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Research note on the criteria for the optimal solution of the inventory model with a mixture of partial backordering and lost sales

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Abstract

Perishable products are commonly seen in inventory management. By allowing shortages and backlogging, the impact on the cost from the decay of the products can be balanced out. In a recent paper published in Computers and Industrial Engineering [P.L. Abad, Optimal lot size for a perishable good under conditions of finite production and partial backordering and lost sale, Comput. Ind. Eng. 38 (2000) 457–465] considered a problem in such context. However, his algorithm was incomplete due to flaws in his solution procedure. The purpose of this note is to explore the same production inventory models with a mixture of partial backordering and lost sales for deteriorated items. We find the criteria for the optimal solution for different cases and derive a formulated minimum value. By theoretical analysis, we develop a few lemmas to reveal parameter effects and optimal solution procedure. The solutions are illustrated by solving the same examples from Abad's paper to illustrate the accuracy and completeness of our procedure.

Keywords: Inventory; Perishable items; Backorder; Lost sales

1. Introduction

Permitting limited planned shortages can reduce the pressure on high production capacity and hence result in a smoother production schedule. Firms are able to maintain a backlog of orders to certain loyal customers without losing their business. However, the costs of shortages or lost sales should not be exorbitant to

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facilitate the feasibility of the strategy. If the cost of holding inventory is significantly higher than the shortage cost, permitting occasional brief shortages to lower the average inventory level may be a sound business practice to reduce the total cost.

There has been a great amount of research considering partial backordering in the inventory model. Montgomery et al. [1] established continuous review and periodic review inventory models that considered a mixture of backorders and lost sales. Kim and Park [2] considered a continuous review system with constant lead-time where a fraction of the unfilled demand was backordered and the backorder cost was assumed to be proportional to the length of the shortage period. Padmanabhan and Vrat [3] developed an inventory model with a mixture of backorders and lost sales such that the backlogged demand rate was dependent upon the negative inventory level during the stock out period. Raafat et al. [4] also derived an alternative method for finding the optimal replenishment schedule for Mak's [5] model in which an inventory model with Weibull distributed deterioration and backlogging is considered. Wee [6] developed an economic production lot size model for deteriorating items with partial backordering and obtained the time intervals and cycle times that minimize the total cost function. Padmanabhan and Vrat [7] presented inventory models for deteriorating items with stock-dependent selling rates and derived the profit functions with and without backlogging and complete backlogging cases. DeCroix and Arreola-Risa [8] explored the potential benefits of offering economic incentives to backorder as a strategy for inventory management when the system involves an unreliable supply. Chung et al. [9] considered the Padmanabhan and Vrat [7] problem and developed the necessary and sufficient conditions for the optimal profit per unit time function solutions. Abad [10] considered the problem of determining the lot size for perishable goods under finite production with exponential decay, partial backordering and lost sales. Zeng [11] studied the effects of using a partial backordering approach to control inventory under deterministic and stochastic demands, respectively. Wu and Ouyang [12] investigated the lot size, reorder point inventory model, including variable lead-time with partial backorders and an imperfect production process.

The model studied by this note is identical to that of Wee [6] and Abad [10] where they showed that the inventory model is a constrained, non-linear problem with convexity characteristics. Abad used the Solver in MS/Excel to solve for the solutions. However, there are critical flaws in his analytical process, rendering the resulting solutions incorrect. We will point out the questionable proofs in his model and establish the necessary and sufficient conditions for the minimum solution to occur inside the interior section of the cost function. A theorem to determine the criterion for the existence and uniqueness of the minimum solution is subsequently developed. We will derive simple formulated optimal solutions for each case, respectively. Numerical examples are given to illustrate all results obtained in this note.

2. Notations and assumptions

Notations and assumptions from Wee [6] and Abad [10] are adopted except for a few minor modifications from Abad and simplification of some expressions. We outline these notations in the following for the sake of completeness and easy reference.

Notations

$I(t)$	net stock (on hand – backorders) level at time t
p	production rate for the item (units/period)
d	demand rate (units/period)
θ	wastage coefficient, assumed to be constant (i.e., exponential decay)
$\theta I(t)$	wastage rate at time t
Γ	duration of inventory cycle when there is positive inventory
λ	duration of inventory cycle for which there exists stock out
β, ψ	interim time-spans
C	unit product cost
c_1	setup cost
c_2	inventory carrying cost/unit/period

c_3	backordering cost/unit/period
c_4	penalty cost of lost sale including lost profit ($/unit)
B	the fraction of demand backordered
(Γ^*,λ^*)	the pair of minimum solution of the inventory model
$B(\Gamma)$	$\frac{c_3}{2}Bdm(\lambda^2(\Gamma)+2\Gamma\lambda(\Gamma))+c_4(1-B)dm\Gamma-c_1-(c+\frac{c_2}{\theta})(p\beta(\Gamma)-d\Gamma)$
m	$\frac{p-d}{p-d+Bd}$
$A(\Gamma)$	$\frac{d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}$
Δ	$(c+\frac{c_2}{\theta})(p-d)-c_4(1-B)dm$
Γ_0	$\frac{1}{\theta}\ln\left[1+(\frac{p}{d})\frac{c_4(1-B)dm\theta}{(c\theta+c_2)(p-d)-c_4(1-B)dm\theta}\right]$, when $\Delta>0$
$\lambda(\Gamma)$	the solution of $(c+\frac{c_2}{\theta})\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}=c_3Bdm\lambda+c_4(1-B)dm$, when $\Delta>0$, for $\Gamma_0\leqslant\Gamma<\infty$
$B(\Gamma_0)$	$c_4(1-B)dm\Gamma_0-c_1-(c+\frac{c_2}{\theta})[p\beta(\Gamma_0)-d\Gamma_0]$
$B(\infty)$	$\frac{p}{\theta}(c+\frac{c_2}{\theta})\ln\frac{p}{d}-c_1+\frac{[(c\theta+c_2)(p-d)-c_4(1-B)dm\theta]^2}{2c_3Bdm\theta^2}$
$F(\Gamma,\lambda)$	$c_1+(c+\frac{c_2}{\theta})(p\beta(\Gamma)-d\Gamma)+\frac{c_3}{2}Bdm\lambda^2+c_4(1-B)dm\lambda$
$C(\Gamma,\lambda)$	$(\Gamma+\lambda)(c+\frac{c_2}{\theta})\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}-F(\Gamma,\lambda)$
$\widetilde{\Gamma}$	if $(c+\frac{c_2}{\theta})\frac{p}{\theta}\ln\frac{p}{d}-c_1>0$, then $\widetilde{\Gamma}$ satisfies that $C(\widetilde{\Gamma},0)=0$

