Constant elasticity of variance (CEV) option pricing model: Integration and detailed derivation[☆]

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Hsu and Lin would like to dedicate this paper to honor and memory our Ph.D. advisor, Prof. Jack C. Lee, who died from cardiovascular disease on 2 March 2007.

Abstract

In this paper we review the renowned constant elasticity of variance (CEV) option pricing model and give the detailed derivations. There are two purposes of this article. First, we show the details of the formulae needed in deriving the option pricing and bridge the gaps in deriving the necessary formulae for the model. Second, we use a result by Feller to obtain the transition probability density function of the stock price at time T given its price at time t with $t < T$. In addition, some computational considerations are given for the facilitation of computing the CEV option pricing formula.
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1. Introduction

Cox [4] has derived the renowned constant elasticity of variance (CEV) option pricing model and Schroder [10] has subsequently extended the model by expressing the CEV option pricing formula in terms of the noncentral Chi-square distribution. However, neither of them has given details of their derivations as well as the mathematical and statistical tools in deriving the formulae. There are two purposes of this article. First, we integrate the results obtained by Cox [4] and Schroder [10] and bridge the gaps in deriving the necessary formulae for the model. Second, we use a result by Feller [7] to obtain the transition probability density function of the stock price at time T given its price at time t with $t < T$. We also show the details of the formulae needed in deriving the option pricing. A proof of Feller's result is given in Appendix A.

2. The CEV diffusion and its transition probability density function

The CEV option pricing model assumes that the stock price is governed by the diffusion process:

$$dS = \mu S \, dt + \sigma S^{\beta/2} \, dZ, \quad \beta < 2, \tag{1}$$

where dZ is a Wiener process and σ is a positive constant. The elasticity is $\beta - 2$ since the return variance $\upsilon(S, t) = \sigma^2 S^{\beta-2}$ with respect to price S has the following relationship:

$$\frac{d\upsilon(S, t)/dS}{\upsilon(S, t)/S} = \beta - 2,$$

which implies that $d\upsilon(S, t)/\upsilon(S, t) = (\beta - 2) \, dS/S$. Upon integration on both sides, we have $\log \upsilon(S, t) = (\beta - 2) \log S + \log \sigma^2$, or $\upsilon(S, t) = \sigma^2 S^{\beta-2}$.

If $\beta = 2$, then the elasticity is zero and the stock prices are lognormally distributed as in the Black and Scholes model. If $\beta = 1$, then (1) is the model proposed by Cox and Ross [5].

In this article, we will focus on the case of $\beta < 2$ since many empirical evidences (see Campbell [2], Glosten et al. [8], Brandt and Kang [1]) have shown that the relationship between the stock price and its return volatility is negative. The transition density for $\beta > 2$ is given by Emanuel and Macbeth [6] and the corresponding CEV option pricing formula can be derived through a similar strategy. For more details, see Chen and Lee [3].

In order to derive the CEV option pricing model, we need the transition probability density function $f(S_T|S_t, T > t)$ of the stock price at time T given the current stock price S_t. For the transition probability density function $f(S_T|S_t)$, we will start with the Kolmogorov forward and backward equations.

Assume X_t follows the diffusion process:

$$dX = \mu(X, t) \, dt + \sigma(X, t) \, dZ, \tag{2}$$

and $P = P(X_t, t)$ is the function of X_t and t, then P satisfies the partial differential equations of motion. From (2), we have the Kolmogorov backward equation:

$$\frac{1}{2}\sigma^2(X_0, t_0)\frac{\partial^2 P}{\partial X_0^2} + \mu(X_0, t_0)\frac{\partial P}{\partial X_0} + \frac{\partial P}{\partial t_0} = 0 \tag{3}$$

and the Kolmogorov forward (or Fokker–Planck) equation:

$$\frac{1}{2}\frac{\partial^2}{\partial X_t^2}[\sigma^2(X_t, t)P] - \frac{\partial P}{\partial X_t}[\mu(X_t, t)P] - \frac{\partial P}{\partial t} = 0. \tag{4}$$

Consider the following parabolic equation

$$(P)_t = (axP)_{xx} - ((bx + h)P)_x, \quad 0 < x < \infty, \tag{5}$$

where $P = P(x, t)$, and a, b, h are constants with $a > 0$, $(P)_t$ is the partial derivative of P with respect to t, $(\ )_x$ and $(\ )_{xx}$ are the first and second partial derivatives of $(\ )$ with respect to x. This can be interpreted as the Fokker–Planck equation of a diffusion problem in which $bx + h$ represents the drift, and ax represents the diffusion coefficient.

Lemma (Feller [7]). *Let $f(x, t|x_0)$ be the probability density function for x and t conditional on x_0. The explicit form of the fundamental solution to the above parabolic equation is given by*

$$f(t, x|x_0) = \frac{b}{a(e^{bt} - 1)}\left(\frac{e^{-bt}x}{x_0}\right)^{(h-a)/2a} \exp\left\{\frac{-b(x + x_0 e^{bt})}{a(e^{bt} - 1)}\right\} I_{1-h/a}\left(\frac{2b}{a(1 - e^{-bt})}(e^{-bt}xx_0)^{1/2}\right), \tag{6}$$

where $I_k(x)$ is the modified Bessel function of the first kind of order k and is defined as

$$I_k(x) = \sum_{r=0}^{\infty}\frac{(x/2)^{2r+k}}{r!\Gamma(r + 1 + k)}. \tag{7}$$

Proof. See Appendix A. $\square$

Before pursuing further, we will first consider the special case in which $\beta = 1$ which is the model considered by Cox and Ross [5]. In this situation we have

$$dS = \mu(S, t)\, dt + \sigma(S, t)\, dZ, \tag{8}$$

where $\sigma(S, t) = \sigma\sqrt{S}$.

