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The linear 3-arboricity of $K_{n,n}$ and K_n

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Abstract

A linear k-forest is a forest whose components are paths of length at most k. The linear k-arboricity of a graph G, denoted by $\mathrm{la}_k(G)$, is the least number of linear k-forests needed to decompose G. In this paper, we completely determine $\mathrm{la}_k(G)$ when G is a balanced complete bipartite graph $K_{n,n}$ or a complete graph K_n, and $k = 3$.

Keywords: Linear k-forest; Linear k-arboricity; Balanced complete bipartite graph; Complete graph

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer to [15] for terminology in Graph Theory. A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \ldots, G_d$, then we say that $G_1, G_2, \ldots, G_d$ decompose G, or G can be decomposed into $G_1, G_2, \ldots, G_d$. Furthermore, a *linear k-forest* is a forest whose components are paths of length at most k. The *linear k-arboricity* of a graph G, denoted by $\mathrm{la}_k(G)$, is the least number of linear k-forests needed to decompose G.

The linear k-arboricity of a graph was first introduced by Habib and Peroche [10]. It is a natural generalization of *edge coloring*. Clearly, a linear 1-forest is induced by a matching, and $\mathrm{la}_1(G)$ is the *edge chromatic number*, or *chromatic index*, $\chi'(G)$ of a graph G. Moreover, the linear k-arboricity $\mathrm{la}_k(G)$ is also a refinement of the ordinary *linear arboricity* $\mathrm{la}(G)$ (or $\mathrm{la}_\infty(G)$) of a graph G, which is the case when every component of each forest is a path with no length constraint.

In 1982, Habib and Peroche [9] proposed the following conjecture for an upper bound on $\mathrm{la}_k(G)$.

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Conjecture 1.1. If G is a graph with maximum degree $\Delta(G)$ and $k \geqslant 2$, then

$$\mathrm{la}_k(G) \leqslant \begin{cases} \left\lceil \dfrac{\Delta(G)\cdot|V(G)|}{2\left\lfloor \dfrac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{when } \Delta(G)=|V(G)|-1 \text{ and} \\ \left\lceil \dfrac{(\Delta(G)\cdot|V(G)|)+1}{2\left\lfloor \dfrac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{when } \Delta(G)<|V(G)|-1. \end{cases}$$

So far, quite a few results on the verification of Conjecture 1.1 have been obtained in the literature, especially for graphs with particular structures, such as trees [4,5,10], cubic graphs [3,12,14], regular graphs [1,2], planar graphs [13], balanced complete bipartite graphs [7,8] and complete graphs [3,6,7,16]. As for a lower bound on $\mathrm{la}_k(G)$, since any vertex in a linear k-forest has degree at most 2 and a linear k-forest in a graph G has at most $\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor$ edges, we can obtain the following result.

Proposition 1.2.

$$\mathrm{la}_k(G) \geqslant \max\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\left\lfloor \dfrac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil \right\}.$$

In this paper, we completely determine $\mathrm{la}_k(G)$ when G is a balanced complete bipartite graph $K_{n,n}$ or a complete graph K_n, and $k=3$. The results are coherent with the corresponding cases of Conjecture 1.1.

2. Linear 3-arboricity of $K_{n,n}$

Let $G(X,Y)$ be a bipartite graph with partite sets $X=\{x_0, x_1, \ldots, x_{r-1}\}$ and $Y=\{y_0, y_1, \ldots, y_{s-1}\}$. Suppose that $|Y|=s \geqslant r=|X|$. We define the *bipartite difference* of an edge $x_p y_q$ in $G(X,Y)$ as the value $q-p \pmod s$. It is not difficult to see that a set consists of those edges in $G(X,Y)$ with the same bipartite difference must be a matching. In particular, such a set is a perfect matching if $G(X,Y)$ is a $K_{n,n}$. Furthermore, we can partition the edge set of $K_{n,n}$ into n pairwise disjoint perfect matchings $M_0, M_1, \ldots, M_{n-1}$ such that M_i is exactly the set of edges of bipartite difference i in $K_{n,n}$, $i=0,1,\ldots,n-1$.

Lemma 2.1. *If $n \geqslant 4$ is even and α belongs to $\{0,1,\ldots,n-3\}$, then the edges of bipartite differences α, $\alpha+1$ and $\alpha+2$ in $K_{n,n}$ can form two edge-disjoint linear 3-forests.*

Proof. Consider $K_{n,n}$ with partite sets $X=\{x_0,x_1,\ldots,x_{n-1}\}$ and $Y=\{y_0,y_1,\ldots,y_{n-1}\}$. Let $M_\alpha=\{x_i y_{i+\alpha \pmod n} \mid i=0,1,\ldots,n-1\}$, $M_{\alpha+1}=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=0,1,\ldots,n-1\}$ and $M_{\alpha+2}=\{x_i y_{i+(\alpha+2) \pmod n} \mid i=0,1,\ldots,n-1\}$ be the sets of edges of bipartite differences α, $\alpha+1$ and $\alpha+2$ in $K_{n,n}$, respectively. Each of M_α, $M_{\alpha+1}$ and $M_{\alpha+2}$ is a perfect matching in $K_{n,n}$. Then, by partitioning $M_{\alpha+1}$ into two disjoint matchings $W_1=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=0,2,4,\ldots,n-2\}$ and $W_2=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=1,3,5,\ldots,n-1\}$, the edges in $M_\alpha \cup W_1$ can form one linear 3-forest in $K_{n,n}$ and the edges in $M_{\alpha+2} \cup W_2$ can form another one. Thus the assertion holds. □