Assumptions

1. The planning horizon is infinite.
2. Demand occurs at a known steady constant rate d.
3. The production rate is a constant p and is strictly greater than the demand rate, i.e., $p>d$.
4. The goods decays at an exponential rate θ.
5. When stockout occurs, demand is partially backlogged. The fraction of demand backordered B is assumed to be between zero and one, i.e., $0\leqslant B\leqslant 1$.
6. Production quantity in each cycle is kept constant.
7. The cost of a deteriorated item (taking into account the salvage value) is known.
8. There are no space or budget limitations. Nor are there any limitations in terms of production lot size or number of setups per year.

3. Mathematical model

The problem that Wee [6] and Abad [10] tried to solve is the minimization of the average total cost during the inventory and shortage cycle of time-span $\Gamma+\lambda$. That is

$$\text{Min}\,\Pi(\Gamma,\lambda)=\frac{F(\Gamma,\lambda)}{\Gamma+\lambda} \tag{1}$$

subject to $\Gamma\geqslant 0$, $\lambda\geqslant 0$ and $\Gamma+\lambda$, where

$$F(\Gamma,\lambda)=c_1+\left(c+\frac{c_2}{\theta}\right)(p\beta(\Gamma)-d\Gamma)+\frac{c_3}{2}Bdm\lambda^2+c_4(1-B)dm\lambda \tag{2}$$

is the total cost during the cycle of time-span $\Gamma+\lambda$ with $\beta(\Gamma)=\frac{1}{\theta}\ln[1+\frac{d}{p}(\exp^{\Gamma\theta}-1)]$. Abad [10] constructed Assumption 1 and proved Proposition 1 as follows.

Assumption 1. The set $G=\{(\Gamma,\lambda)|\Gamma+\lambda>0,\ F(\Gamma,\lambda)>0\}$ is not a null set.

In Appendix A of this note, we prove that Assumption 1 of Abad [10] is unnecessary and can be removed.

Proposition 1. *$\Pi(\Gamma,\lambda)$ is a strictly pseudoconvex function on G.*

In Proposition 1 of Abad [10], he proved that the objective function $\Pi(\Gamma,\lambda)$ is a strictly pseudoconvex function on G. In Bazarra et al. [13], after showing that $\Pi(\Gamma,\lambda)$ being strictly pseudoconvex, if (Γ_1,λ_1) is a solution for the first partial derivative system, then (Γ_1,λ_1) will be the global minimum. Abad [10] predicted that owing to the boundaries $\Gamma \geqslant 0$ and $\lambda \geqslant 0$ being linear, the function $\Pi(\Gamma,\lambda)$ will have a unique global minimum.

In the following, we establish the necessary and sufficient conditions for which the minimum solution occurs in the interior section. We will also establish the criteria for which the minimum solution may degenerate to infinity on the boundary when the system of first partial derivatives has no solution. In other words, Abad's paper contains questionable results. Moreover, we derive a formulated minimum value. From the same numerical examples of Abad [10], we demonstrate that our method improves the minimum value with an average saving of 27.28%.

4. Improved mathematical model

Taking the first partial derivatives for $\Pi(\Gamma,\lambda)$ yields

$$\frac{\partial\Pi}{\partial\Gamma}=\frac{1}{(\Gamma+\lambda^2)}\left[(\Gamma+\lambda)\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}-F(\Gamma,\lambda)\right] \tag{3}$$

and

$$\frac{\partial\Pi}{\partial\Gamma}=\frac{1}{(\Gamma+\lambda)^2}[(\Gamma+\lambda)[c_3Bdm\lambda+c_4(1-B)dm]-F(\Gamma,\lambda)]. \tag{4}$$

Solving the system of $\frac{\partial\Pi}{\partial\Gamma}=0$ and $\frac{\partial\Pi}{\partial\Gamma}=0$ results in

$$\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}=c_3Bdm\lambda+c_4(1-B)dm. \tag{5}$$

Assume that for $A(\Gamma)=\frac{d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}$ for $\Gamma\geqslant 0$, which yields $\frac{\mathrm{d}A(\Gamma)}{\mathrm{d}\Gamma}=\frac{pd\theta\exp^{\Gamma\theta}}{[p+d(\exp^{\Gamma\theta}-1)]^2}>0$, $A(0)=0$ and $\lim_{\Gamma\to\infty}A(\Gamma)=1$. On the other hand, assume that Γ_0 satisfies the equation

$$\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma_0\theta}-1)}{p+d(\exp^{\Gamma_0\theta}-1)}=c_4(1-B)dm \tag{6}$$

and by solving Eq. (6), we have $\Gamma_0=\frac{1}{\theta}\ln\left[1+\left(\frac{p}{d}\right)\frac{c_4(1-B)dm\theta}{(c\theta+c_2)(p-d)-c_4(1-B)dm\theta}\right]$. The necessary and sufficient condition with respect to a given λ for Γ can thus be established in the following lemma.

