國立交通大學

理學院網路學習學程

碩士論文

以 Cabri Geometry 動態幾何軟體探索 Steiner's Porism

Exploring Steiner's Porism with Cabri Geometry.

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中 華 民 國 九 十 六 年 六 月

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摘要

本研究是利用 "Cabri Geometry" 動態幾何軟體來探索 Steiner's Porism ，主要的討論有反演和Steiner's Porism 兩部份。第一部份在介紹反演的定義、性質和應用；第二部份是藉由反演來建構 Steiner Chain 進而探索 Steiner's Porism。本研究的內容也藉由Cabri II Plus Plug-in以網頁的形式，動態呈現。網址如下：

http://home.educities.edu.tw/iamalumi/ 。

關鍵詞：動態幾何、Cabri Geometry、反演、Steiner Chain、Steiner's Porism、Jakob Steiner

Exploring Steiner's Porism with Cabri Geometry.

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Abstract

The purpose of this thesis is to explore Steiner's Porism by the software "Cabri Geometry". There are two discussions referring to inversion and Steiner's Porism. Definition, properties and applications of inversion are introduced in the first part and constructing Steiner Chain by inversion to explore Steiner's Porism is the second one. The contents of this thesis are also presented dynamically on the following website by Cabri II Plus Plug-in developed by the Cabrilog company : http://home.educities.edu.tw/iamalumi/ .

Keywords: Dynamic Geometry, Cabri Geometry, Inversion, Steiner Chain, Steiner's Porism, Jakob Steiner

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首先誠摯的感謝指導教授全任重教授及黃大原教授，老師悉心的教導、對於我的問題幾乎是有求必應，使我在這兩年中獲益匪淺，老師的學者風範更是我今後在教學上的最佳指標。

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季軒寶貝，你的笑容是爸爸的最大動力，爸爸終於可以好好的陪你了。

親愛的爸媽，謝謝您們，讓我無後顧之憂，能夠專心的完成學業，謹將此論文獻給您們！

呂建勳 謹致

2007 年 6 月

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Chapter 1. Inversion

1.1 Introduction

Euclidean geometry deals mainly with points and straight lines. Its basic transformation is the reflection, which leaves fixed all the points on one line and interchanges certain pairs of points on opposite sides of this "mirror". All other isometries (or "congruent transformations" or "motions") are expressible in terms of reflections. (For simplicity, we are describing only plane geometry. In space we would reflect in a plane instead of a line.)

Analogously, inversive geometry deals with points and circles. Its basic transformation (invented by L. J. Magnus in 1831) is the inversion, which leaves fixed all the points on one circle and interchanges the inside and outside of this "circle of inversion". All other circle-preserving transformations (including similarities as a special case) are expressible in terms of inversions. This kind of geometry is worthy of attention not only for the sake of its intrinsic beauty but also because it is the geometry of complex numbers and because the point pairs and circles of the real inversive plane provide an isomorphic model for the lines and planes of hyperbolic (non-Euclidean) space.

All the geometric constructions are constructed by Cabri Geometry II plus. You can download the evaluation version and manuals from the website http://www.cabri.com/v2/pages/en/downloads_cabri2plus.php.

1.2 Definition

As Figure 1.2.1 shows below. Inversion is the process of transforming points P to a corresponding set of points P' known as their inverse points. These two points P and P' are said to be inverses with respect to an inversion circle having inversion center O if P' is the perpendicular foot of the altitude of ΔOQP, where Q is a point on the circle such that $\overline{OQ} \perp \overline{PQ}$. It means that triangles ΔOQP and $\Delta OP'Q$ are similar and $\overline{OQ}^2 = \overline{OP'} \times \overline{OP}$.

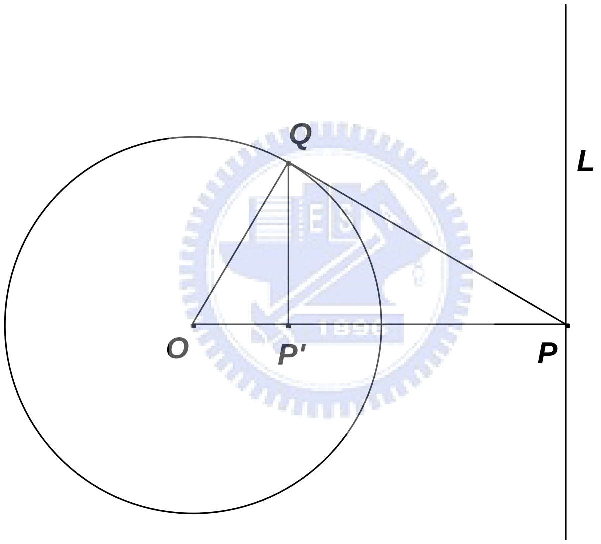


Figure 1.2.1

If P and P' are inverse points , then the line L through P and perpendicular to $\overline{OP}$ is sometimes called a "polar" with respect to point P, and P' is called the "pole" of the polar. In addition, the curve to which a given curve is transformed under inversion is called its inverse curve (or more simply, its "inverse"). This sort of inversion was first systematically investigated by Jakob Steiner (1796-1863).

1.3 Properties of Inversion

Prop. 1 Circles are mapped to circles.

When treating lines as circles of infinite radius, all circles invert to circles (Lachlan, 1893). As Figure 1.3.1 shows, P and P' are inverse points and the circle L' is the inverse of straight line L with respect to circle O. Actually, circle L' is the locus of point P' while point P moves on straight line L.

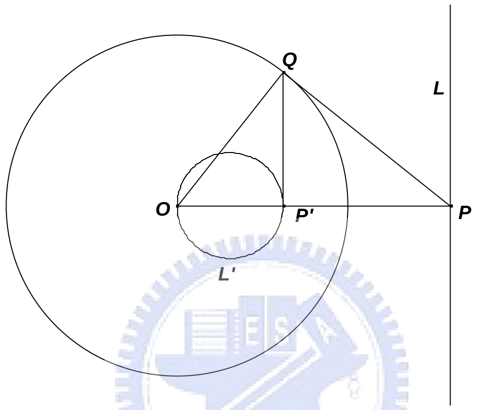


Figure 1.3.1

Constructing steps of Figure 1.3.1:

step 1: Construct a circle as the inversion circle with inversion center O.

step 2: Construct a straight line L and a point P moves on L.

step 3: Connect the segment $\overline{OP}$ and construct a circle C with the diameter $\overline{OP}$.

step 4: The intersections of circle O and circle C named Q and R are the points of tangency with respect to point P.

step 5: The intersection of straight lines $\overline{QR}$ and $\overline{OP}$ named P' is the inverse point of point P with respect to circle O.

step 6: Construct the locus L' of point P' as point P moves on L.

Then the inverse of straight line L with respect to circle O is the locus circle L'.

Build a Macro

Fortunately, the "inverse" function is embedded in Cabri Geometry II plus, so we can supersede step 3, step 4 and step 5 by the "inverse" function. Here are the simplified constructing steps.

step 1: Construct a circle as the inversion circle with inversion center O.

step 2: Construct a straight line L and a point P moves on L.

step 3: Click the "inverse" icon and then click point P and circle O. Now we have the inverse point P' of point P with respect to circle O.

step 4: Construct the locus L' of point P' as point P moves on L.

Then we have the inverse of straight line L.

As a matter of convenience, we can build a macro which helps us simplify the construction of inversion. Here are the steps: (following the above steps)

step 5: Click the "initial object" icon and then click point P and circle O

step 6: Click the "final object" icon and then click locus L'.

step 7: Click the "define macro" icon and give this macro a name. (ex: I&L)

step 8: Save this macro.

Now, we have the most simplified constructing steps of Figure 1.3.1.1:

step 1: Construct a circle as the inversion circle with inversion center O.

step 2: Construct a straight line L and a point P moves on L.

step 3: Click "I&L" macro icon, click point P and circle O.

Then we have the inverse of straight line L.