Lemma 2.2. *If $n \geqslant 3$ is odd, α belongs to $\{0,1,\ldots,n-3\}$ and e is an edge of bipartite difference $\alpha+1$ in $K_{n,n}$, then the edges other than e of bipartite differences α, $\alpha+1$ and $\alpha+2$ in $K_{n,n}$ can form two edge-disjoint linear 3-forests.*

Proof. It is analogous to the proof of Lemma 2.1. Suppose that $e=x_r y_{r+(\alpha+1) \pmod n}$ for some $r \in \{0,1,\ldots,n-1\}$. If r is even, then we partition $M_{\alpha+1}$ into three pairwise disjoint matchings $\{e\}$, $W_1=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=0,2,\ldots,r-2,r+1,r+3,\ldots,n-2\}$ and $W_2=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=1,3,\ldots,r-1,r+2,r+4,\ldots,n-1\}$. If r is odd, then we partition $M_{\alpha+1}$ into three pairwise disjoint matchings $\{e\}$, $W_1=\{x_i y_{i+(\alpha+1) \pmod n} \mid i=1,3,\ldots,r-2,r+1,r+3,\ldots,n-1\}$

and $W_2 = \{x_i y_{i+(\alpha+1) \pmod n} \mid i = 0, 2, \ldots, r-1, r+2, r+4, \ldots, n-2\}$. Then the edges in $M_\alpha \cup W_1$ can form one linear 3-forest in $K_{n,n}$ and the edges in $M_{\alpha+2} \cup W_2$ can form another one. Thus the assertion holds. □

Now, we are ready to show our main results in this section.

Proposition 2.3. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 0 \pmod 6$.

Proof. Since $n \equiv 0 \pmod 6$, by Lemma 2.1, the edges of bipartite differences $0, 1, \ldots, n-1$ in $K_{n,n}$ can form $(\frac{n}{3}) \cdot 2 = \frac{2n}{3} = \lceil \frac{2n}{3} \rceil$ pairwise edge-disjoint linear 3-forests. Thus $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ when $n \equiv 0 \pmod 6$. □

Proposition 2.4. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 4 \pmod 6$.

Proof. Since $n \equiv 4 \pmod 6$, by Lemma 2.1, the edges of bipartite differences $0, 1, \ldots, n-2$ in $K_{n,n}$ can form $(\frac{n-1}{3}) \cdot 2 = \frac{2(n-1)}{3}$ pairwise edge-disjoint linear 3-forests. Also, the edges of bipartite difference $n-1$ in $K_{n,n}$ can form one linear 3-forest. Thus $\mathrm{la}_3(K_{n,n}) \leqslant \frac{2(n-1)}{3} + 1 = \frac{2n+1}{3} = \lceil \frac{2n}{3} \rceil$ when $n \equiv 4 \pmod 6$. □

Proposition 2.5. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 2 \pmod 6$.

Proof. Since $n \equiv 2 \pmod 6$, by Lemma 2.1, the edges of bipartite differences $0, 1, \ldots, n-3$ in $K_{n,n}$ can form $(\frac{n-2}{3}) \cdot 2 = \frac{2(n-2)}{3}$ pairwise edge-disjoint linear 3-forests. Also, the edges of bipartite difference $n-2$ in $K_{n,n}$ can form one linear 3-forest, and the edges of bipartite difference $n-1$ in $K_{n,n}$ can form another one. Thus $\mathrm{la}_3(K_{n,n}) \leqslant \frac{2(n-2)}{3} + 2 = \frac{2n+2}{3} = \lceil \frac{2n}{3} \rceil$ when $n \equiv 2 \pmod 6$. □

Proposition 2.6. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 5 \pmod 6$.

Proof. If H is a subgraph of a graph G, then $\mathrm{la}_k(H) \leqslant \mathrm{la}_k(G)$. Thus, $\mathrm{la}_3(K_{n,n}) \leqslant \mathrm{la}_3(K_{n+1,n+1}) \leqslant \lceil \frac{2(n+1)}{3} \rceil = \lceil \frac{2n+2}{3} \rceil = \lceil \frac{2n}{3} \rceil$ by Proposition 2.3 and $n \equiv 5 \pmod 6$. □

Proposition 2.7. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n+2}{3} \rceil$ *when* $n \equiv 3 \pmod 6$.

Proof. Since $K_{n,n}$ is a subgraph of $K_{n+1,n+1}$, $\mathrm{la}_3(K_{n,n}) \leqslant \mathrm{la}_3(K_{n+1,n+1}) \leqslant \lceil \frac{2(n+1)}{3} \rceil = \lceil \frac{2n+2}{3} \rceil$ by Proposition 2.4 and $n \equiv 3 \pmod 6$. □

Proposition 2.8. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 1 \pmod 6$.

Proof. Consider $K_{n,n}$ with partite sets $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. First, let $e_t = x_{n-t} y_{n-t+3(t-1)+1 \pmod n}$ be an edge of bipartite difference $3(t-1)+1$ in $K_{n,n}$, $t = 1, 2, \ldots, \frac{n-1}{3}$. Then, by Lemma 2.2, the edges other than e_t of bipartite differences $3(t-1)$, $3(t-1)+1$ and $3(t-1)+2$ in $K_{n,n}$ can form two edge-disjoint linear 3-forests, for all $t \in \{1, 2, \ldots, \frac{n-1}{3}\}$. Hence, the edges other than $e_1, e_2, \ldots, e_{(n-1)/3}$ of bipartite differences $0, 1, \ldots, n-2$ in $K_{n,n}$ can form $(\frac{n-1}{3}) \cdot 2 = \frac{2(n-1)}{3}$ pairwise edge-disjoint linear 3-forests.

