consider that the condition (1) in Theorem 3 is violated in G^0. Then, either condition (2) or condition (3) in Theorem 3 is satisfied in G^0. Since G^0 is t-regular and $t > 2$, one node v in $V^{0'}$ is adjacent to at least three nodes in $V^0 - V^{0'}$. Now, let u, w and x be three nodes in $V^0 - V^{0'}$ such that $u, w \in S_1$ and $x \in S_2$. Since $u, w \in S_1 - S_2$, $v \in V - S_1 - S_2$ and $p = 0$, condition (2) in Theorem 3 is also satisfied in G. The theorem follows. □

Let G_i be a t_i-regular interconnection network $i = 1, 2$, and let $G = G_1 \times G_2$ be the product network of G_1 and G_2. Then, the order of each node v in G is estimated from the following lemma.

Lemma 18 *Let $G_i = (V_i, E_i)$ be a t_i-regular network with $t_i > 2$. Suppose each node of G_i has order at least t_i, $i = 1, 2$. Then each node of the product network $G = G_1 \times G_2$ has order $t_1 + t_2$.*

Proof: Let $v = \langle x, y \rangle$ be an arbitrary node of G and let $G(v)$ be the order graph of v in G. According to the definition of product networks, is a node of V_1 and is a node of V_2. Therefore, the order of x is at least t_1 and the order of y is at least t_2. Let $G_1(x)$ be the order graph of x in G_1 and let $G_2(y)$ be the order graph of y in G_2. $N(x)$ is a node cover of $G_1(x)$ so the order of node x is exactly t_1. Similarly, the order of node y is t_2. Let $G_1^y(v)$ be the order graph of v in the subgraph G_1^y of G and let $G_2^x(v)$ be the order graph of v in the subgraph G_2^x of G. Since $V_1^y \bigcap V_2^x = v$, $G_1^y(v) \bigcap G_2^x(v) = \emptyset$, where $V(G_1^y(v))$ and $V(G_2^x(v))$ are the node sets of $G_1^y(v)$ and $G_2^x(v)$, respectively. $G_1^y(v)$ and $G_2^x(v)$ are observed to be subgraphs of $G(v)$. Thus, every node cover of $G(v)$ must contain a node cover of both $G_1^y(v)$ and $G_2^x(v)$. Since the subgraphs G_1^y and G_2^x of G are isomorphic to G_1 and G_2, respectively, $G_1^y(v)$ is isomorphic to $G_1(x)$ and $G_2^x(v)$ is isomorphic to $G_2(y)$. Therefore, the order of v in $G_1^y(v)$ is t_1 and the order of v in $G_2^x(v)$

is t_2. Since $V(G_1^y(v)) \bigcap V(G_2^x(v)) = \emptyset$, the order of v in $G(v)$ is $t_1 + t_2$. Hence, the lemma follows. □

Corollary 3 was proven; it states that the product network $G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable, in which G_i is t_i connected for $t_i > 2$, $i = 1, 2$. The previous section also established that a t_i-diagnosable network is not equivalent to a t_i-connected network. The following theorem states that the product network $G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable, where G_i is t_i-diagnosable for $t_i > 2$, $i = 1, 2$. We present Theorem 19 and omit the related proof which are completely illustrated in [10].

Theorem 19 *[10] For $t_i > 2$, let $G_i = (V_i, E_i)$ be a t_i-diagnosable and t_i-regular network with N_i nodes, $i = 1, 2$. Let $G = (V, E)$ be the product network of G_1 and G_2. Then the product network $G = G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable under the comparison diagnosis model with regularity $t_1 + t_2$.*

Notice that in Theorem 19 the number of nodes N_i is larger than or equal to $2t_i + 1$ for t_i-diagnosable, $i = 1, 2$. From Theorem 19 and by induction, the following corollary is obtained.

Corollary 5 *Let G be the product network of $G_1, G_2, \ldots,$ and G_k, where each G_i is t_i-diagnosable with regularity t_i and $t_i > 2$ for $1 \leq i \leq k$. Then, the product network G is $(t_1 + t_2 + \ldots + t_k)$-diagnosable with regularity $(t_1 + t_2 + \ldots + t_k)$.*

4.2.3 Diagnosability of heterogeneous product networks

This subsection considers different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks. The diagnosability of the heterogeneous product network G of G_1 and G_2, is considered, in which G_1 is t_1-diagnosable and G_2 is t_2-connected. Although the heterogeneous product network differs from the homogeneous product network, a similar result is obtained as that obtained for the homogeneous product network. Lemmas 15 and 17 immediately yield the following lemma.

Lemma 19 *Let G_1 be a t_1-regular and t_1-diagnosable network with $t_1 > 2$ and let G_2 be a t_2-regular and t_2-connected network with $N_2 \geq 2t_2 + 1$ nodes and $t_2 > 2$. Then, each node of the product network $G = G_1 \times G_2$ has order $t_1 + t_2$.*

Section 4.2.1 presents some examples to show that a t-diagnosable network is not equivalent to a t-connected network. Therefore, the following theorem is not implied by Theorem 19 but it can be proven by a similar technique. Theorem 20 is proven in [10].

Theorem 20 *[10] For $t_1, t_2 > 2$, let $G_1 = (V_1, E_1)$ be a t_1-regular and t_1-diagnosable network with N_1 nodes and let $G_2 = (V_2, E_2)$ be a t_2-regular and t_2-connected network with $N_2 \geq 2t_2 + 1$ nodes. Then the product network $G = G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable under the comparison diagnosis model with regularity $t_1 + t_2$.*

In the above theorem, the factor network G_2 must have at least $2t_2 + 1$ nodes. Therefore, by Corollary 5 and Theorem 20, the following corollary holds.

Corollary 6 *Let G be the product network of $G_1, G_2, \ldots,$ and G_k. Suppose that G_1 is t_1-regular and t_1-connected with $N_1 \geq 2t_1 + 1$ nodes, and suppose that G_i is t_i-regular and t_i-diagnosable, $t_i > 2$ for $2 \leq i \leq k$. Then, the product network G is $(t_1 + t_2 + \ldots + t_k)$-diagnosable with regularity $(t_1 + t_2 + \ldots + t_k)$.*

However, Corollaries 4 and 5 yield the following corollary.

Corollary 7 *Let G be the product network of $G_1, G_2, \ldots,$ and G_k. Suppose that G_i is t_i-regular and t_i-connected, $t_i > 2$ for $1 \leq i \leq m$ where $m > 2$, and suppose that G_j is t_j-regular and t_j-diagnosable, $t_j > 2$ for $m + 1 \leq j \leq k$. Then the product network G is $(t_1 + t_2 + \ldots + t_k)$-diagnosable under the comparison diagnosis model with regularity $(t_1 + t_2 + \ldots + t_k)$.*

Chapter 5

Strongly t-diagnosable Systems

The Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n, are all known to be n-connected but not $(n+1)$-connected. For each of these cubes, every vertex cut of size n has a particular structure as stated in the following lemma.

Lemma 20 *Let $n \geq 2$ and let XQ_n represent any n-dimensional cube which belongs to the cube family. For each set of vertices $S \subset V(XQ_n)$ with $|S| = n$, if $XQ_n - S$ is disconnected, there exists a vertex $v \in V(XQ_n)$ such that $N(v) = S$.*

Proof: We prove this lemma by induction on n. A 2-dimensional cube XQ_2 is simply a cycle of length four. Clearly, this lemma is true for XQ_2. Assume it holds for some $n \geq 2$. We now show that it holds for $n+1$.