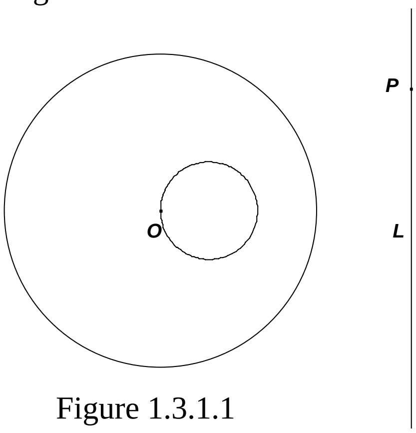


Figure 1.3.1.1

But if the straight line L passes through the inversion center of the inversion circle O, then the inverse of L is L itself. As Figure 1.3.2 shows below. Points P and P' are inverse points with respect to circle O.

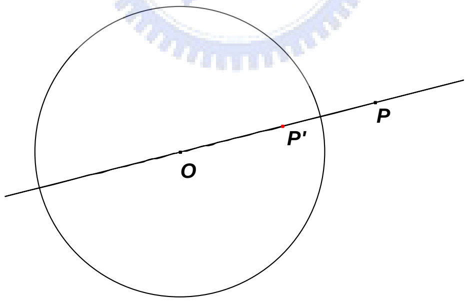


Figure 1.3.2

Constructing steps of Figure 1.3.2:

step 1: Construct a circle with center O and a straight line L through O.

step 2: Given a point P on L.

Step 3: Click "I&L" macro icon, click point P and circle O.

Then the inverse of straight line L is L itself.

As Figure 1.3.3 shows, the circle C' is the inverse of circle C with respect to inversion circle O, and point P' is the inverse of point P with respect to inversion circle O.

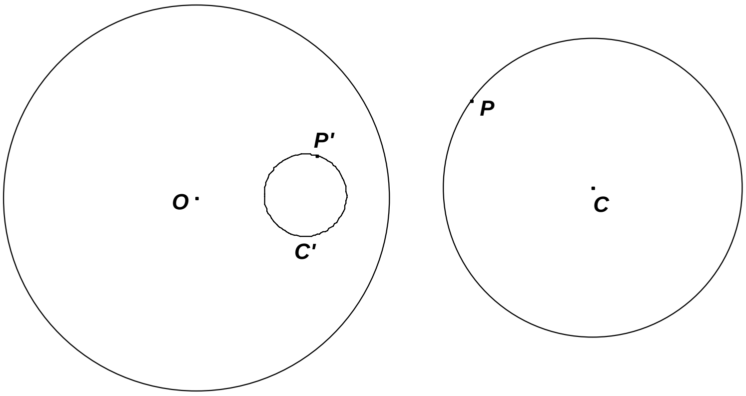


Figure 1.3.3

Constructing steps of Figure 1.3.3:

step 1: Construct circles O and C.

step 2: Given a point P on circle C.

step 3: Click the "inverse" icon, click point P and circle O, then we have the inverse of point P with respect to inversion circle O named P'.

step 4: Click "I&L" macro icon, click point P and circle O, then we have the inverse of circle C with respect to inversion circle O named C'.

There is a little trick to construct Figure 1.3.3 from Figure 1.3.1.1. The function "Redefine Object" is also embedded in Cabri Geometry II plus, it means that we can redefine the dependency characteristics of a previously defined object. So, all we need to do is just redefine the point P to be a point on a circle. Here are the steps:

step 1: Construct a circle C in Figure 1.3.1.1.

step 2: Click the "Redefine Object", click point P, and choose "point on object" then click circle C.

Then we have figure 1.3.3.1.

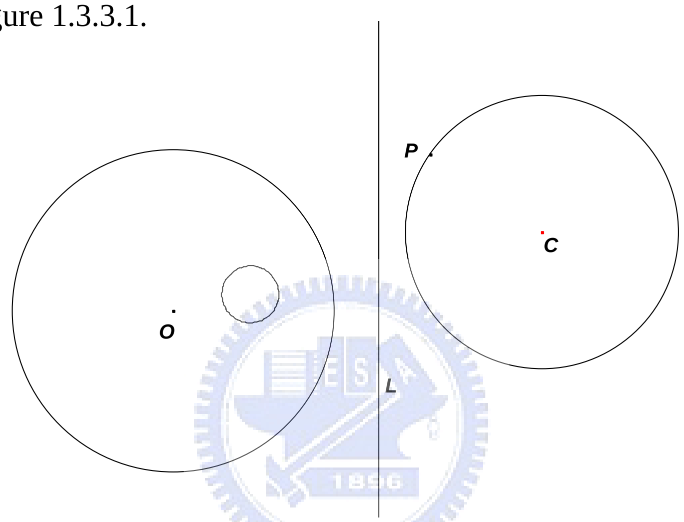


Figure 1.3.3.1

Then we can ask an question: What is the inverse of a triangle with respect to circle O ? Here is the answer.

step 1: Construct a triangle ΔDEF .

step 2: Redefine point P to ΔDEF .

Then we have Figure 1.3.3.2.

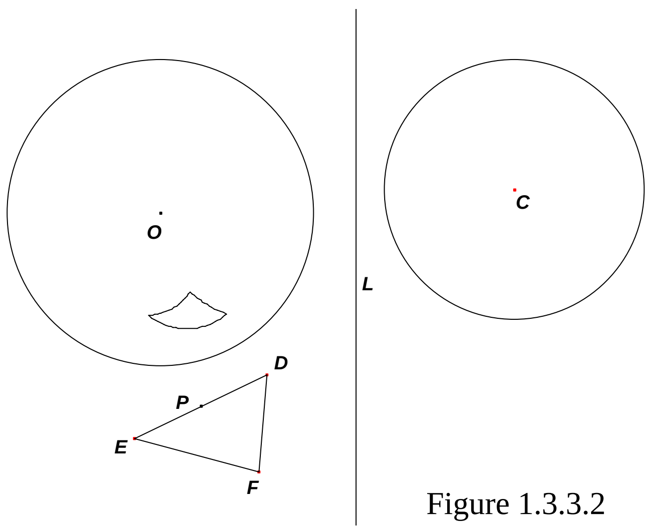


Figure 1.3.3.2

Prop 2. Angles are preserved.

Orthogonal circles invert to orthogonal circles (Coxeter, 1969). As Figure 1.3.4 shows below, the circles A' and B' which are the inverse circles of orthogonal circles A and B with respect to circle O are orthogonal.

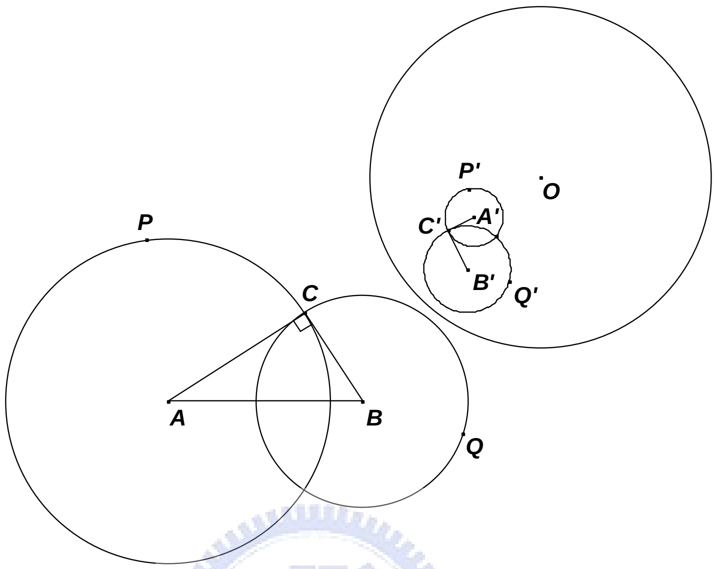


Figure 1.3.4

Constructing steps of Figure 1.3.4:

step 1: Construct a triangle ΔABC with $\angle C = 90°$ and an inversion circle O.

step 2: Construct circles A and B with radius $\overline{AC}$ and $\overline{BC}$ respectively.

step 3: Given a point P on circle A and Q on circle B.

step 4: Click "I&L" macro icon, click point P and circle O, then we have the inverse of circle A with respect to inversion circle O named A'. Also we have circle B' which is the inverse of circle B with respect to inversion circle O.

step 5: Construct inverse of points P and Q named P' and Q' respectively.

step 6: Construct the centers of circles A' and B' respectively.

step 7: Connect the segments $\overline{A'C'}$ and $\overline{B'C'}$ and measure the angle $\angle A'C'B'$ where C' is one intersection of circles A' and B'. Then we have $\angle A'C'B' = 90°$.