Next, let $E = \{e_1, e_2, \ldots, e_{(n-1)/3}\}$, and also let H be the subgraph of $K_{n,n}$ induced by $E \cup M_{n-1}$, where $M_{n-1} = \{x_i y_{i+(n-1) \pmod n} \mid i = 0, 1, \ldots, n-1\}$ is the set of edges of bipartite difference $n-1$ in $K_{n,n}$. Since E is a matching and M_{n-1} is a perfect matching in $K_{n,n}$, each component of H is a path or an even cycle. Now, without loss of generality, assume that $P = u - w_1 - w_2 - v$ is a path of length 3 in H with $w_1 \in X$ and $w_2 \in Y$. Then the edge $w_1 w_2$ in P must belong to E. Otherwise, if $w_1 w_2 \in M_{n-1}$, then $w_1 u \in E$ and $v w_2 \in E$, i.e., $w_1 \in X' = \{x_{n-1}, x_{n-2}, \ldots, x_{n-(n-1)/3}\}$ and $w_2 \in Y' = \{y_0, y_2, \ldots, y_{n-(n-1)/3-3}\}$. But, any edge ab formed by $a \in X'$ and $b \in Y'$ cannot belong to M_{n-1}, it is a contradiction. Therefore, it implies that each component of H is a path of length at most 3, and hence H is a linear 3-forest in $K_{n,n}$.

Accordingly, $\mathrm{la}_3(K_{n,n}) \leqslant \frac{2(n-1)}{3} + 1 = \frac{2n+1}{3} = \lceil \frac{2n}{3} \rceil$ when $n \equiv 1 \pmod 6$. □

Finally, we conclude the work of this section by the following theorem.

Theorem 2.9.

$$\mathrm{la}_3(K_{n,n}) = \begin{cases} \left\lceil \dfrac{2n}{3} \right\rceil & \textit{when } n \equiv 0, 1, 2, 4, 5 \pmod 6 \textit{ and} \\ \left\lceil \dfrac{2n+2}{3} \right\rceil & \textit{when } n \equiv 3 \pmod 6. \end{cases}$$

Proof. $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n}{3} \rceil$ or $\mathrm{la}_3(K_{n,n}) \leqslant \lceil \frac{2n+2}{3} \rceil$, following the value of $n \pmod 6$, via Propositions 2.3–2.8, while $\mathrm{la}_3(K_{n,n}) \geqslant \lceil \frac{2n}{3} \rceil$ or $\mathrm{la}_3(K_{n,n}) \geqslant \lceil \frac{2n+2}{3} \rceil$, following the value of $n \pmod 6$, by Proposition 1.2. This concludes the proof. □

3. Linear 3-arboricity of K_n

In [3], Bermond et al. mentioned briefly the result that $\mathrm{la}_3(K_{12t+4}) = 8t + 2$ for any $t \geqslant 0$ can be obtained by using techniques of resolvable design in the theory of combinatorial design. In this section, we will determine $\mathrm{la}_3(K_n)$ for any $n \in \mathbb{N}$ by specifying how the linear 3-forests decompose K_n are to be found.

Assume that G and H are graphs. A spanning subgraph F of G is called an *H-factor* if each component of F is isomorphic to H. If G is expressible as an edge-disjoint union of H-factors, then this union is called an *H-factorization* of G. Furthermore, we say that a 1-*factor* of a graph G is a spanning 1-regular subgraph of G. A decomposition of a regular graph G into 1-factors is a 1-*factorization* of G. A graph with a 1-factorization is 1-*factorable*. For a complete graph K_n, the following result is well-known.

Theorem 3.1 (*Harary [11]*). *If n is even, then K_n can be decomposed into $n-1$ pairwise edge-disjoint 1-factors, and thus it is 1-factorable.*

Proof. We can obtain simply the $n-1$ pairwise edge-disjoint 1-factors $f_0, f_1, \ldots, f_{n-2}$ of K_n from a circle and $\frac{n}{2}$ chords in it. Let the $n-1$ vertices be placed equally spaced round a circle, and label them $v_0, v_1, \ldots, v_{n-2}$; also label the center v_{n-1}. Then, for each $j \in \{0, 1, \ldots, n-2\}$, the 1-factor f_j is induced by an edge joining vertices v_j and v_{n-1}, and by parallel edges joining the other vertices in pairs. □

We need to specify the proof because such a method can ensure that each 1-factor f_j other than $f_{n/2-1}$ contains an edge $v_k v_{k+1}$ for some $k \in \{0, 1, \ldots, n-3\}$; it cannot be ensured if we label the $n-1$ outside vertices $v_1, v_2, \ldots, v_{n-1}$ and the center v_0.