Let $(n+1)$-dimensional cube XQ_{n+1} be obtained from two n-dimensional cubes XQ_n, denoted by XQ_n^L and XQ_n^R, by adding a perfect matching between them. Let $S \subset V(XQ_{n+1})$, $|S| = n+1$, and, $S_L = V(XQ_n^L) \bigcap S$ and $S_R = V(XQ_n^R) \bigcap S$. In the remainder of this proof, we show that XQ_{n+1} satisfies one of the two conditions: (i) $XQ_{n+1} - S$ is connected, or (ii) $XQ_{n+1} - S$ is disconnected and there is a vertex $v \in V(XQ_{n+1})$ such that $N(v) = S$.

We study three cases: (1) $|S_L| \leq n-1$ and $|S_R| \leq n-1$, (2) either $|S_L| = n$ or $|S_R| = n$, and (3) either $|S_L| = n+1$ or $|S_R| = n+1$.

Case 1: $|S_L| \leq n-1$ and $|S_R| \leq n-1$.

Since XQ_n is n-connected, both $XQ_n^L - S_L$ and $XQ_n^R - S_R$ are connected. For $n \geq 2$, we know that $|V(XQ_n^L) - S_L| \geq 2^n - (n-1) > n - 1 \geq |S_R|$ and $|V(XQ_n^R) - S_R| \geq 2^n - (n-1) > n - 1 \geq |S_L|$. So, the subgraph $XQ_n^L - S_L$ is connected to the other subgraph $XQ_n^R - S_R$. Hence, $XQ_{n+1} - S$ is connected.

Case 2: either $|S_L| = n$ or $|S_R| = n$.

Without loss of generality, suppose that $|S_L| = n$ and $|S_R| = 1$. Suppose $XQ_n^L - S_L$ is connected. Using a similar argument used in case (1), we can prove that $XQ_{n+1} - S$ is connected. Otherwise, $XQ_n^L - S_L$ is disconnected. By induction hypothesis, there exists a vertex $v \in V(XQ_n^L)$ such that $N(\{v\}, XQ_n^L) = S_L$. Now, consider XQ_n^R and consider the matching neighbor u of v in XQ_n^R. Note that $XQ_n^R - S_R$ is connected for $n \geq 2$ and every vertex in XQ_n^R has a matching neighbor in XQ_n^L. Thus, $XQ_{n+1} - S$ is connected if $S_R \neq \{u\}$. If $S_R = \{u\}$, $XQ_{n+1} - S$ is disconnected, and $S = N(v)$. This proves case 2.

Case 3: either $|S_L| = n + 1$ or $|S_R| = n + 1$.

Without loss of generality, suppose that $|S_L| = n + 1$ and $|S_R| = 0$. Since there is one corresponding matched vertex for each vertex $v \in V(XQ_n^L - S_L)$ in $V(XQ_n^R)$, $XQ_{n+1} - S$ is connected.

Consequently, this lemma holds. □

Let F_1 and F_2 be two distinct sets of vertices of XQ_n with $|F_i| \leq n + 1$, $i = 1, 2$ and let $S = F_1 \bigcap F_2$. Then $|S| \leq n$. By the above lemma, either $XQ_n - S$ is connected, or, $XQ_n - S$ is disconnected and there is a vertex $v \in V(XQ_n)$ such that $S = N(v)$. If $XQ_n - S$ is connected, the two sets $V(XQ_n) - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$ both belong to the same component $XQ_n - S$. Thus, there exists one edge connecting $V(XQ_n) - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are distinguishable. Therefore, if F_1 and F_2 are indistinguishable, $|F_i| \leq n+1$, $i = 1, 2$, $XQ_n - S$ is disconnected and there exists a vertex v such that $S = N(v)$. $S = F_1 \bigcap F_2$, so $N(v) \subseteq F_1$ and $N(v) \subseteq F_2$. We then propose the following concept.

Definition 4 *A system G is strongly t-diagnosable if the following two conditions hold:*

(*i*) *G is t-diagnosable, and*

(*ii*) *for any two distinct subsets F_1, $F_2 \subset V(G)$ with $|F_i| \le t+1$, $i = 1, 2$,*

either (*a*) (F_1, F_2) *is a distinguishable pair;*

or (*b*) (F_1, F_2) *is an indistinguishable pair and there exists a vertex $v \in V$ such that*

$N(v) \subseteq F_1$ *and* $N(v) \subseteq F_2$.

A $(t+1)$-diagnosable system is "stronger" than a t-diagnosable system, and of course it is strongly t-diagnosable according to the above definition. However, among all those strongly t-diagnosable systems, we are interested in the one which is t-diagnosable but not $(t+1)$-diagnosable.

Following Lemma 3 and Definition 4, we propose a sufficient condition for verifying if a system G is strongly t-diagnosable.

Proposition 1 *A system $G(V, E)$ with n vertices is strongly t-diagnosable if the following three conditions hold:*

(*i*) $n \ge 2(t+1)+1$,

(*ii*) $\kappa(G) \ge t$, *and*

(*iii*) *for any vertex set $S \subset V$ with $|S| = t$, if $G - S$ is disconnected, there exists a vertex $v \in V$ such that $N(v) \subset S$.*

Proof: With conditions (i) and (ii), by Lemma 3, G is t-diagnosable. Now, we want to prove condition (ii) of Definition 4 holds. Let $F_1, F_2 \subset V$ bet two distinct sets with $|F_i| \le t+1$, $i = 1, 2$ and $S = F_1 \bigcap F_2$. Suppose that $G - S$ is connected. Then there exists one edge connecting $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are distinguishable. That is, condition (ii.a) of Definition 4 holds.

Otherwise, $G - S$ is disconnected. By condition (ii), the connectivity of G is at least t, and $0 \le |S| \le t$, so $|S| = t$. Then by condition (iii), there exists one vertex $v \in V$ such that $N(v) \subset S$. Therefore, $N(v) \subset F_1$ and $N(v) \subset F_2$. So condition (ii.b) of Definition 4 holds. This completes the proof of this proposition. □

Next, we present a necessary and sufficient condition for a system G to be strongly t-diagnosable.

Lemma 21 *A system $G(V, E)$ with $|V| = n$ is strongly t-diagnosable if and only if the following three conditions hold:*

(i) $n \geq 2(t+1)+1$,

(ii) $\delta(G) \geq t$, *and*

(iii) for any two distinct subsets F_1, $F_2 \subset V(G)$ with $|F_i| \leq t+1$, $i = 1, 2$, the pair (F_1, F_2) satisfy condition (ii.a) or (ii.b) of Definition 4.

Proof: We first prove the necessity. To prove condition (i), we show that the assumption $n \leq 2(t+1)$ leads to a contradiction. Assume $n \leq 2(t+1)$. We can partition V into two disjoint vertex sets V_1 and V_2, $V_1 \bigcap V_2 = \emptyset$ and $V = V_1 \bigcup V_2$, with $|V_i| \leq t+1$, $i = 1, 2$. By Lemma 4, V_1 and V_2 are indistinguishable. Since G is strongly t-diagnosable, by Definition 4, $N(v) \subset V_1$ and $N(v) \subset V_2$, for some vertex $v \in V$, contradicting with the assumption $V_1 \bigcap V_2 = \emptyset$.

To prove condition (ii), since G is strongly t-diagnosable, it is t-diagnosable by definition. Then by condition (ii) of Corollary 1, $N(v) \geq t$ for each vertex $v \in V$. So condition (ii) is necessary. Condition (iii) of this lemma is the same as condition (ii) of Definition 4. This proves the necessity.

