# On Cycle Embeddings of Cube Families

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## Abstract

In this dissertation, we investigate fault-tolerant capabilities of the $k$-ary $n$-cubes with respect to the hamiltonian and hamiltonian connected properties. The $k$-ary $n$-cube is a bipartite graph if and only if $k$ is an even integer. Let $F$ be a faulty set with nodes and/or links, and let $k \geq 3$ be an odd integer. When $|F| \leq 2n-2$, we show that there exists a hamiltonian cycle in a wounded $k$-ary $n$-cube. In addition, when $|F| \leq 2n-3$, we prove that, for two arbitrary nodes, there exists a hamiltonian path connecting these two nodes in a wounded $k$-ary $n$-cube. Since the $k$-ary $n$-cube is regular of degree $2n$, the degrees of fault-tolerance $2n-3$ and $2n-2$ respectively, are optimal in the worst case.

The Möbius cube $MQ_n$, crossed cube $CQ_n$, and twisted cube $TQ_n$ are alternatives to the popular hypercube network. However, the diameters of these three interconnection networks are about one half of that of the hypercube. Recently, $MQ_n$ was shown to be pancyclic, i.e., cycles of any length at least four can be embedded into it. Due to the importance of the fault tolerance in the parallel processing area, in this dissertation, we study injured $MQ_n$, $CQ_n$, and $TQ_n$ with mixed node and link faults. We show that they are $(n-2)$-fault-tolerant pancyclic for $n \geq 3$, that is, injured $n$-dimensional $MQ_n$, $CQ_n$, and $TQ_n$ are still pancyclic with up to $(n-2)$ faults. Furthermore, our results are optimal in the sense that if there are $n-1$ faults, there is no guarantee of having a cycle of a certain length in them.

The hypercube $Q_n$ is one of the most popular networks. In this dissertation , we first prove that the $n$-dimensional hypercube is $2n-5$ conditional fault-bipancyclic. That is, an injured hypercube with up to $2n-5$ faulty links has a cycle of length $l$ for every even $4 \leq l \leq 2^n$ when each node of the hypercube is incident with at least two healthy links. In addition, if a certain node is incident with less than two healthy links, we show that an injured hypercube contains cycles of all even lengths except hamiltonian cycles with up to $2n-3$ faulty links. Furthermore, the above two results are optimal. In conclusion, we find cycles of all possible lengths in injured hypercubes with up to $2n-5$ faulty links under all possible fault distributions.

**Keywords:** cycle embedding, Möbius cube, twisted cube, crossed cube, hypercube, $k$-ary $n$-cube, pancyclic, hamiltonian, fault-tolerant.

# Contents

# List of Figures

# Chapter 1

# Introduction and Motivation

## 1.1 Hamiltonian Circuit and Linear Array Embeddings in Faulty $k$-Ary $n$-Cubes

In many parallel computer systems, processors are connected based on an interconnection network. Such networks usually have a regular degree, i.e., every node is incident with the same number of links. Popular instances of interconnection networks include hypercubes, star graphs, meshs, the $k$-ary $n$-cubes, etc.

The $k$-ary $n$-cube, denoted by $Q_n^k$, has a regular degree of $2n$, and is highly symmetric. As a result, it possesses lots of good properties. Some topological properties and hamiltonian cycle embeddings in healthy $k$-ary $n$-cubes were discussed in [6]. Also, various topological properties were studied by using Lee distance [10]. The conditional node connectivity problem of $Q_n^k$ was investigated in [16]. In [5], $n$ edge disjoint hamiltonian cycles were found in $Q_n^k$. Ashir et al. studied the problem of hamiltonian cycle embeddings in $Q_n^k$ with a possibility of link failures [4].

Hamiltonian circuit and linear array embeddings are desired properties in an interconnection network. Many works related to embeddings of longest cycles and paths in various interconnection networks have been studied previously, including hypercubes [11], [38], $k$-ary $n$-cubes [4], stars [25], [51], arrangement graphs [26], [41], etc.

Ashir et al. [4] showed that, with only edge faults and under the condition that every node is incident with at least two fault-free edges, a wounded $k$-ary $n$-cube still has a hamiltonian circuit, provided that faulty edges are no more than $4n-5$. The situation of having both faulty nodes and faulty links remains unanswered, and the hamiltonian linear array embeddings in $Q_n^k$ has not been discussed yet even in a healthy $Q_n^k$.

Since failures are inevitable, fault-tolerance is an important issue in multiprocessor systems. In this dissertation, we consider a possibility of both node and link failures, and discuss the fault-tolerant capabilities of the $k$-ary $n$-cubes with respect to the hamiltonian and hamiltonian connected properties. Let $F$ be a faulty set with nodes and/or links. We observe that $Q_n^k$ is bipartite if and only if $k$ is even. When $k$ is even and there is a faulty node, there exists neither a hamiltonian cycle nor a hamiltonian path between two vertices in different partite sets in a wounded $Q_n^k$. Therefore, throughout this dissertation, we suppose that $k$ is an odd integer with $k \geq 3$. Then, a ring of maximum length, or a hamiltonian cycle, in a wounded $Q_n^k$ can be constructed, provided that $|F| \leq 2n-2$ for $n \geq 2$. On the other hand, if $|F| \leq 2n-3$ for $n \geq 2$, we provide a construction of a linear array of maximum length, or a hamiltonian path, connecting two arbitrary vertices in a wounded $Q_n^k$. In both cases, we have achieved optimal solutions. The reason is as follows. First, any hamiltonian cycles cannot be found in a wounded $Q_n^k$ when there are 2n-1 faulty edges incident to a single node. Second, suppose that there are $2n-2$ edge faults incident to a node $x$. Let $y$ and $z$ be two nodes of $Q_n^k$ incident to $x$. Then, there is no hamiltonian path connecting $y$ and $z$ when all the edges incident to $x$ are faulty except $(x, y)$ and $(x, z)$.

## 1.2 Fault-Tolerant Pancyclicity of the Möbius Cubes, Crossed Cubes, and Twisted Cubes

The topology of an interconnection network is an important issue for parallel and distributed systems. Many topologies have been proposed in the last decades, for example, [7, 15, 18, 39, 45]. Among them, the hypercube is the most popular one, because of its simple and symmetric structure, recursive property, relatively low diameter, and easy routing. However, it was observed that by changing some links of the hypercube, we may obtain different cubes with improved properties; for example, the diameter can be reduced. Some variations of the hypercube have been studied in the literature [1, 12, 15, 19, 39].

The crossed cube $CQ_n$ introduced by Efe [19], the Möbius cube $MQ_n$ proposed by Cull et al. [15], and the twisted cube $TQ_n$ proposed by Hilbers et al. [22] are $n$-regular networks with $2^n$ nodes. Their diameters were shown to be only one half of that of the hypercube. And they are superior to the hypercube in several other properties. Thus they are alternatives to the hypercube network.

Many researches on the topological properties of interconnection networks have been done over the years. Among them, graph embedding of a guest graph into a host graph is useful for the applicability of the known algorithms that were created before [2, 36, 52]. An embedding of a guest graph $G$ into a host graph $H$ consists of two mappings. One is a mapping of each vertex of $G$ into a vertex of $H$, and the other is a mapping of each edge of $G$ into a path of $H$. Two important arguments in evaluating the quality of an embedding are *dilation* and *congestion*. The dilation is the maximum length of a path of $H$ which is mapped by an edge of $G$. The congestion is the maximum number of times that an edge of $H$ is mapped. The smaller the dilation and the congestion, the better the embedding. In particular, if the embedding has dilation 1 and congestion 1, $G$ is a subgraph of $H$. All of the embeddings in this dissertation have dilation 1 and congestion 1.

The ring structure embedding into various topologies have been heavily discussed [24, 44, 46, 47, 51]. However, the above researches all focused on the hamiltonian cycle embedding. In recent years, a lot of people studied the problem of the existence of cycles of arbitrary lengths in various interconnection networks [3, 14, 20, 21, 27, 31]. In particular, Fan [20] showed that the Möbius cube is *pancyclic*. Let $G$ be a simple undirected graph. We use $|V(G)|$ to denote the number of vertices of $G$. $G$ is pancyclic if any cycle of length $l$ can be embedded into it, where $4 \leq l \leq |V(G)|$. However, a pancyclic graph $G$ means that it has any cycle of length $l$, $3 \leq l \leq |V(G)|$ in the literature [8, 9, 42]. Unfortunately, it is easy to check that $CQ_n$, $MQ_n$, and $TQ_n$ have girth 4. Therefore, for ease of discussion, we adapted the previous definition, and we consider an extension of the result of Fan from the fault-tolerant point of view in this dissertation.

The fault tolerance is a crucial matter for parallel computing, especially when the network is large. Therefore, in this dissertation, we study the fault-tolerant capability of the crossed cube, the Möbius cube, and the twisted cube. Using the recursive structure of the three networks, we prove that all of them are $(n-2)$-fault-tolerant pancyclic for

$n \geq 3$. That is, we can embed cycles of any length at least four into injured $CQ_n$, $MQ_n$, and $TQ_n$ with up to $(n-2)$ faults, where faults can be mixed; both nodes and links may be faulty. Furthermore, this result is optimally fault-tolerant in the sense that if there are $(n-1)$ faults around a single node, then there does not exist a hamiltonian cycle in any of them.

## 1.3 Highly fault-tolerant cycle embeddings of hypercubes

Processors of a multiprocessor system are connected according to a given interconnection network. Many interconnection networks have been proposed with their superb merits demonstrated. Among them, the hypercube network is one of the most popular candidates when choosing an interconnection network. Newly proposed properties or measures with respect to interconnection networks are usually studied first on the hypercube because of its symmetric structure and popularity.

In order to speed up computations, a number of processors are grouped together to run a given parallel algorithm. A cycle is a preferred structure for a group of processors to carry out an algorithm because it is branch-free and has low degree. In addition, a ring structure can be used as a control or data flow structure for distributed computations. For more benefits and applications of cycles, refer to [17, 32, 48]. Many researchers have studied the existence of cycle structures in various interconnection networks, for example, [11, 17, 26, 32, 40, 51, 53].

Failures of interconnection network components are inevitable. Accordingly, various fault-tolerant measures have been proposed in the literature, including fault diameter [45], fault hamiltonicity [26], fault bipancyclicity [40], and fault hamiltonian laceability [49]. In particular, Tsai et al. [40] studied the fault-tolerant bipancyclic property on hypercube. A network or a graph $G$ is bipancyclic if $G$ has cycles of all even lengths ranging from 4 to the number of vertices of $G$. They found that an injured hypercube $Q_n$ with up to $n-2$ faulty links is bipancyclic. However, this measure underestimates the fault-tolerant capability of an interconnection network. Although there is no hamiltonian cycle in an injured hypercube if there are $n-1$ faulty links incident to a single node, this is the unique case.

In this paper, we study a kind of fault-tolerant measure, the conditional fault-tolerant bipancyclicity, on the hypercube. By restricting fault distributions, an injured hypercube is still bipancyclic with a large amount of faulty links. We show that an injured hypercube is bipancyclic with up to $2n-5$ faulty links under the condition that every node is incident with at least two healthy links. Normally, this condition makes sense since the probability that a certain node is incident with more than $n-2$ faulty links is low when $n$ is large. Some other conditional properties concerning with connectivity [37], diameter [43], and hamiltonian cycle embeddings [11] have been studied. Furthermore, as mentioned above, there is no hamiltonian cycle with $n-1$ faulty links in the worst case, so we are curious whether cycles of smaller even lengths exist? Our finding is that all of them do exist. When the condition is not satisfied, i.e., a certain node is incident with less than two healthy links, an injured hypercube has cycles of all even lengths except hamiltonian cycles with up to $2n-3$ faulty links. The above two results are optimal, and for details, refer to Chapter 6. Based on these results , we conclude that we can find cycles of all possible lengths in an injured hypercube with up to arbitrary $2n-5$ faulty links.

## 1.4 Basic Definitions and Notation

An interconnection network can be represented by an undirected simple graph $G$. Given a graph $G$, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A *path*, denoted by $\langle v_1, v_2, \ldots, v_k \rangle$, is defined as a sequence of vertices where two successive vertices are adjacent in $G$. A path is said to be a *hamiltonian path* if it traverses all the vertices of $G$ exactly once. A graph $G$ is *hamiltonian connected* if there is a hamiltonian path between any two arbitrarily chosen vertices of $G$. A *cycle* is a path that begins and ends with the same vertex. A *hamiltonian cycle* of $G$ is a cycle which walks through every vertex of $G$ exactly once. A *hamiltonian graph* is a graph that contains a hamiltonian cycle. In this dissertation, a *pancyclic graph* is a graph which contains, for each $l$, $4 \leq l \leq |V(G)|$, a cycle of length $l$.

In this dissertation, the notion of the fault tolerance is as follows. Let $V' \subseteq V(G)$. $G - V'$ is the subgraph of $G$ induced by $V - V'$. Let $F$ be a faulty set which may contain vertex faults and/or edge faults. We use $G - F$ to denote the subgraph of $G$ induced by $V - F$ after removing the edges in $F$. Let $k$ be a positive integer. A graph $G$ is *$k$-fault-tolerant hamiltonian connected* (abbreviated as *$k$-hamiltonian connected*) if $G - F$ is hamiltonian connected for any $F$ with $|F| \leq k$. A graph $G$ is *$k$-fault-tolerant hamiltonian* (abbreviated as *$k$-hamiltonian*) if $G - F$ is hamiltonian for any $F$ with $|F| \leq k$. Let $F_v = V(G) \bigcap F$. A graph $G$ is *$k$-fault-tolerant pancyclic* (abbreviated as *$k$-pancyclic*) if $G - F$ is pancyclic for any $F$ with $|F| \leq k$, i.e., $G - F$ contains every cycle of length $l$, where $4 \leq l \leq |V(G)| - |F_v|$.

## 1.5 Outline of this dissertation

The rest of this dissertation is organized as follows: We show that the $k$-ary $n$-cube is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected for every odd $k \geq 3$ in Chapter 2. Then, we discuss the fault-tolerant pancyclicity of $CQ_n$, $MQ_n$, and $TQ_n$ in Chapter 3, 4, and 5, respectively. After that, we investigate the highly fault-tolerant cycle embeddings of hypercubes in Chapter 6. Finally, the conclusion is given in Chapter 7.

# Chapter 2

# Hamiltonian Circuit and Linear Array Embeddings in Faulty $k$-Ary $n$-Cubes

## 2.1 Preliminaries

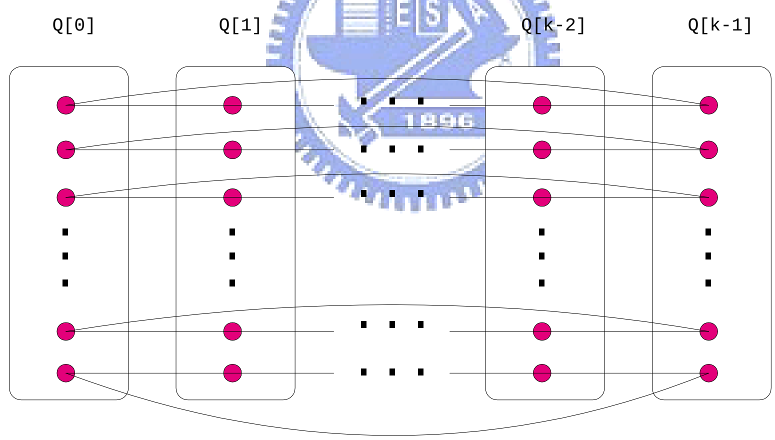
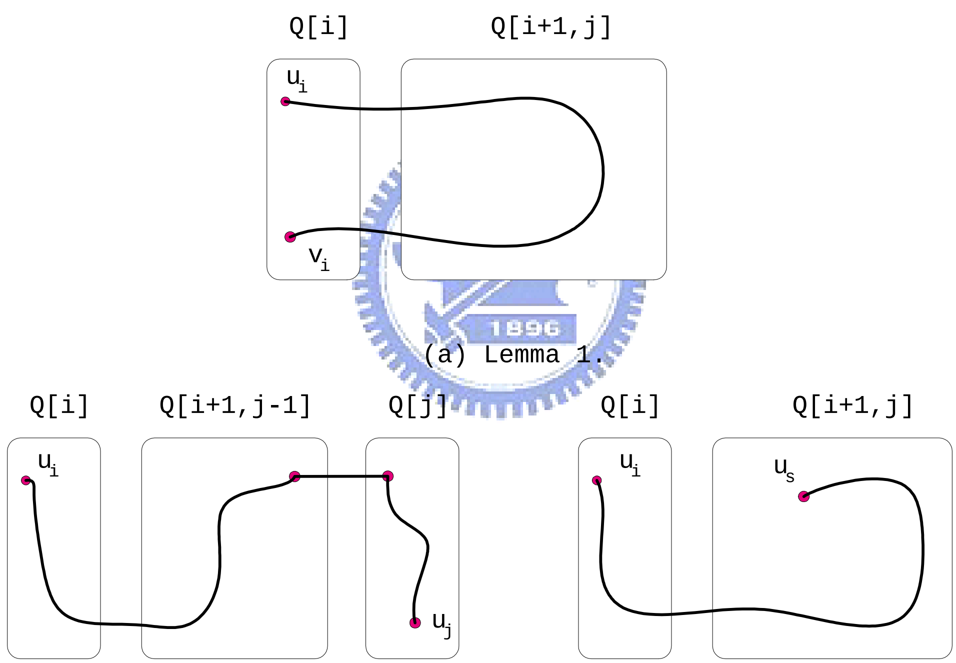



Figure 2.1: $Q_n^k$ is divided into $Q[0]$, $Q[1]$, ..., $Q[k-1]$.

The $k$-ary $n$-cube $Q_n^k$ is a graph consisting of $k^n$ vertices labeled by the integers from 0 to $k^n - 1$. Two vertices are adjacent if and only if the representations of their labels in base $k$ differ by one (modulo k) in exactly one position. We refer to $(x, y) \in E(Q_n^k)$

where $x$ differs from $y$ in the $d$th position, for $0 \le d \le n-1$, as an *edge of dimension* $d$. We say that $Q_n^k$ is divided into $Q_n^k[0]$, $Q_n^k[1]$, …, $Q_n^k[k-1]$ (abbreviated as $Q[0]$, $Q[1]$, …, $Q[k-1]$, if there are no ambiguities) along dimension $d$ for some $0 \le d \le n-1$ if $Q[l]$, for every $0 \le l \le k-1$, is a subgraph of $Q_n^k$ induced by the vertices labeled by $x_{n-1} \dots x_{d+1} l x_{d-1} \dots x_0$ (see Figure 2.1). It is clear that each $Q[l]$ is isomorphic to $Q_{n-1}^k$ for $0 \le l \le k-1$. Note that $Q_n^k$ can be divided into $k$ copies of $Q_{n-1}^k$ along $n$ different dimensions. For $0 \le i, j \le k-1$, we use $[i,j]$ to denote a set of integers: $[i,j] = \{l \mid i \le l \le j\}$ if $i \le j$, and $[i,j] = \{l \mid i \le l \le k-1 \textit{ or } 0 \le l \le j\}$ if $i > j$. $Q_n^k[i,j]$ (abbreviated as $Q[i,j]$ if there is no ambiguity) denotes the subgraph of $Q_n^k$ which is induced by $\{u \mid u \in V(Q[l]);\, l \in [i,j]\}$.

## 2.2 Fundamental tools




(b) Lemma 2.

(c) Lemma 3.

Figure 2.2: Hamiltonian paths in faulty $Q[i,j]$.

Let $k$ be an odd integer with $k \ge 3$, and let $n$ be an integer. Let $F \subseteq V(Q_n^k) \bigcup E(Q_n^k)$

be the set of faulty vertices and/or edges in $Q_n^k$, and let $F^l = F \bigcap (V(Q[l]) \bigcup E(Q[l]))$ for every $0 \leq l \leq k-1$. We refer to an edge $(x, y) \in E(Q_n^k)$ where all of $x$, $y$, and $(x, y)$ are fault-free, as a *safe crossing-edge*.

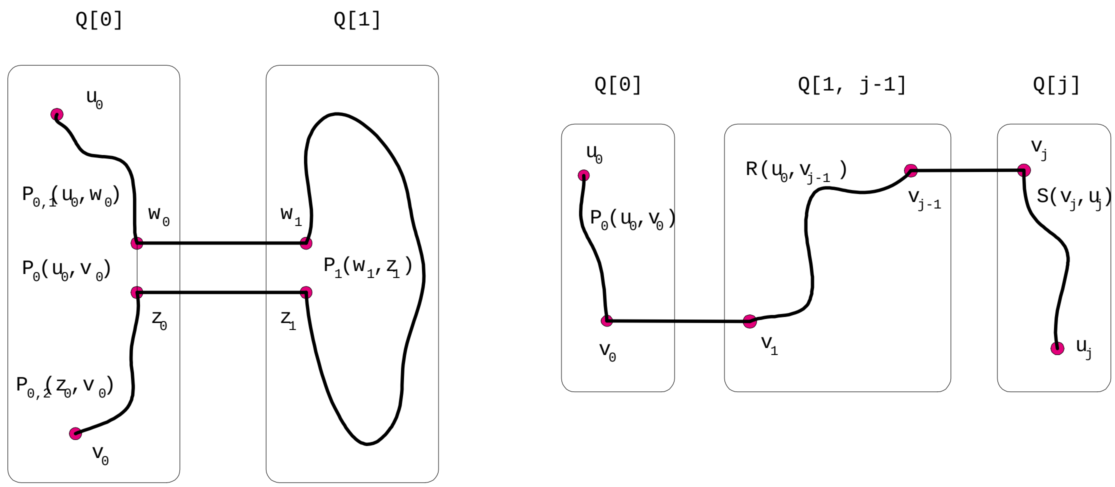
In the following lemmas, namely Lemma 1, Lemma 2, and Lemma 3, we shall construct hamiltonian paths in faulty $Q[i, j]$ for every $i, j \in [0, k-1]$ when each faulty $Q[l]$ is hamiltonian connected for $l \in [i, j]$. These preliminaries would be useful for further discussions.

As a first step, we shall construct a hamiltonian path between two arbitrary vertices belonging to $Q[i]$ in a faulty $Q[i, j]$ (see Figure 2.2 (a)).