Lemma 1. *With a given λ, Eq. (5) has a solution for Γ if and only if $(c+\frac{c_2}{\theta})(p-d)>c_3Bdm\lambda+c_4(1-B)dm$.*

To simplify the expression, we assume that $\Delta=(c+\frac{c_2}{\theta})(p-d)-c_4(1-B)dm$. From Lemma 1, we know that when $\Delta>0$, for each λ satisfying $\frac{\Delta}{c_3Bdm}>\lambda\geqslant 0$, there exists a unique Γ such that the pair (λ,Γ) satisfies Eq. (5). On the other hand, when $\Delta>0$, it means that for each Γ satisfying $\infty>\Gamma\geqslant\Gamma_0$, there exists a λ, say $\lambda(\Gamma)$, such that $(\lambda(\Gamma),\Gamma)$ satisfies Eq. (5). By Eq. (6), it yields that $\lambda(\Gamma_0)=0$. Consequently, there exists a one-to-one and onto relationship between Γ and λ.

Next, we try to solve $\frac{\partial\Pi}{\partial\lambda}=0$, that is, solving

$$[\Gamma+\lambda][c_3Bdm\lambda+c_4(1-B)dm]=c_1+\left(c+\frac{c_2}{\theta}\right)(p\beta(\Gamma)+d\Gamma)+\frac{c_3}{2}Bdm\lambda^2+c_4(1-B)dm\lambda. \tag{7}$$

Motivated by Eq. (7) and $\lambda(\Gamma)$ satisfying Eq. (5) for $\Gamma_0\leqslant\Gamma<\infty$, we define $B(\Gamma)$ as

$$B(\Gamma)=\frac{c_3}{2}Bdm(\lambda^2(\Gamma)+2\Gamma\lambda(\Gamma))+c_4(1-B)dm\Gamma-c_1-\left(c+\frac{c_2}{\theta}\right)(p\beta(\Gamma)-d\Gamma). \tag{8}$$

Using Eq. (5), we rewrite Eq. (8) as

$$B(\Gamma)=\frac{\left[\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}-c_4(1-B)dm\right]^2}{2c_3Bdm}+\left[\frac{(p-d)d(\exp^{\Gamma\theta}-1)\Gamma}{p+d(\exp^{\Gamma\theta}-1)}-(p\beta(\Gamma)-d\Gamma)\right]\left(c+\frac{c_2}{\theta}\right)-c_1. \tag{9}$$

As given in Appendix B, we can show that $B(\Gamma)$ is a strictly increasing function of Γ. Moreover, we know that $B(\Gamma_0) = c_4(1-B)dm\Gamma_0 - c_1 - (c + \frac{c_2}{\theta})[p\beta(\Gamma_0) - d\Gamma_0]$. We now consider $\lim_{\Gamma\to\infty} B(\Gamma)$ and define $B(\infty) = \lim_{\Gamma\to\infty} B(\Gamma)$. Then as given in Appendix C, we can show that

$$B(\infty) = \frac{p}{\theta}\left(c + \frac{c_2}{\theta}\right)\ln\frac{p}{d} - c_1 + \frac{[(c\theta + c_2)(p-d) - c_4(1-B)dm\theta]^2}{2c_3Bdm\theta^2}. \tag{10}$$

In the above discussion, we have found the criteria that ensure the existence of the solution for the first partial derivative system of $\Pi(\Gamma,\lambda)$, which leads to Lemma 2 as established in the following.

Lemma 2. *There exists a solution for the first partial derivative system of $\Pi(\Gamma,\lambda)$ if and only if $\Delta > 0$, $B(\Gamma_0) \leqslant 0$ and $B(\infty) > 0$.*

Next, if we assume that $C(\Gamma,\lambda)$ satisfying $\frac{\partial\Pi}{\partial\Gamma} = \frac{C(\Gamma,\lambda)}{(\Gamma+\lambda)^2}$ and $\lim_{\Gamma\to\infty} C(\Gamma,\lambda) = C(\infty,\lambda)$, we have from Eq. (3)

$$\frac{\partial C(\Gamma,\lambda)}{\partial\Gamma} = (\Gamma+\lambda)\left(c + \frac{c_2}{\theta}\right)\frac{(p-d)pd\theta\exp^{\Gamma\theta}}{[p + d(\exp^{\Gamma\theta} - 1)]^2} > 0. \tag{11}$$

It can be easily verified that

$$C(\infty,0) = \left(c + \frac{c_2}{\theta}\right)\frac{p}{\theta}\ln\left(\frac{p}{d}\right) - c_1. \tag{12}$$

From $C(\Gamma=0,\lambda) = -c_1 - \frac{c_3}{2}Bdm\lambda^2 - c_4(1-B)dm\lambda < 0$ to $C(\infty,\lambda) = (c + \frac{c_2}{\theta})(\lambda(p-d) + \frac{p}{\theta}\ln(\frac{p}{d})) + C(\Gamma=0,\lambda)$, depending on the value of $C(\infty,\lambda)$, we have two cases:

(a) If $C(\infty,\lambda) > 0$, the minimum value of $\{\Pi(\Gamma,\lambda) : \Gamma \geqslant 0\}$ occurs at $\Gamma(\lambda)$, where $C(\Gamma(\lambda),\lambda) = 0$; and
(b) if $C(\infty,\lambda) \leqslant 0$, the minimum value of $\{\Pi(\Gamma,\lambda) : \Gamma \geqslant 0\}$ occurs as Γ approaches to infinity with its minimum value $\lim_{\Gamma\to\infty}\Pi(\Gamma,\lambda) = (c + \frac{c_2}{\theta})(p-d)$.