To prove the sufficiency of conditions (i), (ii) and (iii). We need only to show that G is t-diagnosable. Suppose not, then there exists an indistinguishable pair of sets $F_1, F_2 \subset V$, $F_1 \neq F_2$, and $|F_i| \leq t$, $i = 1, 2$. By condition (ii.b) of Definition 4, there exists a vertex $v \in V$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. By condition (ii), $|N(v)| \geq t$. However, $|F_1| \leq t$ and $|F_2| \leq t$. Hence, $F_1 = F_2 = N(v)$. This contradicts with the fact that $F_1 \neq F_2$. The lemma follows. □

We now give another necessary and sufficient condition for checking whether a system is strongly t-diagnosable. The motivation of these conditions is as follows: Let $G(V, E)$ be a strongly t-diagnosable system. Suppose that G is $(t+1)$-diagnosable. Then by Theorem 2, for every set $S \subset V$, $0 \leq p \leq t$ where $|S| = p$, each component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1$. Otherwise, G is t-diagnosable but not $(t+1)$-diagnosable. Then there exists an indistinguishable pair (F_1, F_2), $F_1 \neq F_2$, with $|F_i| \leq t+1$, $i = 1, 2$. By condition (ii.b) of definition 4, there exists a vertex $v \in V$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$, $i = 1, 2$. Note that $\delta(G) \geq t$, and therefore, $|N(v)| \geq t$. It means that $\{v\}$ is a trivial component of $G - (F_1 \bigcap F_2)$. Setting $S = F_1 \bigcap F_2$ and $|S| = t$, $G - S$ has a trivial component.

Theorem 21 *A system $G = (V, E)$ is strongly t-diagnosable if and only if for each vertex set $S \subset V$ with cardinality $|S| = p$, $0 \leq p \leq t$, the following two conditions are satisfied.*

- *(i) For $0 \leq p \leq t-1$, every component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1$; and*
- *(ii) for $p = t$, either (a) every component C of $G - S$ satisfies $|V_C| \geq 3$; or else, (b) $G - S$ contains at least one trivial component. (Remark: $2((t+1) - p) + 1 = 3$ as $p = t$.)*

Proof: We use Theorem 2 to prove the sufficiency of conditions (i) and (ii). Let S be a set of vertex with $|S| = p$, $0 \leq p \leq t-1$. By condition (i), every component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1 \geq 2(t-p) + 1$. Then by Theorem 2, G is t-diagnosable.

To show that G is strongly t-diagnosable, we need to prove that condition (ii) of Definition 4 holds. Suppose that conditions (i) and (ii.a) are both satisfied. Then by Theorem 2, G is $(t+1)$-diagnosable. Now consider the case that G is not $(t+1)$-diagnosable. Let (F_1, F_2) be an indistinguishable pair, $F_1 \neq F_2$, with $|F_1| \leq t+1$ and $|F_2| \leq t+1$. We let $S = F_1 \bigcap F_2$ and $X = V - (F_1 \bigcup F_2)$, then $0 \leq p \leq t$, where $|S| = p$. Since F_1 and F_2 are indistinguishable, by Lemma 4, there is no edge between X and $F_1 \triangle F_2$. Therefore, in $G - S$, $F_1 \triangle F_2$ is disconnected from the other components. Observe that $|F_1 \triangle F_2| \leq 2((t+1) - p)$, by condition (i), p cannot be in the range from 0 to $t-1$. So $p = t$ and $|F_1 \triangle F_2| \leq 2((t+1) - p) = 2((t+1) - t) = 2$. Then, by condition (ii.b), $G - S$ must have a trivial component $\{v\}$. So $N(v) \subset S$. G is t-diagnosable, by condition (ii) of Corollary 1, $|N(v)| \geq t$. Hence, $S = N(v)$. Since $S = F_1 \bigcap F_2$, $N(v) \subset F_1$ and $N(v) \subset F_2$. Therefore, G is strongly t-diagnosable.

This proves the sufficiency. Next, we show the condition (i) and (ii) are also necessary.

To show condition (i), suppose on the contrary that there exists a set of vertices $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, such that $G - S$ has a component with strictly less than $2((t+1)-p)+1$ vertices. Let C be such a component with $|V_C| \leq 2((t+1)-p)$. We can partition V_C into two disjoint subsets A_1 and A_2, $A_1 \bigcup A_2 = V_C$ and $A_1 \bigcap A_2 = \emptyset$, with $|A_i| \leq (t+1)-p$, $i = 1, 2$. Let $F_1 = A_1 \bigcup S$ and $F_2 = A_2 \bigcup S$. Then $|F_i| \leq t+1$, $i = 1, 2$, and F_1 and F_2 are indistinguishable by Lemma 4. Since G is strongly t-diagnosable, by condition (ii.b) of Definition 4, there exists a vertex v such that $N(v) \subset F_1$ and $N(v) \subset F_2$. G is t-diagnosable, by Corollary 1, each vertex of G has degree at least t. So $|N(v)| \geq t$. However, $N(v) \subset F_1 \bigcap F_2 = S$ and $|S| = p \leq t-1$, this is a contradiction. Thus, condition (i) is necessary.

Now, we prove that condition (ii) is necessary. Let S be a set of vertex with $|S| = p$ and $p = t$. Suppose that G is $(t+1)$-diagnosable. By Theorem 2, for $p = t$, every component C of $G - S$ satisfies $|V_C| \geq 2((t+1)-t)+1 = 3$. That is, condition (ii.a) holds if G is $(t+1)$-diagnosable. Otherwise, G is not $(t+1)$-diagnosable and there exists a component C in $G - S$ with strictly less than three vertices, $|V_C| \leq 2$. We have to show that there is a trivial component in $G - S$. If $|V_C| = 1$, we are done. Assume that $|V_C| = 2$, say $V_C = \{v_1, v_2\}$. Let $F_1 = S \bigcup \{v_1\}$ and $F_2 = S \bigcup \{v_2\}$. Then $|F_1| = t+1$, $|F_2| = t+1$, and F_1 and F_2 are indistinguishable. Since G is strongly t-diagnosable, by condition (ii.b) of Definition 4, there exists a vertex v such that $N(v) \subset F_1$ and $N(v) \subset F_2$. We have $S = F_1 \bigcap F_2$ and $N(v) \subset S$. Therefore, $\{v\}$ is a trivial component in $G - S$, this proves condition (ii.b).

Consequently, the theorem holds. □

The above theorem again states that a strongly t-diagnosable system is almost $(t+1)$-diagnosable, if it is not so. The only case that stops it from being $(t+1)$-diagnosable occurs in the following situation: all the neighboring vertices $N(v)$ of some vertex v are faulty simultaneously.

5.1 Strongly t-diagnosable systems in the MCN networks

In previous studies, the diagnosability of many practical interconnection networks have been explored. Actually, some of them are not only n-diagnosable but also strongly n-diagnosable, for example, the Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are so. In the following, we shall prove that all members in the cube family are strongly n-diagnosable for $n \geq 4$.

Under the comparison model [47, 48], it is proved that a MCN with two t-connected and t-diagnosable M-components is $(t+1)$-diagnosable in [46] and Chapter 3. In the following theorem, we shall show that an MCN with two t-diagnosable M-components is strongly $(t+1)$-diagnosable under PMC model.