**Lemma 1** *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i, j]) \bigcup E(Q[i, j])$ be a faulty set with $|F| \leq 2n-3$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting every two vertices $u_i$ and $v_i \in V(Q[i] - F^i)$ in $Q[i, j] - F$ for every $n \geq 3$ and odd $k \geq 3$.*

**Proof.** If $i = j$, this lemma holds. So we suppose that $i \neq j$. We may assume without loss of generality that $i = 0$ in the following discussion. Since $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, j]$, there is a hamiltonian path, say $P_0(u_0, v_0)$ ($u_0 = u_i$ and $v_0 = v_i$), in $Q[0] - F^0$ (see Figure 2.3 (a)). The length of $P_0(u_0, v_0) = |V(Q[0] - F^0)| - 1$ $\geq k^{n-1} - |F^0| - 1$, and the number of faults outside $Q[0]$ is at most $(2n-3) - |F^0|$. When $n$ and $k \geq 3$, $\frac{k^{n-1} - |F^0| - 1}{2} \geq \frac{3^{n-1} - |F^0| - 1}{2} > (2n-3) - |F^0|$. Hence, we can find two consecutive vertices, say $w_0$ and $z_0$, on $P_0(u_0, v_0)$ such that $(w_0, w_1)$ and $(z_0, z_1)$ are safe crossing-edges where $w_1$ and $z_1$ are the neighbors of $w_0$ and $z_0$ in $Q[1]$ respectively. Let $\langle u_0, P_{0,1}(u_0, w_0), w_0, z_0, P_{0,2}(z_0, v_0), v_0 \rangle = P_0(u_0, v_0)$ , and let $P_1(w_1, z_1)$ be a hamiltonian path in $Q[1] - F^1$. $\langle u_0, P_{0,1}(u_0, w_0),\ w_0, w_1,\ P_1(w_1, z_1),\ z_1, z_0,\ P_{0,2}(z_0, v_0),\ v_0 \rangle$ forms a hamiltonian path in $Q[0, 1] - F$. Repeating the above construction, we have a hamiltonian path in $Q[0, j] - F$. $\square$

In the following lemma, we shall construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^i)$ and $u_j \in V(Q[j] - F^j)$ in a faulty $Q[i, j]$ (see Figure 2.2 (b)). Note that $Q[i, j]$ can tolerate $2n-2$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian cycle embeddings. In addition, we want all the vertices in $Q[j] - F^j$ to form a subpath on this hamiltonian path for proving Lemma 3.




(a) The proof of Lemma 1.

(b) The proof of Lemma 2.

Figure 2.3: The proofs of Lemma 1 and Lemma 2.

**Lemma 2** *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i,j]) \bigcup E(Q[i,j])$ be a faulty set with $|F| \leq 2n-2$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i,j]$, there exists a hamiltonian path connecting two arbitrary vertices $u_i \in V(Q[i]-F^i)$ and $u_j \in V(Q[j]-F^j)$ in $Q[i,j] - F$ such that all the vertices in $Q[j] - F^j$ form a subpath on this hamiltonian path for every $n \geq 3$ and odd $k \geq 3$.*

**Proof.** If $i = j$, the statement follows. Hence, we suppose that $i \neq j$. Without loss of generality, we may assume that $i = 0$ (see Figure 2.3 (b)). Note that $|F| = (2n-2)$ and $|V(Q[0])| = k^{n-1}$. Since $k^{n-1} - (2n-2) \geq 9 - 4 = 5$ for every $n \geq 3$ and odd $k \geq 3$, there exists a safe crossing-edge, say $(v_0, v_1)$, where $v_0 \neq u_0$, $v_0 \in V(Q[0] - F^0)$, $v_1 \neq u_j$, and $v_1 \in V(Q[1] - F^1)$. By assumption, $Q[l] - F^l$ is hamiltonian connected for every $l \in [0, j]$, so we have a hamiltonian path, say $P_0(u_0, v_0)$, in $Q[0] - F^0$. Continuing this process, we can join all hamiltonian paths in $Q[l] - F^l$, for all $l \in [0, j-1]$, to form a hamiltonian path, namely $R(u_0, v_{j-1})$, in $Q[0, j-1] - F$ such that $(v_{j-1}, v_j)$ is a safe crossing-edge where $v_{j-1} \neq u_0$, $v_{j-1} \in V(Q(j-1) - F^{j-1})$, $v_j \neq u_j$, and $v_j \in V(Q[j] - F^j)$. Let $S(v_j, u_j)$ be a hamiltonian path in $Q[j]$. $\langle u_0, R(u_0, v_{j-1}), v_{j-1}, v_j, S(v_j, u_j), u_j \rangle$ is a hamiltonian path in $Q[0, j] - F$, and $S(v_j, u_j)$ contains all vertices in $Q[j] - F^j$. $\square$

In the following lemma, we construct a hamiltonian path between two arbitrary vertices $u_i \in V(Q[i] - F^i)$ and $u_s \in V(Q[s] - F^s)$ with $s \in [i, j]$ in a faulty $Q[i, j]$ (see Figure 2.2 (c)). Note that $Q[i, j]$ can tolerate $2n - 3$ faults in this lemma, which is the maximum degree of the fault-tolerance of hamiltonian path embeddings.

**Lemma 3** *Let $i, j \in [0, k-1]$, and let $F \subseteq V(Q[i,j]) \bigcup E(Q[i,j])$ be a faulty set with $|F| \leq 2n - 3$. If $Q[l] - F^l$ is hamiltonian connected for every $l \in [i, j]$, there exists a hamiltonian path connecting every two vertices $u_i \in V(Q[i] - F^i)$ and $u_s \in V(Q[s] - F^s)$ in $Q[i, j] - F$ with $s \in [i, j]$ for every $n \geq 3$ and odd $k \geq 3$.*

**Proof.** If $i = j$, the statement is true. Therefore, we assume that $i \neq j$. By Lemma 2, there exists a hamiltonian path, say $R(u_i, u_s)$, in $Q[i, s] - F$ such that all the vertices in $Q[s] - F^s$ form a subpath on $R(u_i, u_s)$. Using the counting argument in the proof of Lemma 1, we can find two consecutive vertices, say $u_s$ and $v_s \in V(Q[s])$, on $R(u_i, u_s)$ such that $(u_s, u_{s+1})$ and $(v_s, v_{s+1})$ are safe crossing-edges where $u_{s+1}$ and $v_{s+1} \in V(Q[s+1])$. By Lemma 1, there is a hamiltonian path, namely $S(u_{s+1}, v_{s+1})$, in $Q[s+1, j] - F$. Let $\langle u_i, R_1(u_i, u_s), u_s, v_s, R_2(v_s, v_i), v_i\rangle = R(u_i, u_s)$. Then, $\langle u_i, R_1(u_i, u_s), u_s, u_{s+1}, S(u_{s+1}, v_{s+1}), v_{s+1}, v_s, R_2(v_s, v_i), v_i\rangle$ forms a hamiltonian path in $Q[j] - F$. $\square$

## 2.3 Fault-tolerant hamiltonicity and hamiltonian connectivity

The $m \times n$ torus is a graph of $mn$ vertices labeled as $ab$ where $a$ and $b$ are integers with $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$. Two vertices $ab$ and $cd$ are adjacent if and only if either $a = c$ and $b = d \pm 1_{(mod\ n)}$ or $b = d$ and $a = c \pm 1_{(mod\ m)}$. Therefore, $Q_2^k$ is a $k \times k$ torus for every $k \geq 3$ by the definition. The following theorem related to the fault-tolerant hamiltonicity of the $m \times n$ torus is proved in [33].

**Theorem 1** *[33] If $m \geq 3$, $n \geq 3$, and $n$ is odd, $m \times n$ torus is 2-hamiltonian and 1-hamiltonian connected.*

The following corollary immediately follows by Theorem 1.

**Corollary 1** *If $k$ is odd with $k \geq 3$, $Q_2^k$ is 2-hamiltonian and 1-hamiltonian connected.*

Using the fault-tolerant hamiltonian and hamiltonian connected properties of $Q^k_{n-1}$, we shall show the fault-tolerant hamiltonian property of $Q^k_n$.

**Theorem 2** *Let $k$ be an odd integer with $k \geq 3$. If $Q^k_{n-1}$ is $(2n-4)$-hamiltonian and $(2n-5)$-hamiltonian connected for some $n \geq 3$, $Q^k_n$ is $(2n-2)$-hamiltonian.*

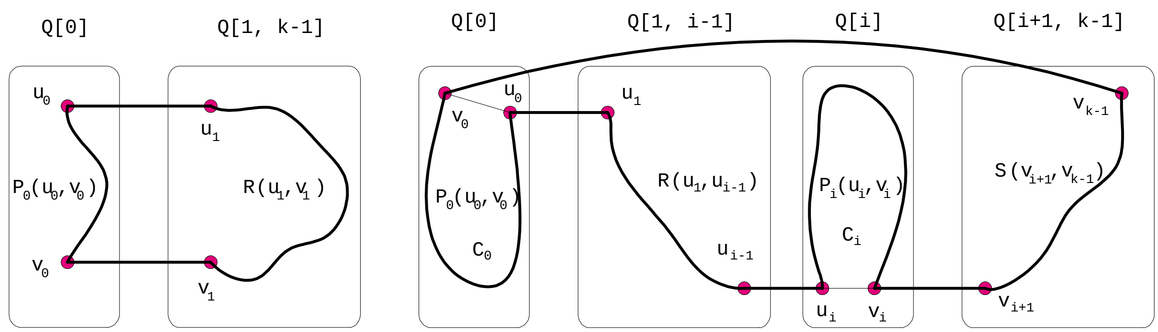
**Proof.** We claim that we can divide $Q^k_n$ into $Q[0]$, $Q[1]$, $\ldots$, $Q[k-1]$ along some dimension such that $|F^l| \leq 2n-3$ for every $0 \leq l \leq k-1$. If $|F| \leq 2n-3$, it is done. So we assume that $|F| = 2n-2$. Then, if there is a faulty edge, we can divide $Q^k_n$ along the dimension of this faulty edge. On the other hand, suppose that $F \subseteq V(Q^k_n)$. Since $|F| \geq 4$ for every $n \geq 3$, picking arbitrarily two faulty vertices in $Q^k_n$, we can divide $Q^k_n$ along some dimension such that these two faulty vertices are in different $Q^k_{n-1}$'s. Hence, the claim follows. Furthermore, without loss of generality, we may assume that $|F^0| \geq |F^l|$ for every $l \in [0, k-1]$. We discuss the existence of a hamiltonian cycle in the following three cases.

**Case 1.** $|F^0| = 2n-3$ (see Figure 2.4 (a)).

By assumption, $Q^k_{n-1}$ is $(2n-4)$-hamiltonian. Therefore, there is a hamiltonian path, namely $P_0(u_0, v_0)$, in $Q[0] - F^0$. Let $u_1$ and $v_1$ be the neighbors of $u_0$ and $v_0$ in $Q[1]$ respectively, and let $u_{k-1}$ and $v_{k-1}$ be the neighbors of $u_0$ and $v_0$ in $Q[k-1]$ respectively. Since there is at most one fault outside $Q[0]$, either the two edges $(u_0, u_1)$ and $(v_0, v_1)$ are safe crossing-edges or the two edges $(u_0, u_{k-1})$ and $(v_0, v_{k-1})$ are safe crossing-edges. Without loss of generality, we may assume that $(u_0, u_1)$ and $(v_0, v_1)$ are safe crossing-edges. By assumption, $Q^k_{n-1}$ is $(2n-5)$-hamiltonian connected and $2n-5 \geq 1$ for $n \geq 3$, so $Q[l] - F^l$ is hamiltonian connected for every $l \in [1, k-1]$ and $n \geq 3$. Since $1 < 2n-3$ for $n \geq 3$, by Lemma 3, there is a hamiltonian path, namely $R(u_1, v_1)$, in $Q[1, k-1] - F$. Therefore, $\langle u_0,\ P_0(u_0, v_0),\ v_0, v_1,\ R(v_1, u_1),\ u_1, u_0 \rangle$ forms a hamiltonian cycle in $Q^k_n - F$.

**Case 2.** $|F^0| = 2n-4$.

By assumption, $Q^k_{n-1}$ is $(2n-4)$-hamiltonian. Therefore, there is a hamiltonian cycle, say $C_0$, in $Q[0] - F^0$. Since there are at most two faults outside $Q[0]$, we can find two consecutive vertices, namely $u_0$ and $v_0$, on $C_0$ for $n \geq 3$ such that $(u_0, u_1)$ and $(v_0, v_1)$ are safe crossing-edges, where $u_1$ and $v_1$ are the neighbors of $u_0$ and $v_0$ in $Q[1]$ respectively. Note that $Q[l] - F^l$ is hamiltonian-connected for every $l \in [1, k-1]$ and $n \geq 4$. In this




(a) Case 1.

(b) Case 2 (when Q[i]-F is not hamiltonian connected).

Figure 2.4: Cases of Theorem 2.

situation, the proof is similar to Case 1.

When $n = 3$, it is possible that in addition to $Q[0]$, there exists another copy of $Q^k_{n-1}$, say $Q[i]$, which contains two faults. Hence, both of $Q[0] - F^0$ and $Q[i] - F^i$ are not hamiltonian connected, but both are hamiltonian. There is a hamiltonian cycle, say $C_i$, in $Q[i]$ (see Figure 2.4 (b)). Note that there is no fault outside $Q[0]$ and $Q[i]$, and $Q[l] - F^l$ is hamiltonian connected for every $l \notin \{0, i\}$. We may assume without loss of generality that $i \neq k - 1$. We can find a safe crossing-edge, say $(u_{i-1}, u_i)$, where $u_{i-1} \in Q[i-1]$ and $u_i \in Q[i]$. By Lemma 3, there is a hamiltonian path, namely $R(u_1, u_{i-1})$, in $Q[1, i-1]$. Let $v_{k-1} \in V(Q[k-1])$ be a neighbor of $v_1$. Let $v_i$ be adjacent to $u_i$ on $C_i$ such that $v_{i+1}$, the neighbor of $v_i$ in $Q[i+1]$, $\neq v_{k-1}$. By Lemma 3, there exists a hamiltonian path, namely $S(v_{i+1}, v_{k-1})$, in $Q[i+1, k-1]$. Furthermore, let $\langle u_0, P_0(u_0, v_0), v_0\rangle = C_0$ and $\langle u_i, P_i(u_i, v_i), v_i\rangle = C_i$. Then, $\langle u_0, u_1, R(u_1, u_{i-1}), u_{i-1}, u_i, P_i(u_i, v_i), v_i, v_{i+1}, S(v_{i+1}, v_{k-1}), v_{k-1}, v_0, P_0(v_0, u_0), u_0\rangle$ is a hamiltonian cycle in $Q^k_3 - F$.

**Case 3.** $|F^0| \leq 2n - 5$.

Since $k^{n-1} > 2n-2$ for $k \geq 3$ and $n \geq 3$, we can find a safe crossing-edge, say $(u_0, u_{k-1})$, where $u_0 \in Q[1]$ and $u_{k-1} \in Q[k-1]$. $|F^0| \leq 2n-5$, and, by assumption, $Q^k_{n-1}$ is $(2n-5)$-hamiltonian connected. Therefore, $Q[l] - F^l$ is hamiltonian connected for every $0 \leq l \leq k-1$. By Lemma 2, there is a hamiltonian path, namely $P(u_0, u_{k-1})$, in $Q[0, k-1]$. Therefore, $\langle u_0, P(u_0, u_{k-1}), u_{k-1}, u_0 \rangle$ is a hamiltonian cycle in $Q^k_n - F$. $\square$

Using the fault-tolerant hamiltonian and hamiltonian connected properties of $Q^k_{n-1}$ again, we shall prove the fault-tolerant hamiltonian connected property of $Q^k_n$ as follows.

**Theorem 3** *Let $k$ be an odd integer with $k \geq 3$. If $Q^k_{n-1}$ is $(2n-4)$-hamiltonian and $(2n-5)$-hamiltonian connected for some $n \geq 3$, $Q^k_n$ is $(2n-3)$-hamiltonian connected.*

**Proof.** We want to prove that there exists a hamiltonian path connecting every two vertices $x$ and $y$ in $Q^k_n - F$ for every $F$ with $|F| \leq 2n-3$. Since $x \neq y$, we can divide $Q^k_n$ into $Q[0]$, $Q[1]$ ..., $Q[k-1]$ along some dimension such that $x$ and $y$ are in different $Q^k_{n-1}$'s. Furthermore, without loss of generality, we may assume that $|F^0| \geq |F^l|$ for every $0 \leq l \leq k-1$. We discuss the existence of a hamiltonian path connecting $x$ and $y$ in the following three cases.

**Case 1.** $|F^0| = 2n-3$.

By assumption, $Q^k_{n-1}$ is $(2n-4)$-hamiltonian. Hence, there is a hamiltonian path, namely $P_0(u_0, v_0)$, in $Q[0] - F^0$. Note that there is no fault outside $Q[0]$. So $Q[l]$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases Case 1.1 and Case 1.2 as follows.

**Case 1.1.** $x \in V(Q[0] - F^0)$ and $y \in V(Q[i] - F^i)$ where $i \neq 0$ (see Figure 2.5 (a)).

We may assume that the distance from $x$ to $u_0$ is at least as far as the distance from $x$ to $v_0$ on $P_0(u_0, v_0)$. Let $\langle u_0, P_{0,1}(u_0, w_0), w_0, x, P_{0,2}(x, v_0), v_0 \rangle = P_0(u_0, v_0)$. $|V(P_0(u_0, v_0))| \geq k^{n-1} - (2n-3) \geq 3^2 - 3 = 6$ for $k$ and $n \geq 3$, so $w_0 \neq u_0$ and $w_0 \neq x$. Since either $u_0$ or $w_0$ is not a neighbor of $y$, we may assume without loss of generality that $u_0$ is not a neighbor of $y$. Furthermore, without loss of generality, we may assume that $i \neq 1$. Then, let $v_1$ and $w_1$ be the neighbors of $v_0$ and $w_0$ in $Q[1]$ respectively. By Lemma 3, there is a hamiltonian path $R(v_1, w_1)$ in $Q[1, i-1]$. Furthermore, let $u_{k-1}$ be the neighbor of

$u_0$ in $Q[k-1]$. By Lemma 3, there exists a hamiltonian path $S(u_{k-1}, y)$ in $Q[i, k-1]$. Then, $\langle x, P_{0,2}(x, v_0), v_0, v_1, R(v_1, w_1), w_1, w_0, P_{0,1}(w_0, u_0), u_0, u_{k-1}, S(u_{k-1}, y), y\rangle$ forms a hamiltonian path in $Q_n^k - F$.

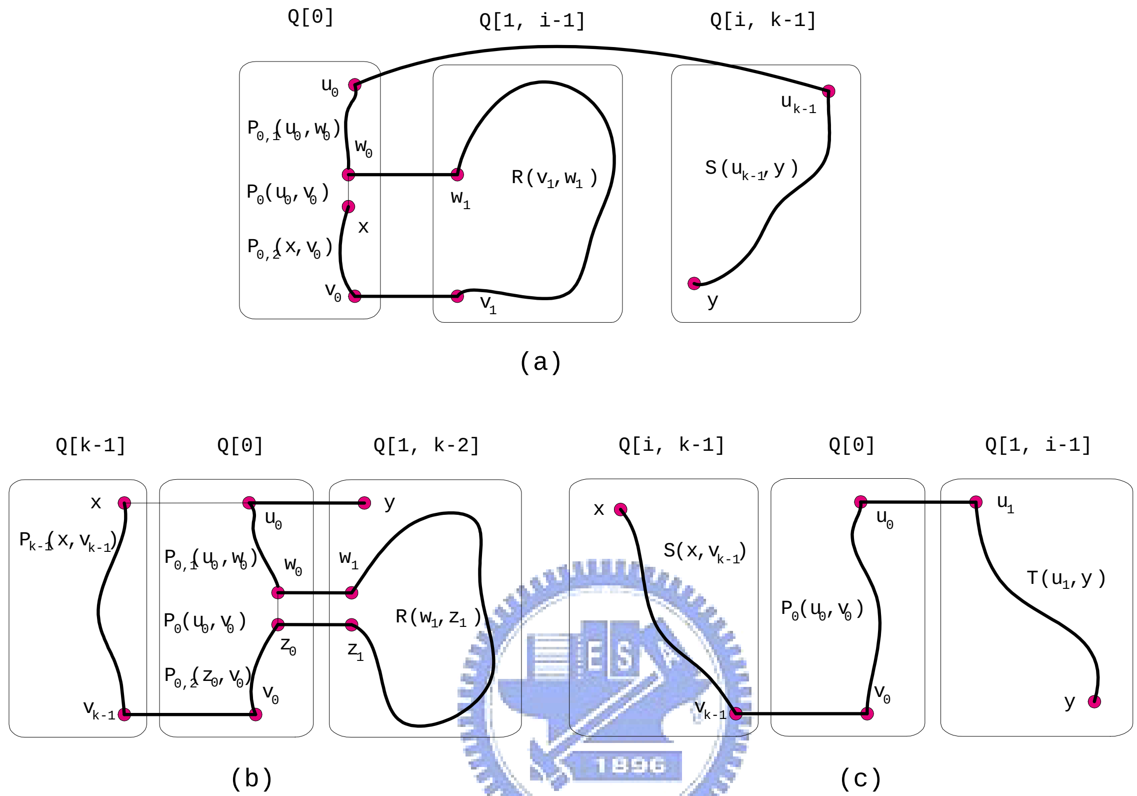



Figure 2.5: Case 1 of Theorem 3.

**Case 1.2.** $x \in V(Q[i] - F^i)$ and $y \in V(Q[j] - F^j)$ where $i, j \neq 0$.

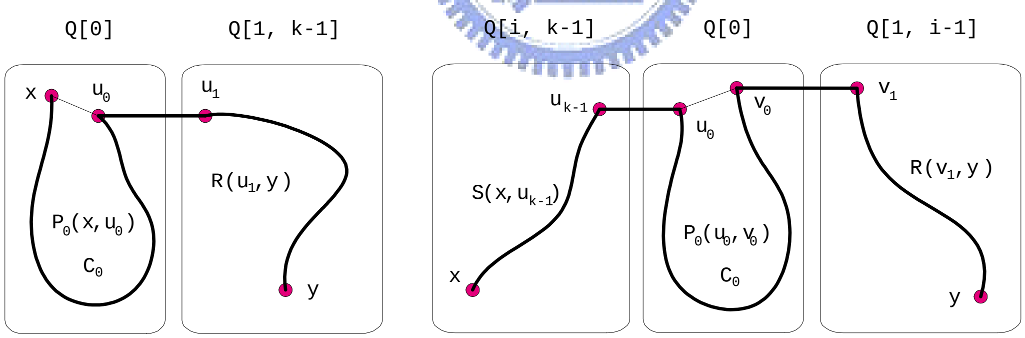
We may assume that $i > j$. Suppose that both of $x$ and $y$ are neighbors of $u_0$(or $v_0$). Then, $x \in Q[k-1]$ and $y \in Q[1]$ (see Figure 2.5 (b)). Let $v_{k-1}$ be the neighbor of $v_0$ in $Q[k-1]$. There exists a hamiltonian path, say $P_{k-1}(x, v_{k-1})$, in $Q[k-1]$. Let $w_0$ and $z_0$ be two consecutive vertices on $P_0(u_0, v_0)$. Also, let $w_1$ and $z_1$ be the neighbors of $w_0$ and $z_0$ in $Q[1]$ respectively. By Lemma 3, there is a hamiltonian path, namely $R(w_1, z_1)$, in $Q[1, k-2] - y$. Let $\langle u_0,\ P_{0,1}(u_0, w_0),\ w_0, z_0,\ P_{0,2}(z_0, v_0),\ v_0\rangle = P_0(u_0, v_0)$. $\langle y, u_0, P_{0,1}(u_0, w_0), w_0, w_1, R(w_1, z_1), z_1, z_0, P_{0,2}(z_0, v_0), v_0, v_{k-1}, P_{k-1}(v_{k-1}, x), x\rangle$ is a hamiltonian path connecting $x$ and $y$ in $Q_n^k - F$. Otherwise, let $u_1 \in Q[1]$ and $v_{k-1} \in Q[k-1]$ be neighbors of $u_0$ and $v_0$ respectively (see Figure 2.5 (c)). We may assume without loss of generality that $u_1 \neq y$ and $v_{k-1} \neq x$. By Lemma 3, there exist hamiltonian paths, say $S(x, v_{k-1})$ and $T(u_1, y)$, in $Q[i, k-1]$ and $Q[1, i-1]$ respectively. As a result, $\langle x,$ $S(x, v_{k-1}), v_{k-1}, v_0, P_0(v_0, u_0), u_0, u_1, T(u_1, y), y\rangle$ is a hamiltonian path connecting $x$ and $y$ in $Q_n^k - F$.