In the following, we consider the minimization problem of $\Pi(\Gamma,\lambda)$. We have shown that Assumption 1 of Abad [10] always holds, and also the minimization problem always has an optimal solution, since it is bounded below by zero. If the interior points of $\{(\Gamma,\lambda) : \Gamma \geqslant 0,\ \lambda \geqslant 0 \text{ and } \Gamma + \lambda > 0\}$ do not have a solution for this minimization problem, the minimum must occur on the boundary. Hence, as shown in Appendix D, we consider the following four cases for the boundary points: (a) $\Gamma = 0$, (b) $\lambda = 0$, (c) $\lambda \to \infty$, and (d) $\Gamma \to \infty$. We list the key results in the next lemma.

Lemma 3. *The minimum value for $\Pi(\Gamma,\lambda)$ along the boundary $\lambda = 0$ satisfies that*

(1) if $C(\infty,0) > 0$, then it occurs at $\widetilde{\Gamma}$ where $C(\widetilde{\Gamma},0) = 0$ with $\frac{\partial\Pi}{\partial\Gamma} = \frac{C(\Gamma,\lambda)}{(\Gamma+\lambda)^2}$ and the minimum value $\Pi(\widetilde{\Gamma},0) = (c + \frac{c_2}{\theta})\frac{(p-d)d(\exp^{\widetilde{\Gamma}\theta}-1)}{p+d(\exp^{\widetilde{\Gamma}\theta}-1)}$; and
(2) if $C(\infty,0) \leqslant 0$, then the minimum can be obtained as $\Gamma \to \infty$ such that the minimum value satisfies $\lim_{\Gamma\to\infty}\Pi(\Gamma,0) = (c + \frac{c_2}{\theta})(p-d)$.

By combining the above results, Theorem 1 can be established as follows.

Theorem 1. *The minimum solution (Γ^*,λ^*) of this inventory model can be divided into the following cases:*

(1) If the conditions $\Delta > 0$, $B(\Gamma_0) \leqslant 0$ and $B(\infty) > 0$ hold, then $(\Gamma^*,\lambda^*) = (\Gamma^\#, \lambda^\#(\Gamma^\#))$ where $\Gamma^\#$ satisfies Eq. (8) as $B(\Gamma^\#) = 0$ and $\lambda^\#(\Gamma^\#)$ satisfies Eq. (5).
(2) otherwise,
(a) if $C(\infty,0) > 0$, then $(\Gamma^*,\lambda^*) = (\widetilde{\Gamma},0)$ where $C(\widetilde{\Gamma},0) = 0$, and
(b) if $C(\infty,0) \leqslant 0$, then $\Gamma^* = \infty$.

Proof. It is sufficient to show that $\Pi(\Gamma^{\#},\lambda^{\#}(\Gamma^{\#})) < \Pi(\widetilde{\Gamma},0)$. Using $\frac{\partial \Pi}{\partial \Gamma}=0$ at $(\Gamma^{\#},\lambda^{\#}(\Gamma^{\#}))$, it can be derived that

$$\Pi(\Gamma^{\#},\lambda^{\#}(\Gamma^{\#})) = \frac{F(\Gamma^{\#},\lambda^{\#}(\Gamma^{\#}))}{\Gamma^{\#}+\lambda^{\#}(\Gamma^{\#})} = \left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma^{\#}\theta}-1)}{p+d(\exp^{\Gamma^{\#}\theta}-1)}. \tag{13}$$

Since $\widetilde{\Gamma}$ satisfies that $C(\widetilde{\Gamma},0)=0$, we have

$$\Pi(\widetilde{\Gamma},0) = \frac{c_1+\left(c+\frac{c_2}{\theta}\right)(p\beta(\widetilde{\Gamma})-d\widetilde{\Gamma})}{\widetilde{\Gamma}} = \left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\widetilde{\Gamma}\theta}-1)}{p+d(\exp^{\widetilde{\Gamma}\theta}-1)}. \tag{14}$$

From Eq. (9), we know that

$$C(\Gamma^{\#},0) = -\frac{\left[\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)d(\exp^{\Gamma^{\#}\theta}-1)}{p+d(\exp^{\Gamma^{\#}\theta}-1)} - c_4(1-B)dm\right]^2}{2c_3Bdm} < 0. \tag{15}$$

Moreover, from $C(\widetilde{\Gamma},0)=0$, Eq. (15) and $C(\Gamma,0)$ being an increasing function of Γ, it follows that $\Gamma^{\#}<\widetilde{\Gamma}$. Observing that $A(\Gamma)$ is an increasing function of Γ and then by Eqs. (13) and (14), we establish that $\Pi(\Gamma^{\#},\lambda^{\#}(\Gamma^{\#})) < \Pi(\widetilde{\Gamma},0)$. □