Consider K_n with vertices $v_0, v_1, \ldots, v_{n-1}$. We define the *difference* d of an edge $v_i v_j \in E(K_n)$ as the value $\min\{|i-j|, n-|i-j|\}$. Then the edge set $E(K_n)$ can be expressed as $\{v_i v_{i+d \pmod n} \mid i = 0, 1, \ldots, n-1 \text{ and } d = 1, 2, \ldots, \frac{n}{2}-1\} \cup \{v_i v_{i+d \pmod n} \mid i = 0, 1, \ldots, \frac{n}{2}-1 \text{ and } d = \frac{n}{2}\}$ when n is even, or $\{v_i v_{i+d \pmod n} \mid i = 0, 1, \ldots, n-1 \text{ and } d = 1, 2, \ldots, \frac{n-1}{2}\}$ when n is odd. Furthermore, if V_r and V_s are any two disjoint subsets of the vertex set of a graph G, then we denote the induced bipartite subgraph with partite sets V_r and V_s in G by a pair (V_r, V_s) $(=(V_s, V_r))$.

Now, we are ready to show our main results.

Proposition 3.2. $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n-2}{3} \rceil$ *when* $n \equiv 0, 4, 8 \pmod{12}$.

Proof. Let $m = \frac{n}{2}$. First, we partition the vertex set of K_n into m pairwise disjoint subsets $V_i = \{x_i, y_i\}, i = 0, 1, \ldots, m-1$. Then K_n can be expressed as a union of the 1-factor induced by the perfect matching $M_0 = \{x_i y_i \mid i = 0, 1, \ldots, m-1\}$ and the spanning subgraph $K_n - M_0$, denoted by H, which is an edge-disjoint union of pairs (V_r, V_s) for all $0 \leqslant r \neq s \leqslant m-1$.

Next, for each V_i of H, we identify x_i with y_i and denote such a vertex by v_i. Then we obtain a complete graph $\tilde{H}$ with vertex set $V(\tilde{H}) = \{v_0, v_1, \ldots, v_{m-1}\}$ and edge set $E(\tilde{H}) = \{v_i v_{i+d \pmod m} \mid i = 0, 1, \ldots, m-1 \text{ and } d = 1, 2, \ldots, \frac{m}{2}-1\} \cup \{v_i v_{i+d \pmod m} \mid i = 0, 1, \ldots, \frac{m}{2}-1 \text{ and } d = \frac{m}{2}\}$. Each edge $v_i v_{i+d \pmod m}$ of $\tilde{H}$ corresponds to a pair $(V_i, V_{i+d \pmod m})$ of H which is isomorphic to $K_{2,2}$, and vice versa.

By $|V(\tilde{H})| = m$, m is even and Theorem 3.1, $\tilde{H}$ can be decomposed into $m-1$ pairwise edge-disjoint 1-factors. Since each 1-factor of $\tilde{H}$ corresponds to a $K_{2,2}$-factor of H, we also have that H can be decomposed into $m-1$ pairwise edge-disjoint $K_{2,2}$-factors. Hence, if we take away an edge $x_i y_{i+d \pmod m}$ from each pair $(V_i, V_{i+d \pmod m})$ of H, then each

$K_{2,2}$-factor of H produces a linear 3-forest in H (or K_n). Therefore, we can obtain $m-1$ pairwise edge-disjoint linear 3-forests in H (or K_n) from the $m-1$ pairwise edge-disjoint $K_{2,2}$-factors of H. Moreover, those edges we took away from H can be partitioned into $\frac{m}{2}-1$ pairwise disjoint perfect matchings $M_d=\{x_i y_{i+d \pmod m} \mid i=0,1,\ldots,m-1\}$, $d=1,2,\ldots,\frac{m}{2}-1$, and one matching $M_{m/2}=\{x_i y_{i+m/2 \pmod m} \mid i=0,1,\ldots,\frac{m}{2}-1\}$.

Now, let $G(X,Y)$ be a complete bipartite graph with partite sets $X=\{x_i \mid i=0,1,\ldots,m-1\}$ and $Y=\{y_i \mid i=0,1,\ldots,m-1\}$. Then $M_0, M_1, M_2, \ldots, M_{m/2-1}$ are exactly the sets of edges of bipartite differences $0,1,2,\ldots,\frac{m}{2}-1$ in $G(X,Y)$, respectively, and $M_{m/2}$ consists of a half of the edges of bipartite difference $\frac{m}{2}$ in $G(X,Y)$. Hence, by $|X|=|Y|=m$, m is even and Lemma 2.1, the edges of $M_1, M_2, \ldots, M_{m/2-1}$ can form

$$\left\lceil \left(\frac{m/2-1}{3}\right)\cdot 2 \right\rceil = \left\lceil \frac{m-2}{3} \right\rceil$$

pairwise edge-disjoint linear 3-forests in $G(X,Y)$ (or K_n). Besides, the edges of M_0 and $M_{m/2}$ also can form one linear 3-forest in $G(X,Y)$ (or K_n).

Accordingly, $\mathrm{la}_3(K_n) \leqslant (m-1)+\lceil\frac{m-2}{3}\rceil+1=\lceil\frac{2n-2}{3}\rceil$ by $m=\frac{n}{2}$ and $n\equiv 0,4,8 \pmod{12}$. □

Proposition 3.3. $\mathrm{la}_3(K_n)\leqslant\lceil\frac{2n}{3}\rceil$ *when* $n\equiv 2,6,10 \pmod{12}$.