**Case 2.** $|F^0| = 2n - 4$.

By assumption, $Q[0]$ is $(2n-4)$-hamiltonian. So there is a hamiltonian cycle, namely $C_0$, in $Q[0] - F^0$. Note that there is at most one fault outside $Q[0]$. Therefore, $Q[l] - F^l$ is hamiltonian connected for every $l \in [1, k-1]$. We divide this case further into two subcases Case 2.1 and Case 2.2 as follows.

**Case 2.1.** $x \in V(Q[0] - F^0)$ and $y \in V(Q[i] - F^i)$ where $i \neq 0$ (see Figure 2.6 (a)).

Let $u_0 \in V(C_0)$ be adjacent to $x$ on $C_0$ such that $u_0$ is not a neighbor of $y$. Let $u_1 \in V(Q[1] - F^1)$ be a neighbor of $u_0$. Since there is at most one fault outside $Q[0]$, we may assume without loss of generality that $(u_0, u_1)$ is a safe crossing-edge. By Lemma 3, there is a hamiltonian path, namely $R(u_1, y)$, in $Q[1, k-1] - F$. Let $\langle x, P_0(x, u_0), u_0, x\rangle$ $= C_0$. $\langle x, P_0(x, u_0), u_0, u_1, R(u_1, y), y\rangle$ forms a hamiltonian path connecting $x$ and $y$ in $Q_n^k - F$.




(a) Case 2.1. (b) Case 2.2.

Figure 2.6: Case 2 of Theorem 3.

**Case 2.2.** $x \in V(Q[i] - F^i)$ and $y \in V(Q[j] - F^j)$ where $i, j \neq 0$ (see Figure 2.6 (b)).

We may assume that $i > j$. Since there is at most one fault outside $Q[0]$, we can choose two adjacent vertices, say $u_0$ and $v_0$, on $C_0$ such that $(u_0, u_{k-1})$ and $(v_0, v_1)$ are safe crossing-edges, $u_{k-1} \neq x$, and $v_1 \neq y$ where $u_{k-1} \in Q[k-1]$ and $v_1 \in Q[1]$ are neighbors of $u_0$ and $v_0$ respectively. By Lemma 3, there exists a hamiltonian path, namely $R(v_1, y)$, in $Q[1, i-1]-F$. Also, there is a hamiltonian path, namely $S(x, u_{k-1})$, in $Q[i, k-1]-F$. Let $\langle u_0, P_0(u_0, v_0), v_0, u_0\rangle = C_0$. Then, $\langle x, S(x, u_{k-1}), u_{k-1}, u_0, P_0(u_0, v_0), v_0, v_1, R(v_1, y), y\rangle$ is a hamiltonian path in $Q_n^k - F$.

**Case 3.** $|F^0| \leq 2n-5$.

As a result, $Q[l]-F^l$ is hamiltonian connected for every $l \in [0, k-1]$. We may assume without loss of generality that $x \in V(Q[0] - F^0)$. Since $|F| \leq 2n-3$, by Lemma 3, there is a hamiltonian path connecting $x$ and $y$ in $Q[0, k-1] - F$. Hence there exists a hamiltonian path connecting $x$ and $y$ in $Q_n^k - F$. □

In conclusion, the fault-tolerant hamiltonicity of $Q_n^k$ is given in the following theorem.

**Theorem 4** *If $k$ is odd with $k \geq 3$ and $n \geq 2$, $Q_n^k$ is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected.*

**Proof.** By Corollary 1, Theorem 2, Theorem 3, and a simple mathematical induction, this theorem is proved. □

# Chapter 3

# Fault-Ttolerant Cycle-Embedding of Crossed Cubes

## 3.1 Crossed Cubes

Given a simple graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex and edge sets of $G$, respectively. In order to define the crossed cube $CQ_n$, as proposed by Efe [19], the pair related set $R$ is introduced. Let $R = \{(00, 00), (10, 10), (11, 01), (01, 11)\}$. Two binary strings $a_1a_2$ and $b_1b_2$ of length 2 are pair related, denoted by $a_1a_2 \sim b_1b_2$, if $(a_1a_2, b_1b_2) \in R$. The following is the recursive definition of the $n$-dimensional crossed cube $CQ_n$. $CQ_n$ has $2^n$ vertices, each labeled by a binary string of length $n$. $CQ_1$ is a complete graph with two vertices labeled 0 and 1, respectively. For $n \geq 2$, $CQ_n$ is obtained by taking two copies of $CQ_{n-1}$, denoted by $CQ^0_{n-1}$ and $CQ^1_{n-1}$, respectively, and adding $2^{n-1}$ edges as follows:

Let $V(CQ^0_{n-1}) = \{0x_{n-2} \ldots x_1x_0 : x_i = 0 \text{ or } 1\}$ and $V(CQ^1_{n-1}) = \{1y_{n-2} \ldots y_1y_0 : y_i = 0 \text{ or } 1\}$. A vertex $0x_{n-2} \ldots x_1x_0 \in V(CQ^0_{n-1})$ and a vertex $1y_{n-2} \ldots y_1y_0 \in V(CQ^1_{n-1})$ are adjacent if

(1) $x_{n-2} = y_{n-2}$ if $n$ is even, and

(2) $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

We take $CQ_3$ and $CQ_4$ as examples and display them in Figures 3.1(a) and (b), respectively. In Figure 3.1(c), we use a different way to draw $CQ_3$ in order to see its vertex-symmetry.

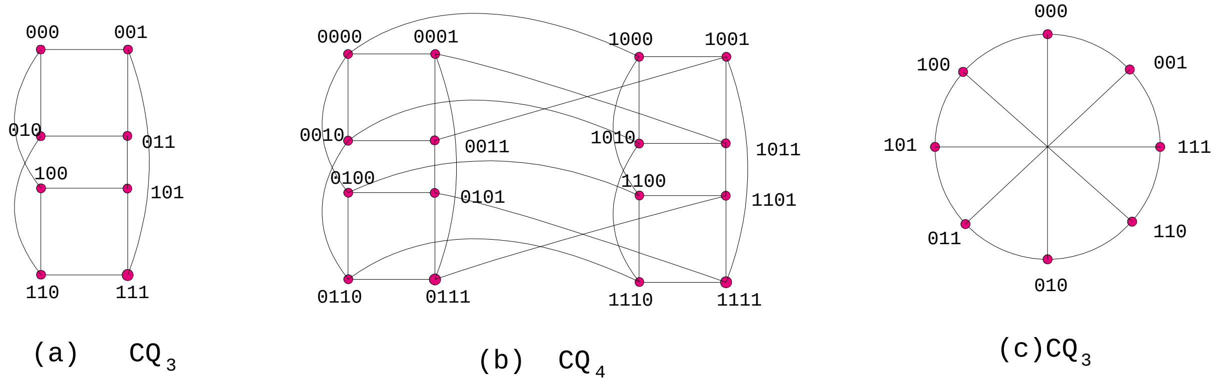



Figure 3.1: (a) $CQ_3$, (b) $CQ_4$, and (c) $CQ_3$ drawn in a different way.

## 3.2 Main Result

We use $CQ^{ij}_{n-2}$ to denote an $(n-2)$-dimensional crossed cube which is a subgraph of $CQ_n$ induced by the vertices labeled $ijx_{n-3}\ldots x_0$. We say that an edge is a *critical edge* of $CQ_n$ if it is an edge in $CQ^i_{n-1}$ with one **endpoint** in $CQ^{i0}_{n-2}$ and the other in $CQ^{i1}_{n-2}$ **for** $i \in \{0,1\}$**.**

**Lemma 4** *Let* $(u_1, u_2)$ *be a critical edge of* $CQ_n$ *which is in* $CQ^0_{n-1}$, *and* $v_1, v_2$ *be the neighbors of* $u_1$ *and* $u_2$ *in* $CQ^1_{n-1}$, *respectively, for* $n \geq 4$. *Then* $(v_1, v_2)$ *is also a critical edge of* $CQ_n$ *in* $CQ^1_{n-1}$.

**Proof.** We discuss two cases: (1) $n$ is even, and (2) $n$ is odd.

**Case 1.** $n$ **is even.** Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}\ldots x_1x_0$ and $u_2 = 01y_{n-3}y_{n-4}\ldots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leq i \leq \lfloor\frac{n-3}{2}\rfloor$. Then $v_1 = 10y_{n-3}y_{n-4}\ldots y_1y_0$, and $v_2 = 11x_{n-3}x_{n-4}\ldots x_1x_0$. By definition, $v_1$ and $v_2$ are adjacent, and $(v_1, v_2)$ is a critical edge in $CQ^1_{n-1}$.

**Case 2.** $n$ **is odd.** Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}$ $x_{n-5}\ldots x_1x_0$. Suppose that $x_{n-3} = 0$. Then $u_1 = 000x_{n-4}\ x_{n-5}\ldots x_1x_0$, $u_2 = 010y_{n-4}$ $y_{n-5}\ldots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leq i \leq \lfloor\frac{n-4}{2}\rfloor$, $v_1 = 100y_{n-4}\ y_{n-5}\ldots y_1y_0$, and $v_2 = 110x_{n-4}\ x_{n-5}\ldots x_1x_0$. Thus, $v_1$ and $v_2$ are adjacent, and $(v_1, v_2)$ is a critical edge in $CQ^1_{n-1}$. It can be checked that the statement is also true for the case $x_{n-3} = 1$. $\square$

It is observed that vertices $u_1, u_2, v_1, v_2$ in the above lemma form a 4-cycle. We call this cycle a *crossed 4-cycle* in $CQ_n$. It is clear that, for each vertex $00x_{n-3}\cdots x_0$, there is exactly one crossed 4-cycle corresponding to the vertex. Thus, there are $2^{n-2}$ disjoint crossed 4-cycles in $CQ_n$. We note that a crossed 4-cycle contains two critical edges.

Huang et al. [29] showed the validity of the following theorem. Based on this theorem, we show the pancyclicity of the crossed cube by induction.

**Theorem 5** *[29] The crossed cube $CQ_n$ is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for $n \geq 3$.*

The base case is $n = 3$, and the proof is given in the following.

**Theorem 6** *$CQ_3$ is 1-pancyclic.*

**Proof.** Note that $CQ_3$ can be redrawn as Figure 3.1(c), and it is vertex-transitive. We consider two cases (1) one faulty vertex, and (2) one faulty edge as follows:

**Case 1. One faulty vertex.** Without loss of generality, we assume that vertex $x = 000$ is faulty. We list cycles of lengths from 4 to 7 as follows: $\langle 001, 111, 101, 011, 001\rangle$, $\langle 001, 111, 110, 010, 011, 001\rangle$, $\langle 001, 111, 110, 100, 101, 011, 001\rangle$, and $\langle 001, 111, 101, 100, 110, 010, 011, 001\rangle$.

**Case 2. One faulty edge.** Without loss of generality, we assume that the faulty edge $e$ is incident to 000. By Case 1, there are cycles of lengths from 4 to 7 in the faulty $CQ_3$. For a cycle of length 8, suppose that $e = (000, 010)$. Then $\langle 000, 001, 111, 110, 010, 011, 101, 100, 000\rangle$ is the desired one. Suppose that $e = (000, 001)$. Then $\langle 000, 010, 110, 111, 001, 011, 101, 100, 000\rangle$ is a cycle of length 8. If $e = (000, 100)$, the case is symmetric to the case $e = (000, 001)$. □

Let $F$ be a set of faults in $CQ_n$. We say that a vertex $u$ in one subcube of $CQ_n$ is a *safe crossing-point* in $CQ_n - F$ if $u$ still connects to the neighbor in the other subcube in $CQ_n - F$, i.e., the corresponding neighbor $u'$ in the other subcube of $u$ is fault-free, and the edge $(u, u')$ is also fault-free. The main result is as follows.

**Theorem 7** *The crossed cube $CQ_n$ is $(n-2)$-pancyclic for $n \geq 3$.*

**Proof.** We prove this by induction on $n$. It follows from Theorem 6 that $CQ_3$ is 1-pancyclic. Now we proceed to the induction step. Suppose that $CQ_{n-1}$ is $(n-3)$-pancyclic for some $n \geq 4$. We will show that $CQ_n$ is $(n-2)$-pancyclic. Let $F \subseteq V(CQ_n) \cup E(CQ_n)$ be the set of faults. We divide $F$ into five disjoint parts: $F_v^0 = F \cap V(CQ_{n-1}^0)$, $F_e^0 = F \cap E(CQ_{n-1}^0)$, $F_v^1 = F \cap V(CQ_{n-1}^1)$, $F_e^1 = F \cap E(CQ_{n-1}^1)$, and $F_e^c = F \cap \{(u,v) \mid (u,v)$ is an edge between $CQ_{n-1}^0$ and $CQ_{n-1}^1\}$. Let $f = |F|$, $f_v^0 = |F_v^0|$, $f_e^0 = |F_e^0|$, $f_v^1 = |F_v^1|$, $f_e^1 = |F_e^1|$, and $f_e^c = |F_e^c|$. For convenience of discussion, we define the following subsets of $F$: $F_v = F \cap V(CQ_n)$, $F_e = F \cap E(CQ_n)$, $F^0 = F_v^0 \cup F_e^0$, and $F^1 = F_v^1 \cup F_e^1$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f_e^c$.

**Case 1.** There is a subcube containing all the $(n-2)$ faults. Without loss of generality, we assume that $f^0 = n-2$. Thus, there is no fault outside $CQ_{n-1}^0$, i.e., $f^1 = f_e^c = 0$. We discuss the existence of cycles of lengths from 4 to $2^n - f_v$ according to the following cases.

**Case 1.1. Cycles of lengths from** $4$ **to** $2^{n-1}$ **.** Since $CQ_{n-1}$ is $(n-3)$-pancyclic , $CQ_{n-1}^1$ contains cycles of lengths from 4 to $2^{n-1}$ for $n \geq 4$. Clearly, $CQ_n - F$ also contains cycles of these lengths.

**Case 1.2. A cycle of length** $2^{n-1}+1$**.** (See Figure 3.2(a).) We want to construct a cycle containing $2^{n-1}-1$ vertices in $CQ_{n-1}^1$ and two vertices in $CQ_{n-1}^0$. To avoid faults in $CQ_{n-1}^0$, we introduce a term called the shadows of the faults. Let $\langle u_1, u_2, v_2, v_1, u_1\rangle$ be a crossed 4-cycle with $u_1, u_2$ in $CQ_{n-1}^0$ and $v_1, v_2$ in $CQ_{n-1}^1$, respectively. If there is a fault on this cycle but the fault is not in $CQ_{n-1}^1$, we call edge $(v_1, v_2)$ a *shadow fault* of $F$ on $CQ_{n-1}^1$. (Similarly, we may define a shadow fault on $CQ_{n-1}^0$.) Let $F^s = \{e \mid$ edge $e$ is a shadow fault of $F$ on $CQ_{n-1}^1\}$. Since all crossed 4-cycles are vertex disjoint, $|F^s| \leq n-2$. If $|F^s| = n-2$, we arbitrarily pick an edge $e_1$ in $F^s$ , and let $F' = F^s - e_1$, or else $F' = F^s$. Then $|F'| \leq n-3$ and $CQ_{n-1}^1 - F'$ is still pancyclic. So there is a cycle $C$ of length $2^{n-1}-1$ in $CQ_{n-1}^1 - F'$. Clearly, there are two critical edges on $C$. Let $(a,b) \neq e_1$ be a critical edge on $C$, so $(a,b) \notin F^s$. Let $a', b'$ be the neighbors of $a$ and $b$ in $CQ_{n-1}^0$, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n-1}+1$ in $CQ_n - F$.

**Case 1.3. Cycles of lengths from** $2^{n-1}+2$ **to** $2^n - f_v$**.** (See Figure 3.2(b).) By Theorem 8, $CQ_{n-1}^0$ is $(n-3)$-hamiltonian and $f^0 = n-2$, $CQ_{n-1}^0 - F^0$ still contains a hamiltonian **path**, say $P = \langle u_1, u_2, \ldots, u_{2^{n-1}-f_v^0}\rangle$, where $f_v^0 = f_v$. Let $2 \leq l \leq 2^{n-1} - f_v$. We construct a cycle of length $2^{n-1}+l$ as follows: Suppose that the neighbors of $u_1$ and $u_l$ in $CQ_{n-1}^1$ are $v_1$ and $v_l$, respectively. Since $CQ_{n-1}$ is $(n-4)$-hamiltonian connected and $n \geq 4$, there is a hamiltonian path $Q$ in $CQ_{n-1}^1$ between $v_1$ and $v_l$ containing $2^{n-1}$ vertices. So $\langle u_1, \cdots, u_l, v_l, Q, v_1, u_1\rangle$ forms a cycle of length $2^{n-1}+l$.

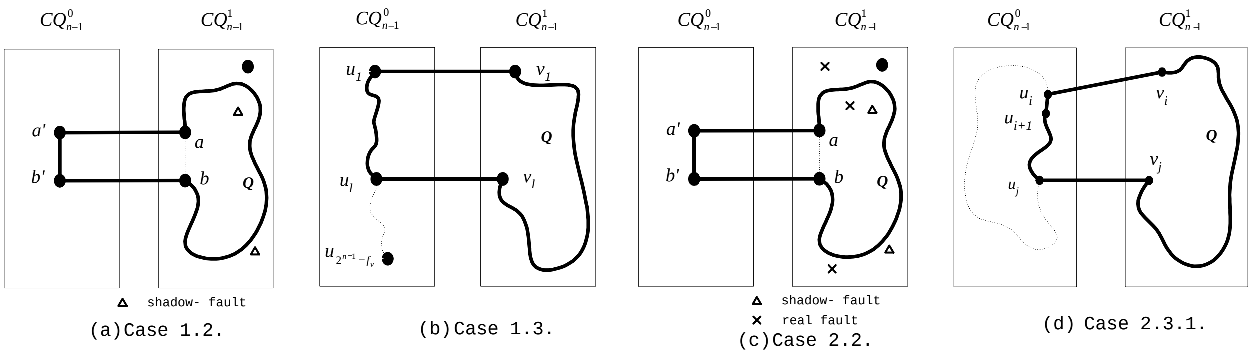



Figure 3.2: Cases of Theorem 9

**Case 2.** Both $f^0$ and $f^1$ are at most $n-3$. By induction hypothesis, $CQ^0_{n-1} - F^0$ and $CQ^1_{n-1} - F^1$ are still pancyclic. We discuss the existence of cycles of all lengths from 4 to $2^n - f_v$ in the following cases.

**Case 2.1. Cycles of lengths from 4 to $2^{n-1} - f^1_v$.** By induction hypothesis, $CQ^1_{n-1}$ is $(n-3)$-pancyclic. Thus, we have cycles of lengths from4 to $2^{n-1} - f^1_v$ in $CQ^1_{n-1} - F^1$.

**Case 2.2. A cycle of length $2^{n-1} - f^1_v + 1$.** (See Figure 3.2(c).) We construct the cycle using a similar way used in Case 1.2. Let $F^s = \{e \mid \text{edge } e \text{ is a shadow fault of } F \text{ on } CQ^1_{n-1}\}$. Then $|F^s \cup F^1| \le n-2$. If $|F^s \cup F^1| = n-2$, we arbitrarily choose an edge $e_1$ in $F^s$, and let $F' = F^s \cup F^1 - e_1$,or else $F' = F^s \cup F^1$ . Then $|F'| \le n-3$ and $CQ^1_{n-1} - F'$ is still pancyclic. Since $F' \cap V(CQ^1_{n-1}) = F^1_v$, there is a cycle $C$ of length $2^{n-1} - f^1_v - 1$ in $CQ^1_{n-1} - F'$. Since $2^{n-1} - f^1_v - 1 > 2^{n-2}$ for $n \ge 4$, $C$ contains two critical edges. Let $(a, b) \ne e_1$ be a critical edge on $C$, so $(a, b) \notin F^s$. Let $a', b'$ be the neighbors of $a$ and $b$ in $CQ^0_{n-1}$, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n-1} - f^1_v + 1$ in $CQ_n - F$.

**Case 2.3. Cycles of lengths from $2^{n-1} - f^1_v + 2$ to $2^n - f_v$.** (See Figure 3.2(d).) Without loss of generality, we assume that $n-3 \ge f^0 \ge f^1$. If $f^1 = n-3$, then $f^0 = n-3$, and $2n-6 \le n-2 = f$, which implies $n \le 4$. Thus, we need to discuss the case $f^1 = n-3$ just for $n = 4$. We leave this particular case to Appendix A, and assume that $f^1 \le n-4$ in the following discussion.