Now suppose also that each unit of the stock pays out in dividends in the continuous stream $b(S, t)$ so that the required mean becomes $\mu(S, t) = rS - b(S, t) = rS - (aS + h)$, where $b(S, t) = aS + h$ and r is the risk-free interest rate. Then $dS = [(r - a)S - h]\, dt + \sigma\sqrt{S}\, dZ$ and the differential option price equation becomes

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 P}{\partial S^2} + [(r - a)S - h]\frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} = rP \tag{9}$$

and the corresponding Kolmogorov forward equation for the diffusion process (8) is

$$\frac{1}{2}\frac{\partial^2}{\partial S_t^2}(\sigma^2 S_T P) + \frac{\partial}{\partial S_T}[((r - a)S - h)P] - \frac{\partial P}{\partial t} = 0, \tag{10}$$

which is obtained by using (4) with $\mu(x_t, t) = (r - a)S - h$.

Comparing with (6), we set $a = \sigma^2/2$, $x = S_T$, $x_0 = S_t$, $b = r - \sigma^2/2$, $h = -h$ and $t = \tau = (T - t)$. Thus, we have the following transition probability density function for the Cox–Ross model:

$$f(S_T | S_t, T > t) = \frac{2(r - \sigma^2/2)}{\sigma^2[e^{(r-\sigma^2/2)\tau} - 1]}\left(\frac{S_t\, e^{(r-\sigma^2/2)\tau}}{S_T}\right)^{(1+2h/\sigma^2)/2}$$

$$\times \exp\left\{\frac{-2(r - \sigma^2/2)[S_T + S_t\, e^{(r-\sigma^2/2)\tau}]}{\sigma^2[e^{(r-\sigma^2/2)\tau} - 1]}\right\} I_{1+2h/\sigma^2}\left(\frac{4(r - \sigma^2/2)(S_t S_T\, e^{(r-\sigma^2/2)\tau})^{1/2}}{\sigma^2[e^{(r-\sigma^2/2)\tau} - 1]\sigma^2}\right). \tag{11}$$

We next consider the constant elasticity of variance diffusion,

$$dS = \mu(S, t) + \sigma(S, t)\, dZ, \tag{12}$$

where

$$\mu(S, t) = rS - aS \tag{13}$$

and

$$\sigma(S, t) = \sigma S^{\beta/2}, \quad 0 \le \beta < 2. \tag{14}$$

Then

$$dS = (r - a)S\, dt + \sigma S^{\beta/2}\, dZ. \tag{15}$$

Let $Y = Y(S, t) = S^{2-\beta}$. By Ito's Lemma with

$$\frac{\partial Y}{\partial S} = (2 - \beta)S^{1-\beta}, \qquad \frac{\partial Y}{\partial t} = 0, \qquad \frac{\partial^2 Y}{\partial S^2} = (2 - \beta)(1 - \beta)S^{-\beta},$$

we have

$$dY = [(r - a)(2 - \beta)Y + \tfrac{1}{2}\sigma^2(\beta - 1)(\beta - 2)]\, dt + \sigma^2(2 - \beta)^2 Y\, dZ. \tag{16}$$

The Kolmogorov forward equation for Y becomes

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial Y^2}[\sigma^2(2 - \beta)YP] - \frac{\partial}{\partial Y}\left\{\left[(r - a)(2 - \beta)Y + \frac{1}{2}\sigma^2(\beta - 1)(\beta - 2)\right]P\right\}. \tag{17}$$

Then $\quad f(S_T|S_t, T > t) = f(Y_T|y_t, T > t)|J| \quad$ where $\quad J = (2 - \beta)S^{1-\beta}$. By Feller's Lemma with $a = (1/2)\sigma^2(2-\beta)^2, b = (r-a)(2-\beta), h = (1/2)\sigma^2(\beta-2)(1-\beta), x = 1/T, x_0 = 1/t \quad$ and $\quad t = \tau = (T - t)$, we have

$$f(S_T|S_t, T > t) = (2 - \beta)k^{*1/(2-\beta)}(xz^{1-\beta})^{1/(2(2-\beta))} e^{-x-z} I_{1/(2-\beta)}(2(xz)^{1/2}),\tag{18}$$

where

$$k^* = \frac{2(r-a)}{\sigma^2(2-\beta)[e^{(r-a)(2-\beta)\tau} - 1]}, \qquad x = k^* S_t^{2-\beta} e^{(r-a)(2-\beta)\tau}, \quad z = k^* S_T^{2-\beta}.$$

Cox [4] obtained the following option pricing formula:

$$C = S_t e^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-x}x^n G(n+1+1/(2-\beta), k^* K^{2-\beta})}{\Gamma(n+1)} - K e^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-x}x^{n+1/(2-\beta)} G(n+1, k^* K^{2-\beta})}{\Gamma(n+1+1/(2-\beta))},\tag{19}$$

where $\quad G(m, v) = [\Gamma(m)]^{-1} \int_v^{\infty} e^{-u}u^{m-1} du \quad$ is the standard complementary gamma distribution function.

For a proof of the above formula, see Chen and Lee [3]. We next present the detailed derivations of the option pricing formula as presented by Schroder [10]. Since the option pricing formula is expressed in terms of the noncentral Chi-square complementary distribution function, a brief review of the noncentral Chi-square distribution is presented in the next section.