Theorem 22 *Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two t-diagnosable systems with the same number of vertices, where $t \geq 2$. Then MCN $G = G_1 \bigoplus_M G_2$ is strongly $(t+1)$-diagnosable.*

Proof: We use Theorem 21 to prove it. Let $G = G(V, E) = G_1 \bigoplus_M G_2$ and $S \subset V$ with $|S| = p$, $0 \leq p \leq t+1$. Let $S_1 = S \bigcap V_1$, $S_2 = S \bigcap V_2$, $|S_1| = p_1$ and $|S_2| = p_2$. In the following proof, we consider two cases: (1) $S_1 = \emptyset$ or $S_2 = \emptyset$, and (2) $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. We shall prove that: (i) $|V_C| \geq 2((t+2) - p) + 1$ for every component C of $G - S$ as $0 \leq p \leq t$, and (ii) for $p = t+1$, either (a) every component C of $G - S$ satisfies $|V_C| \geq 3$; or else, (b) $G - S$ contains at least one trivial component. Then by Theorem 21, G is strongly $(t+1)$-diagnosable.

Case 1: $S_1 = \emptyset$ or $S_2 = \emptyset$.

Without loss of generality, assume $S_1 = \emptyset$ and $S_2 = S$. We know that each vertex of V_2 has an adjacent neighbor in V_1, so, $G - S$ is connected. The only component C of $G - S$ is $G - S$ itself. Hence, $|V_C| = |V - S| = |V_1| + |V_2| - p$. G_i is t-diagnosable, $i = 1, 2$, by Corollary 1, $|V_i| \geq 2t + 1$. So $|V_C| \geq 2(2t+1) - p \geq 2((t+2) - p) + 1$ for $t \geq 2$. That is, conditions (i) and (ii.a) of Theorem 21 are satisfied.

Case 2: $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

$S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, it implies $p_1 \geq 1$ and $p_2 \geq 1$. Then, we divide the case into two subcases: (2.a) both $p_1 \leq t-1$ and $p_2 \leq t-1$, and (2.b) either $p_1 = t$ or $p_2 = t$. Note that $0 \leq p \leq t+1$ and $p = p_1 + p_2$. For subcase (2.a), $1 \leq p_1 \leq t-1$ and $1 \leq p_2 \leq t-1$, and for subcase (2.b), either $p_1 = t$ and $p_2 = 1$, or, $p_2 = t$ and $p_1 = 1$.

Subcase 2.a: $1 \leq p_1 \leq t-1$ and $1 \leq p_2 \leq t-1$.

Let C_1 be a component of $G_1 - S_1$. G_1 is t-diagnosable, by Theorem 2, $|V_{C_1}| \geq 2(t-p_1)+1$. We claim that $2(t-p_1)+1 \geq p_2+1$. Since $p = p_1+p_2$, $2(t-p_1)+1 = 2(t-(p-p_2))+1 = 2p_2+2(t-p)+1$. Suppose $p \leq t$, $|V_{C_1}| \geq 2p_2+1$. Otherwise, $p = t+1$. Since $p_1 \leq t-1$, $p_2 \geq 2$ and $2p_2+2(t-p)+1 \geq p_2+1$. Hence, $|V_{C_1}| \geq 2(t-p_1)+1 \geq p_2+1$. That is, V_{C_1} has at least one adjacent neighbor $v \in V_2$ and $v \notin S_2$. G_2 is t-diagnosable, by Theorem 2, every component of $G_2 - S_2$ has at least $2(t-p_2)+1$ vertices. Let C_2 be the component of $G_2 - S_2$ such that $v \in V_{C_2}$ and let C be the component of $G - S$ such that $V_{C_1} \bigcup V_{C_2} \subset V_C$. Then $|V_C| \geq |V_{C_1}| + |V_{C_2}| \geq (2(t-p_1)+1) + (2(t-p_2)+1) = 2(2t-p+1) \geq 2((t+2)-p)+1$ as $t \geq 2$. So every component of $G-S$ has at least $2((t+2)-p)+1$ vertices in this subcase. It means that conditions (i) and (ii.a) of Theorem 21 are satisfied.

Subcase 2.b: either $p_1 = t$ and $p_2 = 1$, or, $p_2 = t$ and $p_1 = 1$.

Without loss of generality, assume $p_2 = t$ and $p_1 = 1$. Since $p = p_1+p_2 = t+1$, we need only to prove either condition (ii.a) or (ii.b) of Theorem 21 holds. Let C_1 be a component of $G_1 - S_1$. G_1 is t-diagnosable, by Theorem 2, $|V_{C_1}| \geq 2(t-p_1)+1 = 2(t-1)+1$. Since $t \geq 2$, $|V_{C_1}| \geq 2(t-1)+1 \geq 3$. So the component of $G-S$ containing the vertex set V_{C_1} has at least three vertices.

Let C_2 be a component of $G_2 - S_2$, $N(V_{C_2}, V_2) \subset S_2$. If V_{C_2} has some adjacent neighbor $v_1 \in V_1$ and vertex v_1 belongs to some component C_1 of $G_1 - S_1$, then the component C containing the two vertex sets V_{C_1} and V_{C_2} has at least four vertices. Thus, condition (ii.a) of Theorem 21 holds. Otherwise, $N(V_{C_2}, V_1) \subset S_1$. Since $|S_1| = p_1 = 1$, $|N(V_{C_2}, V_1)| = 1$. That is, $|V_{C_2}| = 1$ and $N(V_{C_2}) \subset S_1 \bigcup S_2$. Hence, C_2 is a trivial component of $G-S$; and therefore, condition (ii.b) of Theorem 21 holds.

Consequently, the theorem follows. □

For $t = 1$, the above result is not necessarily true, we give an example shown in Fig. 5.1. Let G_1 and G_2 be two path graphs of length four with vertex sets $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$, respectively. Let G be the Matching Composition Network constructed by adding a perfect matching (the dash lines in Fig. 5.1.i) between G_1 and G_2. By Lemma 3, both G_1 and G_2 are 1-diagnosable and G is 2-diagnosable. See Fig. 5.1.ii, let $F_1 = \{u_1, u_2, v_2\}$ and $F_2 = \{v_1, v_2, u_2\}$. By Lemma 4, F_1 and F_2 are indistinguishable but there doesn't exist any vertex $v \in V(G_i)$, $i = 1, 2$, such that $N(v) \subset F_1$ and $N(v) \subset F_2$. So G is not strongly 2-diagnosable.

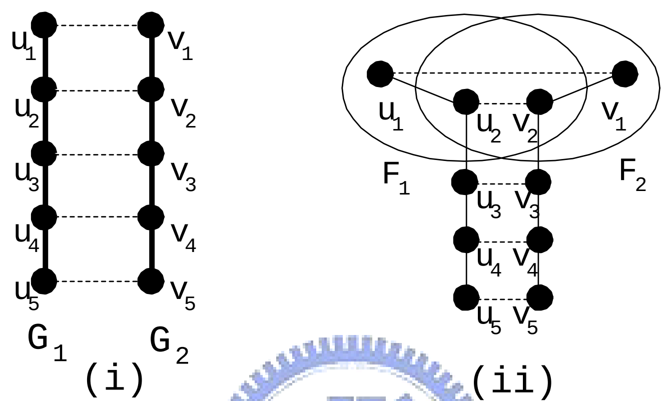

Figure 5.1: An example of non-strongly $(t+1)$-diagnosable as $t = 1$.

Applying Theorem 22, all systems in the cube family are strongly $(t+1)$-diagnosable if their subcubes are t-diagnosable for $t \geq 2$. The Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n are well-known members in the cube family. For $n = 2$, these cubes are all isomorphic to the cycle of length four; they are 1-diagnosable but not 2-diagnosable. For $n = 3$, these cubes are all 3-connected, by Lemma 3, they are 3-diagnosable. So we have the following corollary.