Note that the inverse of a center is not still the center of the inverse circle. As Figure 1.3.5 shows below, the inverse of point C is C' but the center of inverse circle of circle C is C''. Of course, circle O is the inversion circle.

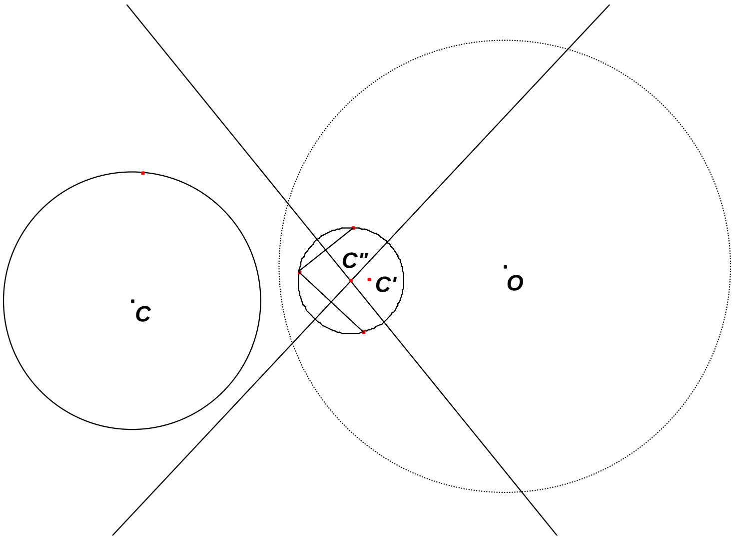


Figure 1.3.5

Furthermore, inversion is a conformal map, so angles are preserved. The two supplementary angles formed by two lines through a point P are equal to the angles formed by their inverse circles, which intersect at O and again at P', as in Figure 1.3.6. In other words, inversion is an angle-preserving (or "conformal") transformation.

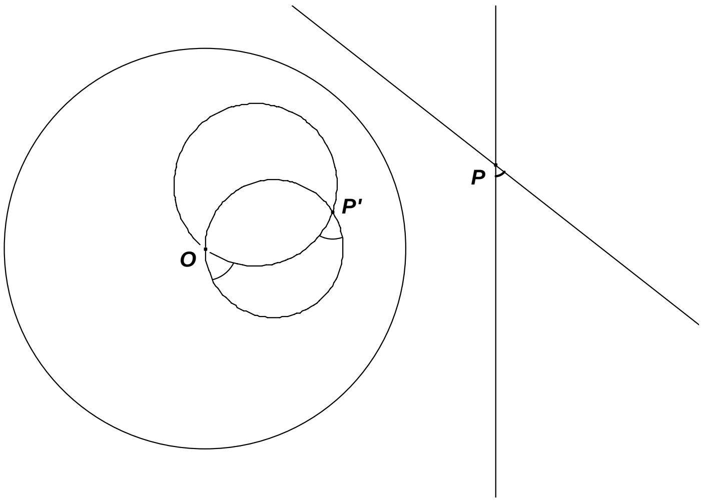


Figure 1.3.6

Note that a point on the circumference of the inversion circle is its own inverse point. In addition, any angle inverts to an opposite angle. As in Figure 1.3.7, the points $1'$, $2'$ and $3'$ are the inverse points of the points 1, 2 and 3 with respect to the inversion circle O.

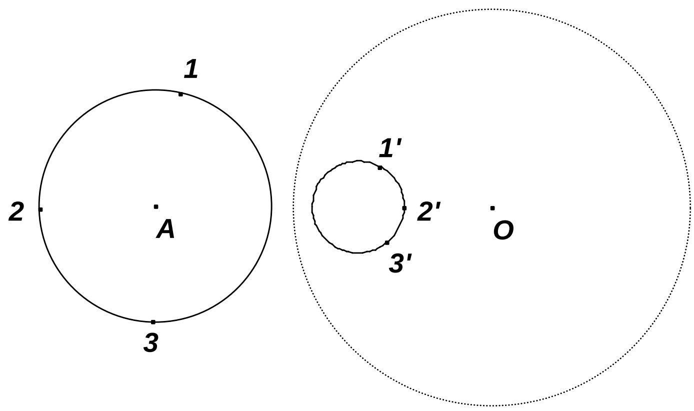


Figure 1.3.7

1.4 The Reduction of Two Circles Theorem

Theorem : Any two non-intersecting circles can be inverted into concentric circles by taking the inversion center at one of the two so-called limiting points of the two circles (Coxeter, 1969).

The proof is constructive. We will show how to find the two points such that a inversion center at either point sends the given circles to concentric circles by a compass and a straight-edge. Before the construction works, introducing some interesting concepts about circles is necessary. These concepts are power, radical line, pencil and limiting point.

Ⅰ. The Power of a Point with respect to a Circle

Definition: If C is a circle of radius r and A is a point with distance d from the center of circle C then the power of A with respect to circle C is $d^2 - r^2$.

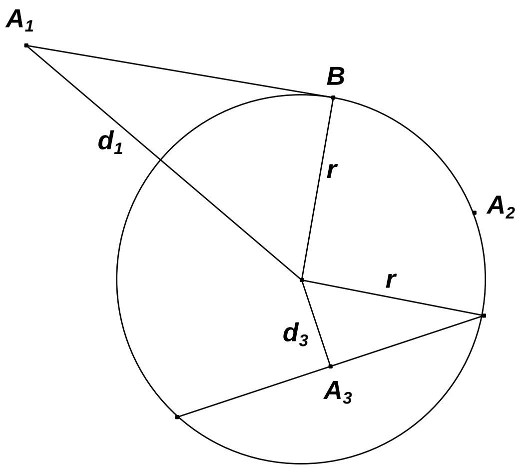


Figure 1.4.1

As Figure 1.4.1 shows, we know three facts:

1. The power of point A_1 is positive and equal to the square of the distance from A_1 to the point of tangency B.

2. The power of point A_2 which is on the circle is zero.

3. The power of point A_3 inside of the circle is negative and equal to the negative of the square of the distance from A_3 to the point where the chord perpendicular to the radius through A_3 intersects the circle.

Ⅱ. The Radical Line of two Non-concentric Circles

Definition: The radical line, also called the radical axis, is the locus of points of equal power with respect to two non-concentric circles. By the chordal theorem, it is perpendicular to the line of centers (Dörrie, 1965).

Proof: Without loss of generality, we introduce a coordinate system with the *x*-axis as the line of centers, the origin at the center of circle O_1 and the center of circle O_2 at the point $(h,0)$ as Figure 1.4.2 shows below.

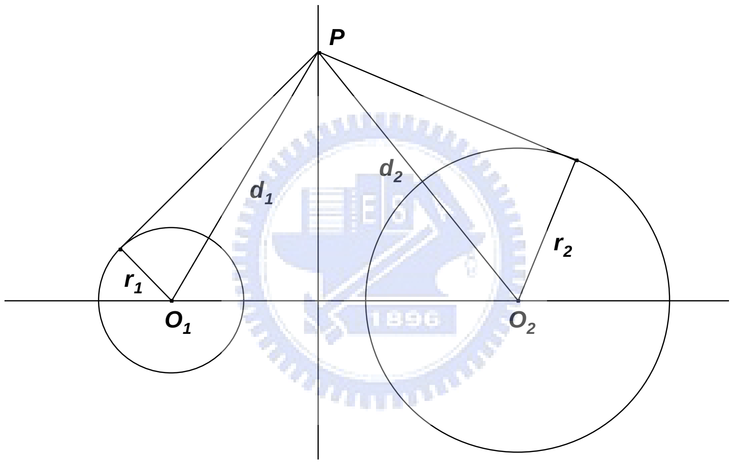


Figure 1.4.2

Let $P(x,y)$ be a point which has the same power of circle O_1 and circle O_2.