3. Review of noncentral Chi-square distribution

If $Z_1, \ldots, Z_v$ are standard normal random variables, and $\delta_1, \ldots, \delta_v$ are constants, then

$$Y = \sum_{i=1}^{v} (Z_i + \delta_i)^2\tag{20}$$

is the noncentral Chi-square distribution with v degrees of freedom and noncentrality parameter $\lambda = \sum_{j=1}^{v} \delta_j^2$, and is denoted as $\chi'^2_v(\lambda)$. When $\delta_j = 0$ for all j, then Y is distributed as the central Chi-square distribution with v degrees of freedom, and is denoted as χ^2_v. The cumulative distribution function of $\chi'^2_v(\lambda)$ is

$$F(x; v, \lambda) = P(\chi'^2_v(\lambda) \le x) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!2^{v/2}\Gamma(v/2+j)} \int_0^x y^{v/2+j-1} e^{-y/2} dy, \quad x > 0.\tag{21}$$

An alternative expression for $F(x; v, \lambda)$ is

$$F(x; v, \lambda) = \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j e^{-\lambda/2}}{j!} \right) P(\chi^2_{v+2j} \le x).\tag{22}$$

The complementary distribution function of $\chi'^2_v(\lambda)$ is

$$Q(x; v, \lambda) = 1 - F(x; v, \lambda),\tag{23}$$

where $F(x; v, \lambda)$ is given in either (21) or (22).

The probability density function of $\chi'^2_v(\lambda)$ can be expressed as a mixture of central Chi-square probability density functions:

$$p_{\chi'^2_v(\lambda)}(x) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(1/2\lambda)^j}{j!} p_{\chi^2_{v+2j}}(x) = \frac{e^{-(x+\lambda)/2}}{2^{v/2}} \sum_{j=0}^{\infty} \frac{x^{v/2+j-1}\lambda^j}{\Gamma(v/2+j)2^{2j}j!}.\tag{24}$$

An alternative expression for the probability density function of $\chi_\nu'^2(\lambda)$ is

$$p_{\chi_\nu'^2(\lambda)}(x) = \frac{1}{2}\left(\frac{x}{\lambda}\right)^{(\nu-2)/4} \exp\left\{-\frac{1}{2}(\lambda + x)\right\} I_{(\nu-2)/2}(\sqrt{\lambda x}), \quad x > 0, \tag{25}$$

where I_k is the modified Bessel function of the first kind of order k and is defined as

$$I_k(z) = \left(\frac{1}{2}z\right)^k \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!\Gamma(k + j + 1)}. \tag{26}$$

It is noted that for integer k:

$$I_k(z) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos(k\theta)} \cos(k\theta)\, d\theta = I_{-k}(z). \tag{27}$$

The noncentral Chi-square distribution satisfies the reproductivity property with respect to n and λ. If $X_1, \ldots, X_k$ are independent random variables with X_t distributed as $\chi_{n_i}'^2(\lambda_i)$, then

$$Y = \sum_{i=1}^{k} X_i \sim \chi_{\sum_{i=1}^{k} n_i}'^2\left(\sum_{i=1}^{k} \lambda_i\right). \tag{28}$$

4. The noncentral Chi-square approach to option pricing model

Following Schroder [10], with the transition probability density function given in (18), the option pricing formula under the CEV model is

$$C = E(\max(0, S_T - K)) = e^{-r\tau} \int_K^{\infty} f(S_T|S_t, T > t)(S_T - K)\, dS_T$$

$$= e^{-r\tau} \int_K^{\infty} S_T f(S_T|S_t, T > t)\, dS_T - e^{-r\tau} K \int_K^{\infty} f(S_T|S_t, T > t)\, dS_T = C_1 - C_2. \tag{29}$$

where $\tau = T - t$

4.1. Detailed derivations of C_1 and C_2

Making the change of variable $w = k^* S_T^{2-\beta}$, we have

$$dS_T = (2 - \beta)^{-1} k^{*-1/(2-\beta)} w^{(\beta-1)/(2-\beta)}\, dw.$$

Thus, with $y = k^* K^{2-\beta}$, we have

$$C_1 = e^{-r\tau} \int_y^{\infty} e^{-x-w}(x/w)^{1/(4-2\beta)} I_{1/(2-\beta)}(2\sqrt{xw})(w/k^*)^{1/(2-\beta)}\, dw$$

$$= e^{-r\tau} \int_y^{\infty} e^{-x-w}(x/w)^{1/(4-2\beta)} I_{1/(2-\beta)}(2\sqrt{xw})(w/x)^{1/(2-\beta)}(x/k^*)^{1/(2-\beta)}\, dw$$

$$= e^{-r\tau}(x/k^*)^{1/(2-\beta)} \int_y^{\infty} e^{-x-w}(x/w)^{-1/(4-2\beta)} I_{1/(2-\beta)}(2\sqrt{xw})\, dw$$

$$= e^{-r\tau} S_t\, e^{(r-a)\tau} \int_y^{\infty} e^{-x-w}(w/x)^{1/(4-2\beta)} I_{1/(2-\beta)}(2\sqrt{xw})\, dw$$

$$= e^{-a\tau} S_t \int_y^{\infty} e^{-x-w}(w/x)^{1/(4-2\beta)} I_{1/(2-\beta)}(2\sqrt{xw})\, dw \tag{30}$$