By Theorem 8, $CQ^1_{n-1} - F^1$ is still hamiltonian connected, i.e., there is a path of length $2^{n-1} - f^1_v - 1$ between any two vertices in $CQ^1_{n-1} - F^1$. As a result, if we can find a path of length $l$ in $CQ^0_{n-1} - F^0$ with the two endpoints being safe crossing-points, then we find a cycle of length $l + 2 + (2^{n-1} - f^1_v - 1)$. Since we want to construct cycles of lengths from

$2^{n-1} - f_v^1 + 2$ to $2^n - f_v$, $1 \le l \le 2^{n-1} - (f_v - f_v^1) - 1 = 2^{n-1} - f_v^0 - 1$. Now we construct a path of length $l$ in $CQ_{n-1}^0$ for each $l$, $1 \le l \le 2^{n-1} - f_v^0 - 1$. By Theorem 8, $CQ_{n-1}^0$ is $(n-3)$-hamiltonian. Thus we have a hamiltonian cycle $C = \langle u_0, u_1, \ldots, u_{2^{n-1}-f_v^0-1}, u_0 \rangle$ of length $2^{n-1} - f_v^0$ in $CQ_{n-1}^0 - F^0$. We claim that there exist two safe crossing-points $u_i$ and $u_j$ on $C$ such that $(j-i)_{(\mathbf{mod}\ 2^{n-1}-f_v^0)} = l$. Suppose on the contrary that there do not exist such $u_i$ and $u_j$. Then there are at least $\lceil \frac{2^{n-1}-f_v^0}{2} \rceil$ faults outside $CQ_{n-1}^0$. However, $\lceil \frac{2^{n-1}-f_v^0}{2} \rceil + f_v^0 \ge 2^{n-2} > n-2$ for $n \ge 2$. We obtain a contradiction. Thus, there exist such two vertices $u_i$ and $u_j$. And then we find a path of length $l$ on $C$.

Hence, the theorem follows. $\square$

# Chapter 4

# Fault-Tolerant Pancyclicity of the Möbius Cubes

## 4.1 The Möbius Cubes

An $n$-dimensional Möbius cube $MQ_n$ consists of $2^n$ vertices and each vertex is labeled by a unique $n$-bit binary string. Each vertex has $n$ neighbors as follows.

A vertex $x = x_1x_2 \ldots x_{i-1}\ x_ix_{i+1} \ldots x_{n-1}x_n$ connects to its $i$th neighbor, for $2 \leq i \leq n$,

$$x^i = x_1x_2 \ldots x_{i-1}\ \bar{x}_ix_{i+1} \ldots x_{n-1}x_n \text{ if } x_{i-1} = 0,$$

$$x_1x_2 \ldots x_{i-1}\ \bar{x}_i\bar{x}_{i+1} \ldots \bar{x}_{n-1}\bar{x}_n \text{ if } x_{i-1} = 1.$$

For $i = 1$, since there is no bit on the left of $x_1$, the first neighbor of $x$ can be defined as $\bar{x}_1x_2 \ldots x_n$ or $\bar{x}_1\bar{x}_2 \ldots \bar{x}_n$. We call the cube 0-$MQ_n$ if the first neighbor of $x$ is $\bar{x}_1x_2 \ldots x_n$, or 1-$MQ_n$ if the first neighbor of $x$ is $\bar{x}_1\bar{x}_2 \ldots \bar{x}_n$. For example, $MQ_4$ is illustrated in Figure 4.1.

Therefore, $MQ_n$ is an $n$-regular graph and can be recursively defined as follows: Both 0-$MQ_1$ and 1-$MQ_1$ are complete graphs $K_2$ with one vertex labeled 0 and the other 1. 0-$MQ_n$ and 1-$MQ_n$ are both composed of a subcube $MQ^0_{n-1}$ which is isomorphic to 0-$MQ_{n-1}$ and a subcube $MQ^1_{n-1}$ which is isomorphic to 1-$MQ_{n-1}$. The first bit of each vertex of $MQ^0_{n-1}$ is 0, and the first bit of each vertex of $MQ^1_{n-1}$ is 1. Each vertex $x = 0x_2x_3 \ldots\ x_{n-1}x_n \in V(MQ^0_{n-1})$ connects to $1x_2x_3 \ldots\ x_{n-1}x_n \in V(MQ^1_{n-1})$ in 0-$MQ_n$

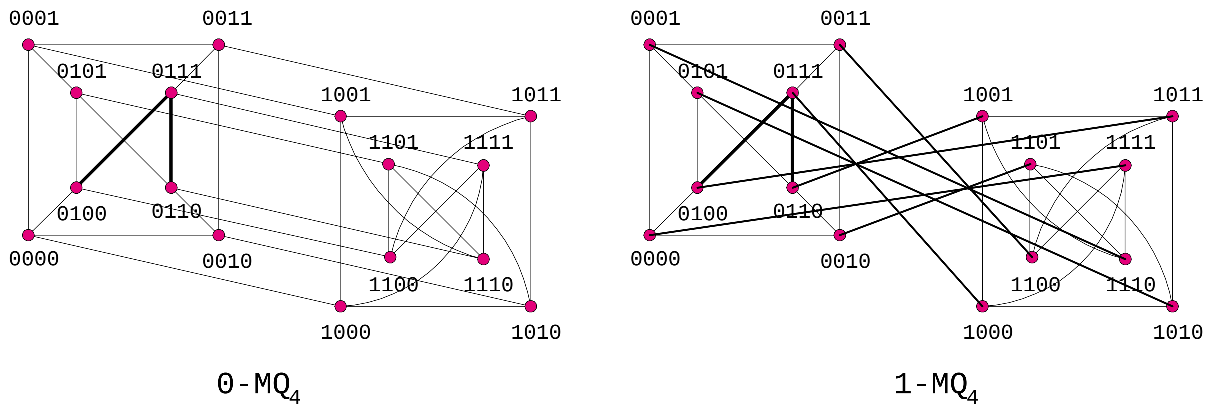



Figure 4.1: The 4-dimensional *Möbius* cube $MQ_4$.

and to $1\bar{x}_2\bar{x}_3 \dots \bar{x}_{n-1}\bar{x}_n \in V(MQ^1_{n-1})$ in 1-$MQ_n$. For convenience of discussion, if there are no ambiguities, $MQ_n$ denotes either 0-$MQ_n$ or 1-$MQ_n$ in the rest of this paper.

## 4.2 Properties of the Möbius Cubes

We say that an edge $(u, v)$ in one subcube of $MQ_n$ is a *shared-edge* in $MQ_n$ if the corresponding neighbors of $u$ and $v$ in the other subcube of $MQ_n$ are also adjacent. Clearly, these neighbors of $u$ and $v$ must be the first neighbors of $u$ and $v$. The following lemma states that, for each vertex in 0-$MQ_n$, all the edges incident to it are shared-edges except two.

**Lemma 5** *Let $x = x_1x_2 \cdots x_n$ be a vertex of* 0-$MQ_n$ *and $x^i$ be the ith neighbor of $x$ for $3 \le i \le n$ and $n \ge 3$. Then $(x, x^i)$ is a shared-edge in* 0-$MQ_n$.

**Proof.** By definition, $(x, x^i)$ is an edge in one subcube of 0-$MQ_n$. Suppose that $x_{i-1} = 0$. Then $x^i = x_1x_2 \dots x_{i-1}\bar{x}_ix_{i+1} \dots x_{n-1}x_n$. And the first neighbors of $x$ and $x^i$ are $\bar{x}_1x_2 \cdots x_n$ and $\bar{x}_1x_2 \dots x_{i-1}\ \bar{x}_ix_{i+1} \dots x_{n-1}x_n$, respectively. Clearly, the two neighbors are adjacent. On the other hand, suppose that $x_{i-1} = 1$, then $x^i = x_1x_2 \dots x_{i-1}\ \bar{x}_i\bar{x}_{i+1} \dots \bar{x}_{n-1}\bar{x}_n$. And the first neighbors of $x$ and $x^i$ are $\bar{x}_1x_2 \cdots x_n$ and $\bar{x}_1x_2 \dots x_{i-1}\ \bar{x}_i\bar{x}_{i+1} \dots \bar{x}_{n-1}\bar{x}_n$, respectively. The two neighbors are also adjacent. Thus, $(x, x^i)$ is a shared-edge, and the lemma follows. $\square$

By Lemma 5, we have the following corollary.

**Corollary 2** *Let $P$ be a path in one subcube $MQ_{n-1}^i$ of $0$-$MQ_n$ for $i = 0, 1$ and $n \geq 3$. For any two consecutive edges on $P$, at least one of them is a shared-edge in $MQ_n$.*

Let $F$ be a set of faults in $MQ_n$. We say that a vertex $u$ is a *safe crossing-point* in $MQ_n - F$ if $u$ still connects to its first neighbor in $MQ_n - F$. And we say that an edge $(u, v)$ is a *safe shared-edge* in $MQ_n - F$ if $(u, v)$ is a shared-edge in $MQ_n - F$, i.e., the corresponding first neighbors $u'$ and $v'$ of $u$ and $v$, respectively are fault-free, and edges $(u, u')$, $(v, v')$ and $(u', v')$ are also fault-free.

**Lemma 6** *Let $C$ be a cycle of length $l$ in one subcube $MQ_{n-1}^i$ of $0$-$MQ_n$ for $i = 0, 1$, $n \geq 3$. Suppose that $f$ is the number of faults outside $MQ_{n-1}^i$, and $3f < l$. Then there is a safe shared-edge on $C$, and there is a cycle of length $l + 2$ in $0$-$MQ_n$.*

**Proof.** Suppose on the contrary that there does not exist any safe shared-edge on $C$. By Corollary 2, there exists a shared-edge $e$ on $C$, which is not safe. Let $C = \langle u_1, u_2, \ldots, u_l, u_1 \rangle$. Let $v_1$ and $v_2$ be the first neighbors of $u_1$ and $u_2$, respectively. Without loss of generality, we assume that $e = (u_1, u_2)$ and that at least one of $(u_1, v_1)$, $v_1$, or $(v_1, v_2)$ is faulty. For each consecutive three vertices $u_{3i+2}$, $u_{3i+3}$ and $u_{3i+4}$ on cycle $C$, by Corollary 2, one of the edges $(u_{3i+2}, u_{3i+3})$ and $(u_{3i+3}, u_{3i+4})$ is a shared-edge in $0$-$MQ_n$, but is not a safe shared-edge in $MQ_n - F$, $0 \leq i \leq \lfloor \frac{l-1}{3} \rfloor - 1$. Each of these corresponds to a distinct fault, so the total number of faults is at least $\lfloor \frac{l-1}{3} \rfloor + 1$. And $3(\lfloor \frac{l-1}{3} \rfloor + 1) \geq l$, which contradicts to the assumption that $3f < l$. Let $(a, b)$ be a safe shared-edge on $C$. Let $C = \langle a, P, b, a \rangle$, where $P$ is a subpath of $C$ between $a$ and $b$. Let $a'$ and $b'$ be the first neighbors of $a$ and $b$, respectively. Then $\langle a, P, b, b', a', a \rangle$ forms a cycle of length $l + 2$. This proves the lemma. $\square$

We have a similar result for a path, and the proof is also similar to that of Lemma 6.

**Lemma 7** *Let $P$ be a path containing $l$ vertices in one subcube $MQ_{n-1}^i$ of $0$-$MQ_n$ for $i = 0, 1$, $n \geq 3$. Suppose that $f$ is the number of faults outside $MQ_{n-1}^i$, and $3f < l - 2$. Then there is a safe shared-edge on $P$.*

Huang et al. [29] proved the following theorem concerning the hamiltonicity of $MQ_n$. We will use it later in our proof of Theorem 9.

**Theorem 8** *[29] The Möbius cube $MQ_n$ is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for $n \geq 3$.*

## 4.3 Fault-tolerant pancyclicity

Now we prove our main result.

**Theorem 9** *The Möbius cube $MQ_n$ is $(n-2)$-pancyclic for $n \geq 3$.*

**Proof.** We prove this by induction on $n$. We show the cases $n = 3, 4$ in Appendix B. In fact, we also use a computer program to verify the statements. Now we shall go directly to the inductive step. Suppose that $MQ_{n-1}$ is $(n-3)$-pancyclic for some $n \geq 5$. We will show that $MQ_n$ is $(n-2)$-pancyclic. Let $F \subseteq V(MQ_n) \cup E(MQ_n)$ be the set of faults. We divide $F$ into five disjoint parts: $F_v^0 = F \cap V(MQ_{n-1}^0)$, $F_e^0 = F \cap E(MQ_{n-1}^0)$, $F_v^1 = F \cap V(MQ_{n-1}^1)$, $F_e^1 = F \cap E(MQ_{n-1}^1)$, and $F_e^c = F \cap \{(u, v) \mid (u, v)$ is an edge between $MQ_{n-1}^0$ and $MQ_{n-1}^1\}$. Let $f = |F|$, $f_v^0 = |F_v^0|$, $f_e^0 = |F_e^0|$, $f_v^1 = |F_v^1|$, $f_e^1 = |F_e^1|$, and $f_e^c = |F_e^c|$. For convenience of discussion, we define the following subsets of $F$: $F_v = F \cap V(MQ_n)$, $F_e = F \cap E(MQ_n)$, $F^0 = F_v^0 \cup F_e^0$, and $F^1 = F_v^1 \cup F_e^1$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f_e^c$.

In addition, throughout this paper, we only considers the case $f = n - 2$, since it is the same as considering all the cases $f \leq n - 2$. The reason is as follows: Suppose that $f \leq n - 2$, and let $F' \subseteq E(MQ_n) - F_e$ with $f + |F'| = n - 2$. So, $MQ_n - F - F'$ is a subgraph of $MQ_n - F$. Trivially, if $MQ_n - F - F'$ has a cycle $C$, $MQ_n - F$ has $C$ . For further discussion, we consider the following cases.

**Case 1.** There is a subcube containing all the $(n-2)$ faults.

Without loss of generality, we assume that $f^0 = n - 2$. Thus, $f^1 = f_e^c = 0$. By Theorem 8, $MQ_{n-1}^0$ is $(n-3)$-hamiltonian. Thus, it is clear that $MQ_{n-1}^0 - F^0$ still contains a hamiltonian path, say, $P = \langle u_1, u_2, \ldots, u_{2^{n-1}-f_v^0} \rangle$, where $f_v^0 = f_v$. We discuss the existence of cycles of every length from 4 to $2^n - f_v$ according to the following cases.

**Case 1.1.** Cycles of lengths from 4 to $2^{n-1}$.

Since $MQ_{n-1}$ is $(n-3)$-pancyclic , $MQ^1_{n-1}$ contains cycles of lengths from 4 to $2^{n-1}$ for $n \geq 5$. Clearly, $MQ_n - F$ also contains cycles of these lengths.

**Case 1.2.** Cycles of lengths from $2^{n-1}+2$ to $2^n - f_v$. (See Figure 4.2 (a).)

Let $2 \leq l \leq 2^{n-1} - f_v$. We construct a cycle of length $2^{n-1}+l$ as follows: Suppose that the first neighbors of $u_1$ and $u_l$ are $v_1$ and $v_l$, respectively, which are in $MQ^1_{n-1}$. Since $MQ_{n-1}$ is $(n-4)$-hamiltonian connected and $n \geq 5$, there is a hamiltonian path $Q$ in $MQ^1_{n-1}$ between $v_1$ and $v_l$ containing $2^{n-1}$ vertices. So $\langle u_1, \cdots, u_l, v_l, Q, v_1, u_1\rangle$ forms a cycle of length $2^{n-1}+l$. Note that there are no faults outside $MQ^0_{n-1}$. Thus all the vertices on $P$ are safe crossing-points.

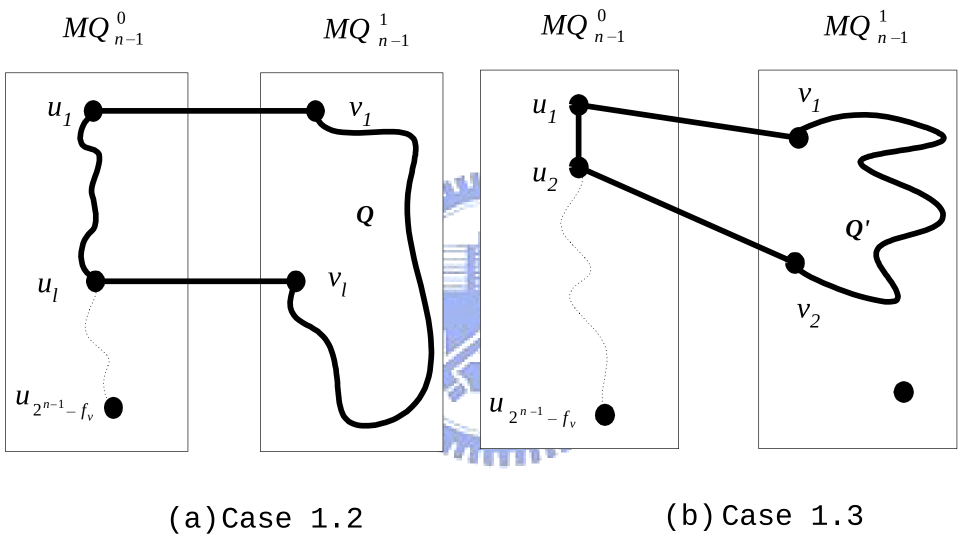



Figure 4.2: Case 1.2 and Case 1.3.

**Case 1.3.** A cycle of length $2^{n-1}+1$. (See Figure 4.2 (b).)

Consider the vertices $u_1$ and $u_2$ on $P$ and their first neighbors $v_1$ and $v_2$, respectively, in $MQ^1_{n-1}$. By Theorem 8, $MQ^1_{n-1}$ is $(n-4)$-hamiltonian connected for $n \geq 5$. Since $f^1 = 0$, we may find a path $Q'$ between $v_1$ and $v_2$ containing $2^{n-1}-1$ vertices in $MQ^1_{n-1}$. Then $\langle u_1, u_2, v_2, Q', v_1, u_1\rangle$ forms a cycle of length $2^{n-1}+1$.

**Case 2.** Both $f^0$ and $f^1$ are at most $n-3$.

By induction hypothesis, $MQ^0_{n-1} - F^0$ and $MQ^1_{n-1} - F^1$ are still pancyclic. Without loss of generality, we assume that $f^0 \geq f^1$. We claim that $f^1 \leq n-4$ for $n \geq 5$. Suppose for the sake of contradiction that $f^1 = n-3$. Since $f^0 \geq f^1$, $f^0 = n-3$. The total number of faults is at most $(n-2)$. Thus $(n-3)+(n-3) \leq n-2$. This implies that $n \leq 4$, which is a contradiction. Thus $MQ^1_{n-1} - F^1$ is still hamiltonian connected. We discuss the existence of cycles of all lengths from 4 to $2^n - f_v$ in the following cases.

**Case 2.1.** Cycles of lengths from 4 to $2^{n-1} - f^1_v$.

By induction hypothesis, $MQ^1_{n-1}$ is $(n-3)$-pancyclic. Thus, we have cycles of lengths from 4 to $2^{n-1} - f^1_v$ in $MQ^1_{n-1} - F^1$.

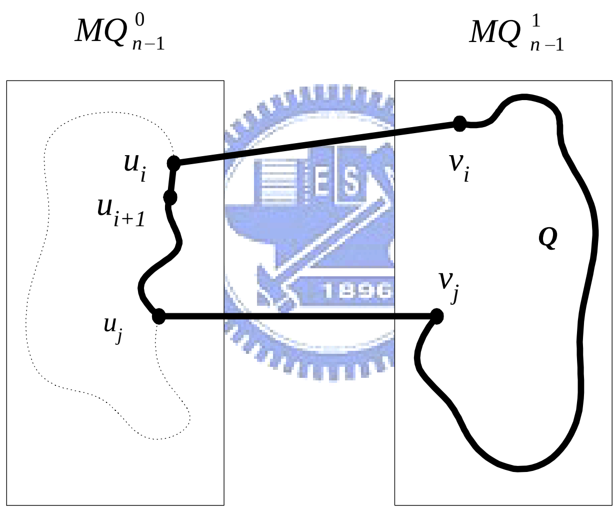



Figure 4.3: Case 2.2.

**Case 2.2.** Cycles of lengths from $2^{n-1} - f^1_v + 2$ to $2^n - f_v$. (See Figure 4.3.)

By Theorem 8, $MQ^0_{n-1}$ is $(n-3)$-hamiltonian. Thus, we have a hamiltonian cycle $C = \langle u_0, u_1, \ldots, u_{2^{n-1}-f^0_v-1}, u_0 \rangle$ of length $2^{n-1} - f^0_v$ in $MQ^0_{n-1} - F^0$. Let $2 \leq l \leq 2^{n-1} - f^0_v$. We construct a cycle of length $2^{n-1} - f^1_v + l$ as follows: First, we claim that there exist two safe crossing-points $u_i$ and $u_j$ on $C$ such that $(j-i)_{(\mathbf{mod}\ 2^{n-1}-f^0_v)} = l-1$. Suppose on the contrary that there do not exist such $u_i$ and $u_j$. Then there are at least $\lceil \frac{2^{n-1}-f_v^0}{2} \rceil$ faults outside $MQ_{n-1}^0$. However, $\lceil \frac{2^{n-1}-f_v^0}{2} \rceil + f_v^0 \geq 2^{n-2} > n-2$ for $n \geq 2$. We obtain a contradiction. Thus, there exist such $u_i$ and $u_j$. Let $v_i$ and $v_j$ be the first neighbors of $u_i$ and $u_j$, respectively. By Theorem 8, $MQ_{n-1}^1 - F^1$ is hamiltonian connected. Therefore, there is a hamiltonian path $Q$ in $MQ_{n-1}^1 - F^1$ between $v_i$ and $v_j$. Clearly, $Q$ contains $(2^{n-1} - f_v^1)$ vertices. Then $\langle u_i, u_{i+1}, \cdots, u_j, v_j, Q, v_i, u_i \rangle$ forms a cycle of length $2^{n-1} - f_v^1 + l$.

**Case 2.3.** A cycle of length $2^{n-1} - f_v^1 + 1$.

We divide this case into two parts:

**Case 2.3.1.** $MQ_n$ is a 0-$MQ_n$.

By Theorem 8, $MQ_{n-1}^1$ is $(n-3)$-hamiltonian, but $f^1 \leq n-4$. Thus we have a cycle of length $2^{n-1} - f_v^1 - 1$ in $MQ_{n-1}^1 - F^1$. Since $3(n-2) < 2^{n-1} - f_v^1 - 1$ for $n \geq 5$, by lemma 6, we can find a cycle of length $2^{n-1} - f_v^1 + 1$ in $MQ_n - F$.

**Case 2.3.2.** $MQ_n$ is a 1-$MQ_n$.

Let $G$ and $H$ be two induced subgraphs of 1-$MQ_n$ where $V(G) = \{v \in V(MQ_n) \mid v_1v_2 = 00 \text{ or } 11\}$ and $V(H) = \{v \in V(MQ_n) \mid v_1v_2 = 01 \text{ or } 10\}$. It is not difficult to check that, using the definition of the Möbius cube, both $G$ and $H$ are isomorphic to 1-$MQ_{n-1}$, and $\{V(G), V(H)\}$ is a partition of $V(MQ_n)$. (See Figure 4.4.)