Then we have $d_1^2 - r_1^2 = d_2^2 - r_2^2$

$$x^2 + y^2 - r_1^2 = (x-h)^2 + y^2 - r_2^2$$

$$x = \frac{h^2 - (r_2^2 - r_1^2)}{2h}$$

So, locus of point P is a line perpendicular to the line of centers.

Ⅲ. Constructing the Radical Line of Two Non-intersecting Circles

We start the construction from the case of two intersecting circles. Let C_1 and C_2 be two circles with two intersecting points A and B. Then the radical line is the straight line $\overleftrightarrow{AB}$ because the power of points A and B with respect to circles C_1 and C_2 are zero and straight line $\overleftrightarrow{AB}$ is perpendicular to the line of centers as Figure 1.4.3 shows below.

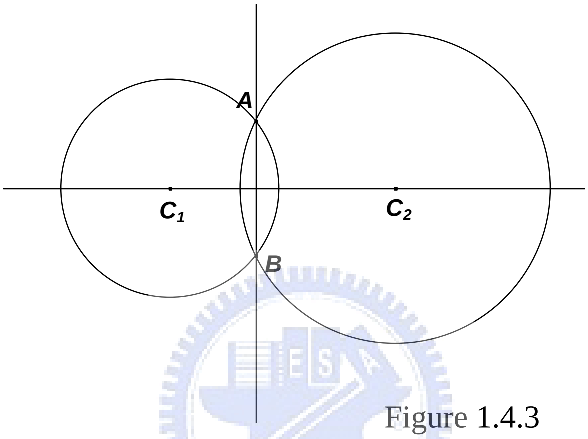


Figure 1.4.3

Let C_1 and C_2 be two non-intersecting circles. We can pick a circle O that intersects both circles C_1 and C_2 whose center is not on their line of centers. Draw L_1, the radical line of circles C_1 and O. Draw L_2, the radical line of circles C_2 and O. L_1 and L_2 intersect at point P that has the same power with respect to circles C_1 ,C_2 and O. So, the line through point P perpendicular to the line of centers of circles C_1 and C_2 is the radical line of circles C_1 and C_2, as Figure 1.4.4 and Figure 1.4.5 show below.

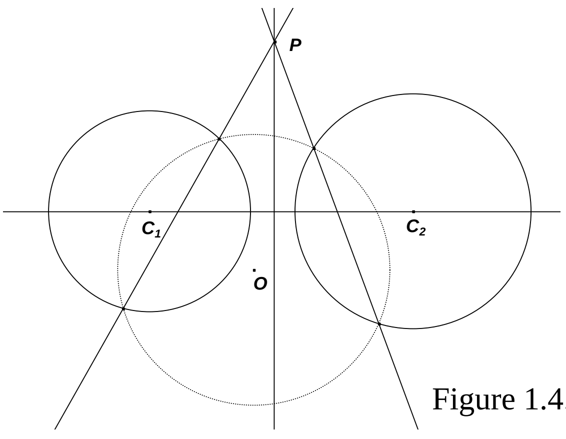


Figure 1.4.4

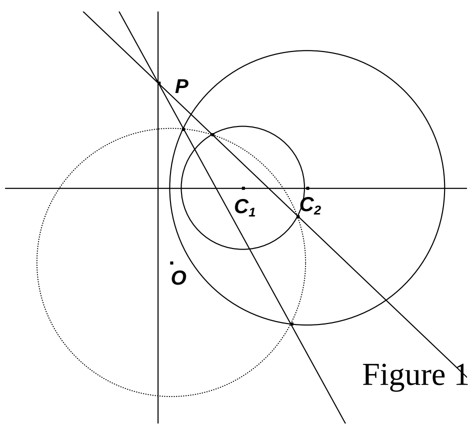


Figure 1.4.5

Ⅳ. Pencils of Coaxial Circles

Definition: Coaxal circles are circles whose centers are collinear and that share a common radical line. The collection of all coaxal circles is called a pencil of coaxal circles (Coxeter and Greitzer, 1967). As Figure 1.4.6 and 1.4.7 show blow, L is the radical line of the circles. Figure 1.4.6 is a intersecting pencil, and Figure 1.4.7 is a non-intersecting pencil.

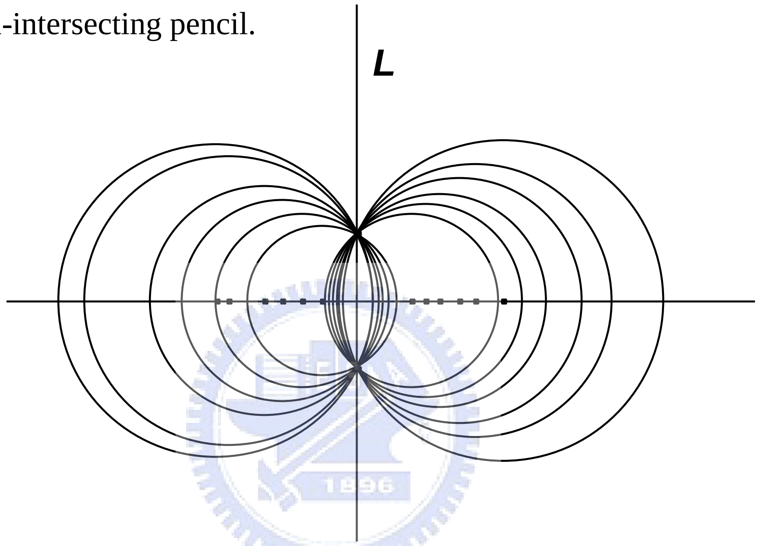


Figure 1.4.6

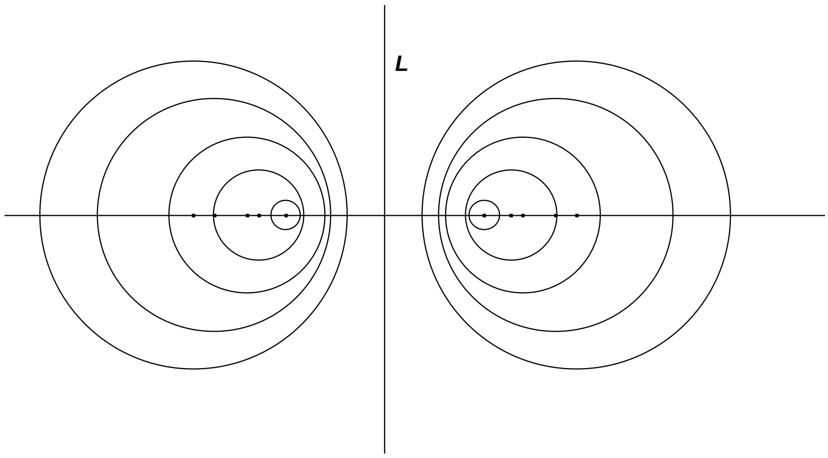


Figure 1.4.7

Ⅴ. Limit Points of Pencils of Non-intersecting Coaxal Circles

Definition: Members of a coaxal system satisfy

$$x^2 + y^2 + 2kx + c = (x + k)^2 + y^2 + c - k^2 = 0 \quad \text{for values of } k.$$

Picking $k^2 = c$ then gives the two circles $(x \pm \sqrt{c})^2 + y^2 = 0$ of zero radius, known as point circles. The two point circles $(\pm\sqrt{c}, 0)$, real or imaginary, are called the limiting points of the coaxal system.