and

$$C_2 = K\,\mathrm{e}^{-r\tau}\int_y^\infty (2-\beta)k^{*1/(2-\beta)}(xw^{1-2\beta})^{1/(4-2\beta)}\,\mathrm{e}^{-x-w}I_{1/(2-\beta)}(2\sqrt{xw})\frac{k^{*-1/(2-\beta)}}{2-\beta}w^{(\beta-1/2-\beta)}\,\mathrm{d}w$$

$$= K\,\mathrm{e}^{-r\tau}\int_y^\infty x^{1/(4-2\beta)}w^{(1-2\beta+2\beta-2)/(4-2\beta)}\,\mathrm{e}^{-x-w}I_{1/(2-\beta)}(2\sqrt{xw})\,\mathrm{d}w$$

$$= K\,\mathrm{e}^{-r\tau}\int_y^\infty \mathrm{e}^{-x-w}(x/w)^{1/(4-\beta)}I_{1/(2-\beta)}(2\sqrt{xw})\,\mathrm{d}w. \tag{31}$$

Recall that the probability density function of the noncentral Chi-square distribution with noncentrality λ and degree of freedom ν is

$$p_{\chi_\nu'^2(\lambda)}(x) = \tfrac{1}{2}(x/\lambda)^{(\nu-2)/4}I_{(\nu-2)/2}(\sqrt{\lambda x})\,\mathrm{e}^{-(\lambda+x)/2} = P(x;\nu,\lambda).$$

Let $Q(x;\nu,\lambda) = \int_x^\infty p_{\chi_\nu'^2(\lambda)}(y)\,\mathrm{d}y$. Then letting $w' = 2w$ and $x' = 2x$, we have

$$C_1 = S_t\,\mathrm{e}^{-a\tau}\int_y^\infty \mathrm{e}^{-(x+w)/2}\left(\frac{w}{x}\right)^{1/(4-2\beta)}I_{1/(2-\beta)}\left(2\sqrt{xw}\right)\,\mathrm{d}w$$

$$= S_t\,\mathrm{e}^{-a\tau}\int_{2y}^\infty \mathrm{e}^{-(x'+w')/2}\left(\frac{w'}{x'}\right)^{1/(4-2\beta)}I_{1/(2-\beta)}\left(2\sqrt{x'w'}\right)\frac{1}{2}\,\mathrm{d}w'$$

$$= S_t\,\mathrm{e}^{-a\tau}Q(2y;\nu,x'), = S_t\,\mathrm{e}^{-a\tau}Q\left(2y;2+\frac{2}{2-\beta},2x\right) \tag{32}$$

obtained by noting that $(\nu-2)/2 = 1/(2-\beta)$, implying $\nu = 2 + 2/(2-\beta)$. Analogously, with $w' = 2w$, $x' = 2x$ and $I_n(z) = I_{-n}(z)$, we have

$$C_2 = K\,\mathrm{e}^{-r\tau}\int_y^\infty \mathrm{e}^{-x-w}\left(\frac{x}{w}\right)^{1/(4-2\beta)}I_{1/(2-\beta)}\left(2\sqrt{xw}\right)\,\mathrm{d}w$$

$$= K\,\mathrm{e}^{-r\tau}\int_{2y}^\infty \mathrm{e}^{-(x'+w')/2}\left(\frac{x'}{w'}\right)^{1/(4-2\beta)}I_{1/(2-\beta)}\left(2\sqrt{x'w'}\right)\frac{1}{2}\,\mathrm{d}w' = Q\left(2y;2-\frac{2}{2-\beta},2x\right), \tag{33}$$

obtained by noting that $(\nu^*-2)/2 = -1/(2-\beta)$, implying $\nu^* = 2 - 2/(2-\beta)$. Thus,

$$C = S_t\,\mathrm{e}^{-a\tau}Q\left(2y;2+\frac{2}{2-\beta},2x\right) - K\,\mathrm{e}^{-r\tau}Q\left(2y;2-\frac{2}{2-\beta},2x\right). \tag{34}$$

It is noted that $2 - 2/(2-\beta)$ can be negative for $\beta < 2$. Thus further work is needed. Using the monotone convergence theorem and the integration by parts, we have

$$\int_y^\infty P(2y;2\nu,2k)\,\mathrm{d}k = \int_y^\infty \mathrm{e}^{-z-k}(z/k)^{\nu-1}\sqrt{kz}^{\nu-1}\sum_{n=0}^\infty \frac{(zk)^n}{n!\Gamma(n+\nu-1+1)}\,\mathrm{d}k$$

$$= \int_y^\infty \frac{\mathrm{e}^{-z}z^{n+\nu-1}}{\Gamma(n+\nu)}\int_y^\infty \frac{\mathrm{e}^{-k}k^n}{\Gamma(n+1)}\,\mathrm{d}k = \sum_{n=0}^\infty g(n+\nu,z)G(n+1,y)$$

$$= \sum_{n=0}^\infty g(n+\nu-1,z)\sum_{i=1}^\infty g(i,y). \tag{35}$$

Now we also have the result $G(n,y) = \sum_{i=1}^\infty g(i,y)$, which can be shown by observing that

$$G(n,y) = \int_y^\infty \frac{\mathrm{e}^{-k}k^{n-1}}{\Gamma(n)}\,\mathrm{d}k = -\int_y^\infty \frac{k^{n-1}}{\Gamma(n)}d\mathrm{e}^{-k} = \frac{y^{n-1}\mathrm{e}^{-y}}{\Gamma(n)} + \int_y^\infty \frac{k^{n-2}\mathrm{e}^{-k}}{\Gamma(n-1)}\,\mathrm{d}k = \sum_{i=1}^n \frac{y^{i-1}\mathrm{e}^{-y}}{\Gamma(i)} = \sum_{i=1}^n g(i,y).$$