Next, we consider Case (2) with $C(\infty,0)\leqslant 0$. We know that by Appendix D, Case (d), when $\Gamma\to\infty$, $\lim_{\Gamma\to\infty}\Pi(\Gamma,\lambda) = (c+\frac{c_2}{\theta})(p-d)$ is the minimum value. However, owing to some operational constraints, for examples, capacity of storage spaces and limited budget, we may only extend the inventory period to, say Γ_1. Hence, we turn to minimizing $\Pi(\Gamma_1,\lambda)$ for $0\leqslant\lambda<\infty$. According to Eqs. (4), (6) and (7), solving $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Pi(\Gamma_1,\lambda)=0$ is equivalent to solving

$$\lambda^2+2\Gamma_1\lambda+a_0=0, \tag{16}$$

where $a_0=\frac{2}{c_3Bdm}\left[c_4(1-B)dm\Gamma_1-c_1-\left(c+\frac{c_2}{\theta}\right)(p\beta(\Gamma_1)-d\Gamma_1)\right]$. Therefore, the complex solutions for Eq. (16) are $\lambda=-\Gamma_1\pm\sqrt{\Gamma_1^2-a_0}$. To simplify the expression, we assume $\lambda_1=-\Gamma_1+\sqrt{\Gamma_1^2-a_0}$, and then divide the minimization problem for $\Pi(\Gamma_1,\lambda)$ into the following two cases: (i) $a_0\leqslant 0$, and (ii) $a_0>0$.

For Case (i), it yields that $\lambda_1\geqslant 0$. Moreover, as $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Pi(\Gamma_1,\lambda)<0$, for $0\leqslant\lambda<\lambda_1$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Pi(\Gamma_1,\lambda)>0$, for $\lambda_1<\lambda<\infty$, λ_1 is therefore the minimum solution for $\Pi(\Gamma_1,\lambda)$.

For Case (ii), as it implies that there is no nonnegative solution for Eq. (16) and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Pi(\Gamma_1,\lambda)>0$, for $0\leqslant\lambda<\infty$, $\lambda=0$ is therefore the minimum solution for $\Pi(\Gamma_1,\lambda)$.

These findings are summarized by Lemma 4.

Lemma 4. *When Case* (2) *with* $C(\infty,0)\leqslant 0$ *happens and the largest possible inventory period is represented as* Γ_1, *the minimum solution for* $\Pi(\Gamma_1,\lambda)$ *can be divided into the following two cases*:

(i) *If* $a_0\leqslant 0$, *then* $\lambda_1=-\Gamma_1+\sqrt{\Gamma_1^2-a_0}$ *is the minimum solution.*
(ii) *If* $a_0>0$, *then* $\lambda=0$ *is the minimum solution.*

Finally, we consider the instantaneous case where $p\to\infty$. From $m\to 1$, Eq. (5) can be revised as

$$(c\theta+c_2)(\exp^{\Gamma\theta}-1) = c_3B\theta\lambda+c_4(1-B)\theta. \tag{17}$$

Moreover, since $\lim_{p\to\infty}p\beta(\Gamma)=\frac{d}{\theta}(\exp^{\Gamma\theta-1})$, we obtain the special case of $B(\Gamma)$ as

$$B(\Gamma) = \frac{d\theta^2}{2c_3B}\left[\left(c+\frac{c_2}{\theta}\right)(\exp^{\Gamma\theta}-1)-c_4(1-B)\right]^2+\left(c+\frac{c_2}{\theta}\right)\frac{d}{\theta}(\Gamma\theta\exp^{\Gamma\theta}+1-\exp^{\Gamma\theta})-c_1. \tag{18}$$

On the other hand, it implies that

$$C(\Gamma,0) = \left(c+\frac{c_2}{\theta}\right)\frac{d}{\theta}(\Gamma\theta\exp^{\Gamma\theta}+1-\exp^{\Gamma\theta})-c_1. \tag{19}$$

From $\Delta = (c+\frac{c_2}{\theta})(p-d) - c_4(1-B)dm$ and Eqs. (10) and (12), it yields that when $p \to \infty$, then $\Delta > 0$, $B(\infty) > 0$, and $C(\infty, 0) > 0$. In Appendix E, we show the procedure for simplifying the expression of $B(\Gamma_0)$. It implies that when $p \to \infty$

$$B(\Gamma_0) = \frac{d}{\theta}\left(c + \frac{c_2}{\theta}\right)[(1+z)\ln(1+z) - z] - c_1, \tag{20}$$

where $z = \frac{c_4(1-B)\theta}{c\theta + c_2}$. Consequently, the results for the infinite production rate can be established in Lemma 5.

Lemma 5. *The minimum solution* (Γ^*, λ^*) *of the inventory model for the instantaneous replenishment case can be divided into the following two cases*:

(1) *If the conditions of Eq.* (20), $B(\Gamma_0) \leqslant 0$, *holds, then* $(\Gamma^*, \lambda^*) = (\Gamma^\#, \lambda^\#(\Gamma^\#))$, *where* $\Gamma^\#$ *satisfies Eq.* (18) *as* $B(\Gamma^\#) = 0$ *and* $\lambda^\#(\Gamma^\#)$ *satisfies Eq.* (17).
(2) *Otherwise*, $(\Gamma^*, \lambda^*) = (\widetilde{\Gamma}, 0)$, *where* $C(\widetilde{\Gamma}, 0) = 0$ *from Eq.* (19).