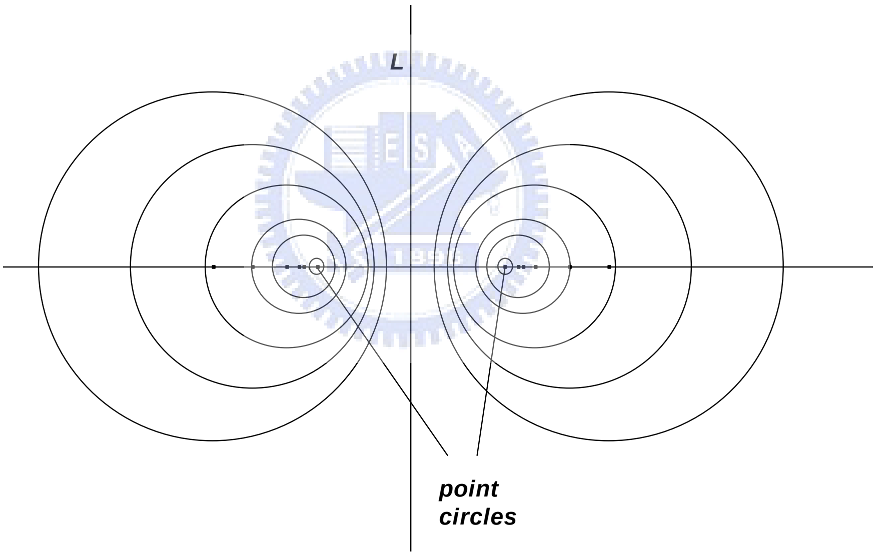
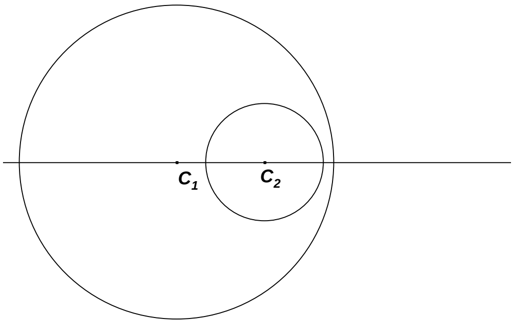
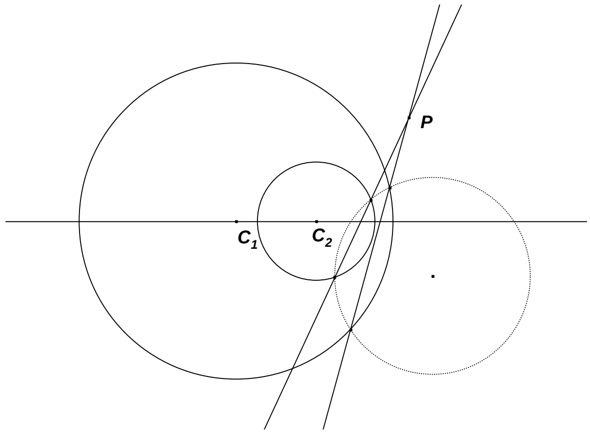


Figure 1.4.8

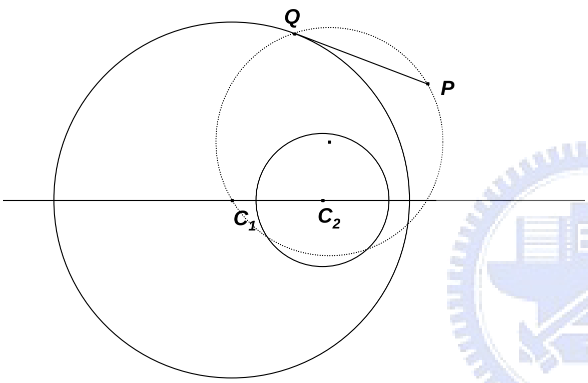
Ⅵ. Proof of the Reduction of Two Circles Theorem



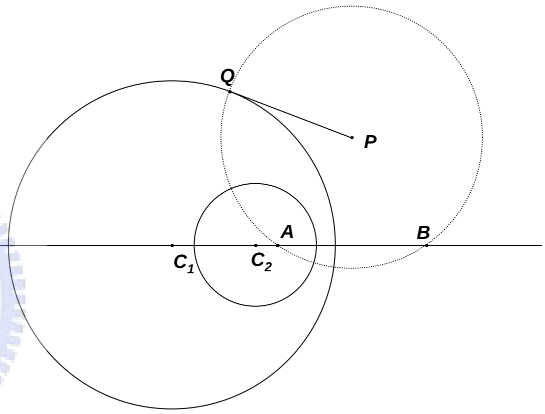
step 1. Given two non-intersecting circles



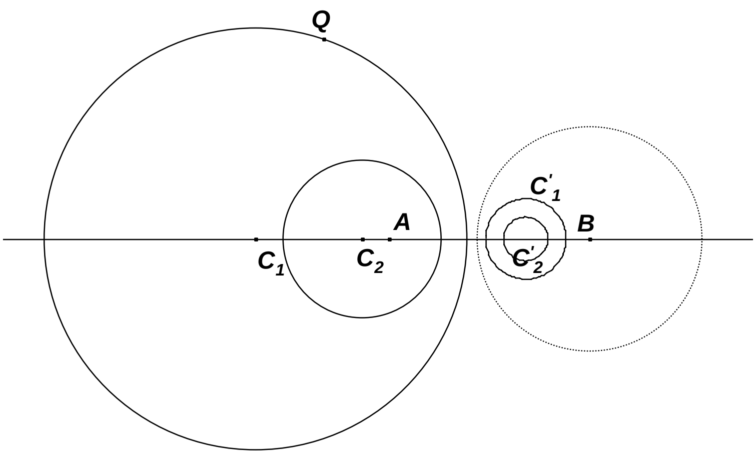
step 2. Find a point P on the radical line



step 3. Find the point of tangency Q



step 4. Points A and B are the limiting points



step 5. Circles $C_1^{'}$ and $C_2^{'}$ are concentric circles by taking the inversion center at point B .

1.5 Some applications of Inversion

I.

The property that inversion transforms circles and lines to circles or lines makes it an extremely important tool of plane analytic geometry. By picking a suitable inversion circle, it is often possible to transform one geometric configuration into another simpler one in which a proof is more easily effected. The illustration below shows examples of the results of geometric inversion.

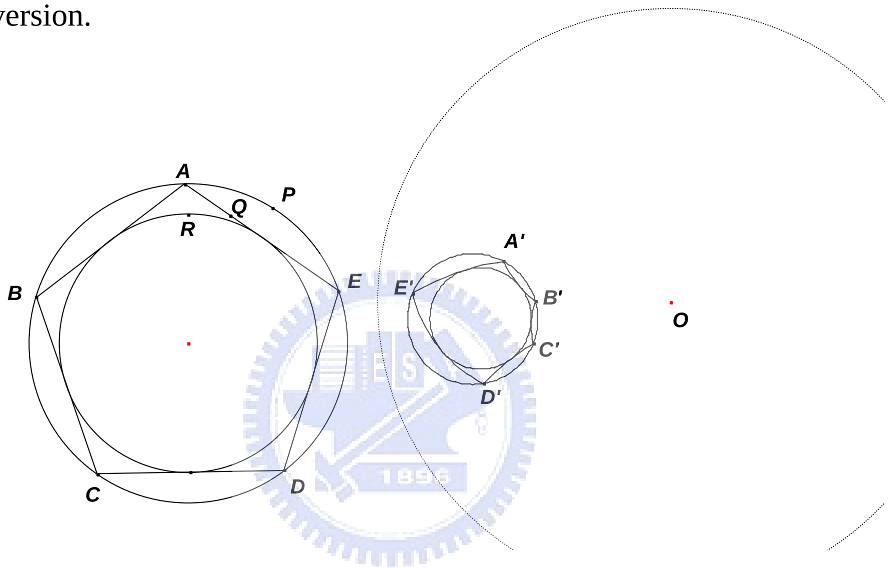


Figure 1.5.1 : O is the inversion center

Constructing steps of Figure 1.5.1:

step 1: Construct a regular pentagon $ABCDE$, a circumcircle and a incircle.

step 2: Given points P, Q and R on the circumcircle, regular pentagon and incircle respectively.

step 3: Click "I&L" macro icon, click point P and circle O, click point Q and circle O, click point R and circle O.

Then we have Figure 1.5.1.

II.

Figure 1.5.2 shows a chessboard centered at (0, 0) and its inverse about a small circle also centered at (0, 0) (Gardner, 1984; Dixon, 1991).