We divide $F$ into five parts: $F_v^G = F \cap V(G)$, $F_e^G = F \cap E(G)$, $F_v^H = F \cap V(H)$, $F_e^H = F \cap E(H)$, and $F_e^{GH} = F \cap \{(u,v) \mid (u,v) \text{ is an edge between } G \text{ and } H\}$. Let $f_v^G = |F_v^G|$, $f_e^G = |F_e^G|$, $f_v^H = |F_v^H|$, $f_e^H = |F_e^H|$, and $f_e^{GH} = |F_e^{GH}|$. We shall also use the following notation: $F^G = F_v^G \cup F_e^G$ and $F^H = F_v^H \cup F_e^H$. And let $f^G = |F^G|$ and $f^H = |F^H|$. Since $G$ and $H$ are isomorphic, there are two situations: (1) All the faults are within $G$, and (2) $f^H \leq f^G \leq n-3$.

**(1) All the faults are within $G$.**

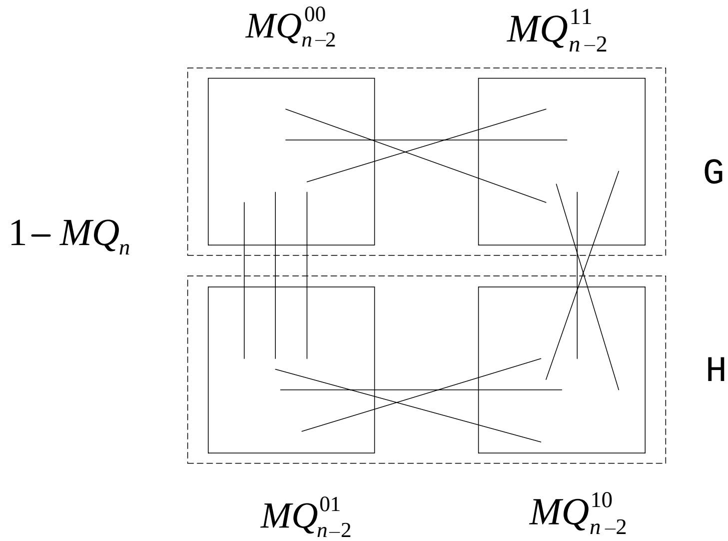



Figure 4.4: $G$ and $H$.

Thus, there is no fault outside $G$. Suppose that $f_v^1 \geq 1$. Since $H$ is isomorphic to 1-$MQ_{n-1}$, which is $(n-3)$-pancyclic, $H - F^H$ contains a cycle of length $2^{n-1} - f_v^1 + 1$. Suppose that $f_v^1 = 0$. Treating $G$ and $H$ as $MQ_{n-1}^0$ and $MQ_{n-1}^1$, respectively, we may construct a cycle of length $2^{n-1}+1$ by the same method used in Case 1.3.

**(2)** $f^H \leq f^G \leq n-3$**.**

By induction hypothesis, $G - F^G$ and $H - F^H$ are still pancyclic. Suppose that $f_v^1 > f_v^H$. Then $H - F^H$ contains a cycle of length $2^{n-1} - f_v^1 + 1$. Second, suppose that $f_v^1 < f_v^H$. Let $l = (2^{n-1} - f_v^1 + 1) - (2^{n-1} - f_v^H)$. Then $l \geq 2$. Since $f^H \leq n-4$ for $n \geq 5$, by Theorem 8, $H - F^H$ is still hamiltonian connected. We may construct a cycle of length $2^{n-1} - f_v^H + l$ by the same method used in Case 2.2. Finally, suppose that $f_v^1 = f_v^H$. We may virtually suppose that one vertex $v$ of $MQ_{n-2}^{10} - F$ is faulty. That is, let $F' = F^H \cup \{v\}$. Hence, if $H - F'$ has a cycle $C$, $H - F^H$ has $C$. Since $|F'| \leq n-3$ $(f^H \leq n-4)$ and $H$ is isomorphic to 1-$MQ_{n-1}$, by the same method used in Case 1.2 or Case 2.2, $H - F'$ has a cycle $C$ of length $2^{n-1} - f_v^1 - 1$, where all the vertices in $MQ_{n-2}^{01} - F$ are consecutive in $C$ for $n \geq 6$. Let $P$ be the maximal subpath of $C$ contained in $MQ_{n-2}^{01}$.

We claim that there exists a safe shared-edge on $P$ in $MQ^0_{n-1} - F^0$. Then we have a cycle of length $2^{n-1} - f_v^1 + 1$ as illustrated in Figure 4.5.

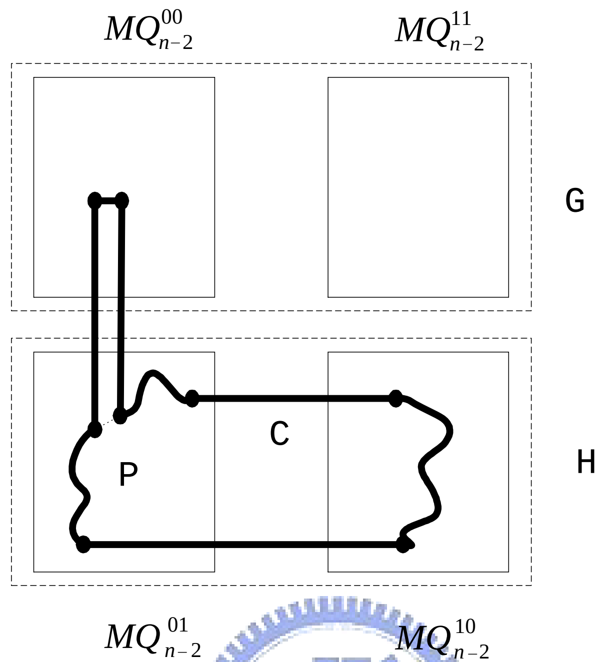



Figure 4.5: Case 2.3.2 for $n \geq 6$.

Now we prove the claim. Let $F_v^{01} = F \bigcap V(MQ^{01}_{n-2})$, $F_e^{01} = F \bigcap E(MQ^{01}_{n-2})$ and $F^{01} = F_v^{01} \bigcup F_e^{01}$. Let $f_v^{01} = |F_v^{01}|$, $f_e^{01} = |F_e^{01}|$ and $f^{01} = |F^{01}|$. $P$ has $2^{n-2} - f_v^{01}$ vertices. The number of faults outside $MQ^{01}_{n-2}$ in $MQ^0_{n-1}$ is at most $(n-3) - f_v^{01}$. Since $3(n - 3 - f_v^{01}) < 2^{n-2} - f_v^{01} - 2$ for $n \geq 6$ ,by lemma 7, we have a safe shared-edge on $P$.

Suppose that $n = 5$ and $f^{01} = 1$. Since $f^H \leq n - 4$, we may virtually suppose that one vertex of $MQ^{01}_3 - F$ is faulty. By the same method used in case 1.2, we have a cycle $C$ of length $2^{n-1} - f_v^1 - 1$ in $H - F^H$, where all the vertices in $MQ^{01}_3 - F$ are consecutive in $C$. Let $P$ be the maximal subpath of $C$ contained in $MQ^{01}_{n-2}$. $P$ has at least 6 vertices. Since $(n-3) - f^{01} = 1$, by lemma 7, we have a cycle of length $2^{n-1} - f_v^1 + 1$.

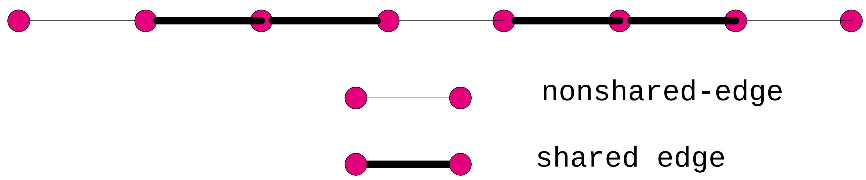



Figure 4.6: $P$ for $n = 5$.

Suppose that $n = 5$ and $f_v^{01} = 0$. Since $f^H \leq n - 4$, we may virtually suppose that one vertex of $MQ_3^{10} - F$ is faulty. By the same method used in case 1.2, we have a cycle $C$ of length $2^{n-1} - f_v^1 - 1$ in $H - F^H$, where all the vertices in $MQ_3^{01}$ are consecutive in $C$. Let $P$ be the maximal subpath of $C$ contained in $MQ_{n-2}^{01}$. $P$ has 8 vertices. Since $(n-3) - f^{01} = 2$, by Corollary 2, there is a safe shared-edge on $P$ except the case that $P$ has exactly three nonshared-edges at the head, tail and in the middle of $P$, respectively, and the other four edges are all shared-edges. (See Figure 4.6.) But such path does not exist between any two vertex $x$ and $y$ in $MQ_3$. Assume that such path exists, and we traverse it starting from $x = x_1x_2x_3$. In 0-$MQ_3$, $x_1x_2x_3 \rightarrow \bar{x}_1x_2x_3 \rightarrow \bar{x}_1\bar{x}_2x_3$ (or $\bar{x}_1x_2\bar{x}_3$) $\rightarrow \bar{x}_1\bar{x}_2\bar{x}_3 \rightarrow x_1\bar{x}_2\bar{x}_3 \rightarrow x_1\bar{x}_2x_3$ (or $x_1x_2\bar{x}_3$) $\rightarrow x_1x_2x_3$. In 1-$MQ_3$, $x_1x_2x_3 \rightarrow \bar{x}_1\bar{x}_2\bar{x}_3 \rightarrow \bar{x}_1x_2\bar{x}_3$ (or $\bar{x}_1\bar{x}_2x_3$) $\rightarrow \bar{x}_1x_2x_3 \rightarrow x_1\bar{x}_2\bar{x}_3 \rightarrow x_1x_2\bar{x}_3$ (or $x_1\bar{x}_2x_3$) $\rightarrow x_1x_2x_3$. In either case, we have a cycle of length 6, and it is a contradiction. So, the theorem is proved. $\square$

# Chapter 5

# On Eembedding Cycles Into Faulty Twisted Cubes

## 5.1 Twisted Cubes

The twisted cube was firstly proposed by Hilbers et al. in [22]. In the following, we give the recursive definition of the $n$-dimensional twisted cube $TQ_n$ for any odd integer $n \geq 1$. $TQ_n$ has $2^n$ vertices, and each of them is labeled by a binary string of length $n$. To define $TQ_n$, first of all, a parity function $P_i(x)$ is introduced. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0 \in V(TQ_n)$. For $0 \leq i \leq n-1$, $P_i(u) = u_i \oplus u_{i-1} \oplus \ldots u_1 \oplus u_0$, where $\oplus$ is the exclusive-or operation. $TQ_1$ is a complete graph with two vertices labeled by 0 and 1, respectively. For an odd integer $n \geq 3$, $TQ_n$ is obtained by taking four copies of $TQ_{n-2}$ and adding some additional edges to connect them. We use $TQ^{ij}_{n-2}$ to denote an $(n-2)$-dimensional twisted cube which is a subgraph of $TQ_n$ induced by the vertices labeled by $iju_{n-3}\ldots u_0$, where $i, j \in \{0, 1\}$. Each vertex $u = u_{n-1}u_{n-2}\ldots u_1u_0 \in V(TQ_n)$ is adjacent to $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$ and $\bar{u}_{n-1}\bar{u}_{n-2}\ldots u_1u_0$ if $P_{n-3}(u) = 0$; and to $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$ and $u_{n-1}\bar{u}_{n-2}\ldots u_1u_0$ if $P_{n-3}(u) = 1$. Figure 5.1 illustrates $TQ_3$.

## 5.2 Properties of the Twisted Cubes

**Lemma 8** *Let $G$ be a graph. $G$ is $k$-pancyclic if $G - F$ is pancyclic for every faulty set $F$ with $|F| = k$.*

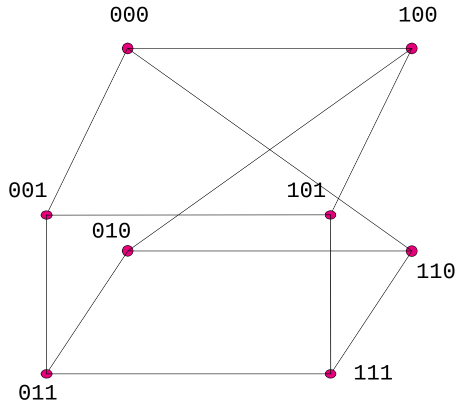



Figure 5.1: $TQ_3$.

**Proof.** Suppose that $|F| \leq k$, and let $F' \subseteq E(G) - F_e$ with $|F| + |F'| = k$. So, $(G - F) - F'$ is a subgraph of $G - F$. Trivially, if $(G - F) - F'$ has a cycle $C$, $G - F$ contains $C$ . This implies that if $(G - F) - F'$ is pancyclic, $G - F$ is also pancyclic. $\square$

Therefore, throughout this paper, whenever we prove that a graph $G$ is $k$-pancyclic, we only consider the case $|F| = k$.

A *matching* $M$ of a graph $G$ is a set of pairwise disjoint edges. $M$ is a *perfect matching* if each vertex of $G$ belongs to some edge in $M$.

**Lemma 9** *[30] For $n \geq 1$, both of the subgraphs induced by $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$ are isomorphic to $TQ_n \times K_2$. Furthermore, the edges joining $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$ is a perfect matching of $TQ_{n+2}$.*

Let $G$ and $H$ be two graphs having the same number of vertices. $G \bigoplus_M H$ denotes a graph which has copies of $G$ and $H$ connected by a matching $M$. Let $G_{n+1}^0$ and $G_{n+1}^1$ be the subgraphs induced by $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$, respectively. Then by Lemma 9, both of $G_{n+1}^0$ and $G_{n+1}^1$ are isomorphic to $TQ_n \times K_2$, and $G_{n+1}^0 \bigoplus_M G_{n+1}^1$ is isomorphic to $TQ_{n+2}$ for a specific matching $M$. In addition, $TQ_n \times K_2$ has two copies of $TQ_n$, and we use $TQ_n^0$ and $TQ_n^1$ to denote them, respectively. For convenience of discussion, we add 0 to every vertex $v \in V(TQ_n^0)$ and 1 to every vertex $u \in V(TQ_n^1)$, respectively, as the leading bits. As a result, each vertex of $TQ_n \times K_2$ is represented by a binary string of length $n+1$.

Let $F$ be a set of faults in $TQ_n \times K_2$ ($TQ_{n+2}$, respectively). We say that a vertex $u$ in $TQ_n^0$ ($G_{n+1}^0$, respectively) is a *safe crossing-point* in $TQ_n \times K_2 - F$ ($TQ_{n+2} - F$, respectively) if $u$ still connects to the neighbor $v$ in $TQ_n^1$ ($G_{n+1}^1$, respectively) in $TQ_n \times K_2 - F$ ($TQ_{n+2} - F$, respectively), i.e., vertices $u, v$ and edge $(u, v)$ are fault-free. If $u$ is in $TQ_n^1$ ($G_{n+1}^1$, respectively), we may define safe crossing-point in the same way.

Huang et al. [30] proved the following theorem concerning fault hamiltonicity and fault hamiltonian connectivity of $TQ_n$, and we shall use it in the proof of Theorem 12.

**Theorem 10** *[30] $TQ_n$ is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for any odd integer $n \geq 3$.*

## 5.3 Base Case

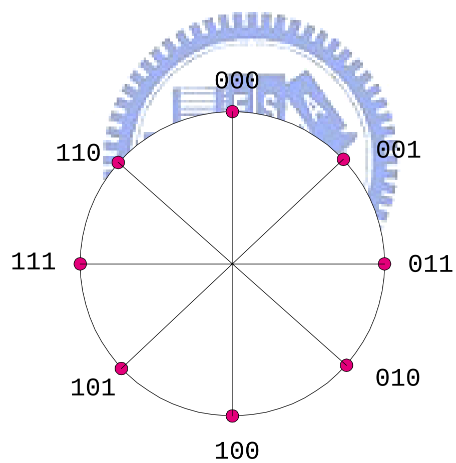



Figure 5.2: Another layout of $TQ_3$.

**Theorem 11** *$TQ_3$ is 1-pancyclic.*

**Proof.** Figure 5.2 is another layout of $TQ_3$, and it is vertex-transitive. We consider two cases (1) one faulty vertex and (2) one faulty edge as follows:

**Case 1. One faulty vertex.** Without loss of generality, we assume that vertex 000 is faulty. We list cycles of lengths from 4 to 7 as follows: $\langle 001, 101, 111, 011, 001\rangle$, $\langle 010, 100, 101, 111, 011, 010\rangle$, $\langle 001, 101, 111, 110, 010, 011, 001\rangle$, and $\langle 001, 101, 100, 010, 110, 111, 011, 001\rangle$.

**Case 2. One faulty edge.** We may assume that the faulty edge $e$ is incident to 000 because of the symmetry of $TQ_3$. By Case 1, there are cycles of lengths from 4 to 7 in the faulty $TQ_3$. For a cycle of length 8, suppose that $e = (000, 100)$. Then $\langle 000, 001, 011, 010, 100, 101, 111, 110, 000\rangle$ is a desired one. Suppose that $e = (000, 001)$. Then $\langle 000, 110, 111, 101, 001, 011, 010, 100, 000\rangle$ is a cycle of length 8. If $e = (000, 100)$, this case is symmetric to the case $e = (000, 110)$. $\square$

## 5.4 Inductive case

**Theorem 12** *Let $n \geq 3$ be an odd integer. If $TQ_n$ is $(n-2)$-pancyclic, $TQ_n \times K_2$ is $(n-1)$-pancyclic.*

**Proof.** Suppose that $TQ_n$ is $(n-2)$-pancyclic for some $n \geq 3$. We will show that $TQ_n \times K_2$ is $(n-1)$-pancyclic. Let $F \subseteq V(TQ_n \times K_2) \cup E(TQ_n \times K_2)$ be a set of faults. We divide $F$ into five disjoint parts: $F_v^0 = F \cap V(TQ_n^0)$, $F_e^0 = F \cap E(TQ_n^0)$, $F_v^1 = F \cap V(TQ_n^1)$, $F_e^1 = F \cap E(TQ_n^1)$, and $F_e^c = F \cap \{(u, v) \mid (u, v)$ is an edge between $TQ_n^0$ and $TQ_n^1\}$. Let $f = |F|$, $f_v^0 = |F_v^0|$, $f_e^0 = |F_e^0|$, $f_v^1 = |F_v^1|$, $f_e^1 = |F_e^1|$, and $f_e^c = |F_e^c|$. For convenience of discussion, we define the following subsets of $F$: $F_v = F \cap V(TQ_n \times K_2)$, $F_e = F \cap E(TQ_n \times K_2)$, $F^0 = F_v^0 \cup F_e^0$, and $F^1 = F_v^1 \cup F_e^1$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f_e^c$.

For further discussion, we consider the following cases.

**Case 1.** There is a subcube containing all the $n-1$ faults.

Without loss of generality, we assume that $TQ_n^0$ contains all the faults, i.e., $f^0 = n-1$. Thus, $f^1 = f_e^c = 0$. We discuss the existence of cycles of all lengths from 4 to $2^{n+1} - f_v$ according to the following cases.

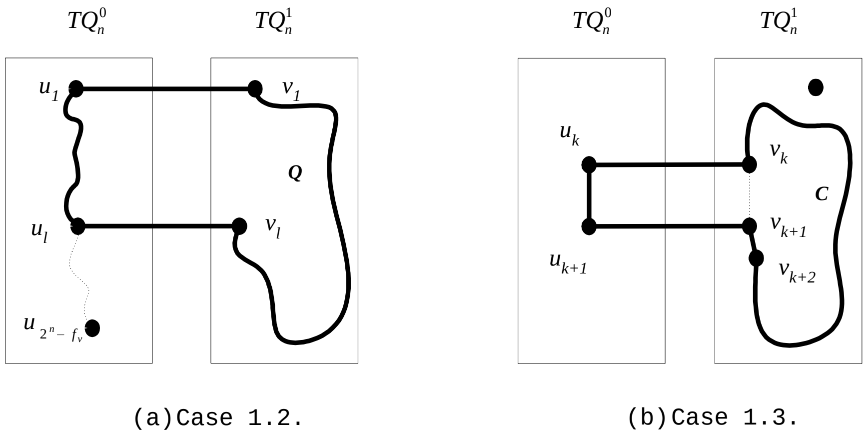



Figure 5.3: Case 1.2 and Case 1.3 of Theorem 12.

**Case 1.1.** Cycles of lengths from 4 to $2^n$.

Since $TQ_n$ is $(n-2)$-pancyclic , $TQ_n^1$ contains cycles of lengths from 4 to $2^n$ for $n \geq 3$. Thus, $TQ_n \times K_2 - F$ also contains cycles of these lengths.

**Case 1.2.** Cycles of lengths from $2^n + 2$ to $2^{n+1} - f_v$ (see Figure 5.3(a)).

$TQ_n^0$ is $(n-2)$-pancyclic, and hence $(n-2)$-hamiltonian. Clearly, $TQ_n^0 - F^0$ still contains a hamiltonian path, say, $P = \langle u_1, u_2, \ldots, u_{2^n - f_v^0} \rangle$, where $f_v^0 = f_v$. Let $2 \leq l \leq 2^n - f_v$. We construct a cycle of length $2^n + l$ as follows: Suppose that $v_1$ and $v_l$ are the neighbors in $TQ_n^1$ of $u_1$ and $u_l$, respectively. By Theorem 10, $TQ_n$ is $(n-3)$-hamiltonian connected and $n \geq 3$. Therefore, there is a hamiltonian path $Q$ in $TQ_n^1$ between $v_1$ and $v_l$ containing $2^n$ vertices, and $\langle u_1, \cdots, u_l, v_l, Q, v_1, u_1 \rangle$ forms a cycle of length $2^n + l$. Note that there are no faults outside $TQ_n^0$. Thus, all the vertices on $P$ are safe crossing-points.

**Case 1.3.** A cycle of length $2^n + 1$ (see Figure 5.3(b)).

Since $TQ_n^1$ is $(n-2)$-pancyclic and fault-free, we have a cycle $C = \langle v_1, v_2, \ldots, v_{2^n-1}, v_1 \rangle$ of length $2^n - 1$ in $TQ_n^1$. There are $n-1$ faults in total, and $\frac{2^n-1}{2} > n-1$ for $n \geq 3$. So there exist two safe crossing-points $v_k$ and $v_{k+1}$ on $C$, and also their neighbors in $TQ_n^0$, say, $u_k$

and $u_{k+1}$, respectively are connected in $TQ_n^0 - F^0$. $\langle v_{k+1}, v_{k+2}, \ldots, v_{2^n-1}, \ldots, v_k, u_k, u_{k+1}, v_{k+1}\rangle$ is a fault-free cycle of length $2^n + 1$.