Corollary 8 *The Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are all strongly n-diagnosable for $n \geq 4$.*

We now give some examples which are not strongly t-diagnosable. Consider the 3-dimensional Hypercube Q_3, it is 3-diagnosable but not strongly 3-diagnosable due to the fact that $|V(Q_3)| = 8 \leq 2(t+1)+1$ as $t = 3$, which contradicts the condition (i) of Lemma

21. Let C_n be a cycle of length n, $n \geq 7$. By Lemma 3, C_n is 2-diagnosable, but it is not strongly 2-diagnosable. Another nontrivial example is presented in Fig. 5.2. This graph G is 3-regular, 2-connected, and by Theorem 2, it is 3-diagnosable. As shown in Fig. 5.2, $F_1 = \{1, 2, 5, 6\}$ and $F_2 = \{3, 4, 5, 6\}$. (F_1, F_2) is an indistinguishable pair, but there does not exist any vertex v in $V(G)$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. By Definition 4, the graph is not strongly 3-diagnosable.

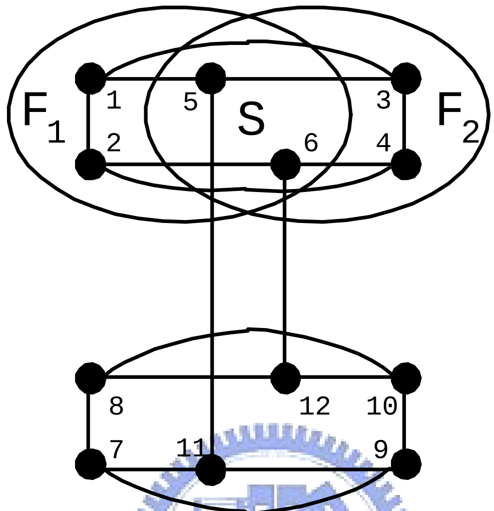

Figure 5.2: An example of non-strongly 3-diagnosable system.

Chapter 6

Conditional Diagnosability

Consider a system G with diagnosability $t(G) = t$; so G is t-diagnosable but not $(t+1)$-diagnosable. In previous research on diagnosability, the investigated networks are often strongly t-diagnosable, for examples, members in the cube family are so. Given a system G, suppose that it is strongly t-diagnosable but not $(t+1)$-diagnosable. As we mentioned before, the only case that stops it from being $(t+1)$-diagnosable is that there exists a vertex v whose neighboring vertices are faulty simultaneously. We are, therefore, led to the following question: How large the maximum value of t can be such that G remains t-diagnosable under the condition that every faulty set F satisfies $N(v) \nsubseteq F$ for each vertex $v \in V$.

For classical measurement of diagnosability, it is usually assumed that processor failures are statically independent. It does not reflect the total number of processors in the system and the probabilities of processor failures. In [51], Najjar and Gaudiot have proposed *fault resilience* as the maximum number of failures that can be sustained while the network remains connected with a reasonably high probability. For Hypercube, the fault resilience is shown as 25% for the 4-dimensional cube Q_4 and it increases to 33% for the 10-dimensional cube Q_{10}. More particularly, for the 10-dimensional cube Q_{10}, 33% processors can fail and the network still remains connected with a probability of 99%. They also gave a conclusion that large-scale systems with a constant degree are more susceptible to failures by disconnection than smaller networks. With the observation of Lemma 4, a connected network gives higher probability to diagnosis faulty processors and has better ability of distinguishing any two sets of processors.

Motivated by the deficiency of the classical measurement of diagnosability and the broadness of a system being strongly t-diagnosable, we introduce a measure of conditional diagnosability by claiming the property that any faulty set cannot contain all neighbors of any processor. We formally introduce some terms related to the conditional diagnosability. A faulty set $F \subset V$ is called a *conditional faulty set* if $N(v) \nsubseteq F$ for any vertex $v \in V$. A system $G(V, E)$ is *conditionally t-diagnosable* if F_1 and F_2 are distinguishable, for each pair of conditional faulty sets F_1, $F_2 \subset V$ and $F_1 \neq F_2$, with $|F_1| \leq t$ and $|F_2| \leq t$. The *conditional diagnosability* of a system G, written as $t_c(G)$, is defined to be the maximum value of t such that G is conditionally t-diagnosable. It is clear that $t_c(G) \geq t(G)$.

Lemma 22 *Let G be a network system. Then $t_c(G) \geq t(G)$.*

Let $F_1, F_2 \subset V$ and $F_1 \neq F_2$. We say (F_1, F_2) is a *distinguishable conditional-pair* (an indistinguishable conditional-pair respectively) if F_1 and F_2 are conditional faulty sets and are distinguishable (indistinguishable respectively).

It follows from the definition that a strongly t-diagnosable system is clearly conditionally $(t + 1)$-diagnosable. However, the conditional diagnosability of some strongly t-diagnosable systems can be far greater than $t + 1$. This motivates us to study the conditional diagnosability of the Hypercube.

Lemma 23 *Let G be a strongly t-diagnosable system. Then G is conditionally $(t + 1)$-diagnosable.*

6.1 Conditional Diagnosability of Q_n

Before discussing the conditional diagnosability, we have some observations as follows. Let $F_1, F_2 \subset V$ be an indistinguishable conditional-pair. Let $X = V - (F_1 \bigcup F_2)$. Then there is no edge between X and $F_1 \triangle F_2$. So $N(F_1 \triangle F_2, X) = \emptyset$ and $N(X, F_1 \triangle F_2) = \emptyset$. Let vertex $v \in F_1 - F_2$ (or $v \in F_2 - F_1$). Then $N(v) \subset (F_1 \bigcup F_2)$. F_1 is a conditional faulty set; so $N(v) \nsubseteq F_1$ and $N(v) \bigcap (F_2 - F_1) \neq \emptyset$. Similarly, F_2 is a conditional faulty set, $N(v) \nsubseteq F_2$ and $N(v) \bigcap (F_1 - F_2) \neq \emptyset$. So $|N(v) \bigcap (F_1 - F_2)| \geq 1$ and $|N(v) \bigcap (F_2 - F_1)| \geq 1$ for every vertex $v \in F_1 \triangle F_2$. Now consider a vertex $u \in X = V - (F_1 \bigcup F_2)$. Since F_1 and F_2 are indistinguishable conditional-pair, $N(u) \bigcap (F_1 \triangle F_2) = \emptyset$, $N(u) \nsubseteq F_1$ and $N(u) \nsubseteq F_2$. So

$N(u) \nsubseteq (F_1 \bigcup F_2)$. Therefore, every vertex $u \in X$ has at least one neighbor in X(See Fig. 6.1). We state this fact in the following lemma.

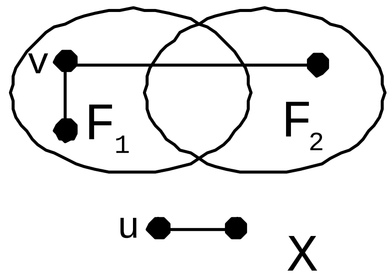

Figure 6.1: Illustration for Lemma 24.

Lemma 24 *Let $G(V, E)$ be a system. Given an indistinguishable conditional-pair (F_1, F_2), $F_1 \neq F_2$, the following two conditions hold:*

(i) $|N(u) \bigcap (V - (F_1 \bigcup F_2))| \geq 1$ *for* $u \in (V - (F_1 \bigcup F_2))$*, and*

(ii) $|N(v) \bigcap (F_1 - F_2)| \geq 1$ *and* $|N(v) \bigcap (F_2 - F_1)| \geq 1$ *for* $v \in F_1 \triangle F_2$*.*

Let (F_1, F_2) be an indistinguishable conditional-pair, and let $S = F_1 \bigcap F_2$. By the above observations, every component of $G-S$ is nontrivial. Moreover, for each component C_1 of $G - S$, if $V_{C_1} \bigcap (F_1 \triangle F_2) = \emptyset$, $deg_{C_1}(v) \geq 1$ for $v \in V_{C_1}$; for each component C_2 of $G - S$, if $V_{C_2} \bigcap (F_1 \triangle F_2) \neq \emptyset$, $deg_{C_2}(v) \geq 2$ for $v \in V_{C_2}$. To find the conditional diagnosability of the Hypercube Q_n, we need to study the cardinality of the set S.