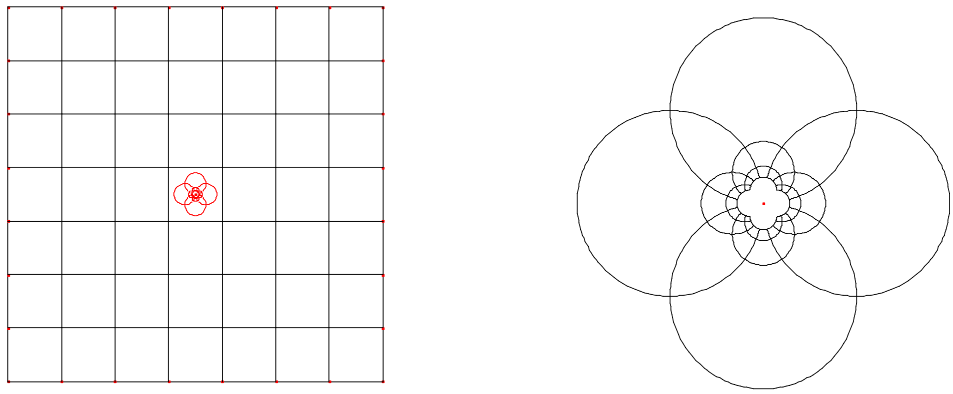


Figure 1.5.2

Constructing steps of Figure 1.5.2:

step 1: Construct a chessboard.

step 2: Construct an inversion circle which has the same center as the chessboard.

step 3: Construct the inverse of the 16 segments with respect to the inversion circle respectively.

Then we have Figure 1.5.2.

III.

Figure 1.5.3 shows an inversion with the center O at the common point of tangency of two circles maps the two circles onto two parallel lines, and maps a chain of circles which are tangent to the two circles onto equal circles inscribed between the two parallel lines.

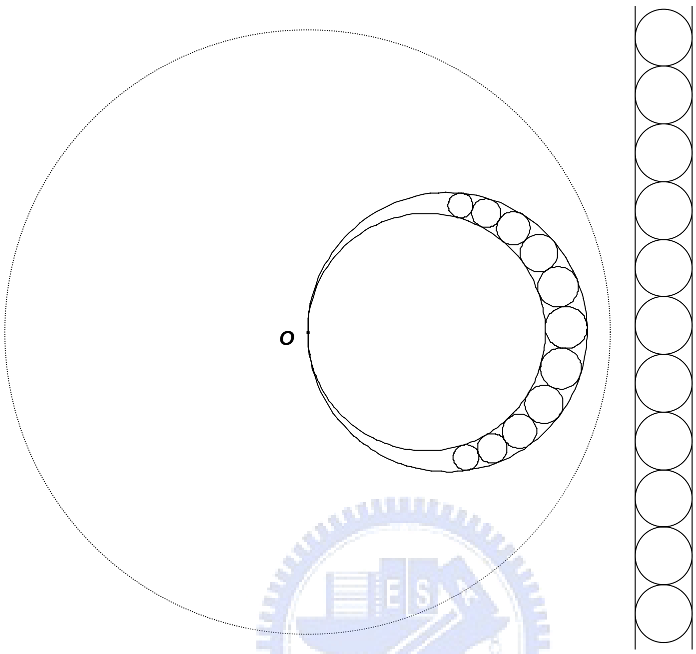


Figure 1.5.3

Constructing steps of Figure 1.5.3:

step 1: Construct two parallel lines and some circles which are tangent to these two parallel lines.

step 2: Construct an inversion circle O.

step 3: Construct the inverse of the two parallel lines and the circles tangent to the parallel lines.

Then we have Figure 1.5.3.

IV. Four Touching Circles

Let S_1, S_2, S_3, and S_4 be four circles each tangent cyclically to its neighbors, so that S_1 touches S_2 and S_4, S_2 also touches S_3, and S_3 touches S_4. Then the four tangency points are concyclic, i.e. lie on a circle. As Figure 1.5.4 shows below.

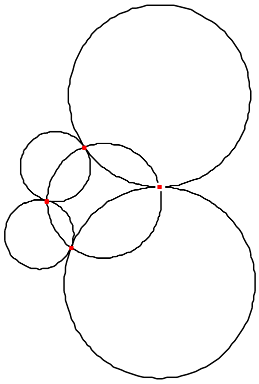


Figure 1.5.4

Actually, when we make an inversion with any one point of tangency as the center, two circles tangent at that point will map on two parallel lines, the other two circles map onto tangent circles each touching one of the parallel lines. (Because of the angle preservation property, tangent curves are mapped onto tangent curves.) As Figure 1.5.5 shows below.

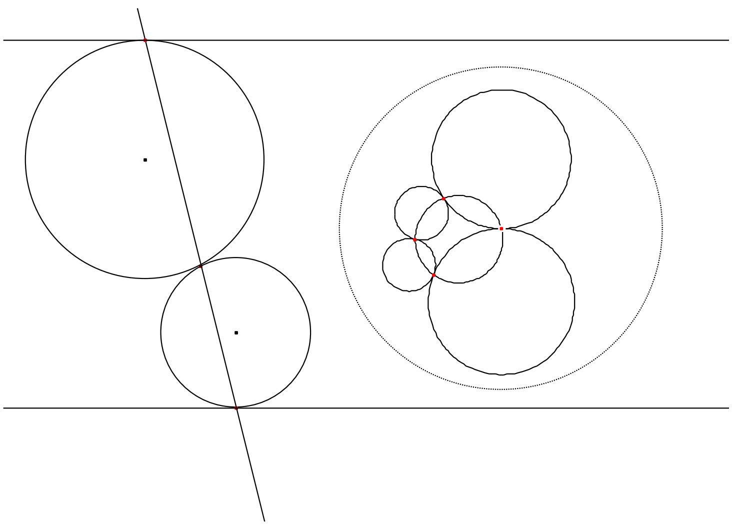


Figure 1.5.5

Chapter 2. Steiner's Porism

2.1 Porism

The term "porism" is an archaic type of mathematical proposition whose historical purpose is not entirely known. It is used instead of "theorem" by some authors for a small number of results for historical reasons.

However, two meanings predominate in nonhistorical usage. The first is "corollary", a usage now mostly superseded by that term itself. The second (which may now be considered the 'modern' usage) is, "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions" (Playfair, 1792). Unfortunately, this definition is slightly inaccurate, because the proposition actually states the conditions, rather than affirming the possibility of finding them.

2.2 Steiner Chain

Given two circles f and g with one interior to the other, if small tangent circles c_1, c_2, c_3, …, and c_n, with each touching the next and c_n touching c_1 can be inscribed around the region between circles f and g, then such a sequence of circles c_i is called a Steiner n-chain.

The simplest way to construct a Steiner chain is to perform an inversion on a symmetrical arrangement on n circles packed between two concentric circles.