Proof. Let $m=\frac{n-2}{2}$. First, we partition the vertex set of K_n into $m+2$ pairwise disjoint subsets $V_i=\{x_i,y_i\}$, $i=0,1,\ldots,m-1$, $V_m=\{v_m\}$ and $V_{m+1}=\{v_{m+1}\}$. Then K_n can be expressed as a union of the subgraph induced by the matching $M_0=\{x_i y_i \mid i=0,1,\ldots,m-1\}$ and the spanning subgraph K_n-M_0, denoted by H, which is an edge-disjoint union of pairs (V_r,V_s) for all $0\leqslant r\neq s\leqslant m+1$.

Next, for each V_i other than V_m and V_{m+1} of H, we identify x_i with y_i and denote such a vertex by v_i. Then we obtain a complete graph $\tilde{H}$ with vertex set $V(\tilde{H})=\{v_0,v_1,\ldots,v_{m+1}\}$ and edge set $E(\tilde{H})=\{v_i v_{i+d \pmod{m+2}} \mid i=0,1,\ldots,m+1$ and $d=1,2,\ldots,\frac{m+2}{2}-1\}\cup\{v_i v_{i+d \pmod{m+2}} \mid i=0,1,\ldots,\frac{m+2}{2}-1$ and $d=\frac{m+2}{2}\}$. Each edge $v_i v_{i+d \pmod{m+2}}$ of $\tilde{H}$ corresponds to a pair $(V_i, V_{i+d \pmod{m+2}})$ of H which is isomorphic to $K_{2,2}$, $K_{1,2}$ or $K_{1,1}$, and vice versa.

By $|V(\tilde{H})|=m+2$, $m+2$ is even and Theorem 3.1, $\tilde{H}$ can be decomposed into $m+1$ pairwise edge-disjoint 1-factors. Since each 1-factor of $\tilde{H}$ corresponds to a spanning subgraph of H which satisfies that every component is isomorphic to $K_{2,2}$, $K_{1,2}$ or $K_{1,1}$, we also have that H can be decomposed into $m+1$ pairwise edge-disjoint spanning subgraphs $H_0,H_1,\ldots,H_m$ such that in each of which every component is isomorphic to $K_{2,2}$, $K_{1,2}$ or $K_{1,1}$. Hence, if we take away an edge $x_i y_{i+d \pmod{m+2}}$ from each pair $(V_i, V_{i+d \pmod{m+2}})$ of H which is isomorphic to $K_{2,2}$, then each of $H_0,H_1,\ldots,H_m$ produces a linear 3-forest in H (or K_n). Therefore, we can obtain $m+1$ pairwise edge-disjoint linear 3-forests in H (or K_n) from $H_0,H_1,\ldots,H_m$. Moreover, those edges we took away from H can be partitioned into $\frac{m}{2}$ pairwise disjoint matchings $M_d=\{x_i y_{i+d \pmod m} \mid i=0,1,\ldots,m-1\}$, $d=1,2,\ldots,\frac{m}{2}-1$, and $M_{m/2}=\{x_i y_{i+m/2 \pmod m} \mid i=0,1,\ldots,\frac{m}{2}-1\}$.

Now, let $G(X,Y)$ be a complete bipartite graph with partite sets $X=\{x_i \mid i=0,1,\ldots,m-1\}$ and $Y=\{y_i \mid i=0,1,\ldots,m-1\}$. Then $M_0, M_1, M_2, \ldots, M_{m/2-1}$ are exactly the sets of edges of bipartite differences $0,1,2,\ldots,\frac{m}{2}-1$ in $G(X,Y)$, respectively, and $M_{m/2}$ consists of a half of the edges of bipartite difference $\frac{m}{2}$ in $G(X,Y)$. Hence, by $|X|=|Y|=m$, m is even and Lemma 2.1, the edges of $M_1, M_2, \ldots, M_{m/2-1}$ can form

$$\left\lceil \left(\frac{m/2-1}{3}\right)\cdot 2 \right\rceil = \left\lceil \frac{m-2}{3} \right\rceil$$

pairwise edge-disjoint linear 3-forests in $G(X,Y)$ (or K_n). Besides, the edges of M_0 and $M_{m/2}$ also can form one linear 3-forest in $G(X,Y)$ (or K_n).

Accordingly, $\mathrm{la}_3(K_n) \leqslant (m+1)+\lceil\frac{m-2}{3}\rceil+1=\lceil\frac{2n}{3}\rceil$ by $m=\frac{n-2}{2}$ and $n\equiv 2,6,10 \pmod{12}$. □

Proposition 3.4. $\mathrm{la}_3(K_n)\leqslant\lceil\frac{2n}{3}\rceil$ *when* $n\equiv 1,9 \pmod{12}$.

Proof. Let $m=\frac{n-1}{2}$. First, we partition the vertex set of K_n into m pairwise disjoint subsets $V_i=\{x_i,y_i\}$, $i=0,1,\ldots,m-1$, and one subset $\{u\}$. Then K_n can be expressed as a union of the star induced by the edge subset $U=\{ux_i, uy_i \mid i=0,1,\ldots,m-1\}$, the subgraph induced by the matching $M_0=\{x_i y_i \mid i=0,1,\ldots,m-1\}$ and the subgraph $(K_n-u)-M_0$, denoted by H, which is an edge-disjoint union of pairs (V_r,V_s) for all $0\leqslant r\neq s\leqslant m-1$.