The above result can also be expressed as

$$G(m + 1, t) = g(m + 1, t) + G(m, t). \tag{36}$$

Next, applying the monotone convergence theorem, we have

$$
\begin{aligned}
Q(z; v, k) &= \int_z^\infty \frac{1}{2} \left(\frac{y}{k} \right)^{(v-2)/4} I_{v-2/2} \left(\sqrt{ky} \right) e^{-(k+y)/2} \, dy \\
&= \int_z^\infty \frac{1}{2} \left(\frac{y}{k} \right)^{(v-2)/4} \left(\frac{1}{2} \sqrt{ky} \right)^{(v-2)/2} \sum_{n=0}^\infty \frac{(ky/4)^n}{n! \, \Gamma(v + 2n/2)} e^{-(k+y)/2} \, dy \\
&= \sum_{n=0}^\infty e^{-k/2} \frac{k^n (1/2)^n}{\Gamma(n+1)} \int_z^\infty \frac{e^{-y/2} y^{v+2n/2-1}}{(1/2)^{-(v+2n)/2} \Gamma(v + 2n/2)} \, dy \\
&= \sum_{n=0}^\infty e^{-k/2} \frac{(k/2)^n}{\Gamma(n+1)} \int_z^\infty \frac{(1/2)^{(v+2n)/2}}{\Gamma(v + 2n/2)} e^{-y/2} y^{(v+2n/2)-1} \, dy \\
&= \sum_{n=0}^\infty e^{-k/2} \frac{(k/2)^n}{\Gamma(n+1)} Q(z; v + 2n, 0), \tag{37}
\end{aligned}
$$

where

$$
\begin{aligned}
Q(z; v + 2n, 0) &= \int_z^\infty \frac{(1/2)^{(v+2n)/2}}{\Gamma(v + 2n/2)} e^{-y/2} y^{v+2n/2-1} \, dy \\
&= \int_{z/2}^\infty \frac{1}{\Gamma(v + 2n/2)} e^{-y} y^{(v+2n/2)-1} \, dy = G(n + v/2, z/2).
\end{aligned}
$$

Furthermore, from the property of $G(\cdot, \cdot)$ as shown in (36), we have

$$Q(z; v, k) = \sum_{n=0}^\infty g \left(n + 1, \frac{k}{2} \right) G \left(n + \frac{v}{2}, \frac{z}{2} \right) = \sum_{n=0}^\infty g \left(n, \frac{k}{2} \right) G \left(n + \frac{v-2}{2}, \frac{z}{2} \right). \tag{38}$$

Hence:

$$Q(2z; 2v, 2k) = \sum_{n=0}^\infty g(n, k) G(n + v - 1, z).$$

Using the property of $G(\cdot, \cdot)$ as given in (36) again, we have

$$
\begin{aligned}
Q(2z; 2v, 2k) &= g(1, k) G(v, z) + g(2, k) G(v + 1, z) + g(3, k) G(v + 2, z) + \cdots \\
&= g(1, k)[G(v - 1, z) + g(v, z)] + g(2, k)[G(v - 1, z) + g(v + 1, z)] + g(3, k)[G(v - 1, z) \\
&\quad + g(v, z) + g(v + 1, z) + g(v + 2, z)] + \cdots = [G(v - 1, z) + g(v, z)] \sum_{n=1}^\infty g(n, k) \\
&\quad + g(v + 1, z) \sum_{n=2}^\infty g(n, k) + g(v + 2, z) \sum_{n=3}^\infty g(n, k) + \cdots = G(v - 1, z) + g(v, z) \\
&\quad + g(v + 1, z)[1 - g(1, k)] + g(v + 2, z)[1 - g(1, k) - g(2, k)] + \cdots \\
&= G(v - 1, z) + \sum_{n=0}^\infty g(v + n, z) - g(v + 1, z)[g(1, k)] - g(v + 2, z)[g(1, k) + g(2, k)] + \cdots \\
&= 1 - g(v + 1, z)[g(1, k)] - g(v + 2, z)[g(1, k) + g(2, k)] - \cdots.
\end{aligned}
$$

We conclude that

$$Q(2z; 2\nu, 2k) = 1 - \sum_{n=1}^{\infty} g(n + \nu, z) \sum_{i=1}^{n} g(i, k). \tag{39}$$

From (35) and (39) we observe that

$$\int_y^{\infty} P(2z; 2\nu, 2k)\,\mathrm{d}k = 1 - Q(2z; 2(\nu - 1), 2y). \tag{40}$$

Thus, we can write C_2 as

$$C_2 = K\,\mathrm{e}^{-r\tau} \int_y^{\infty} P\left(2x; 2 + \frac{2}{2-\beta}, 2w\right)\,\mathrm{d}w = K\,\mathrm{e}^{-r\tau} Q\left(2y; 2 - \frac{2}{2-\beta}, 2x\right)$$

$$= K\,\mathrm{e}^{-r\tau}\left(1 - Q\left(2x; \frac{2}{2-\beta}, 2y\right)\right). \tag{41}$$

From (41) we immediately obtain

$$Q\left(2y; 2 - \frac{2}{2-\beta}, 2x\right) + Q\left(2x; \frac{2}{2-\beta}, 2y\right) = 1 \tag{42}$$

implying that

$$Q(z; 2n, k) + Q(k; 2 - 2n, z) = 1, \tag{43}$$

with degrees of freedom $2 - 2n$ of $Q(k; 2 - 2n, z)$ can be a non-integer.