First, we give an example to show that the conditional diagnosability of the Hypercube Q_n is no greater than $4(n-2)+1$. As shown in Fig. 6.2, we take a cycle of length four in Q_n, let $\{v_1, v_2, v_3, v_4\}$ be the four consecutive vertices on this cycle and let $F_1 = N(\{v_1, v_2, v_3, v_4\}) \bigcup \{v_1, v_2\}$ and $F_2 = N(\{v_1, v_2, v_3, v_4\}) \bigcup \{v_3, v_4\}$. It is a simple matter to check that (F_1, F_2) is an indistinguishable conditional-pair. Note that the Hypercube Q_n has no triangle and any two vertices have at most two common neighbors. As we can see that, $|F_1 - F_2| = |F_2 - F_1| = 2$ and $|F_1 \bigcap F_2| = 4(n-2)$. Hence, Q_n is not conditionally $(4(n-2)+2)$-diagnosable and $t_c(Q_n) \leq 4(n-2)+1$. Then, we shall show that Q_n is in fact conditionally t-diagnosable, where $t = 4(n-2)+1$.

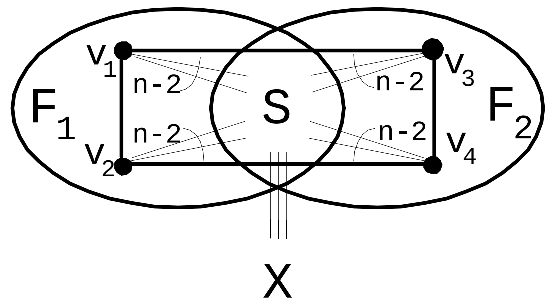

Figure 6.2: Illustration for an indistinguishable conditional-pair (F_1, F_2), where $|F_1| = |F_2| = 4(n-2)+2$.

Lemma 25 $t_c(Q_n) \leq 4(n-2)+1$ *for* $n \geq 3$.

Let S be a set of vertices, $S \subset V(Q_n)$. Suppose that $Q_n - S$ is disconnected and C is a component of $Q_n - S$. We need some results on the cardinalities of S and V_C under some restricted conditions. The results are listed in Lemmas 26 and 27.

These two lemmas are both proved by dividing Q_n into two Q_{n-1}'s, denoted by Q^L_{n-1} and Q^R_{n-1}. To simplify the explanation, we define some symbols as follows: $V_L = V(Q^L_{n-1})$, $V_R = V(Q^R_{n-1})$, $C_L = Q^L_{n-1} \bigcap C$, $C_R = Q^R_{n-1} \bigcap C$, $V_{C_L} = V(C_L)$, $V_{C_R} = V(C_R)$, $S_L = V_L \bigcap S$, and $S_R = V_R \bigcap S$.

The following result is also implicit in [43].

Lemma 26 *Let* Q_n *be the* n*-dimensional Hypercube,* $n \geq 3$*, and let* S *be a set of vertices* $S \subset V(Q_n)$*. Suppose that* $Q_n - S$ *is disconnected. Then the following two conditions hold:*

(i) $|S| \geq n$*, and*

(ii) if $n \leq |S| \leq 2(n-1)-1$*, then* $Q_n - S$ *has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of* $Q_n - S$ *contains* $2^n - |S| - 1$ *vertices.*

Proof: Since $\kappa(Q_n) = n$ [57], condition (i) hold. We need only to prove condition (ii) is true. Because $Q_n - S$ is disconnected, there are at least two components in $Q_n - S$. We consider three cases: (1) $Q_n - S$ contains at least two trivial components, (2) $Q_n - S$ has at least two nontrivial components, (3) there are exactly one trivial component and one nontrivial component in $Q_n - S$. In cases (1) and (2), we shall prove that $|S| \geq 2(n-1)$. Then $n \leq |S| \leq 2(n-1)-1$, it implies $Q_n - S$ belongs to case (3).

Case 1: $Q_n - S$ contains at least two trivial components.

Let $v_i \in V$, $i = 1, 2$, and $\{v_1\}, \{v_2\} \subset V(Q_n)$ be two trivial components of $Q_n - S$. It means that $N(v_1) \subset S$ and $N(v_2) \subset S$. For Q_n, it is not difficult to see that any two vertices have at most two common neighbors. That is, $|N(v_1) \bigcap N(v_2)| \leq 2$. Hence, $|S| \geq |N(v_1) \bigcup N(v_2)| = |N(v_1)| + |N(v_2)| - |N(v_1) \bigcap N(v_2)| \geq 2n - 2 = 2(n-1)$.

Case 2: $Q_n - S$ has at least two nontrivial components.

We prove, by induction on n, that $|S| \geq 2(n-1)$. For $n = 3$, suppose $n \leq |S| \leq 2(n-1)-1$, it implies that $|S| = 3$. The connectivity of Q_3 is 3. By Lemma 20 ,the only vertex cut S with $|S| = 3$ in Q_3 is $S = N(v)$ for some vertex $v \in V(Q_3)$. It follows that $Q_3 - S$ has exactly two components, one is trivial and the other is nontrivial. Therefore, if $Q_3 - S$ has at least two nontrivial components, $|S| \geq 2(n-1)$, where $n = 3$. Assume the case holds for some $n-1$, $n-1 \geq 3$. We now show that it holds for n.

Let C and C' be two nontrivial component of $Q_n - S$. So $|V_C| \geq 2$. It is feasible to divide Q_n into the two disjoint Q_{n-1}'s, denoted by Q^L_{n-1} and Q^R_{n-1}, such that $|V_{C_L}| \geq 1$ and $|V_{C_R}| \geq 1$. There is another component C' of $Q_n - S$, so at least one of the two graphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected.

Suppose that both $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected. Since $\kappa(Q_{n-1}) = n-1$, $|S_L| \geq n-1$ and $|S_R| \geq n-1$. Then $|S| = |S_L| + |S_R| \geq 2(n-1)$. Otherwise, one of the two subgraphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is connected. Without loss of generality, assume that $Q^L_{n-1} - S_L$ is connected and $Q^R_{n-1} - S_R$ is disconnected. Then $V_L = V_{C_L} \bigcup S_L$ and the other nontrivial component C' of $Q_n - S$ is completely contained in $Q^R_{n-1} - S_R$. Since $V_{C'}$ is disconnected from V_{C_L}, the corresponding matched vertices of $V_{C'}$ in Q^L_{n-1} are in S_L. That is, $N(V_{C'}, Q^L_{n-1}) \subseteq S_L$. Hence, $|S_L| \geq |V_{C'}| \geq 2$.

If $|S_R| \geq 2(n-2)$, then $|S| = |S_L| + |S_R| \geq 2 + 2(n-2) = 2(n-1)$. Otherwise, $n - 1 \leq |S_R| \leq 2(n-2) - 1$, by induction hypothesis that $Q_R - S_R$ cannot have two nontrivial components and by the result of case (1), $Q_R - S_R$ has exactly two components, one is trivial and the other is nontrivial. We know that $Q^R_{n-1} - S_R$ has C_R and C' as its components and C' is a nontrivial component. So C_R must be a trivial component of $Q^R_{n-1} - S_R$, and $|V_{C'}| = 2^{n-1} - |S_R| - 1$. Note that $N(V_{C'}, Q^L_{n-1}) \subseteq S_L$. Then $|S| = |S_L| + |S_R| \geq |V_{C'}| + |S_R| = 2^{n-1} - |S_R| - 1 + |S_R| = 2^{n-1} - 1 \geq 2(n-1)$ for $n \geq 4$.