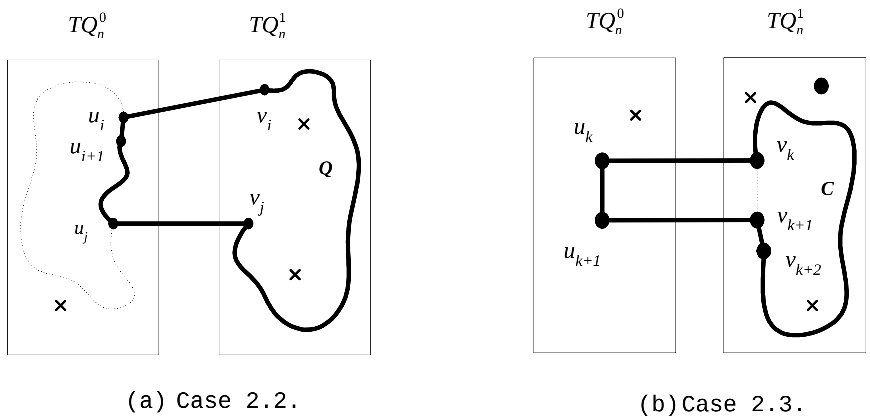



Figure 5.4: Case 2.2 and Case 2.3 of Theorem 12.

**Case 2.** Both $f^0$ and $f^1$ are at most $n-2$.

Since $f^i \leq n-2$ for any $i \in \{0,1\}$, $TQ_n^0 - F^0$ and $TQ_n^1 - F^1$ are still pancyclic. Without loss of generality, we assume that $f^0 \geq f^1$. We discuss the existence of cycles of all lengths from 4 to $2^{n+1} - f_v$ in the following cases.

**Case 2.1.** Cycles of lengths from 4 to $2^n - f_v^1$.

Since $TQ_n^1$ is $(n-2)$-pancyclic, we have cycles of lengths from 4 to $2^n - f_v^1$ in $TQ_n^1 - F^1$. Hence, $TQ_n \times K_2 - F$ also has cycles of these lengths.

**Case 2.2.** Cycles of lengths from $2^n - f_v^1 + 2$ to $2^{n+1} - f_v$ (see Figure 5.4(a)).

For the case $f^0 = f^1 = n-2$, we leave it to Appendix C because of its tediousness. For $f^1 \leq n-3$, the proof is as follows: $TQ_n^0 - F^0$ is pancyclic, and hence hamiltonian. We have a hamiltonian cycle $C = \langle u_1, u_2, \ldots, u_{2^n - f_v^0}, u_1\rangle$ of length $2^n - f_v^0$ in $TQ_n^0 - F^0$. Let $2 \leq l \leq 2^n - f_v^0$. We construct a cycle of length $2^n - f_v^1 + l$ as follows: We claim that there exist two safe crossing-points $u_i$ and $u_j$ on $C$ such that $(j-i) = l - 1 (\textbf{mod } 2^n - f_v^0)$.

Suppose on the contrary that there do not exist such $u_i$ and $u_j$. Then there are at least $\lceil\frac{2^n-f_v^0}{2}\rceil$ faults outside $TQ_n^0$. However, $\lceil\frac{2^n-f_v^0}{2}\rceil + f_v^0 \geq 2^{n-1} > n-1$ for $n \geq 1$. We obtain a contradiction. Thus, there exist such $u_i$ and $u_j$. By Theorem 10, $TQ_n^1$ is $(n-3)$-hamiltonian connected and $f^1 \leq n-3$, so $TQ_n^1 - F^1$ is still hamiltonian connected. Let $v_i$ and $v_j$ be the neighbors in $TQ_n^1$ of $u_i$ and $u_j$, respectively. There is a hamiltonian path $Q$ in $TQ_n^1 - F^1$ between $v_i$ and $v_j$. Clearly, $Q$ contains $(2^n - f_v^1)$ vertices. Then $\langle u_i, u_{i+1}, \cdots, u_j, v_j, Q, v_i, u_i\rangle$ forms a cycle of length $2^n - f_v^1 + l$.

**Case 2.3.** A cycle of length $2^n - f_v^1 + 1$ (see Figure 5.4(b)).

Since $TQ_n^1 - F^1$ is pancyclic, there is a cycle $\langle v_1, v_2, \ldots, v_{2^n-f_v^1-1}, v_1\rangle$ of length $2^n - f_v^1 - 1$ in $TQ_n^1 - F^1$. Furthermore, there are $(n-1) - f^1$ faults outside $TQ_n^1$, and $\frac{2^n-f_v^1-1}{2} \geq \frac{2^n-1}{2} - f^1 > (n-1) - f^1$ for $n \geq 3$. Thus there exist two safe crossing-points $v_k$ and $v_{k+1}$ on $C$, and also their neighbors in $TQ_n^0$, say, $u_k$ and $u_{k+1}$, respectively are adjacent in $TQ_n^0 - F^0$. $\langle v_{k+1}, v_{k+2}, \ldots, v_{2^n-f_v^1-1}, \ldots, v_k, u_k, u_{k+1}, v_{k+1}\rangle$ is a fault-free cycle of length $2^n - f_v^1 + 1$ in $TQ_n \times K_2 - F$. This completes the proof of the theorem. □

For the following discussion, we recall that $G_{n+1}^0$ and $G_{n+1}^1$ are the subgraphs induced by $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$, respectively. We say that an edge is a *critical edge* of $TQ_{n+2}$ if it is an edge in $G_{n+1}^i$ with one **endpoint** in $TQ_n^{i0}$ and the other in $TQ_n^{i1}$ for $i \in \{0,1\}$.

**Lemma 10** *Let $n \geq 3$ be an odd integer, and $(u_1, u_2)$ be a critical edge of $TQ_{n+2}$ which is in $G_{n+1}^0$, and $v_1, v_2$ be the neighbors in $G_{n+1}^1$ of $u_1$ and $u_2$, respectively. Then $(v_1, v_2)$ is also a critical edge of $TQ_{n+2}$ in $G_{n+1}^1$.*

**Proof.** Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}\ldots x_1x_0$. If $P_{n-3}(u_1) = 0$, $u_2 = 10x_{n-3}x_{n-4}\ldots x_1x_0$, $v_1 = 11x_{n-3}x_{n-4}\ldots x_1x_0$, and $v_2 = 01x_{n-3}x_{n-4}\ldots x_1x_0$. By definition, $v_1$ and $v_2$ are adjacent, and $(v_1, v_2)$ is a critical edge in $G_{n+1}^1$. It can be checked that the statement is also true if $P_{n-3}(u_1) = 1$. □

It is observed that vertices $u_1, u_2, v_1, v_2$ in the above lemma form a 4-cycle. We call this cycle a *crossed* 4-*cycle* in $TQ_{n+2}$. It is clear that, for each vertex $00x_{n-3}\cdots x_0$, there is exactly one crossed 4-cycle corresponding to this vertex. Thus, there are $2^n$ disjoint crossed 4-cycles in $TQ_{n+2}$. We note that a crossed 4-cycle contains two critical edges.

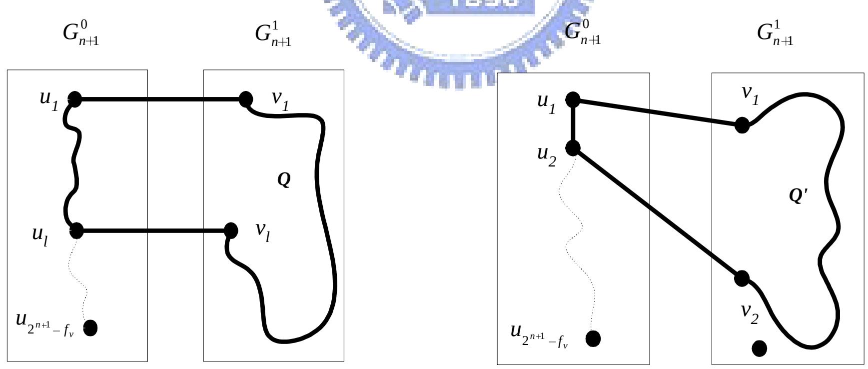
Huang et al. [30] proved the following theorem.

**Theorem 13** *[30] $TQ_n \times K_2$ is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected for any odd integer $n \geq 3$.*

**Theorem 14** *Let $n \geq 3$ be an odd integer. If $TQ_n$ is $(n-2)$-pancyclic, $TQ_{n+2}$ is $n$-pancyclic.*

**Proof.** Suppose that $TQ_n$ is $(n-2)$-pancyclic for some $n \geq 3$. By Theorem 12, $TQ_n \times K_2$ is $(n-1)$-pancyclic. That is, both $G^0_{n+1}$ and $G^1_{n+1}$ in $TQ_{n+2}$ are $(n-1)$-pancyclic. We will show that $TQ_{n+2}$ is $n$-pancyclic. Let $F \subseteq V(TQ_{n+2}) \cup E(TQ_{n+2})$ be a set of faults. We divide $F$ into five disjoint parts: $F^0_v = F \cap V(G^0_{n+1})$, $F^0_e = F \cap E(G^0_{n+1})$, $F^1_v = F \cap V(G^1_{n+1})$, $F^1_e = F \cap E(G^1_{n+1})$, and $F^c_e = F \cap \{(u,v) \mid (u,v)$ is an edge between $G^0_{n+1}$ and $G^1_{n+1}\}$. Let $f = |F|$, $f^0_v = |F^0_v|$, $f^0_e = |F^0_e|$, $f^1_v = |F^1_v|$, $f^1_e = |F^1_e|$, and $f^c_e = |F^c_e|$. For convenience of discussion, we define the following subsets of $F$: $F_v = F \cap V(TQ_{n+2})$, $F_e = F \cap E(TQ_{n+2})$, $F^0 = F^0_v \cup F^0_e$, and $F^1 = F^1_v \cup F^1_e$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f^c_e$.

For further discussion, we consider the following cases.




(a)Case 1.2.　　(b)Case 1.3

Figure 5.5: Case 1.2 and Case 1.3 of Theorem 14.

**Case 1.** There is a subcube containing all the $n$ faults.

Without loss of generality, we assume that $f^0 = n$. Thus, $f^1 = f_e^c = 0$. $G_{n+1}^0$ is $(n-1)$-pancyclic, and hence $(n-1)$-hamiltonian. Clearly, $G_{n+1}^0 - F^0$ still contains a hamiltonian path, say, $P = \langle u_1, u_2, \ldots, u_{2^{n+1}-f_v^0} \rangle$, where $f_v^0 = f_v$. We discuss the existence of cycles of all lengths from 4 to $2^{n+2} - f_v$ according to the following cases.

**Case 1.1.** Cycles of lengths from 4 to $2^{n+1}$.

Since $G_{n+1}^1$ is $(n-1)$-pancyclic , $G_{n+1}^1$ contains cycles of lengths from 4 to $2^{n+1}$ for $n \geq 3$. So, $TQ_{n+2} - F$ also contains cycles of these lengths.

**Case 1.2.** Cycles of lengths from $2^{n+1} + 2$ to $2^{n+2} - f_v$ (see Figure 5.5(a)).

Let $2 \leq l \leq 2^{n+1} - f_v$. We construct a cycle of length $2^{n+1} + l$ as follows: Suppose that the neighbors in $G_{n+1}^1$ of $u_1$ and $u_l$ are $v_1$ and $v_l$, respectively. By Theorem 13, $G_{n+1}^1$ is $(n-2)$-hamiltonian connected and $n \geq 3$. Hence there is a hamiltonian path $Q$ in $G_{n+1}^1$ between $v_1$ and $v_l$ containing $2^{n+1}$ vertices. $\langle u_1, \cdots, u_l, v_l, Q, v_1, u_1 \rangle$ forms a cycle of length $2^{n+1} + l$. Note that there are no faults outside $G_{n+1}^0$. So all the vertices on $P$ are safe crossing-points.

**Case 1.3.** A cycle of length $2^{n+1} + 1$ (see Figure 5.5(b)).

Consider the vertices $u_1$ and $u_2$ on $P$ and their neighbors in $G_{n+1}^1$, say $v_1$ and $v_2$, respectively. By Theorem 13, $G_{n+1}^1$ is $(n-2)$-hamiltonian connected for $n \geq 3$. Since $f^1 = 0$, we may find a path $Q'$ between $v_1$ and $v_2$ containing $2^{n+1} - 1$ vertices in $G_{n+1}^1$. Then $\langle u_1, u_2, v_2, Q', v_1, u_1 \rangle$ forms a cycle of length $2^{n+1} + 1$.

**Case 2.** Both $f^0$ and $f^1$ are at most $n - 1$.

Since both of $G_{n+1}^0$ and $G_{n+1}^1$ are $(n-1)$-pancyclic for $n \geq 3$, both of $G_{n+1}^0 - F^0$ and $G_{n+1}^1 - F^1$ are still pancyclic. Without loss of generality, we assume that $f^0 \geq f^1$. We discuss the existence of cycles of all lengths from 4 to $2^n - f_v$ in the following cases.

**Case 2.1.** Cycles of lengths from 4 to $2^{n+1} - f_v^1$.

Since $G_{n+1}^1 - F^1$ is pancyclic for $n \geq 3$, we have cycles of lengths from 4 to $2^{n+1} - f_v^1$ in $G_{n+1}^1 - F^1$.

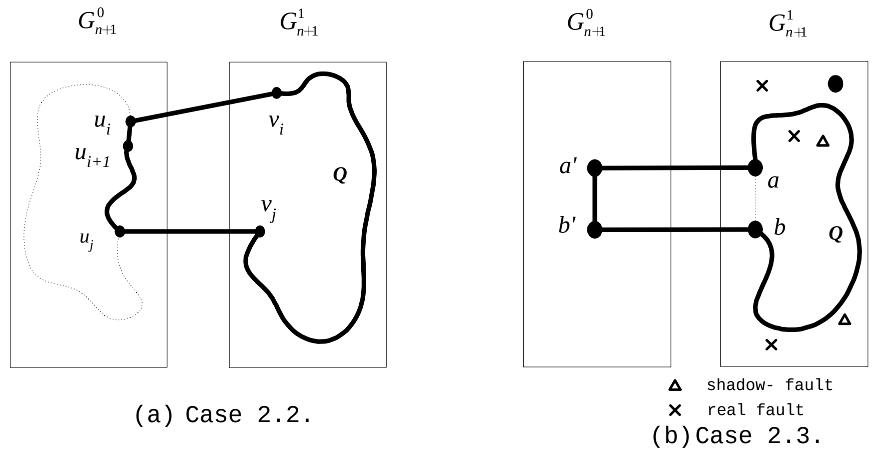



Figure 5.6: Case 2.2 and Case 2.3 of Theorem 14.

**Case 2.2.** Cycles of lengths from $2^{n+1} - f_v^1 + 2$ to $2^{n+2} - f_v$ (see Figure 5.6(a)).

Since $G_{n+1}^0 - F^0$ is pancyclic, we have a hamiltonian cycle $C = \langle u_0, u_1, \ldots, u_{2^{n+1}-f_v^0-1}, u_0 \rangle$ of length $2^{n+1} - f_v^0$ in $G_{n+1}^0 - F^0$. Let $2 \leq l \leq 2^{n+1} - f_v^0$. We construct a cycle of length $2^{n+1} - f_v^1 + l$ as follows: First, we claim that there exist two safe crossing-points $u_i$ and $u_j$ on $C$ such that $(j - i) = l - 1(\mathbf{mod}\ 2^{n+1} - f_v^0)$. Suppose on the contrary that there do not exist such $u_i$ and $u_j$. Then there are at least $\lceil \frac{2^{n+1}-f_v^0}{2} \rceil$ faults outside $G_{n+1}^0$. However, $\lceil \frac{2^{n+1}-f_v^0}{2} \rceil + f_v^0 \geq 2^n > n$ for $n \geq 0$. We obtain a contradiction. Thus, there exist such $u_i$ and $u_j$, and our claim is true. Secondly, we claim that $f^1 \leq n - 2$ for $n \geq 3$. Suppose for the sake of contradiction that $f^1 = n - 1$. Since $f^0 \geq f^1$, $f^0 = n - 1$. The total number of faults is at most $n$. Thus $(n - 1) + (n - 1) \leq n$. This implies that $n \leq 2$, which is a contradiction. This completes the proof of our second claim. By Theorem 13, $G_{n+1}^1$ is $(n-2)$-hamiltonian connected and $f^1 \leq n-2$ for $n \geq 3$. Hence $G_{n+1}^1 - F^1$ is hamiltonian connected. Let $v_i$ and $v_j$ be the neighbors in $G_{n+1}^1$ of $u_i$ and $u_j$, respectively. There is a hamiltonian path $Q$ in $G_{n+1}^1 - F^1$ between $v_i$ and $v_j$. Clearly, $Q$ contains $(2^{n+1} - f_v^1)$ vertices. Then $\langle u_i, u_{i+1}, \cdots, u_j, v_j, Q, v_i, u_i \rangle$ forms a cycle of length $2^{n+1} - f_v^1 + l$.

**Case 2.3.** A cycle of length $2^{n+1} - f_v^1 + 1$ (see Figure 5.6(b)).

We want to construct a cycle containing $2^{n+1} - f_v^1 - 1$ vertices in $G_{n+1}^1 - F^1$ and two vertices in $G_{n+1}^0 - F^0$. To avoid faults in $G_{n+1}^0$, we introduce a term called shadows of faults. Let $\langle u_1, u_2, v_2, v_1, u_1\rangle$ be a crossed 4-cycle with $u_1, u_2$ in $G_{n+1}^0$ and $v_1, v_2$ in $G_{n+1}^1$, respectively. If there is a fault on this cycle but the fault is not in $G_{n+1}^1$, we call edge $(v_1, v_2)$ a *shadow fault* of $F$ on $G_{n+1}^1$ (similarly, we may define a shadow fault on $G_{n+1}^0$). Let $F^s = \{e \mid \text{edge } e \text{ is a shadow fault of } F \text{ on } G_{n+1}^1\}$. Then $|F^s \cup F^1| \leq n$. If $|F^s \cup F^1| = n$, we arbitrarily choose an edge $e_1$ in $F^s$, and let $F' = F^s \cup F^1 - e_1$, otherwise let $F' = F^s \cup F^1$ . Then $|F'| \leq n - 1$ and $G_{n+1}^1 - F'$ is still pancyclic. Since $F' \cap V(G_{n+1}^1) = F_v^1$, there is a cycle $C$ of length $2^{n+1} - f_v^1 - 1$ in $G_{n+1}^1 - F'$. Since $2^{n+1} - f_v^1 - 1 > 2^n$ for $n \geq 3$, $C$ contains two critical edges. Let $(a, b) \neq e_1$ be a critical edge on $C$, so $(a, b) \notin F^s$. Let $a', b'$ be the neighbors of $a$ and $b$ in $G_{n+1}^0$, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n+1} - f_v^1 + 1$ in $TQ_{n+2} - F$. □

By Theorem 11, Theorem 14 and using the mathematical induction, we obtain the following theorem.

**Theorem 15** *The twisted cube $TQ_n$ is $(n-2)$-pancyclic for any odd integer $n \geq 3$.*

# Chapter 6

# Highly fault-tolerant cycle embeddings of hypercubes

## 6.1 Preliminaries

The hypercube $Q_n$ is a graph with $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$. Vertices are assigned binary strings of length $n$ ranging from 0 to $2^n - 1$. Two vertices are adjacent if they differ only in one bit position.

A bipartite graph is *hamiltonian laceable* if, for two arbitrary vertices $x$ and $y$ in different partite sets, there is a hamiltonian path connecting $x$ and $y$. A bipartite graph $G$ is *bipancyclic* if, for every even integer $l$ with $4 \leq l \leq |V(G)|$, $G$ has a cycle of length $l$.

Let $F \subseteq E(G)$ be a faulty set containing edges of $G$. $G - F$ denotes the subgraph of $G$ obtained by deleting the edges in $F$ from $G$. Let $k$ be a positive integer. A bipartite graph $G$ is $k$ *edge fault-tolerant hamiltonian laceable* (abbreviated as $k$ *fault-hamiltonian laceable* in this dissertation) if $G - F$ is hamiltonian laceable for every $F$ with $|F| \leq k$. A bipartite graph $G$ is $k$ *edge fault-tolerant bipancyclic* (abbreviated as $k$ *fault-bipancyclic*) if $G - F$ is bipancyclic for every $F$ with $|F| \leq k$. A bipartite graph $G$ is $k$ *conditional edge fault-tolerant bipancyclic* (abbreviated as $k$ *conditional fault-bipancyclic*) if $G - F$ is bipancyclic for every $F$ with $|F| \leq k$ under the condition that every vertex is incident with at least two nonfaulty edges.

The following lemma is proved in [49].

**Theorem 16** *[49] $Q_n$ is $(n-2)$ edge fault-tolerant hamiltonian laceable for $n \geq 2$.*

For convenience of further discussion, we say that $Q_n$ is divided into $Q^0_{n-1}$ and $Q^1_{n-1}$ along dimension $k$ for $0 \leq k \leq n-1$ if $Q^i_{n-1}$ is an $(n-1)$-dimensional hypercube which is a subgraph of $Q_n$ induced by the vertices labeled by $x_{n-1} \ldots x_{k+1} i x_{k-1} \ldots x_0$. We say that $(x, y) \in E(Q_n)$ is a $k$-*dimensional* edge if $x$ differs from $y$ in the $k$th position for $0 \leq k \leq n-1$. In addition, let $F \subset E(Q_n)$ be the set of faulty edges, $F_0 = F \bigcap E(Q^0_{n-1})$, and $F_1 = F \bigcap E(Q^1_{n-1})$.

## 6.2 Conditional fault bipancyclic

The following theorem states that, under the condition that each node of $Q_n$ is incident with at least two healthy links, an injured $Q_n$ is still bipancyclic with $F \leq 2n-5$ for $n \geq 3$. We note that this condition implies that the number of faulty edges incident to any vertex is at most $n-2$.

Tsai [50] proved the following theorem at the same time we did. We had not been informed until this work was completed. We obtained the result independently.

**Theorem 17** *The hypercube $Q_n$ is $(2n-5)$ conditional fault-bipancyclic for $n \geq 3$.*

**Proof.** We prove this by induction on $n$. It is straightforward to see that $Q_3$ is 1 edge fault-tolerant bipancyclic. Since $2 \times 3 - 5 = 1$, the theorem holds for $n = 3$. Assume that $Q_{n-1}$ is $2(n-1)-5 = 2n-7$ conditional fault-bipancyclic for some $n \geq 4$. We shall prove that $Q_n$ is $(2n-5)$ conditional fault-bipancyclic. There are three possible fault distributions:

(1) There is only one vertex incident with $n-2$ faulty edges. Without loss of generality, we may assume that one of these $n-2$ faulty edges is an $(n-1)$-dimensional edge.

(2) There are two vertices which share a faulty edge and are both incident with $n-2$ faulty edges. Without loss of generality, we may assume that the faulty edge they share is an $(n-1)$-dimensional edge.