Next, for each V_i of H, we identify x_i with y_i and denote such a vertex by v_i. Then we obtain a complete graph $\tilde{H}$ with vertex set $V(\tilde{H}) = \{v_0, v_1, \ldots, v_{m-1}\}$ and edge set $E(\tilde{H}) = \{v_i v_{i+d \pmod m} \mid i = 0, 1, \ldots, m-1$ and $d = 1, 2, \ldots, \frac{m}{2} - 1\} \cup \{v_i v_{i+d \pmod m} \mid i = 0, 1, \ldots, \frac{m}{2} - 1$ and $d = \frac{m}{2}\}$. Each edge $v_i v_{i+d \pmod m}$ of $\tilde{H}$ corresponds to a pair $(V_i, V_{i+d \pmod m})$ of H which is isomorphic to $K_{2,2}$, and vice versa.

By $|V(\tilde{H})| = m$, m is even and the proof of Theorem 3.1, $\tilde{H}$ can be decomposed into $m-1$ pairwise edge-disjoint 1-factors $f_0, f_1, \ldots, f_{m-2}$ such that each 1-factor f_j other than $f_{m/2-1}$ contains an edge $v_k v_{k+1}$ for some $k \in \{0, 1, \ldots, m-3\}$. Since each 1-factor of $\tilde{H}$ corresponds to a $K_{2,2}$-factor of H, we also have that H can be decomposed into $m-1$ pairwise edge-disjoint $K_{2,2}$-factors $H_0, H_1, \ldots, H_{m-2}$ such that each $K_{2,2}$-factor H_j other than $H_{m/2-1}$ contains a pair (V_k, V_{k+1}) for some $k \in \{0, 1, \ldots, m-3\}$.

Hence, if we take away an edge $x_i y_{i+d \pmod m}$ from each pair $(V_i, V_{i+d \pmod m})$ of H, then each $K_{2,2}$-factor of H produces a linear 3-forest in H (or K_n). Therefore, from $H_0, H_1, \ldots, H_{m-2}$, we can obtain $m-1$ pairwise edge-disjoint linear 3-forests $L_0, L_1, \ldots, L_{m-2}$ in H (or K_n) such that each linear 3-forest L_j other than $L_{m/2-1}$ contains a 3-path $x_k - x_{k+1} - y_k - y_{k+1}$ for some $k \in \{0, 1, \ldots, m-3\}$. Moreover, those edges we took away from H can be partitioned into $\frac{m}{2} - 1$ pairwise disjoint perfect matchings $M_d = \{x_i y_{i+d \pmod m} \mid i = 0, 1, \ldots, m-1\}$, $d = 1, 2, \ldots, \frac{m}{2} - 1$, and one matching $M_{m/2} = \{x_i y_{i+m/2 \pmod m} \mid i = 0, 1, \ldots, \frac{m}{2} - 1\}$.

Now, for each L_j other than $L_{m/2-1}$, we replace the 3-path $x_k - x_{k+1} - y_k - y_{k+1}$ inside L_j by another 3-path $y_k - u - x_k - y_{k+1}$ induced by $\{ux_k, uy_k, x_k y_{k+1}\} \subseteq U \cup M_1$. Then, let those replaced 3-paths $x_k - x_{k+1} - y_k - y_{k+1}$ for all $k \in \{0, 2, \ldots, m-4\}$ and the 3-path $y_{m-2} - u - x_{m-2} - y_{m-1}$ induced by $\{ux_{m-2}, uy_{m-2}, x_{m-2} y_{m-1}\} \subseteq U \cup M_1$ form one linear 3-forest T_1 in K_n; also let those replaced 3-paths $x_k - x_{k+1} - y_k - y_{k+1}$ for all $k \in \{1, 3, \ldots, m-3\}$ and the 3-path $y_{m-1} - u - x_{m-1} - y_0$ induced by $\{ux_{m-1}, uy_{m-1}, x_{m-1} y_0\} \subseteq U \cup M_1$ form another linear 3-forest T_2 in K_n. It is worthy of noting that all edges of U and M_1 are being used to form linear 3-forests T_1 and T_2 in K_n.

Furthermore, let $G(X, Y)$ be a complete bipartite graph with partite sets $X = \{x_i \mid i = 0, 1, \ldots, m-1\}$ and $Y = \{y_i \mid i = 0, 1, \ldots, m-1\}$. Then $M_0, M_2, M_3, \ldots, M_{m/2-1}$ are exactly the sets of edges of bipartite differences $0, 2, 3, \ldots, \frac{m}{2} - 1$ in $G(X, Y)$, respectively, and $M_{m/2}$ consists of a half of the edges of bipartite difference $\frac{m}{2}$ in $G(X, Y)$. Hence, by $|X| = |Y| = m$, m is even and Lemma 2.1, the edges of $M_2, M_3, \ldots, M_{m/2-1}$ can form

$$\left\lceil \left(\frac{m/2 - 2}{3}\right) \cdot 2 \right\rceil = \left\lceil \frac{m-4}{3} \right\rceil$$

pairwise edge-disjoint linear 3-forests in $G(X, Y)$ (or K_n). Besides, the edges of M_0 and $M_{m/2}$ also can form one linear 3-forest in $G(X, Y)$ (or K_n).

Accordingly, $\mathrm{la}_3(K_n) \leqslant (m-1) + 2 + \lceil \frac{m-4}{3} \rceil + 1 = \lceil \frac{2n}{3} \rceil$ by $m = \frac{n-1}{2}$ and $n \equiv 1, 9 \pmod{12}$. □

Proposition 3.5. $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 3, 7 \pmod{12}$.