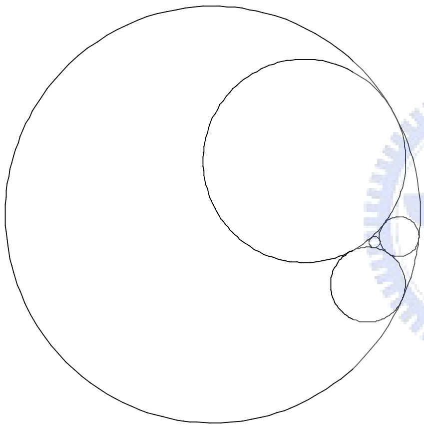


Figure 2.2.1 $n = 3$

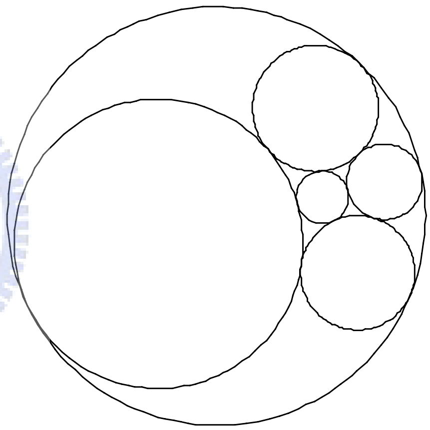


Figure 2.2.2 $n = 4$

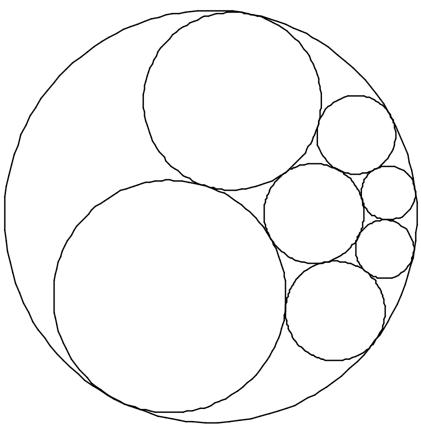


Figure 2.2.3 $n = 6$

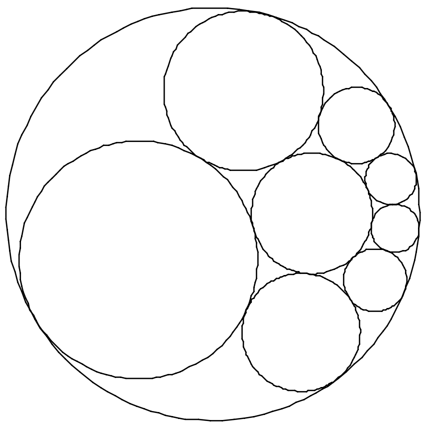
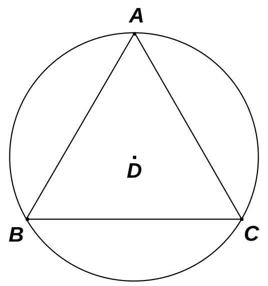
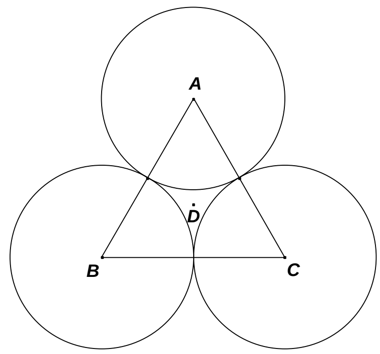


Figure 2.2.4 $n = 7$

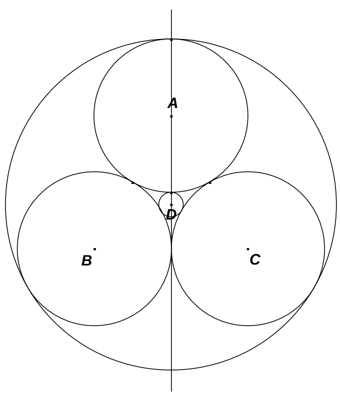
The constructing steps of distinct n are almost the same, so we show the easy one for $n=3$. Here are the constructing steps:



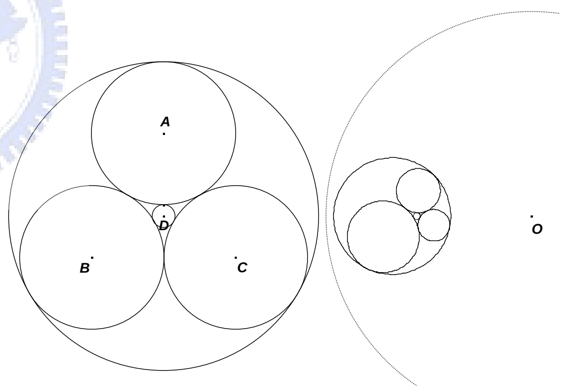
step 1: Construct a regular triangle ΔABC inscribed in a circle with center at point D by the "Regular polygon" function.



step 2: Construct circles A, B and C with the same radius of $\frac{1}{2}\overline{AB}$ so that they are tangent to the others.



step 3: Construct two concentric circles with center at point D such that these two circles are tangent to circles A, B and C at the same time.



Step 4: Construct an inversion circle O, construct the inverse of circles A, B, C and the two concentric circles. Then we have a Steiner's chain with $n=3$.

In the following properties, let C_j denote the center of c_j ,and let c_j touching c_{j+1} at T_j .

Prop. 1 The centers C_j lie on an ellipse (Ogilvy, 1990).

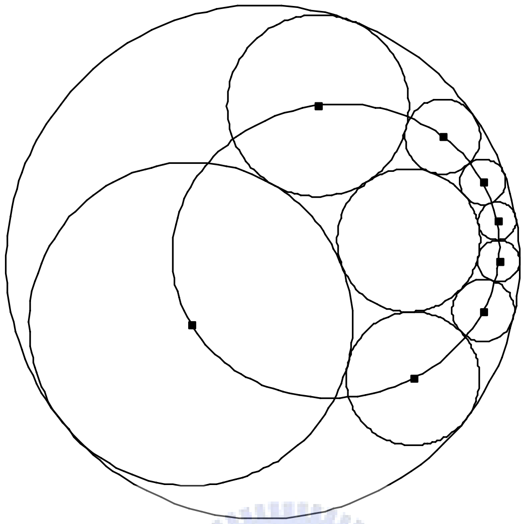


Figure 2.2.5

Prop. 2 The contact points of neighboring circles T_j lie on a circle.

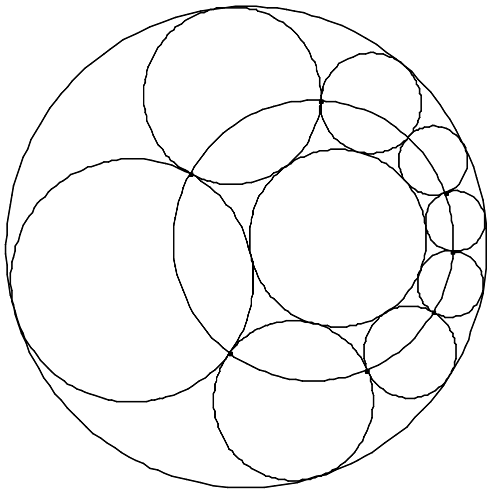


Figure 2.2.6

Prop. 3 The lines of tangency passing through the contact points of neighboring circles T_j in the chain are concurrent in a point. Furthermore, this is the same point at which the lines through the contact points of the inner and outer circles also concur (Wells, 1991).

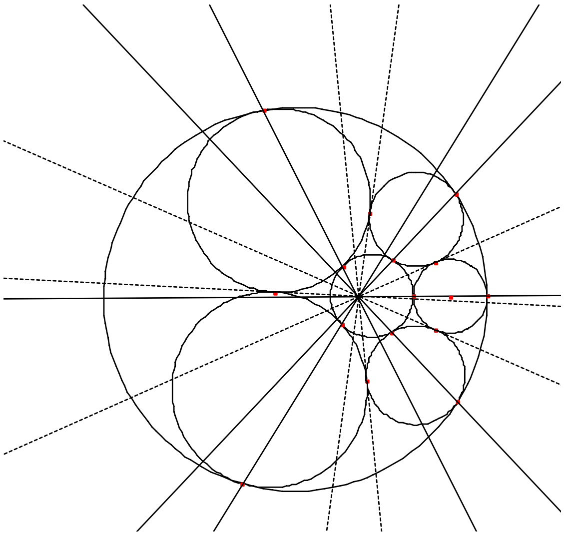


Figure 2.2.7

Prop. 4 If n is even, $n = 2m$ say, the lines $T_j\, T_{m+j}$ and $C_j\, C_{m+j}$ (j =1, 2, 3,…, m; all suffixes taken modulo n), i.e. the joins of opposite tangent-points and opposite centers, are all concurrent. Here are two examples with $n = 6$ and $n = 8$.