From (42), we can obtain that the noncentral Chi-square $Q(2y; 2 - 2/(2 - \beta), 2x)$ with $2 - 2/(2 - \beta)$ degrees of freedom and the noncentrality parameter $2x$ can be represented by another noncentral Chi-square distribution $1 - Q(2x; 2/(2 - \beta), 2y)$ with degrees of freedom $2/(2 - \beta)$ and the noncentrality parameter $2y$. The standard definition of noncentral Chi-square distribution in Section 3 has integer degrees of freedom. If the degree of freedom is not an integer, we can use (43) to transfer the original noncentral Chi-square distribution into another noncentral Chi-square distribution. Thus, we obtain an option pricing formula for the CEV model in terms of the complementary noncentral Chi-square distribution function $Q(z; \nu, k)$ which is valid for any value of β less than 2, as required by the model.

Substituting (41) into (34), we obtain

$$C = S_t\,\mathrm{e}^{-a\tau} Q\left(2y; 2 + \frac{2}{2-\beta}, 2x\right) - K\,\mathrm{e}^{-r\tau}\left(1 - Q\left(2x; \frac{2}{2-\beta}, 2y\right)\right), \tag{44}$$

where $y = k^* K^{2-\beta}$, $x = k^* S_t^{2-\beta}\,\mathrm{e}^{(r-a)(2-\beta)\tau}$, $k^* = 2(r - a)/(\sigma^2(2 - \beta)(\mathrm{e}^{(r-a)(2-\beta)\tau} - 1))$ and a is the continuous proportional dividend rate. The corresponding CEV option pricing formula for $\beta > 2$ can be derived through the same fashion. When $\beta > 2$ (see, Emanuel and MacBeth [6] and Chen and Lee [3]), the call option formula is as follows:

$$C = S_t\,\mathrm{e}^{-a\tau} Q\left(2x; \frac{2}{\beta-2}, 2y\right) - K\,\mathrm{e}^{-r\tau}\left(1 - Q\left(2y; 2 + \frac{2}{\beta-2}, 2x\right)\right). \tag{45}$$

We note that from the evaluation of the option pricing formula C, especially C_2, as given in (34), we have

$$2k^* S_T^{2-\beta} \sim {\chi'}_\nu^{\,2}(\lambda), \tag{46}$$

where

$$\nu = 2 - \frac{2}{2-\beta}, \qquad \lambda = 2k^* S_t^{2-\beta}\,\mathrm{e}^{(r-a)(2-\beta)\tau}.$$

Thus, the option pricing formula for the CEV model (44) can be obtained directly from the payoff function:

$$\max(S_T - K, 0) = \begin{cases} S_T - K, & \text{if } S_T > K \\ 0, & \text{otherwise} \end{cases} \tag{47}$$

by taking the expectation of (47), with S_T having the distribution given by (46).

Before concluding this subsection we consider that the noncentral Chi-square distribution will approach log-normal as β tends to 2. When either λ or ν approaches to infinity, the standardized variable

$$\frac{\chi_\nu'^2(\lambda) - (\nu + \lambda)}{\sqrt{2(\nu + 2\lambda)}}$$

tends to $N(0, 1)$ as either $\nu \to \infty$ or $\lambda \to \infty$. Using the fact that $(x^a - 1)/a$ will approach to $\ln x$ as $a \to 0$, it can be verified that

$$
\begin{aligned}
\lim_{\beta \to 2^-} \frac{\chi_\nu'^2(\lambda) - (\nu + \lambda)}{\sqrt{2(\nu + 2\lambda)}} &= \lim_{\beta \to 2^-} \frac{2k^* S_T^{2-\beta} - (\nu + \lambda)}{\sqrt{2(\nu + 2\lambda)}} \\
&= \lim_{\beta \to 2^-} \frac{2r^* S_T^{2-\beta} - (1-\beta)\sigma^2(e^{r^*\tau(2-\beta)} - 1) - 2r^* S_t^{2-\beta} e^{r^*\tau(2-\beta)}}{\sigma^2(e^{r^*\tau(2-\beta)} - 1)} \\
&\quad \times \sqrt{\frac{\sigma^2(e^{r^*\tau(2-\beta)} - 1)/(2-\beta)}{(1-\beta)\sigma^2(e^{r^*\tau(2-\beta)} - 1) + 4r^* S_t^{2-\beta} e^{r^*\tau(2-\beta)}}} = \frac{\ln S_T - [\ln S_t + (r^* - \sigma^2/2)\tau]}{\sigma\sqrt{\tau}},
\end{aligned}
$$

where $r^* = r - a$. Thus:

$$\ln S_T \mid \ln S_t \sim N\left(\ln S_t + \left(r - a - \frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right) \tag{48}$$

as $\beta \to 2^-$. Similarly, (48) also holds when $\beta \to 2^+$. From (45), we have $2k^* S_T^{2-\beta} \sim \chi_\nu'^2(\lambda)$, where $\nu = 2 + 2/(\beta - 2)$ if $\beta > 2$. Thus, we clarify the result of (48).