Proof. Since K_n is a subgraph of K_{n+1}, $\mathrm{la}_3(K_n) \leqslant \mathrm{la}_3(K_{n+1}) \leqslant \lceil \frac{2(n+1)-2}{3} \rceil = \lceil \frac{2n}{3} \rceil$ by Proposition 3.2 and $n \equiv 3, 7 \pmod{12}$. □

Proposition 3.6. $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n}{3} \rceil$ *when* $n \equiv 5 \pmod{12}$.

Proof. Since K_n is a subgraph of K_{n+1}, $\mathrm{la}_3(K_n) \leqslant \mathrm{la}_3(K_{n+1}) \leqslant \lceil \frac{2(n+1)}{3} \rceil = \lceil \frac{2n+2}{3} \rceil = \lceil \frac{2n}{3} \rceil$ by Proposition 3.3 and $n \equiv 5 \pmod{12}$. □

Proposition 3.7. $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n-2}{3} \rceil$ *when* $n \equiv 11 \pmod{12}$.

Proof. Let $m = \frac{n-1}{2}$. First, we partition the vertex set of K_n into $m+1$ pairwise disjoint subsets $V_i = \{x_i, y_i\}$, $i = 0, 1, \ldots, m-1$, and $V_m = \{v_m\}$. Then K_n can be expressed as a union of the subgraph induced by the matching $M_0 = \{x_i y_i \mid i = 0, 1, \ldots, m-1\}$ and the spanning subgraph $K_n - M_0$, denoted by H, which is an edge-disjoint union of pairs (V_r, V_s) for all $0 \leqslant r \neq s \leqslant m$.

Next, for each V_i other than V_m of H, we identify x_i with y_i and denote such a vertex by v_i. Then we obtain a complete graph $\tilde{H}$ with vertex set $V(\tilde{H}) = \{v_0, v_1, \ldots, v_m\}$ and edge set $E(\tilde{H}) = \{v_i v_{i+d \pmod{m+1}} \mid i = 0, 1, \ldots, m$ and $d = 1, 2, \ldots, \frac{m+1}{2} - 1\} \cup \{v_i v_{i+d \pmod{m+1}} \mid i = 0, 1, \ldots, \frac{m+1}{2} - 1$ and $d = \frac{m+1}{2}\}$. Each edge $v_i v_{i+d \pmod{m+1}}$ of $\tilde{H}$ corresponds to a pair $(V_i, V_{i+d \pmod{m+1}})$ of H which is isomorphic to $K_{2,2}$ or $K_{1,2}$, and vice versa.

By $|V(\tilde{H})| = m+1$, $m+1$ is even and the proof of Theorem 3.1, $\tilde{H}$ can be decomposed into m pairwise edge-disjoint 1-factors $f_0, f_1, \ldots, f_{m-1}$ such that each 1-factor f_j contains the edge $v_j v_m$. Since each 1-factor of $\tilde{H}$ corresponds to a spanning subgraph of H which satisfies that every component is isomorphic to $K_{2,2}$ or $K_{1,2}$, we also have that H can be decomposed into m pairwise edge-disjoint spanning subgraphs $H_0, H_1, \ldots, H_{m-1}$ such that in each of which every component is isomorphic to $K_{2,2}$ or $K_{1,2}$, and each spanning subgraph H_j contains the pair (V_j, V_m).

Hence, if we take away an edge $x_i y_{i+d \pmod{m+1}}$ from each pair $(V_i, V_{i+d \pmod{m+1}})$ of H which is isomorphic to $K_{2,2}$, then each of $H_0, H_1, \ldots, H_{m-1}$ produces a linear 3-forest in H (or K_n). Therefore, from $H_0, H_1, \ldots, H_{m-1}$, we can obtain m pairwise edge-disjoint linear 3-forests $L_0, L_1, \ldots, L_{m-1}$ in H (or K_n) such that each linear 3-forest L_j contains the 2-path $x_j - v_m - y_j$. Moreover, those edges we took away from H can be partitioned into $\frac{m-1}{2}$ pairwise disjoint matchings $M_d = \{x_i y_{i+d \pmod m} \mid i = 0, 1, \ldots, m-1\}$, $d = 1, 2, \ldots, \frac{m-1}{2}$.

Now, let $G(X, Y)$ be a complete bipartite graph with partite sets $X = \{x_i \mid i = 0, 1, \ldots, m-1\}$ and $Y = \{y_i \mid i = 0, 1, \ldots, m-1\}$. Then $M_0, M_1, \ldots, M_{(m-1)/2}$ are exactly the sets of edges of bipartite differences $0, 1, \ldots, \frac{m-1}{2}$ in $G(X, Y)$, respectively. Moreover, let $a_1 = m-1$ and $a_t = (m-2) - 2(t-2)$ for all $t \in \{2, 3, \ldots, \frac{m+1}{6}\}$; also let $b_t = a_t + 3(t-1) + 1 \pmod m$, $t = 1, 2, \ldots, \frac{m+1}{6}$, then $e_t = x_{a_t} y_{b_t} \in M_{3(t-1)+1}$ is an edge of bipartite difference $3(t-1)+1$ in $G(X, Y)$. Hence, by $|X| = |Y| = m$, m is odd and Lemma 2.2, the edges other than e_t of $M_{3(t-1)}$, $M_{3(t-1)+1}$ and $M_{3(t-1)+2}$ can form two edge-disjoint linear 3-forests ℓ_{2t-1} and ℓ_{2t} in $G(X, Y)$ (or K_n), for each $t \in \{1, 2, \ldots, \frac{m+1}{6}\}$. Therefore, the edges other than $e_1, e_2, \ldots, e_{(m+1)/6}$ of $M_0, M_1, \ldots, M_{(m-1)/2}$ can form $(\frac{m+1}{6}) \cdot 2 = \frac{m+1}{3}$ pairwise edge-disjoint linear 3-forests $\ell_1, \ell_2, \ldots, \ell_{(m+1)/3-1}, \ell_{(m+1)/3}$ in $G(X, Y)$ (or K_n).