Consequently, condition (ii) is true and the lemma holds. □

Suppose that $Q_n - S$ is disconnected, every component of $Q_n - S$ is nontrivial, and there exists one component C of $Q_n - S$ such that $deg_C(v) \geq 2$ for every vertex v in C. In view of the example given in Fig. 6.1 and Lemma 24, we shall prove that either $|S|$ is sufficiently large or else $|V_C|$ is large as stated in the following lemma.

Lemma 27 *Let Q_n be the n-dimensional Hypercube and $n \geq 5$, and let S be a vertex set $S \subseteq V(Q_n)$. Suppose that $Q_n - S$ is disconnected and every component of $Q_n - S$ is nontrivial, and suppose that there exists one component C of $Q_n - S$ such that $deg_C(v) \geq 2$ for every vertex v in C. Then one of the following two conditions holds:*

(i) $|S| \geq 4(n-2)$, *or*

(ii) $|V_C| \geq 4(n-2) - 1$.

Proof: Since $deg_C(v) \geq 2$ for every vertex v in C, it is feasible to divide Q_n into two disjoint Q_{n-1}'s, denoted by Q^L_{n-1} and Q^R_{n-1}, such that $V(Q^L_{n-1} \bigcap C) \neq \emptyset$ and $V(Q^R_{n-1} \bigcap C) \neq \emptyset$. Let $C_L = Q^L_{n-1} \bigcap C$ and $C_R = Q^R_{n-1} \bigcap C$. For each vertex x in C_L (y in C_R, respectively), it has at most one neighbor in C_R (C_L, respectively). Hence, $deg_{C_L}(x) \geq 1$ and $deg_{C_R}(y) \geq 1$ for $x \in V_{C_L}$ and $y \in V_{C_R}$, respectively.

$Q_n - S$ is disconnected, there are at least two components in $Q_n - S$. Let $S_L = V_L \bigcap S$ and $S_R = V_R \bigcap S$. Note that both Q^L_{n-1} and Q^R_{n-1} contain some nonempty part of the component C. So at least one of the two subgraphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected. In the following proof, we investigate two cases: (1) one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is connected, (2) both $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected.

Case 1: One of $Q_{n-1}^L - S_L$ and $Q_{n-1}^R - S_R$ is connected, and the other is disconnected.

Without loss of generality, assume $Q_{n-1}^L - S_L$ is connected and $Q_{n-1}^R - S_R$ is disconnected. Let C' be another component of $Q_n - S$ other than C. Then $V_L = S_L \bigcup V_{C_L}$ and the component C' of $Q_n - S$ is in $Q_{n-1}^R - S_R - V_{C_R}$. Since C_R and C' are both nontrivial component, by Lemma 26, $|S_R| \geq 2(n-2)$. If $|S_L| \geq 2(n-2)$, then $|S| = |S_L| + |S_R| \geq 4(n-2)$ and condition (i) holds. Otherwise, $|S_L| \leq 2(n-2) - 1$. Then $|V_{C_L}| = 2^{n-1} - |S_L| \geq 2^{n-1} - 2(n-2) + 1$. That is, $|V_C| = |V_{C_L}| + |V_{C_R}| \geq (2^{n-1} - 2(n-2) + 1) + 2 = 2^{n-1} - 2(n-2) + 3 \geq 4(n-2) - 1$ for $n \geq 4$ and condition (ii) holds.

Case 2: Both $Q_{n-1}^L - S_L$ and $Q_{n-1}^R - S_R$ are disconnected.

By Lemma 26, we consider the following three subcases: (2.a) $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$, (2.b) $n-1 \leq |S_L| \leq 2(n-2) - 1$ and $n-1 \leq |S_R| \leq 2(n-2) - 1$, and (2.c) either $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2) - 1$; or, $|S_R| \geq 2(n-2)$, $n-1 \leq |S_L| \leq 2(n-2) - 1$.

Subcase 2.a: $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$.

Since $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$, $|S| = |S_L| + |S_R| \geq 4(n-2)$. Hence, condition (i) holds.

Subcase 2.b: $n-1 \leq |S_L| \leq 2(n-2) - 1$ and $n-1 \leq |S_R| \leq 2(n-2) - 1$.

In this subcase, $|V_{C_L}| = 2^{n-1} - |S_L| - 1$ and $|V_{C_R}| = 2^{n-1} - |S_R| - 1$. So $|V_C| = |V_{C_L}| + |V_{C_R}| = 2^n - |S| - 2$. Suppose $|S| \geq 4(n-2)$. Then condition (i) holds. Otherwise, $|S| \leq 4(n-2) - 1$. Then $|V_C| = 2^n - |S| - 2 \geq 2^n - (4(n-2) - 1) - 2 = 2^n - 4(n-2) - 1 \geq 4(n-2) - 1$ for $n \geq 4$. Hence, condition (ii) holds.

Subcase 2.c: Either $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2) - 1$; or, $|S_R| \geq 2(n-2)$, $n-1 \leq |S_L| \leq 2(n-2) - 1$.

Without loss of generality, assume that $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2) - 1$. Then $|V_{C_R}| = 2^{n-1} - |S_R| - 1 \geq 2^{n-1} - 2(n-2)$. Since $deg_{C_L}(x) \geq 1$ for each vertex

$x \in V_{C_L}$, we have $|V_{C_L}| \geq 2$. Thus, $|V_C| = |V_{C_L}| + |V_{C_R}| \geq 2 + (2^{n-1} - 2(n-2)) = 2^{n-1} - 2(n-2) + 2 \geq 4(n-2) - 1$ for $n \geq 5$.

This completes the proof of the Lemma. □

We are now ready to show the conditional diagnosability of Q_n is $4(n-2)+1$ for $n \geq 5$. Let $F_1, F_2 \subset V(Q_n)$ be an indistinguishable conditional-pair, $n \geq 5$. We shall show our result by proving that either $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$. Let $S = F_1 \bigcap F_2$. We consider two cases: (1) $Q_n - S$ is connected, and (2) $Q_n - S$ is disconnected.

Lemma 28 *Let Q_n be the n-dimensional Hypercube, $n \geq 5$. Let $F_1, F_2 \subset V(Q_n)$, $F_1 \neq F_2$, be an indistinguishable conditional-pair and $S = F_1 \bigcap F_2$. Then either $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$.*

Proof: Suppose that $Q_n - S$ is connected. Then $F_1 \triangle F_2 = V(Q_n - S)$ and $V(Q_n) = F_1 \bigcup F_2$. Suppose on the contrary that $|F_1| \leq 4(n-2)+1$ and $|F_2| \leq 4(n-2)+1$. Then $2^n = |F_1| + |F_2| - |F_1 \bigcap F_2| \leq (4(n-2)+1) + (4(n-2)+1) - 0 = 8(n-2)+2$. This contradicts the fact that $2^n > 8(n-2)+2$ for $n \geq 5$. Hence, the result holds as $Q_n - S$ is connected.

Now we consider the case that $Q_n - S$ is disconnected, by Lemma 24, $Q_n - S$ has a component C with $deg_C(v) \geq 2$ for every vertex $v \in V_C$. By Lemma 27, we have $|S| \geq 4(n-2)$ or $|V_C| \geq 4(n-2) - 1$.