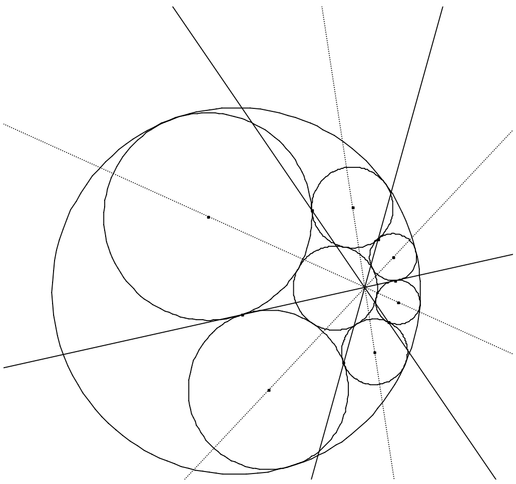


Figure 2.2.8 $n = 6$

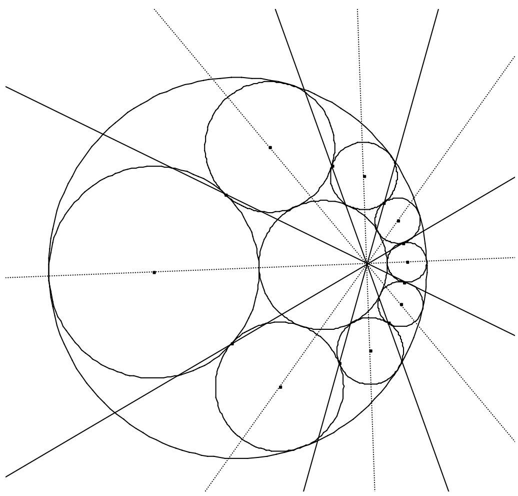


Figure 2.2.9 $n = 8$

Prop. 5 If n is even, $n = 2m$ say, the intersection points of lines $C_j\,C_{j+1}$ and $C_{m+j}\,C_{m+j+1}$ (j =1, 2, 3,…, m ; all suffixes taken modulo n) are all colinear. Here are two examples with $n = 6$ and $n = 8$.

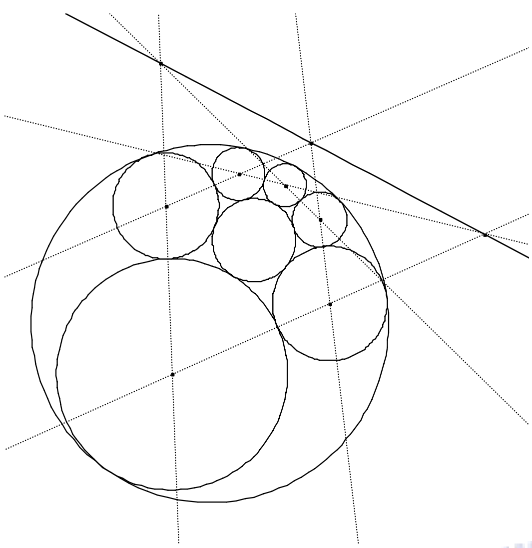


Figure 2.2.10 $n = 6$

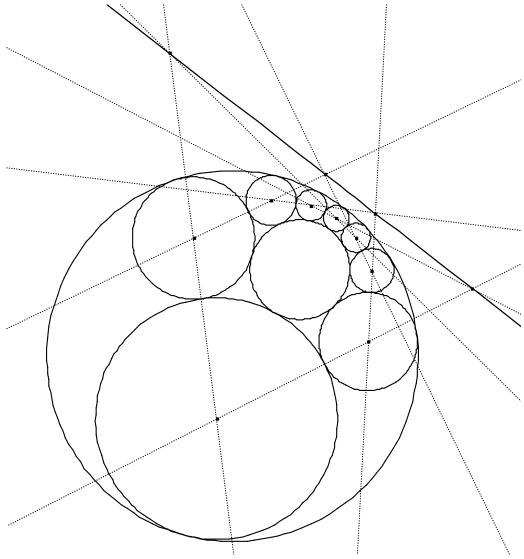


Figure 2.2.11 $n = 8$

Prop. 6 Let U_j be the intersection points of outer circle and circle c_j. If n is even, $n = 2m$ say, the intersection points of lines $U_j\,U_{j+1}$ and $U_{m+j}\,U_{m+j+1}$ (j =1, 2, 3,…, m ; all suffixes taken modulo n) are all colinear. Actually, intersection points of lines $U_j\,U_{j+1}$ and $U_{m+j}\,U_{m+j+1}$ are the same with intersection points of lines $C_j\,C_{j+1}$ and $C_{m+j}\,C_{m+j+1}$. Here is a example with $n = 6$.

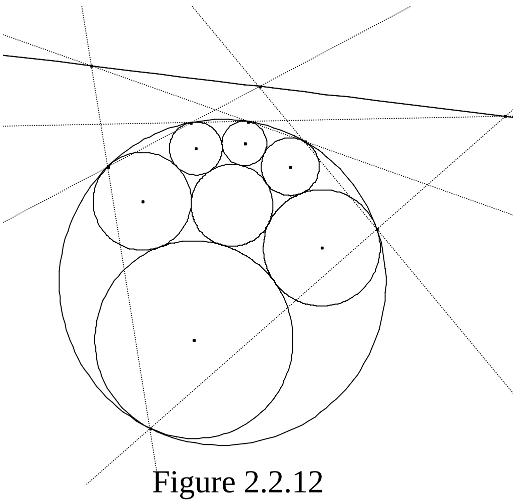


Figure 2.2.12

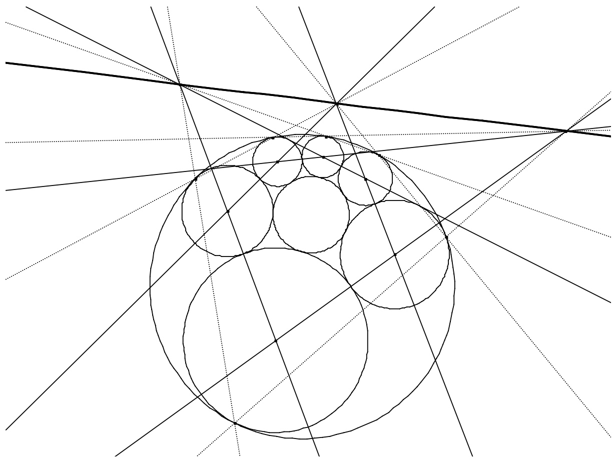


Figure 2.2.13

2.3 Steiner’s Porism

If a Steiner chain is formed from one starting circle, then a Steiner chain is formed from any other starting circle. In other words, given two circles with one interior to the other, draw circles successively touching them and each other. If the last touches the first, this will also happen for any position of the first circle.

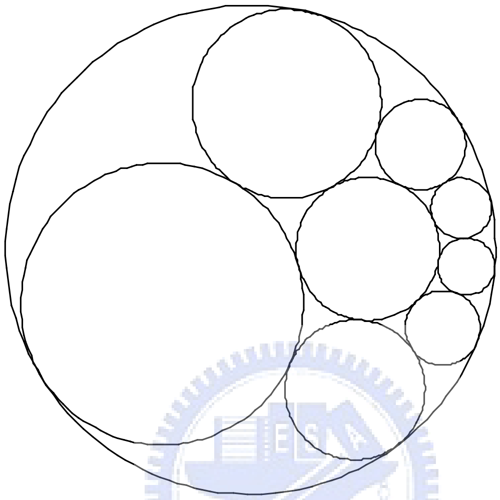


Figure 2.3.1

Steiner found that if you could not fill a particular circle pair starting at one spot you never could from any starting spot.

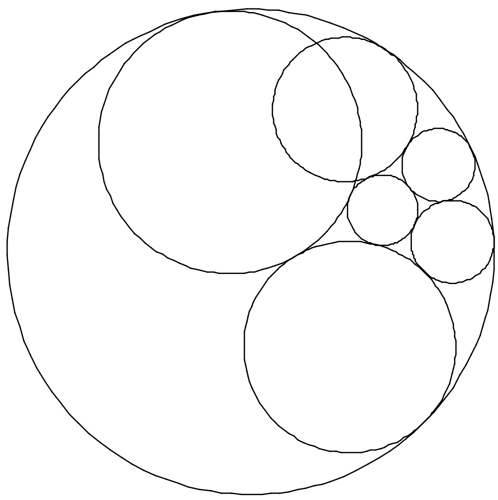


Figure 2.3.2

Chapter 3. Conclusions

The appearance of dynamic geometry software has already revolutionized the learning and the research of geometry. The technique of making a construction with Cabri is similar to what we do with a paper. A geometrical figure or an equation becomes an object to manipulate. With this powerful tool, now we can explore mathematics in an active and interactive way. It also facilitates the construction of figure in less time than using traditional construction techniques. All the constructions can be shared by web pages containing "Cabrijava Applets" or "Cabri II Plus Plug-in" and users can access the work without Cabri geometry.

If you are interested in this topic and want to get more information, you can visit the following website: http://www.math.nthu.edu.tw/~jcchuan/ .

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