In the following, for each $t \in \{1, 2, \ldots, \frac{m+1}{6}\}$, we will replace two edges e'_t and e''_t of ℓ_1 by e_t, then the replaced edges e'_t and e''_t will be moved into linear 3-forests L'_t and L''_t, respectively; moreover, we will also move edges w'_t and w''_t of L'_t and L''_t into another linear 3-forests ℓ'_t and ℓ''_t, respectively. The above processes of replacing and moving edges from some linear 3-forests will finally let K_n be decomposed into $m + \frac{m+1}{3}$ pairwise edge-disjoint linear 3-forests. Note that ℓ_1 is consisting of 3-paths $y_i - x_i - y_{i+1} - x_{i+1}$, $i = 0, 2, \ldots, m-3$, and one edge $x_{m-1} y_{m-1}$.

If $t = 1$, then we first replace two edges $x_{a_1} y_{a_1}$ and $x_{b_1+1} y_{b_1+1}$ of ℓ_1 by $e_1 = x_{a_1} y_{b_1}$ such that the 3-path $y_{b_1} - x_{b_1} - y_{b_1+1} - x_{b_1+1}$ in ℓ_1 is replaced by another 3-path $x_{a_1} - y_{b_1} - x_{b_1} - y_{b_1+1}$. Next, we move the replaced edge $x_{a_1} y_{a_1}$ into L_{a_1} and move the edge $v_m y_{a_1}$ of L_{a_1} into ℓ_1 such that the 2-path $x_{a_1} - v_m - y_{a_1}$ in L_{a_1} is replaced by another 2-path $v_m - x_{a_1} - y_{a_1}$, and ℓ_1 contains the isolated edge $v_m y_{a_1}$. Also, we move the replaced edge $x_{b_1+1} y_{b_1+1}$ into L_{b_1+1} and move the edge $v_m y_{b_1+1}$ of L_{b_1+1} into ℓ_2 such that the 2-path $x_{b_1+1} - v_m - y_{b_1+1}$ in L_{b_1+1} is replaced by another 2-path $v_m - x_{b_1+1} - y_{b_1+1}$, and ℓ_2 contains the 2-path $x_{a_1} - y_{b_1+1} - v_m$. It is not difficult to see that, after we replaced and moved some edges, all modified linear 3-forests still satisfy that each component is a path of length at most 3.

Similarly, if $t \geqslant 2$ and t is even, then we first replace two edges $x_{a_t} y_{a_t}$ and $x_{b_t+1} y_{b_t+1}$ of ℓ_1 by $e_t = x_{a_t} y_{b_t}$. Next, we move $x_{a_t} y_{a_t}$ into L_{a_t} and move the edge $v_m x_{a_t}$ of L_{a_t} into ℓ_{2t-1}; also move $x_{b_t+1} y_{b_t+1}$ into L_{b_t+1} and move the edge $v_m y_{b_t+1}$ of L_{b_t+1} into ℓ_{2t}. Otherwise, if $t \geqslant 2$ and t is odd, then we first replace two edges $x_{b_t-1} y_{b_t-1}$ and $x_{a_t} y_{a_t}$ of ℓ_1 by $e_t = x_{a_t} y_{b_t}$. Next, we move $x_{b_t-1} y_{b_t-1}$ into L_{b_t-1} and move the edge $v_m y_{b_t-1}$ of L_{b_t-1} into ℓ_{2t-1}; also move $x_{a_t} y_{a_t}$ into L_{a_t} and move the edge $v_m x_{a_t}$ of L_{a_t} into ℓ_{2t}.

Accordingly, $\mathrm{la}_3(K_n) \leqslant m + \frac{m+1}{3} = \lceil \frac{2n-1}{3} \rceil = \lceil \frac{2n-2}{3} \rceil$ by $m = \frac{n-1}{2}$ and $n \equiv 11 \pmod{12}$. □

From the propositions obtained above, we conclude this paper by the following theorem.

Theorem 3.8.

$$\mathrm{la}_3(K_n) = \begin{cases} \left\lceil \dfrac{2n-2}{3} \right\rceil & \text{when } n \equiv 0, 4, 8, 11 \pmod{12} \text{ and} \\ \left\lceil \dfrac{2n}{3} \right\rceil & \text{when } n \equiv 1, 2, 3, 5, 6, 7, 9, 10 \pmod{12}. \end{cases}$$

Proof. $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n-2}{3} \rceil$ or $\mathrm{la}_3(K_n) \leqslant \lceil \frac{2n}{3} \rceil$, following the value of $n \pmod{12}$, via Propositions 3.2–3.7, while $\mathrm{la}_3(K_n) \geqslant \lceil \frac{2n-2}{3} \rceil$ or $\mathrm{la}_3(K_n) \geqslant \lceil \frac{2n}{3} \rceil$, following the value of $n \pmod{12}$, by Proposition 1.2. This concludes the proof. □

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