4.2. Some computational considerations

As noted by Schroder [10], (39) allows the following iterative algorithm to be used in computing the infinite sum when z and k are not large. First initialize the following four variables (with $n = 1$)

$$gA = \frac{e^{-z} z^\nu}{\Gamma(1 + \nu)} = g(1 + \nu, z), \qquad gB = e^{-k} = g(1, k), \quad Sg = gB, \qquad R = 1 - (gA)(Sg).$$

Then repeat the following loop beginning with $n = 2$ and increase increment n by one after each iteration. The loop is terminated when the contribution to the sum, R, is declining and is very small.

$$gA = gA\left(\frac{z}{n + \nu - 1}\right) = g(n + \nu, z), \qquad gB = gB\left(\frac{k}{n-1}\right) = g(n, k),$$

$$Sg = Sg + gB = g(1, k) + g(n, k), \qquad R = R - (gA)(Sg) = \text{the } n\text{th partial sum.}$$

At each iteration, gA equals $g(n + \nu, z)$, gB equals $g(n, k)$ and Sg equals $g(1, k) + \cdots + g(n, k)$. The computation is easily done.

As for an approximation, Sankaran [9] showed that the distribution of $(\chi_\nu'^2/(\nu + k))^h$ is approximately normal with the expected value $\mu = 1 + h(h-1)P - h(2-h)mP^2/2$ and variance $\sigma^2 = h^2 P(1 + mP)$, where $h = 1 - (2/3)(\nu + k)(\nu + 3k)(\nu + 2k)^{-2}$, $P = (\nu + 2k)/(\nu + k)^2$ and $m = (h-1)(1-3h)$. Therefore we have

$$Q(z; \nu, k) = Pr(\chi'^2 z) = Pr\left(\frac{\chi'^2}{\nu + k} > \frac{z}{\nu + k}\right) = Pr\left(\left(\frac{\chi'^2}{\nu + k}\right)^h > \left(\frac{z}{\nu + k}\right)^h\right)$$

$$\doteq \Phi\left(\frac{1 - hP[1 - h + 0.5(2-h)mP] - (z/\nu + k)^h}{h\sqrt{2P(1 + mP)}}\right).$$

5. Concluding remarks

The option pricing formula under the CEV model is quite complex because it involves the cumulative distribution function of the noncentral Chi-square distribution $Q(z; v, k)$. Some computational considerations given in the article which will facilitate the computation of the CEV option pricing formula. Hence, the computation will no longer be a difficult problem in practice.

Acknowledgements

Appendix A. Proof of Feller's Lemma

We need some preliminary results in order to prove (6).

Proposition 1. $f(z) = e^{Av/z}z^{-1}$ *is the Laplace transformation of* $I_0(2(Avx)^{1/2})$*, where* $I_k(x)$ *is the Bessel function*:

$$I_k(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{2r+k}}{r!\Gamma(r+1+k)}.$$

Proof. By the definition of Laplace transformation and the monotone convergence theorem, we have

$$f(z) = \int_0^{\infty} e^{-zx} I_0(2(Avx)^{1/2}) \, dx = \int_0^{\infty} e^{-zx} \sum_{r=0}^{\infty} \frac{(Avx)^{1/2}}{r!\Gamma(r+1)}$$

$$= \int_0^{\infty} e^{-zx} \left\{ 1 + \frac{(Avx)}{\Gamma(2)} + \frac{(Avx)^2}{2!\Gamma(3)} + \cdots + \frac{(Avx)^n}{n!\Gamma(n+1)} + \cdots \right\}$$

$$= \frac{1}{z} + \frac{Av}{z^2} + \frac{(Av)^2}{2!z^3} + \cdots + \frac{(Av)^n}{n!z^{n+1}} + \cdots$$

$$= \frac{1}{z} \left\{ 1 + \frac{Av}{z} + \frac{(Av)^2}{2!z^2} + \cdots + \frac{(Av)^n}{n!z^n} + \cdots \right\} = e^{Av/z}z^{-1}. \quad \square$$

Proposition 2. *Consider the parabolic differential equation*

$$P_t = (axP)_{xx} - ((bx+h)P)_x, \quad 0 < x < \infty, \tag{A.1}$$

where a, b, h *are constants*, $0 < h < a$*, then the Laplace transformation of* $f(t, x, x_0)$ *with respect to* x *takes the form*

$$w(t, s; x_0) = \int_0^{\infty} e^{-sx} f(t, x; x_0) \, dx = \left(\frac{b}{sa(e^{bt}-1)+b} \right)^{h/a}$$

$$\times \exp \left\{ \frac{-sbx_0 \, e^{bt}}{sa(e^{bt}-1)+b} \right\} \Gamma \left(1 - \frac{h}{a}; \frac{b^2 x_0 \, e^{bt}}{a(e^{bt}-1)(sa(e^{bt}+1)+b)} \right), \tag{A.2}$$

where $\Gamma(n; z) = \Gamma^{-1}(n) \int_0^z e^{-x} x^{n-1} \, dx$.

Proof. The proof of (A.2) is referred to Lemma 7 of Feller [7]. $\quad \square$

We now turn to prove (6). From (A.2), let

$$A = \frac{bx_0}{a(1 - \mathrm{e}^{-bt})} \quad \text{and} \quad z = \frac{1}{b}(sa(\mathrm{e}^{bt} - 1) + b).$$