5. Numerical examples

To demonstrate the advantage of our method, we consider the same numerical example as Abad [10] with the following data: $p = 750$ units/week; $d = 400$ units/week; $c = \$20$/units; $c_1 = \$1000$/production run; $c_2 = \$2$/unit/week; $\theta = 0.1$; $c_3 = \$4$/unit/week backordered; $c_4 = \$10$/unit lost sale and $B = 0.7$. Abad [10] examined the sensitivity analyses with respect to various relevant parameters. We quote his results in Table 1.

Using our method, for example, $B = 0.7$, we have $\Delta = 13{,}300 > 0$, $\Gamma_0 = 0.896$, $B(\Gamma_0) = -701 < 0$ and $B(\infty) = 334{,}000 > 0$, from Theorem 1 case (1), so the minimum solution is the solution for the first partial derivative system, $(\Gamma^\#, \lambda^\#(\Gamma^\#))$. When $B = 0.3$, we have $\Delta = 11{,}910 > 0$, $\Gamma_0 = 2.838$, $B(\Gamma_0) = 1939 > 0$, $(c+\frac{c_2}{\theta})\frac{p}{\theta}\ln\frac{p}{d} = 188,582$ and $c_1 = 1000$ from Case (2) of Theorem 1, then the minimum solution is the minimum solution for the boundary along $\lambda = 0$ $(\widetilde{\Gamma}, 0)$. We compute the same examples and list the results in Table 2.

Comparison of Tables 1 and 2 indicates that our method can find the optimal solution. The range of our saving for these five examples from 40.85% to 15.44% and its average is 27.28%.

Finally, we illustrate the extreme case where setup costs become very big, for example, $c_1 = 4 \times 10^5$. We list some possibilities to demonstrate that the optimal solution will be attained when $\Gamma \to \infty$. When Γ is chosen, the value of λ is derived according to Eq. (7).

Table 1
Sensitivity analysis with respect to B (reproduced from Abad [10, Table 1])

B	0.1	0.3	0.7	0.85	1
Γ	1.175	1.175	0.921	0.791	0.674
λ	0	0	1.105	1.251	1.335
$\Pi(\Gamma,\lambda)$	1717.163	1717.163	1354.421	1167.58	996.75

Table 2
Sensitivity analysis with respect to B by our method

B	0.1	0.3	0.7	0.85	1
Γ	1.645	1.645	1.505	1.334	1.162
λ	0	0	0.722	0.994	1.156
$\Pi(\Gamma,\lambda)$	1219.115	1219.115	1116.081	990.223	863.421

Table 3
The extreme case for $B = 0.7$ and $c_1 = 4 \times 10^5$

Γ	$\Gamma = 1$	$\Gamma = 10$	$\Gamma = 100$	$\Gamma = 1000$	$\Gamma = 5000$	$\Gamma = 7000$
λ	34.444	25.480	16.433	15.157	15.032	15.023
$\Pi(\Gamma,\lambda)$	22357.35	18435.93	14655.61	14040.25	14016.21	14011.59

When Γ is chosen, the value of λ is derived according to Lemma 4.

From Table 3, we observe that when Γ approaches infinity then the value of $\Pi(\Gamma,\lambda)$ will decrease to its minimum value $(c+\frac{c_2}{\theta})(p-d)=14,000$. When having a high setup cost, it implies inventory holding duration should be extended as long as possible to lessen average total cost. From a practical view, the product types must be simplified to decrease the effect on setup cost for rearranging production procedures, including equipment preparation and adjustment. Stock must be held continuously and the machine should operate uninterruptedly to reduce shutdown loss (because of setup and shortage) in regular procedure, to gain minimum average total cost.

6. Conclusion

We had pointed out in this note the questionable results in Abad's paper and provided a new and correct solution. From our theorem, the decision maker can decide where to search for the optimal solution. Reviewing the sensitivity analysis in Abad's paper, he examined 25 examples. However, he was not aware that sometimes the optimal solution approaches infinity. Therefore, our detailed analytical work patches the leak in Abad's paper, with a variety of proposed examples explaining the background and strategy that we meet in real cases. Finally, we deduced the optimal values via complete procedures that are mathematically sound.

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Appendix A. The proof for Assumption 1 of Abad [10]

Here we prove that Assumption 1 of Abad [10] is always valid. The verification is divided into the following cases: (1) $\Gamma>0$, and (2) $\Gamma=0$.

Under the condition of $\Gamma>0$, to show that $F(\Gamma,\lambda)$ is positive for any combination of constants c_1, $c+\frac{c_2}{\theta}$, c_3 and c_4, it is sufficient to prove that $p\beta(\Gamma)-d\Gamma>0$ for $\Gamma>0$, when $\theta>0$ and $\lim_{\theta\to 0}\frac{p\beta(\Gamma)-d\Gamma}{\theta}$ exists and is positive. We will divide it into two cases: (a): $\theta>0$ and (b): $\theta=0$.

For Case (a), since $p\beta(\Gamma)-d\Gamma>0$ is equivalent to $\frac{p}{\theta}\ln(1+\frac{d}{p}\exp^{\Gamma\theta}-1)>d\Gamma$, that is, equivalent to $\frac{d}{p}(\exp^{\Gamma\theta}-1)>\exp^{\frac{d}{p}\Gamma\theta}-1$, we assume that $\frac{d}{p}=x$ and $\theta\Gamma=y$ with $0<x<1$ and $y>0$. By fixing x with $0<x<1$ and letting $f(y)=x(\exp^{y}-1)-(\exp^{xy}-1)$ for $y\geqslant 0$, with $f(0)=0$ and $\frac{\mathrm{d}f}{\mathrm{d}y}=x(\exp^{y}-\exp^{xy})>0$, it yields that $f(y)>0$ for $y>0$. Consequently, for $\theta>0$, we derive that $\frac{d}{p}(\exp^{\Gamma\theta}-1)>\exp^{\frac{d}{p}\Gamma\theta}-1$.