(3) Every vertex is incident with less than $n-2$ faulty edges. We may assume without loss of generality that one of them is an $(n-1)$-dimensional edge.

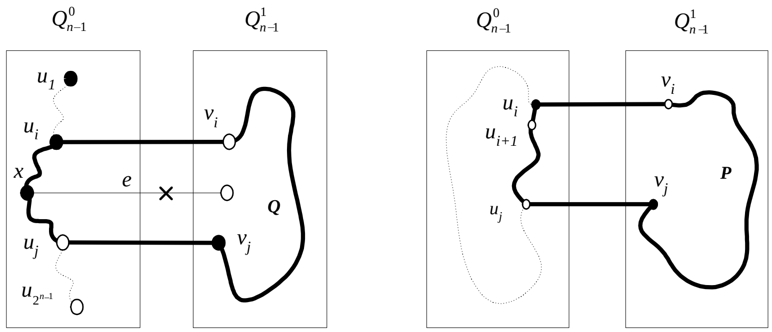
Note that there cannot be more than two vertices which are incident with $n-2$ faulty edges for $n \geq 3$. Then, we can divide $Q_n$ into $Q^0_{n-1}$ and $Q^0_{n-1}$ along dimension $n-1$. So both of $Q^0_{n-1}$ and $Q^1_{n-1}$ satisfy the condition that every vertex is incident with at least two nonfaulty edges. Furthermore, we may assume without loss of generality that $|F^0| \geq |F^1|$. Since $\frac{(2n-5)-1}{2} = n-3$, $|F^1| \leq n-3$. We discuss the existence of cycles of all even lengths from 4 to $2^n$ in the following two cases.

**Case 1.** Cycles of even lengths from 4 to $2^{n-1}$. Note that $|F^1| \leq n-3 \leq 2n-7$ for $n \geq 4$. By induction hypothesis, $Q^1_{n-1}$ is $(2n-7)$ conditional fault-bipancyclic, so $Q^1 - F^1$ contains cycles of all even lengths from 4 to $2^{n-1}$.

**Case 2.** Cycles of even lengths from $2^{n-1}+2$ to $2^n$. We divide this case further into two subcases.

**Case 2.1.** $|F^0| = 2n-6$ (see Figure 6.1 (a)). Hence, there is only one $(n-1)$-dimensional faulty edge, say $e$. Let $x \in V(Q^0_{n-1})$ be the vertex incident with $e$. Notice that there are at most $n-3$ faulty edges incident with $x$ in $Q^0_{n-1}$. Since $2n-6 > n-3$ for $n \geq 4$, there must be a faulty edge in $Q^0_{n-1}$, say $e'$, such that it is not incident to $x$. Let $F' = F^0 - e'$. Clearly, $|F'| = 2n-7$. By induction hypothesis, $Q^0_{n-1}$ is $(2n-7)$ conditional fault-bipancyclic, so $Q^0_{n-1} - F'$ contains a hamiltonian cycle, say $C$. Then, $Q^0_{n-1} - F^0$ contains a hamiltonian path on $C$, say $P = \langle u_1, u_2, \ldots, u_{2^{n-1}} \rangle$, such that $u_1 \neq x$ and $u_{2^{n-1}} \neq x$. Let $2 \leq l \leq 2^{n-1}$ be an even integer. We construct a cycle of length $2^{n-1}+l$ as follows. Since the edge $e$ is the only faulty edge in $(n-1)$-dimension, there must exist two vertices $u_i$ and $u_j$ such that the two $(n-1)$-dimensional edges incident to $u_i$ and $u_j$, respectively, are nonfaulty and $j - i = l - 1$. Let $v_i$ and $v_j$ be the neighbors of $u_i$ and $u_j$ in $Q^1_{n-1}$, respectively. Since $j-i$ is odd, $u_i$ and $u_j$ are in different partite sets, and then $v_i$ and $v_j$ are also in different partite sets. By Theorem 16, $Q^1_{n-1}$ is $(n-3)$ fault-hamiltonian laceable. Since $|F^1| \leq n-3$, $Q^1_{n-1} - F^1$ contains a hamiltonian path, say $Q$. Then, $\langle u_i, u_{i+1}, \cdots, u_j, v_j, Q, v_i, u_i \rangle$ is a cycle of length $2^{n-1}+l$ in $Q_n - F$.

**Case 2.2.** $|F^0| \leq 2n-7$ (see Figure 6.1 (b)). By induction hypothesis, $Q^0_{n-1}$ is $(2n-7)$ conditional fault-bipancyclic. Therefore, $Q^0_{n-1} - F^0$ contains a hamiltonian cycle,




(a)Case 2.1 (c) Case 2.2

Figure 6.1: Case 2.1 and Case 2.2 of Theorem 17.

say $C = \langle u_1, u_2, \ldots, u_{2^{n-1}}, u_1\rangle$. Let $l$ be an even integer for $2 \leq l \leq 2^{n-1}$. We construct a cycle of length $2^{n-1} + l$ as follows: First, we claim that there exist two vertices $u_i$ and $u_j$ on $C$ such that the two $(n-1)$-dimensional edges incident to $u_i$ and $u_j$, respectively are nonfaulty, and $(j - i) = l - 1(\mathbf{mod}\ 2^{n-1})$. Suppose on the contrary that there do not exist such $u_i$ and $u_j$. Then there are at least $\frac{2^{n-1}}{2} = 2^{n-2}$ $(n-1)$-dimensional faulty edges. However, $2^{n-2} > 2n - 5$ for $n \geq 4$. We obtain a contradiction. Thus, there exist such $u_i$ and $u_j$. By Theorem 16, $Q^1_{n-1}$ is $(n-3)$ fault-hamiltonian laceable. Since $|F^1| \leq n - 3$, $Q^1_{n-1} - F^1$ is still hamiltonian laceable. Let $v_i$ and $v_j$ be the neighbors of $u_i$ and $u_j$ in $Q^1_{n-1}$, respectively. Since $u_i$ and $u_j$ are in different partite sets, $v_i$ and $v_j$ are also in different partite sets. There is a hamiltonian path $P$ in $Q^1_{n-1} - F^1$ between $v_i$ and $v_j$. Then $\langle u_i, u_{i+1}, \cdots, u_j, v_j, P, v_i, u_i\rangle$ forms a cycle of length $2^{n-1} + l$. □

The above result is optimal in the sense that if there are more than $2n-5$ faulty edges, there is no guarantee to have a fault-free cycle in an injured hypercube. For example, let $\langle a, b, c, d, a\rangle$ be a 4-cycle in $Q_n$ (see Figure 6.2). Assume that all the edges incident to $b$ are faulty except $(a, b)$ and $(b, c)$, and all the edges incident to $d$ are faulty except $(a, d)$ and $(d, c)$. Then, there are $(n - 2) + (n - 2) = 2n - 4$ faulty edges, and there is no hamiltonian cycle in the injured hypercube.

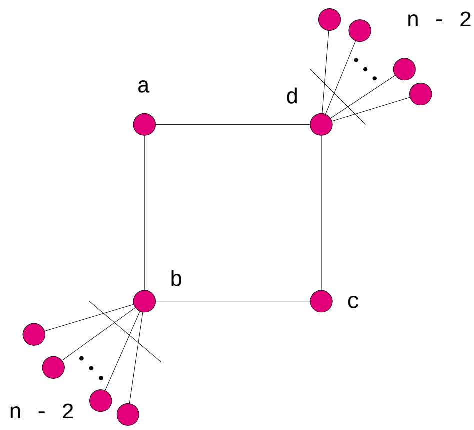



Figure 6.2: There are $n-2$ fault edges incident to $b$ and $d$, respectively, and the injured hypercube has no hamiltonian cycle.

## 6.3 Smaller lengths of cycles

If the condition is not satisfied, i.e., there are more than $n-2$ faulty edges incident to a certain vertex, there cannot be a hamiltonian cycle in an injured hypercube. But what about the other cycles of even lengths? The following two theorems address this problem. Our finding is that the hamiltonian cycle is the only missing cycle.

**Theorem 18** *If there are* 2 *or* 3 *faulty edges incident to some vertex in* $Q_3$*, then* $Q_3 - F$ *contains cycles of lengths* 4 *and* 6 *with* $|F| \leq 3$.

**Proof.** Since $Q_3$ is vertex symmetric (see Figure 6.3), we may assume that 000 is incident with 2 or 3 faulty edges. First, suppose that there are two faulty edges incident to 000, and the other faulty edge, say $e$, is not. We observe that $e$ is not on one of the 4-cycles (see Figure 6.3 (a), (b), and (c)): $\langle 100, 101, 111, 110, 100\rangle$, $\langle 001, 101, 111, 011, 001\rangle$, and $\langle 010, 011, 111, 110, 010\rangle$. Also, $e$ is not on one of the 6-cycles (see Figure 6.3 (d), (e), and (f)): $\langle 010, 011, 001, 101, 111, 110, 010\rangle$, $\langle 001, 101, 100, 110, 111, 011, 001\rangle$, and $\langle 010, 011, 111, 101, 100, 110, 010\rangle$. Therefore, we have cycles of lengths 4 and 6 in $Q_3 - F$. Second, suppose that there are three faulty edges incident to 000. Since 000 is not on all the above cycles, $Q_3 - F$ contains all the above cycles. The proof is complete. $\square$

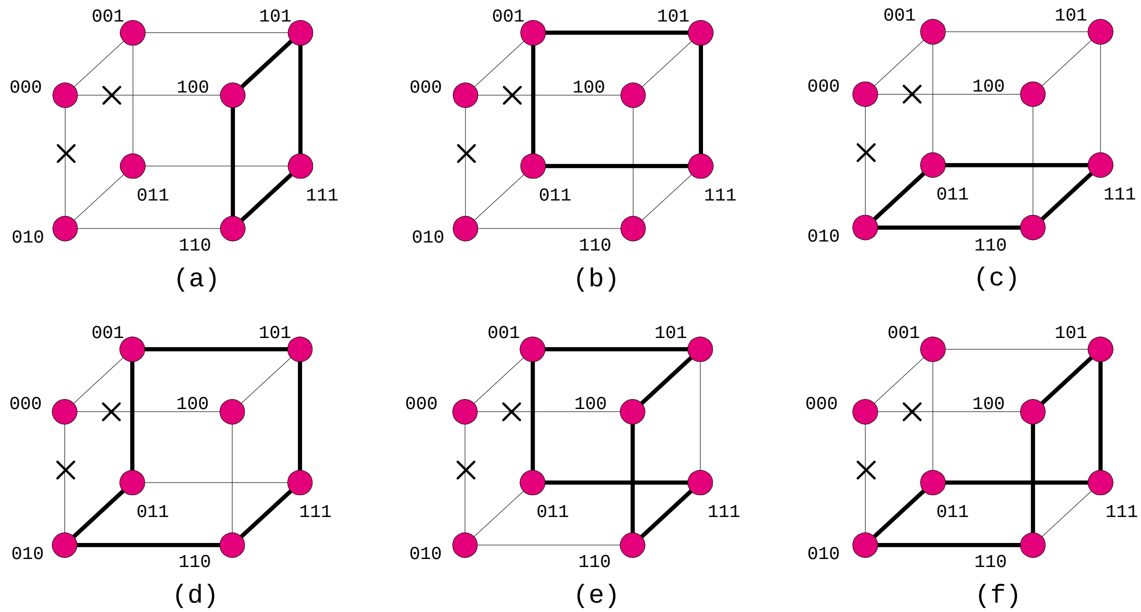



Figure 6.3: 4-cycles and 6-cycles in $Q_3 - F$.

**Theorem 19** *If there are more than $n-2$ faulty edges incident to some vertex in $Q_n$, then $Q_n - F$ contains cycles of all even lengths from 4 to $2^n - 2$ with $|F| \leq 2n-3$ for $n \geq 3$.*

**Proof.** This theorem is proved by induction on $n$. By Theorem 18, the theorem holds for $n = 3$. Assume that the theorem is true for $Q_{n-1}$ with some $n \geq 4$. We shall prove that $Q_n - F$ contains cycles of all even lengths from 4 to $2^n - 2$ with $|F| \leq 2n-3$. Note that we only need to consider the case $|F| = 2n-3$. There is only one vertex, say $a$ which is incident with at least $n-1$ faulty edges. Since $(2n-3) - n = n-3 \geq 1$ for $n \geq 4$, there is a faulty edge which is not incident to $a$. Without loss of generality, we may assume that it is an $(n-1)$-dimensional edge. Then, we divide $Q_n$ into $Q^0_{n-1}$ and $Q^0_{n-1}$ along dimension $n-1$. Without loss of generality, assume that $a \in V(Q^0_{n-1})$. Then, $|F^1| \leq (2n-3) - (n-1) - 1 = n-3$. By Theorem 16, $Q^1_{n-1}$ is $(n-3)$ fault-hamiltonian laceable, so $Q^1_{n-1} - F^1$ is still hamiltonian laceable. We discuss the existence of cycles of all even lengths from 4 to $2^n - 2$ as follows.

**Case 1.** Cycles of even lengths from 4 to $2^{n-1}$. By Theorem 17, $Q^1_{n-1}$ is $(2n-7)$ conditional fault-bipancyclic, Since $|F^1| \leq n-3$, we have cycles of all even lengths from 4 to $2^{n-1}$ in $Q^1 - F^1$.

**Case 2.** Cycles of even lengths from $2^{n-1}+2$ to $2^n-2$. The proof is similar to the one in Case 2.2 of Theorem 17. First, if $|F^0| \le 2n-5$, by induction hypothesis, $Q^0_{n-1} - F^0$ contains a cycle of length $2^{n-1}-2$. Second, if $|F^0| = 2n-4$, there is only one $(n-1)$-dimensional faulty edge, and the $(n-1)$-dimensional edge incident to $a$ is nonfaulty. Hence, all the $n-1$ edges incident to $a$ in $Q^0_{n-1}$ is faulty. Let $e'$ be one of this $n-1$ faulty edges. Let $F' = F^0 - e'$. Then, $|F'| = 2n-5$. By induction hypothesis, $Q^0_{n-1} - F'$ contains a cycle of length $2^{n-1}-2$. Since $a$ is not on this cycle, $Q^0_{n-1} - F^0$ also contains this cycle. Let $l$ be an even integer for $2 \le l \le 2^{n-1}-2$. Repeating the argument in the proof of Case 2.2 of Theorem 17, if $(n-1)$-dimensional faulty edges are less than $\frac{2^{n-1}-2}{2} = 2^{n-2}-1$, we have a cycle of length $2^{n-1}+l$ in $Q_n - F$. $2^{n-2}-1 \ge (2n-3)-(n-2)$ ($a$ is incident with at least $n-2$ faulty edges in $Q^0_{n-1}$) for $n \ge 4$, and the equality holds only when $n=4$, i.e., $Q_4$ contains three $(n-1)$-dimensional faulty edges. In this situation, there are three faulty edges incident to $a$, and the other two faulty edges, say $e_1$ and $e_2$, are $(n-1)$-dimensional. We can divide $Q_4$ into $Q^0_3$ and $Q^1_3$ along a dimension $k$, $0 \le k \le 3$, such that $e_1$ and $e_2$ are in different $(n-1)$-dimensional subcubes. Hence, $|F^1| = 1$, and there are at most 2 $k$-dimensional faulty edges. And either $|F^0| \le 3$, or $a$ is incident with 3 faulty edges in $Q^0_3$. Hence, this theorem is proved. □

Let $x, y \in V(Q_n)$ be two vertices in the same partite set. Suppose that there are $n-1$ faulty edges incident to $x$ and $y$, respectively. There cannot be a cycle of length $2^n-2$ in $Q_n - F$. Therefore, the number of faulty edges, $2n-3$, provided in the above theorem is maximum.

In fact, we have found cycles of all possible lengths in $Q_n - F$ with $|F| \le 2n-5$ under all possible fault distributions. By Theorem 17 and Theorem 19 , the following theorem follows.

**Theorem 20** *Suppose that $|F| \le 2n-5$ and $n \ge 4$. If the condition that every vertex in $Q_n$ has at least two nonfaulty edges is satisfied, $Q_n - F$ contains cycles of all even lengths from 4 to $2^n$. Otherwise, $Q_n - F$ contains cycles of all even lengths from 4 to $2^n-2$.*

# Chapter 7

# Conclusion, Discussion, and Future Work

In Chapter 2, we show how to find a hamiltonian cycle and a hamiltonian path joining two arbitrary vertices in a wounded $k$-ary $n$-cube. When $k$ is an odd integer, we have proved that $Q_n^k$ is $(2n-2)$-hamiltonian and $(2n-3)$-hamiltonian connected. Furthermore, our results are optimal. For even integer $k$, $Q_n^k$ is a bipartite graph, and it is easy to check that $Q_n^k$ contains a hamiltonian cycle. However, with one single vertex fault, the remaining network does not contain any hamiltonian cycle. Therefore, for the fault-tolerant hamiltonian and hamiltonian connected properties of $Q_n^k$, we can only consider edge faults. Let $F_e \subseteq E(Q_n^k)$ be the set of faulty edges in $Q_n^k$ with $|F_e| \leq 2n-2$ (not $2n-3$). For even integer $k$, we propose to show that $Q_n^k - F_e$ has a hamiltonian path connecting two arbitrary vertices belonging to different partite sets and a path of maximum length, $k^n - 2$, connecting two arbitrary vertices in the same partite set for every $n \geq 2$ and even $k \geq 4$. This problem has not yet been resolved.

The fault-tolerant hamiltonian and hamiltonian connected properties are fundamental tools for exploring further properties concerning cycle or path embedding problems. For example, we have studied the fault-tolerant pancyclicity of Möbius cubes by using the fault-tolerant hamiltonian and hamiltonian connected properties of Möbius cubes. In addition, by employing hamiltonian cycles and paths in faulty hypercubes, linear array and cycle embeddings in conditional faulty hypercubes were resolved [50].

The crossed cube $CQ_n$, the Möbius cube $MQ_n$, and the twisted cube $TQ_n$ are alternatives to the hypercube architecture in the parallel computing area. From Chapter 3 to Chapter 5, we provide a superior property of the above three networks to the hypercube, the fault-tolerant pancyclicity. With $f_e$ faulty links and $f_v$ faulty nodes, where $f_e + f_v \leq n-2$, we can find cycles of all lengths except 3 in these networks. However, the hypercube, as a bipartite graph, does not contain any odd cycles even if it is fault-free. Furthermore, if there exist $n-1$ faulty elements around a single vertex, then they cannot have a hamiltonian cycle. Hence, the number of $n-2$ faults is the most that $CQ_n$, $MQ_n$, and $TQ_n$ can be tolerant with respect to pancyclic property. The above results show that all of them have a very good fault-tolerant property.

In chapter 6, we extend the result of [40] by restricting fault distributions to increase the degree of fault tolerance, and we prove that the hypercube is $2n-5$ conditional fault-bipancyclic. Then, we show that with up to $2n-3$ faulty edges if a certain vertex is incident with less than two nonfaulty edges, an injured $Q_n$ has a cycle of length $l$ for every even $l$, $4 \leq l \leq 2^n - 2$. Therefore, the degree of fault tolerance doubles that of [40].

We propose some interesting problems that can be studied further:

1. We have studied the fault-tolerant pancyclic property on some cube networks. Accordingly, we have a question here. Given an arbitrary vertex (or edge), do all lengths of cycles containing the vertex (or edge) exist? That is, the fault-tolerant vertex (or edge) pancyclicity is a problem to be considered.

2. Conditional bipancyclicity have been studied on hypercube in this dissertation. What about conditional pancyclicity on nonbipartite networks?

3. Other conditional properties can be addressed, for example, conditional connectivity.

4. When considering the bipancyclic problem on hypercube, we observed that the degree of fault-tolerance is limited by the worst case. Furthermore, we have broken the limitation by restricting the fault distribution or obtaining a weaker result, in this case, abandoning the hamiltonian cycle. It is interesting to observe other limitations about other properties.

5. By restricting that each vertex is incident with at least two healthy edges, the degree of fault-tolerance is almost doubled with respect to bipancyclicity. We derive a question

here. Are there other reasonable conditions?

# Appendix A

In the following, we construct cycles of lengths from $2^{n-1} - f_v^1 + 2$ to $2^n - f_v$ for the case $f^0 = f^1 = n - 3$. Since $f^0 + f^1 = 2n - 6 \leq n - 2$, $n \leq 4$. Thus, we need only to discuss the case $f^0 = f^1 = 1$ for $n = 4$ here. We shall use some symmetric properties of $CQ_3$ to reduce the cases.

For convenience of discussion, (See Figure 7.1(a).) we call $(000, 010)$, $(001, 011)$, $(111, 101)$, $(110, 100)$ as *inner edges* of $CQ_3$ and $(000, 001)$, $(001, 111)$, $(111, 110)$, $(110, 010)$, $(010, 011)$, $(011, 101)$, $(101, 100)$, $(100, 000)$ as *outer edges* of $CQ_3$ , respectively. Let $x$ be a vertex of $CQ_3$. An inner edge $e$ is said to be an *$N$-edge* of $x$ if $x$ connects to one of the endpoints of $e$. Hence $CQ_3$ has two $N$-edges of $x$. An inner edge $e$ is said to be an *$H$-edge* of $x$ if $x$ is not incident to $e$, and $e$ is not an $N$-edge of $x$. Therefore, $CQ_3$ has one $H$-edge of $x$.

To explore the pancyclicity of $CQ_4 - F$, we need an observation, and it is stated in the following lemma.

**Lemma 11** *Let $x$ be a faulty vertex in $CQ_3$. Then the two $N$-edges of $x$, $e_1$ and $e_2$, are on cycles of lengths from 4 to 7 in $CQ_3 - x$. And the $H$-edge of $x$ is on cycles of lengths 4, 5, and 7 in $CQ_3 - x$.*

**Proof.** Without loss of generality, we may assume that $x = 000$. Then the two $N$-edges of $x$ are $(001, 011)$ and $(110, 100)$, and the $H$-edge of $x$ is $(111, 101)$. We list all the cycles as follows: $\langle 001, 111, 101, 011, 001\rangle$, $\langle 111, 110, 100, 101, 111\rangle$, $\langle 001, 111, 110, 010, 011, 001\rangle$, $\langle 111, 110, 010, 011, 101, 111\rangle$, $\langle 110, 010, 011, 101, 100, 110\rangle$ , $\langle 001, 111, 110, 100, 101, 011, 001\rangle$, and $\langle 001, 111, 101, 100, 110, 010, 011, 001\rangle$. $\square$

We continue to discuss cycles in $CQ_4 - F$, and consider two situations: (1) cycles of lengths from 8 to 14 and (2) cycles of lengths 15 and 16.