Suppose $|S| \geq 4(n-2)$. Since $deg_C(v) \geq 2$ for every vertex v in C, and Q_n does not contain any cycle of length three, so $|V_C| \geq 4$. With the observation that $V_C \subset F_1 \triangle F_2$, we conclude that either $(F_1 - F_2) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2$ or $(F_2 - F_1) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2$. Therefore, either $|F_1| = |S| + |F_1 - F_2| \geq 4(n-2)+2$ or $|F_2| = |S| + |F_2 - F_1| \geq 4(n-2)+2$.

Otherwise, $|V_C| \geq 4(n-2) - 1$. Then either $(F_1 - F_2) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2(n-2)$ or $(F_2 - F_1) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2(n-2)$. Because there are at least two nontrivial components in $Q_n - S$, by Lemma 26, $|S| \geq 2(n-1)$. Hence, $|F_1| = |S| + |F_1 - F_2| \geq 4(n-2)+2$ or $|F_2| = |S| + |F_2 - F_1| \geq 4(n-2)+2$.

Therefore, for any indistinguishable conditional-pair $F_1, F_2 \subset V(Q_n)$, it implies that $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$. This proves the lemma. □

By Lemma 25, $t_c(Q_n) \leq 4(n-2)+1$, and by Lemmas 6 and 28, Q_n is conditionally $(4(n-2)+1)$-diagnosable for $n \geq 5$. Hence, $t_c(Q_n) = 4(n-2)+1$ for $n \geq 5$. For Q_3 and Q_4, we observe that Q_3 is not conditionally 4-diagnosable and Q_4 is not conditionally 8-diagnosable, as shown in Fig. 6.3.i and 6.3.ii. So $t_c(Q_3) \leq 3$ and $t_c(Q_4) \leq 7$. Hence, the conditional diagnosabilities of Q_3 and Q_4 are both strictly less than $4(n-2)+1$.

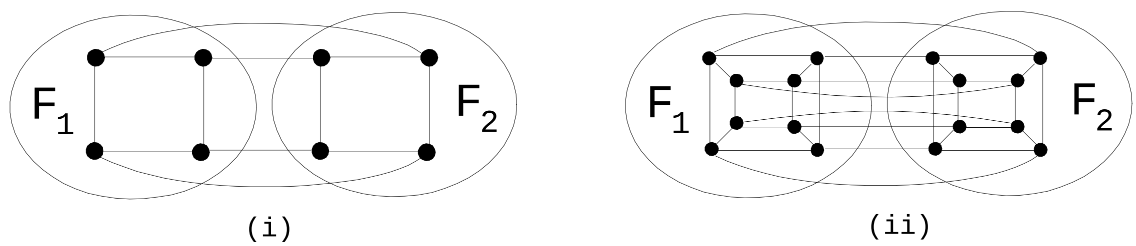

Figure 6.3: Illustration for two indistinguishable conditional-pairs for Q_3 and Q_4.

Q_3 is 3-diagnosable and it is not conditionally 4-diagnosable. It follows from Lemma 22 that $t_c(Q_3) = 3$. For Q_4, we prove that $t_c(Q_4) = 7$ in the following Lemma.

Lemma 29 $t_c(Q_4) = 7$.

Proof: We already know $t_c(Q_4) \leq 7$. Suppose on the contrary that Q_4 is not conditionally 7-diagnosable. Let $F_1, F_2 \subset V(Q_4)$ be an indistinguishable conditional-pair with $|F_i| \leq 7$, $i = 1, 2$, and let $S = F_1 \bigcap F_2$. It follows from Lemmas 24 and 26 that $|S| \geq 2(n-1) = 6$ for $n = 4$. Furthermore, $|F_1 - F_2| \geq 2$ and $|F_2 - F_1| \geq 2$. Then $|F_1| \geq 8$ and $|F_2| \geq 8$, which is a contradiction. So $t_c(Q_4) = 7$. □

Finally, the conditional diagnosability of Hypercube Q_n is stated as follows:

Theorem 23 *The conditional diagnosability of* Q_n *is* $t_c(Q_n) = 4(n-2)+1$ *for* $n \geq 5$, $t_c(Q_3) = 3$ *and* $t_c(Q_4) = 7$.

Chapter 7

Conclusion, discussion, and future work

In this thesis, we propose a sufficient theorem, Theorem 6, to verify the diagnosability of multiprocessor systems under the comparison-based model. The conditions of this theorem include all the cases of the original necessary and sufficient condition stated in Theorem 3. Therefore, it is more suitable for verifying the diagnosability of a system. Then we propose a family of interconnection networks which are recursively constructed, called the Matching Composition Networks.

Each member $G_1 \bigoplus_M G_2$ of this family are constructed from a pair G_1 and G_2 of lower dimensional networks with the same number of nodes, joining by a perfect matching M between the two. Applying Theorem 9 in this thesis, we show that the diagnosability of $G_1 \bigoplus_M G_2$ is one larger than those of the G_1 and G_2, provided some regular conditions, as stated in Theorem 9, are satisfied. Many well-known interconnection networks, such as the Hypercubes Q_n, the Crossed cubes CQ_n, the Twisted cubes TQ_n, and the Möbious cubes MQ_n, belong to our proposed family.

We note here that these special cases all satisfy the condition of Theorem 9 for $n \geq 4$. Thus, their diagnosabilities are n, for $n \geq 4$. In particular, the diagnosability of the 4-dimensional Hypercube Q_4 is 4. Also, the diagnosabilities of the Twisted cube TQ_n and the Möbious cubes MQ_n are first time proposed to be n for $n \geq 4$.

The diagnosability of the product networks under the comparison diagnosis model is also studied in thesis. We show that homogeneous product network G of G_1 and G_2 is $(t_1 + t_2)$-diagnosable, in which G_i is either t_i-diagnosable or t_i-connected with regularity t_i for $i = 1, 2$. Furthermore, we use different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks under the comparison diagnosis model. We prove that the heterogeneous product network G of G_1 and G_2 is $(t_1 + t_2)$-diagnosable, in which G_1 is t_1-diagnosable with regularity t_1, and G_2 is t_2-regular and t_2-connected with $2t_2 + 1$ nodes. We also show that the product network G is $(t_1 + t_2 + \ldots + t_k)$-diagnosable with at least two factor networks t_i-connected, where G is the product of G_1, G_2, $\ldots$, and G_k, each with regularity t_i, and each G_i is either t_i-diagnosable or t_i-connected for $1 \leq i \leq k$.

In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. As a consequence, the diagnosability of a system is upper bounded by its minimum degree. In probabilistic models of a multiprocessor system, processors fail independently but with different probabilities. In other words, the probability that all faulty processors are neighbors of one processor is very small.

In this thesis, we propose the concept of strongly t-diagnosable system and derive some conditions for verifying whether a system is strongly t-diagnosable. To grant more accurate measurement of diagnosability for large-scale processing system, we also introduce the conditional diagnosability of a system under PMC model. We consider the measure by restricting that for each processor v in the network, all the processors which are directly connected to v do not fail at the same time. Moreover, we show that the conditional diagnosability of Q_n is $4(n - 2) + 1$, which is about four times larger than the classical diagnosability.

Some ongoing research on diagnosis problems are described as follows. We are interested in exploring more generalized measures for better reflecting fault patterns in a real system than the existing ones. For example, how much more the diagnosability would increase if more neighbors are claimed to be non-faulty for every vertex. In practice, to design an efficient algorithm to identify the conditional faulty-set of a system would be useful. Also, it would be interesting to study the conditional diagnosability of a system under the comparison model.

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