The $w(t, s; x_0)$ in (A.2) can be rewritten as

$$w(t, s; x_0) = \frac{z^{-h/a}\,\mathrm{e}^{-(1-1/z)A}}{\Gamma(1 - h/a)} \int_0^{A/z} \mathrm{e}^{-x} x^{-h/a}\,\mathrm{d}x = \frac{z^{-h/a}\,\mathrm{e}^{-(1-1/z)A}}{\Gamma(1 - h/a)} \int_0^1 \mathrm{e}^{-Ax'/z}(Ax'/z)^{-h/a}(A/z)^{1-h/a}\,\mathrm{d}x'$$

$$= \frac{A^{1-h/a}\,\mathrm{e}^{-A}}{\Gamma(1 - h/a)} \int_0^1 \mathrm{e}^{A(1-x')/z}(x')^{-h/a} z^{-1}\,\mathrm{d}x' = \frac{A^{1-h/a}\,\mathrm{e}^{-A}}{\Gamma(1 - h/a)} \int_0^1 \mathrm{e}^{Av/z}(1 - v)^{-h/a} z^{-1}\,\mathrm{d}v. \tag{A.3}$$

By Proposition 1 now that $f(z) = \mathrm{e}^{Av/z} z^{-1}$ is the Laplace transformation of $I_0(2(Avx)^{1/2})$ and by the Fubini theorem, we have

$$w(t, s; x_0) = \frac{A^{1-h/a}\,\mathrm{e}^{-A}}{\Gamma(1 - h/a)} \int_0^1 (1 - v)^{-h/a} \left[\int_0^\infty \mathrm{e}^{-zx} I_0(2(Avx)^{1/2})\,\mathrm{d}x \right]\,\mathrm{d}v$$

$$= \frac{A^{1-h/a}\,\mathrm{e}^{-A}}{\Gamma(1 - h/a)} \int_0^\infty \int_0^1 (1 - v)^{-h/a}\,\mathrm{e}^{-zx} I_0(2(Avx)^{1/2})\,\mathrm{d}x\,\mathrm{d}v$$

$$= \frac{A^{2-h/a}\,\mathrm{e}^{-A}}{\Gamma(1 - h/a)x_0\mathrm{e}^{bt}} \int_0^\infty \int_0^1 (1 - v)^{-h/a}\,\mathrm{e}^{-sx'}\,\mathrm{e}^{-Ax'/(x_0\mathrm{e}^{bt})} \times I_0(2A(\mathrm{e}^{-bt}vx'/x_0)^{1/2})\,\mathrm{d}v\,\mathrm{d}x'. \tag{A.4}$$

Hence, upon comparing the two formulae for $w(t, s; x_0)$ and by the monotone convergence theorem, we have

$$f(t, x; x_0) = \frac{b}{\Gamma(1 - h/a)a(\mathrm{e}^{bt} - 1)} \left[\frac{bx_0}{a(\mathrm{e}^{bt} - 1)} \right]^{1-h/a}$$

$$\times \exp\left\{ \frac{-b(x + x_0\,\mathrm{e}^{bt})}{a(\mathrm{e}^{bt} - 1)} \right\} \int_0^1 (1 - v)^{-h/a} I_0\left(\frac{2b(\mathrm{e}^{-bt}vxx_0)^{1/2}}{a(1 - \mathrm{e}^{-bt})} \right)\,\mathrm{d}v$$

$$= \frac{b}{\Gamma(1 - h/a)a(\mathrm{e}^{bt} - 1)} \left[\frac{bx_0}{a(\mathrm{e}^{bt} - 1)} \right]^{1-h/a}$$

$$\times \exp\left\{ \frac{-b(x + x_0\mathrm{e}^{bt})}{a(\mathrm{e}^{bt} - 1)} \right\} \int_0^1 (1 - v)^{-h/a} \sum_{r=0}^\infty \frac{[b(\mathrm{e}^{-bt}vxx_0)^{1/2}/(a(1 - \mathrm{e}^{-bt}))]^{2r}}{r!\Gamma(r + 1)}\,\mathrm{d}v$$

$$= \frac{b}{\Gamma(1 - h/a)a(\mathrm{e}^{bt} - 1)} \left[\frac{bx_0}{a(\mathrm{e}^{bt} - 1)} \right]^{1-h/a}$$

$$\times \exp\left\{ \frac{-b(x + x_0\,\mathrm{e}^{bt})}{a(\mathrm{e}^{bt} - 1)} \right\} \sum_{r=0}^\infty \frac{[b(\mathrm{e}^{-bt}xx_0)^{1/2}/(a(1 - \mathrm{e}^{-bt}))]^{2r}}{r!\Gamma(r + 1)} \int_0^1 (1 - v)^{-h/a}v^r\,\mathrm{d}v$$

$$= \frac{b}{\Gamma(1 - h/a)a(\mathrm{e}^{bt} - 1)} \left[\frac{bx_0}{a(\mathrm{e}^{bt} - 1)} \right]^{1-h/a}$$

$$\times \exp\left\{ \frac{-b(x + x_0\,\mathrm{e}^{bt})}{a(\mathrm{e}^{bt} - 1)} \right\} \sum_{r=0}^\infty \frac{[b(\mathrm{e}^{-bt}xx_0)^{1/2}/(a(1 - \mathrm{e}^{-bt}))]^{2r}}{r!\Gamma(r + 1)} \frac{\Gamma(r + 1)\Gamma(1 - h/a)}{\Gamma(r + 1 + 1 - h/a)}$$

$$= \frac{b}{a(\mathrm{e}^{bt} - 1)} \left(\frac{\mathrm{e}^{-bt}x}{x_0} \right)^{(h-a)/2a} \exp\left\{ \frac{-b(x + x_0\mathrm{e}^{bt})}{a(\mathrm{e}^{bt} - 1)} \right\} I_{1-h/a}\left(\frac{2b}{a(1 - \mathrm{e}^{bt})}(\mathrm{e}^{-bt}xx_0)^{1/2} \right). \tag{A.5}$$

This completes the proof.

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