For case (b), we compute

$$\begin{aligned}
\lim_{\theta\to 0}\frac{p\beta(\Gamma)-d\Gamma}{\theta} &= \lim_{\theta\to 0}\frac{p\ln\left[1+\frac{d}{p}(\exp^{\Gamma\theta}-1)\right]-d\Gamma}{\theta^2}\\
&= \lim_{\theta\to 0}\frac{p\left[1+\frac{d}{p}(\exp^{\Gamma\theta})\right]^{-1}\frac{d}{p}\Gamma\exp^{\Gamma\theta}-d\Gamma}{2\theta}\\
&= \lim_{\theta\to 0}d\Gamma\frac{\exp^{\Gamma\theta}-1-\frac{d}{p}(\exp^{\Gamma\theta}-1)}{2\theta\left[1+\frac{d}{p}(\exp^{\Gamma\theta}-1)\right]}\\
&= \lim_{\theta\to 0}\frac{d\Gamma(1-dp^{-1})p\exp^{\Gamma\theta}}{2\left[1+\frac{d}{p}(\exp^{\Gamma\theta}-1)\right]+2\theta\frac{d}{p}\Gamma\exp^{\Gamma\theta}}\\
&= \frac{d}{2}\Gamma^2\left(1-\frac{d}{p}\right).
\end{aligned}$$

On the other hand, we find $\frac{c_2}{2}(p-d)\beta^2 + \frac{c_2}{2}d(\Gamma-\beta)^2$ with

$$\beta(\Gamma,0) = \lim_{\theta\to 0}\beta(\Gamma,\theta) = \lim_{\theta\to 0}\frac{\ln\left[1+\frac{d}{p}(\exp^{\Gamma\theta}-1)\right]}{\theta} = \lim_{\theta\to 0}\frac{dp^{-1}\Gamma\exp^{\Gamma\theta}}{1+\frac{d}{p}(\exp^{\Gamma\theta}-1)} = \frac{d}{p}\Gamma.$$

Then, we have $\frac{c_2}{2}(p-d)(\frac{d}{p}\Gamma)^2 + \frac{c_2}{2}d(1-\frac{d}{p})^2\Gamma^2 = \frac{c_2}{2}\frac{d}{p}(p-d)\Gamma^2$, which means when $\theta = 0$

$$F(\Gamma,\lambda) = c_1 + \frac{c_2}{2}\frac{d}{p}(p-d)\Gamma^2 + \frac{c_3}{2}Bdm\lambda^2 + c_4(1-B)dm\lambda,$$

which is the total cost per cycle in the model without deterioration.

For the case of $\Gamma = 0$, from $\beta(0) = 0$, it yields that $F(0,\lambda) = c_1 + \frac{c_3}{2}Bdm\lambda^2 + c_4(1-B)dm\lambda$ such that $F(0,\lambda)$ has the desired property.

Therefore, we finish the proof for Assumption 1 of Abad [10] that $F(\Gamma,\lambda) > 0$ is always valid. Consequently, Assumption 1 of Abad's [10] should thus be dropped from the discussion.

Appendix B. The proof for B(Γ) being a strictly increasing function of Γ

From $\frac{d}{d\Gamma}B(\Gamma) = \frac{c_3}{2}Bdm[2\lambda(\Gamma)\frac{d\lambda(\Gamma)}{d\Gamma} + 2\lambda(\Gamma) + 2\lambda\frac{d\lambda(\Gamma)}{d\Gamma}] + c_4(1-B)dm - (c+\frac{c_2}{\theta}) + \frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)}$, since $\lambda(\Gamma)$ satisfies Eq. (5), we may simplify $\frac{dB(\Gamma)}{d\Gamma} = c_3Bdm(\Gamma+\lambda(\Gamma))\frac{d\lambda(\Gamma)}{d\Gamma}$. Using Eq. (5), we obtain

$$\frac{d\lambda(\Gamma)}{d\Gamma} = \frac{(c\theta+c_2)(p-d)}{c_3Bdm\theta}\frac{dA(\Gamma)}{d\Gamma} = \frac{(c\theta+c_2)(p-d)}{c_3Bdm\theta}\frac{pd\theta\exp^{\Gamma\theta}}{[p+d(\exp^{\Gamma\theta}-1)]^2}.$$

Therefore, we derive that

$$\frac{dB(\Gamma)}{d\Gamma} = (\Gamma+\lambda(\Gamma))\left(c+\frac{c_2}{\theta}\right)\frac{(p-d)pd\theta\exp^{\Gamma\theta}}{[p+d(\exp^{\Gamma\theta}-1)]^2} > 0.$$

We therefore have the condition that $B(\Gamma)$ is a strictly increasing function of Γ.