**Case 1. Cycles of lengths from** $8$ **to** $14$**.** Suppose that there is a faulty vertex $x$ in $CQ_3^0$ or a faulty edge $e_1$ which is incident to $x$. Let $(u_1, u_2)$ and $(u_3, u_4)$ be the two $N$-edges of $x$. By Lemma 11, $(u_1, u_2)$ and $(u_3, u_4)$ are on cycles of lengths from 4 to 7 in $CQ_3^0 - F^0$. Let $v_1$, $v_2$, $v_3$, and $v_4$ be the neighbors of $u_1$, $u_2$, $u_3$, and $u_4$ in $CQ_3^1$, respectively. It is not difficult to check that both $(v_1, v_2)$ and $(v_3, v_4)$ are inner edges of $CQ_3^1$. (See Figure 7.1(b).) And $(v_1, v_2)$ can not reach $(v_3, v_4)$ via exactly one edge of $CQ_3^1$. Suppose that the other fault is a faulty vertex $y$ in $CQ_3^1$, or a faulty edge $e_2$ which is incident to $y$. Then $(v_1, v_2)$ or $(v_3, v_4)$, say $(v_1, v_2)$ is an $N$-edge or $H$-edge of $y$ in $CQ_3^1 - F^1$. By Lemma 11, $(v_1, v_2)$ is on cycles of lengths 4, 5, and 7 in $CQ_3^1 - F^1$. We use $C_i$ to denote a cycle of length $i$. Let $C_i$ and $C_j$ be cycles containing $(u_1, u_2)$ and $(v_1, v_2)$, respectively, $4 \le i \le 7$, $j = 4$, 5, or 7. Then we can construct a cycle $C_l$ from $C_i$ and $C_j$ by adding $(u_1, v_1)$ and $(u_2, v_2)$, and deleting $(u_1, u_2)$ and $(v_1, v_2)$ for $8 \le l \le 14$. For example, Figure 7.1(b) shows a cycle of length 12 in $CQ_4$ with one faulty vertex 0000 and one faulty edge $(1100, 1110)$.

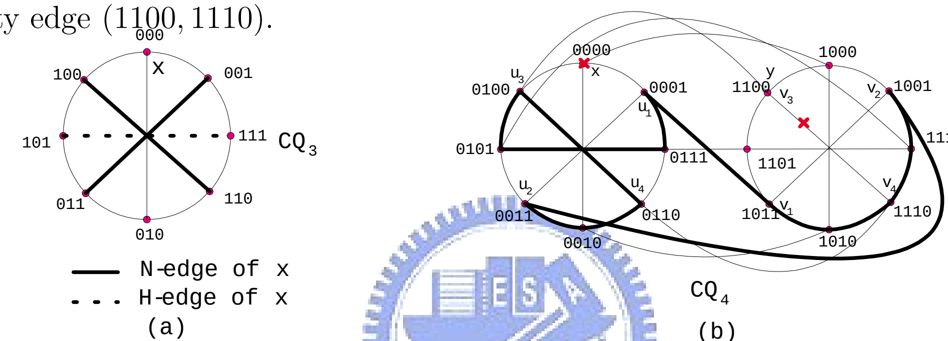



Figure 7.1: (a) $N$-edge and $H$-edge of $x$ and (b) a cycle of length 12 with one faulty vertex 0000 and one faulty edge $(1100, 1110)$.

**Case 2. Cycles of lengths from** $15$ **to** $16$**.** If $f_v = 2$, we need only to find cycles of lengths from 4 to 14 which we did in the previous cases. If $f_v = 1$ and $f_e = 1$, we have to find a cycle of length 15. By Theorem 8, $CQ_4$ is 2-hamiltonian, so there is a cycle of length 15 in $CQ_4 - F$. If $f_e = 2$, since $CQ_4$ is 2-hamiltonian, there is also a cycle of length 16. Suppose that $F = \{(x_1, y_1), (x_2, y_2)\}$. Let $F' = \{(x_1, y_1), x_2\}$. Then there is a cycle of length 15 in $CQ_4 - F'$. This cycle is also fault-free in $CQ_4 - F$.

# Appendix B

In this appendix, we show that $MQ_3$ is 1-pancyclic, and $MQ_4$ is 2-pancyclic.

**Theorem 21** *$MQ_3$ is 1-pancyclic.*

**Proof.** 0-$MQ_3$ and 1-$MQ_3$ are displayed in Figure 7.2. We remark that these two are isomorphic, and both are vertex symmetric. Therefore, we may consider only 0-$MQ_3$. We discuss two cases: (1) one vertex fault and (2) one edge fault as follows.

**Case 1.** One vertex fault.

Since $MQ_3$ is vertex symmetric, we may assume that vertex $x = 000$ is faulty , and $x$ is in 0-$MQ_3$. We list cycles of lengths from 4 to 7 as follows: $\langle 100, 101, 110, 111, 100\rangle$, $\langle 100, 101, 001, 011, 111, 100\rangle$, $\langle 100, 101, 110, 010, 011, 111, 100\rangle$, and $\langle 100, 101, 001, 011, 010, 110, 111, 100\rangle$.

**Case 2.** One edge fault.

Without loss of generality, we assume that the faulty edge $e$ is incident to $x = 000$. In Case 1, we have cycles of lengths from 4 to 7. For a cycle of length 8, if $e = (000, 001)$, then $\langle 000, 100, 111, 011, 001, 101, 110, 010, 000\rangle$ is the desired one. If $e = (000, 010)$, then $\langle 000, 100, 111, 011, 010, 110, 101, 001, 000\rangle$ is a cycle of length 8. For $e = (000, 100)$, the case is symmetric to the case $e = (000, 010)$. $\square$

We say that an edge $(u, v)$ is a *$k$-dimensional edge* in $MQ_n$ if $v$ is the $k$th neighbor of $u$ in $MQ_n$. To show that $MQ_4$ is 2-pancyclic, we need one observation.

**Lemma 12** *Let $x$ be a faulty vertex in $MQ_3$. Let $u_1$ and $u_2$ be the first and 2nd neighbors of $x$, respectively. Then the two 3-dimensional edges $(u_1, u_3)$ and $(u_2, u_4)$ are on cycles of lengths from 4 to 7 in $MQ_3 - x$.*

**Proof.** Without loss of generality, we may assume that $x = 000$, which is in 0-$MQ_3$. Then $u_1 = 100$, $u_2 = 010$, $u_3 = 101$, and $u_4 = 011$. We list all the cycles as follows: $\langle 100, 101, 110, 111, 100\rangle$, $\langle 010, 011, 111, 110, 010\rangle$, $\langle 100, 101, 001, 011, 111, 100\rangle$, $\langle 010, 011, 001, 101, 110, 010\rangle$, $\langle 100, 101, 110, 010, 011, 111, 100\rangle$, and $\langle 100, 101, 001, 011, 010, 110, 111, 100\rangle$. $\square$

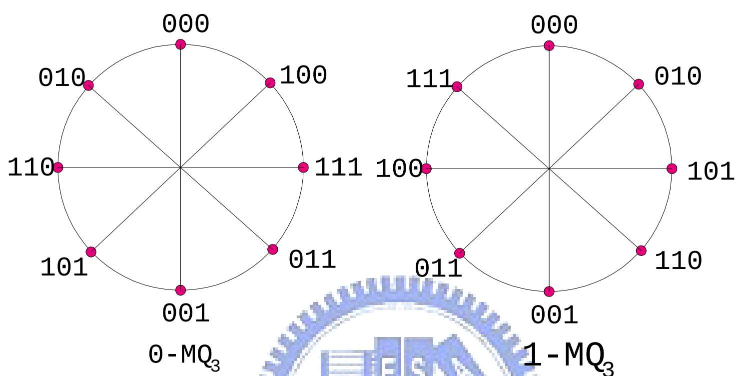



Figure 7.2: $MQ_3$.

**Theorem 22** *$MQ_4$ is 2-pancyclic.*

**Proof.** We consider two situations: (1) There is a subcube containing 2 faults, and (2) both $f^0$ and $f^1$ are at most 1.

**Case 1.** There is a subcube containing 2 faults.

Without loss of generality, we assume that $f^0 = 2$. Thus, $f^1 = f_e^c = 0$. We discuss the existence of cycles of every length from 4 to $16 - f_v$ according to the following cases.

**Case 1.1.** Cycles of lengths from 4 to 8.

By Theorem 21, $MQ_3$ is 1-pancyclic. Thus $MQ_3^1$ contains cycles of lengths from 4 to 8. Clearly, $MQ_n - F$ also contains cycles of these lengths.

**Case 1.2.** Cycles of lengths from 10 to $16 - f_v$.

By the same method used in Case 1.2 of Theorem 9, we have cycles of lengths from 10 to $16 - f_v$.

**Case 1.3.** A cycle of length 9.

We claim that, for any two vertices $x$ and $y$ in 0-$MQ_3$, there exists a path containing 7 vertices connecting $x$ and $y$ in 0-$MQ_3$. Due to the symmetry of 0-$MQ_3$, it is sufficient to consider the following four cases, $x = 000$, and $y = 100$, 111, 011, or 001 in 0-$MQ_3$. For each case, we list the path as follows: $\langle 000, 001, 011, 010, 110, 101, 100\rangle$, $\langle 000, 010, 110, 101, 001, 011, 111\rangle$, $\langle 000, 010, 110, 101, 100, 111, 011\rangle$, and $\langle 000, 100, 111, 110, 010, 011, 001\rangle$. Then we consider $MQ_4$. Since both $MQ_3^0$ and $MQ_3^1$ are isomorphic to 0-$MQ_3$, the above claim holds in $MQ_3^0$ and $MQ_3^1$. Suppose that $MQ_3^0$ contains 2 faults. Then, we can find a fault-free edge $(u_1, u_2)$ in the injured $MQ_3^0$. Let $v_1$ and $v_2$ be the first neighbors of $u_1$ and $u_2$, respectively. We have a path $Q$ containing 7 vertices in $MQ_3^1$ between $v_1$ and $v_2$. Then $\langle u_1, v_1, Q, v_2, u_2, u_1\rangle$ forms a cycle of length 9.

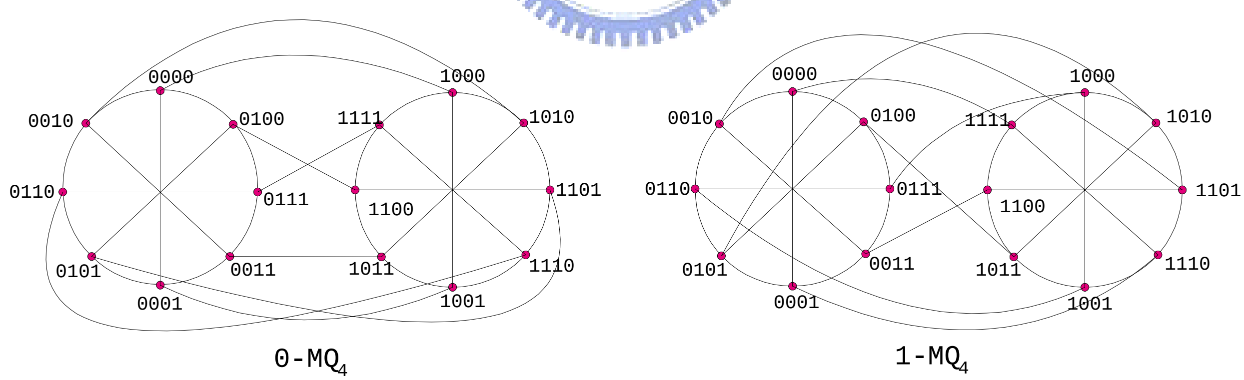



Figure 7.3: An equivalent form of $MQ_4$.

**Case 2.** Both $f^0$ and $f^1$ are at most 1.

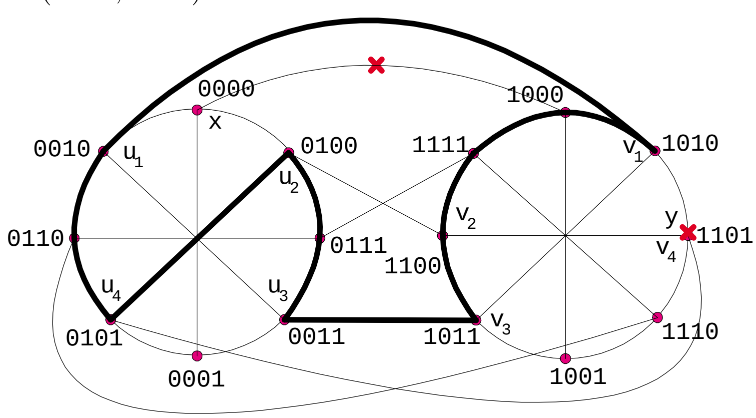
We discuss the existence of cycles of every length from 4 to $16 - f_v$ according to the following cases.

**Case 2.1.** Cycles of lengths from 4 to 7.

By Theorem 21, $MQ_3^0$ is 1-pancyclic. Thus $MQ_3^0 - F^1$ contains cycles of lengths from 4 to 7.

**Case 2.2.** Cycles of lengths from 8 to 14.

Suppose that there is a vertex fault $x$ in $MQ_3^0$ or an edge fault $e_1$ which is incident to $x$. Let $u_1$ and $u_2$ be the 2nd and 3rd neighbors of x, respectively. By Lemma 12, the two 4-dimensional edges $(u_1, u_3)$ and $(u_2, u_4)$ are on cycles of lengths from 4 to 7 in $MQ_3^0 - F^0$. We note that these two 4-dimensional edges are in fact 3-dimensional edges in $MQ_3$. Let $v_1$, $v_2$, $v_3$, and $v_4$ be the first neighbors of $u_1$, $u_2$, $u_3$, and $u_4$, respectively. It is not difficult to check that both $(v_1, v_3)$ and $(v_2, v_4)$ are 4-dimensional edges, and $v_1$ is a neighbor of $v_2$ or $v_4$. (See Figure 7.3 and Figure 7.4.) Suppose that the other fault is a vertex fault $y$ in $MQ_3^1$, or an edge fault $e_2$ which is incident to $y$. Then $y$ is either the 2nd or 3rd neighbor of one of $\{v_1, v_2, v_3, v_4\}$, say, $v_1$. By Lemma 12, $(v_1, v_3)$ is on cycles of lengths from 4 to 7 in $MQ_3^1 - F^1$. We use $C_i$ to denote a cycle of length $i$. Let $C_i$ and $C_j$ be cycles containing $(u_1, u_3)$ and $(v_1, v_3)$, respectively, $4 \leq i, j \leq 7$. Then we can construct a cycle $C_l$ from $C_i$ and $C_j$ by adding $(u_1, v_1)$ and $(u_3, v_3)$, and deleting $(u_1, u_3)$ and $(v_1, v_3)$ for $8 \leq l \leq 14$. For example, Figure 7.4 shows a cycle of length 11 in 0-$MQ_4$ with one vertex fault 1101 and one edge fault $(0000, 1000)$.




0-$MQ_4$

Figure 7.4: Illustration of $C_{11}$ in $MQ_4 - F$.

**Case 2.3.** Cycles of lengths from 15 to 16.

Suppose that there are one edge fault and one vertex fault in $MQ_4$. Then we need to find cycles of lengths from 4 to 15. By Theorem 8, $MQ_4$ is 2-hamiltonian. Thus $MQ_4 - F$ contains a cycle of length 15. Suppose that there are two edge faults $e_1$ and $e_2$. Since $MQ_4$ is 2-hamiltonian, $MQ_4 - F$ has a cycle of length 16. For a cycle of length 15, we may virtually suppose that one vertex $x$, which is incident to $e_1$, is faulty. Taking vertex $x$ and edge $e_2$ as faults, and using the fact that $MQ_4$ is 2-hamiltonian, $MQ_4 - F$ has a cycle of length 15. Suppose that there are two vertex faults. We only need to find cycles of lengths from 4 to 14, and we have shown them in the previous cases.

# Appendix C

In the following, we construct cycles of lengths from $2^n - f_v^1 + 2$ to $2^{n+1} - f_v$ in $TQ_n \times K_2$ for the case $f^0 = f^1 = n - 2$. Since $f^0 + f^1 = 2n - 4 \leq n - 1$, $n \leq 3$. Thus, we need only to discuss the case $f^0 = f^1 = 1$ for $n = 3$ here.

First, we show the case $f_v^0 = f_v^1 = 1$, and thus find cycles of lengths from 9 to 14 in $TQ_3 \times K_2 - F$. Let $F = \{u, v\}$ for $u \in V(TQ_3^0)$ and $v \in V(TQ_3^1)$. We need only discuss five cases due to the symmetry of $TQ_3 \times K_2$ (see Figure 7.5): (1) $u = 0000$, $v = 1000$, (2) $u = 0000$, $v = 1110$, (3) $u = 0000$, $v = 1111$, (4) $u = 0000$, $v = 1101$, and (5) $u = 0000$, $v = 1100$. They are listed one by one in Table 7.1.

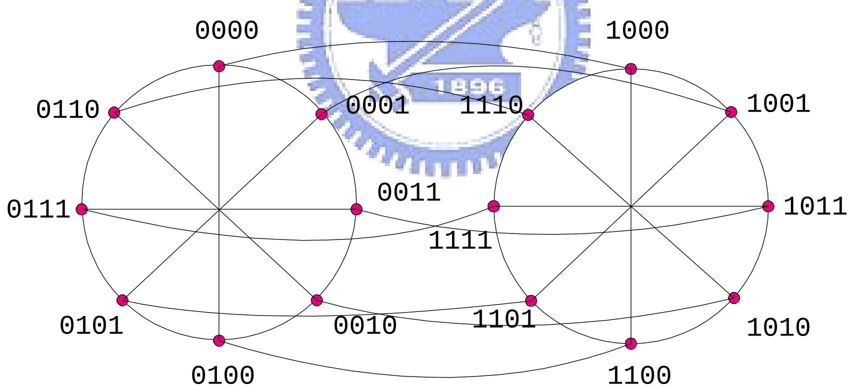



Figure 7.5: $TQ_3 \times K_2$

Secondly, consider that $f_v = 1$ and $f_e = 1$. We find cycles of lengths from 8 to 14 as follows. Let $F = \{u_1, (u_2, v_2)\}$ and $F' = \{u_1, u_2\}$. From the above discussion, there are cycles of lengths from 9 to 14 in $TQ_3 \times K_2 - F'$, which are also in $TQ_3 \times K_2 - F$. Furthermore, since $TQ_3 \times K_2$ is 2-hamiltonian, there is a cycle of length 15 in $TQ_3 \times K_2 - F$.

| Length | $u = 0000,\ v = 1000$ |
|---:|---|
| 9 | $\langle 0001, 0011, 0010, 0100, 1100, 1101, 1111, 1011, 1001, 0001 \rangle$ |
| 10 | $\langle 0001, 0011, 0010, 0100, 0101, 1101, 1100, 1010, 1011, 1001, 0001 \rangle$ |
| 11 | $\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$ |
| 12 | $\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$ |
| 13 | $\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1100, 1010, 1110, 1111, 1011, 1001, 0001 \rangle$ |
| 14 | $\langle 0001, 0011, 0010, 0100, 0101, 0111, 0110, 1110, 1111, 1101, 1100, 1010, 1011, 1001, 0001 \rangle$ |
| Length | $u = 0000,\ v = 1110$ |
| 9 | $\langle 0111, 0110, 0010, 0011, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$ |
| 10 | $\langle 0111, 0101, 0100, 0010, 0011, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$ |
| 11 | $\langle 0111, 0101, 0100, 0010, 0011, 1011, 1001, 1000, 1100, 1101, 1111, 0111 \rangle$ |
| 12 | $\langle 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$ |
| 13 | $\langle 0101, 0111, 0110, 0010, 0011, 0001, 1001, 1000, 1100, 1010, 1011, 1111, 1101, 0101 \rangle$ |
| 14 | $\langle 0111, 0110, 0010, 0100, 0101, 0001, 0011, 1011, 1010, 1100, 1000, 1001, 1101, 1111, 0111 \rangle$ |
| Length | $u = 0000,\ v = 1111$ |
| 9 | $\langle 0110, 0111, 0101, 0100, 0010, 1010, 1100, 1000, 1110, 0110 \rangle$ |
| 10 | $\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1000, 1110, 0110 \rangle$ |
| 11 | $\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1110, 0110 \rangle$ |
| 12 | $\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1101, 1100, 1010, 1110, 0110 \rangle$ |
| 13 | $\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1100, 1000, 1110, 0110 \rangle$ |
| 14 | $\langle 0010, 0100, 0101, 0001, 0011, 0111, 0110, 1110, 1000, 1100, 1101, 1001, 1011, 1010, 0010 \rangle$ |
| Length | $u = 0000,\ v = 1101$ |
| 9 | $\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1011, 1001, 0001 \rangle$ |
| 10 | $\langle 0001, 0011, 0010, 0110, 0111, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$ |
| 11 | $\langle 0001, 0011, 0010, 0110, 0111, 1111, 1011, 1010, 1100, 1000, 1001, 0001 \rangle$ |
| 12 | $\langle 0001, 0011, 0010, 0110, 0111, 1111, 1110, 1000, 1100, 1010, 1011, 1001, 0001 \rangle$ |
| 13 | $\langle 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1000, 1110, 1111, 1011, 1001, 0001 \rangle$ |
| 14 | $\langle 0010, 0100, 0101, 0001, 0011, 0111, 0110, 1110, 1111, 1011, 1001, 1000, 1100, 1010, 0010 \rangle$ |
| Length | $u = 0000,\ v = 1100$ |
| 9 | $\langle 0101, 0001, 0011, 0111, 1111, 1110, 1000, 1001, 1101, 0101 \rangle$ |
| 10 | $\langle 0001, 0011, 0111, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$ |
| 11 | $\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1111, 1110, 1000, 1001, 0001 \rangle$ |
| 12 | $\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1011, 1010, 1110, 1000, 1001, 0001 \rangle$ |
| 13 | $\langle 0110, 0010, 0100, 0101, 0001, 0011, 0111, 1111, 1101, 1001, 1011, 1010, 1110, 0110 \rangle$ |
| 14 | $\langle 0011, 0001, 0101, 0100, 0010, 0110, 0111, 1111, 1101, 1001, 1000, 1110, 1010, 1011, 0011 \rangle$ |

Table 7.1: Fault-free cycles of lengths from 9 to 14 in $TQ_3 \times K_2$ with two faulty vertices.

Finally, in the same way, we can deal with the case $f_e = 2$. In this case, cycles of lengths from 10 to 16 have to be found. Assume that $F = \{(u_1, v_1), (u_2, v_2)\}$. Then let $F' = \{u_1, (u_2, v_2)\}$. From the above discussion, there are cycles of lengths from 9 to 15 in $TQ_3 \times K_2 - F'$, which are also in $TQ_3 \times K_2 - F$. In addition, since $TQ_3 \times K_2$ is 2-hamiltonian, there is a cycle of length 16. This completes our proof.

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