Appendix C. The value of *B*(∞)

We know that $\lim_{\Gamma\to\infty}\frac{\exp^{\Gamma\theta}}{\exp^{\theta\beta(\Gamma)}} = \frac{\exp^{\Gamma\theta}}{1+\frac{d}{p}(\exp^{\Gamma\theta}-1)} = \frac{p}{d}$ and $\lim_{\Gamma\to\infty}\lambda(\Gamma) = \frac{(c+\frac{c_2}{\theta})(p-d)-c_4(1-B)dm}{c_3Bdm}$. Since

$$\begin{aligned} B(\Gamma) &= \frac{c_3}{2}Bdm\lambda^2(\Gamma) + \Gamma[c_3Bdm\lambda(\Gamma) + c_4(1-B)dm] - c_1 - \left(c+\frac{c_2}{\theta}\right)(p\beta(\Gamma)-d\Gamma) \\ &= \frac{c_3}{2}Bdm\lambda^2(\Gamma) + \left[\Gamma\frac{(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)} - (p\beta(\Gamma)-d\Gamma)\right]\left(c+\frac{c_2}{\theta}\right) - c_1, \end{aligned}$$

we consider

$$\begin{aligned} \lim_{\Gamma\to\infty}\frac{\Gamma(p-d)d(\exp^{\Gamma\theta}-1)}{p+d(\exp^{\Gamma\theta}-1)} - (p\beta(\Gamma)-d\Gamma) &= \lim_{\Gamma\to\infty}\frac{(p-d)d(\exp^{\Gamma\theta}-1)\Gamma - (p\beta(\Gamma)-d\Gamma)[p+d(\exp^{\Gamma\theta}-1)]}{p+d(\exp^{\Gamma\theta}-1)} \\ &= \lim_{\Gamma\to\infty}\frac{(p-d)d\theta\exp^{\Gamma\theta}\Gamma - (p\beta(\Gamma)-d\Gamma)d\theta\exp^{\Gamma\theta}}{d\theta\exp^{\Gamma\theta}} \\ &= \lim_{\Gamma\to\infty}(p-d)\Gamma - (p\beta(\Gamma)-d\Gamma) \\ &= \lim_{\Gamma\to\infty}\frac{p}{\theta}(\theta\Gamma - \theta\beta(\Gamma)) \\ &= \frac{p}{\theta}\ln\lim_{\Gamma\to\infty}\frac{\exp^{\Gamma\theta}}{\exp^{\beta(\Gamma)\theta}} \\ &= \frac{p}{\theta}\ln\frac{p}{d}. \end{aligned}$$

Consequently, we derive that

$$B(\infty) = \frac{p}{\theta}\left(c + \frac{c_2}{\theta}\right)\ln\frac{p}{d} - c_1 + \frac{[(c\theta + c_2)(p-d) - c_4(1-B)dm\theta]^2}{2c_3Bdm\theta^2}.$$

Appendix D. For the boundary cases

For Case (a), along the boundary $\Gamma = 0$, recall Eq. (11). We have shown that when λ is fixed, the minimum of $\Pi(\Gamma,\lambda)$ will at (a1) $\Gamma(\lambda)$, where $\Gamma(\lambda)$ satisfies the equation $C(\Gamma(\lambda),\lambda) = 0$, or (a2) $\Gamma \to \infty$, when $C(\Gamma(\lambda),\lambda) = 0$ does not have solution. Therefore, the minimum solution will not occur along the boundary $\Gamma = 0$.

For case (b), $\lambda = 0$, it is the special situation for case (a) with $\lambda = 0$. Then we know that

(b1) if $(c + \frac{c_2}{\theta})\frac{p}{\theta}\ln\frac{p}{d} - c_1 > 0$, there is a unique point, say $\widetilde{\Gamma}$, such that $C(\widetilde{\Gamma},0) = 0$ and $\widetilde{\Gamma}$ is the optimal solution for $\Pi(\Gamma,0)$ and $\Pi(\widetilde{\Gamma},0) = (c + \frac{c_2}{\theta})\frac{(p-d)d(\exp^{\widetilde{\Gamma}\theta}-1)}{p+d(\exp^{\widetilde{\Gamma}\theta}-1)}$; and

(b2) on the other hand, if $(c + \frac{c_2}{\theta})\frac{p}{\theta}\ln\frac{p}{d} - c_1 \leqslant 0$, the minimum will occur when $\Gamma \to \infty$ such that $\lim_{\Gamma\to\infty}\Pi(\Gamma,0) = (c + \frac{c_2}{\theta})(p-d)$.

For Case (c), $\lambda \to \infty$, since we can rewrite $\Pi(\Gamma,\lambda)$ as

$$\Pi(\Gamma,\lambda) = \frac{c_3}{2}Bdm(\lambda - \Gamma) + c_4(1-B)dm + \frac{c_1 + (c + \frac{c_2}{\theta})(p\beta(\Gamma) - d\Gamma) - c_4(1-B)dm\Gamma + \frac{c_3}{2}Bdm\Gamma^2}{\Gamma + \lambda}.$$

When Γ is fixed, if we take $\lambda \to \infty$, the value of $\Pi(\Gamma,\lambda) \to \infty$. Then, we do not need to consider the case $\lambda \to \infty$ for this minimum problem.

For Case (d), $\Gamma \to \infty$, we have

$$\begin{aligned}\lim_{\Gamma\to\infty}\Pi(\Gamma,\lambda) &= \lim_{\Gamma\to\infty}\frac{(c + \frac{c_2}{\theta})[p\frac{1}{\theta}\ln(1 + \frac{d}{p}(\exp^{\Gamma\theta} - 1)) - d\Gamma]}{\Gamma + \lambda}\\ &= \lim_{\Gamma\to\infty}\left(c + \frac{c_2}{\theta}\right)\left[\frac{p}{\theta}\left(\frac{dp^{-1}\theta\exp^{\Gamma\theta}}{1 + dp^{-1}(\exp^{\Gamma\theta} - 1)}\right) - d\right]\\ &= \left(c + \frac{c_2}{\theta}\right)(p-d).\end{aligned}$$

Since $\Pi(\widetilde{\Gamma},0) < \lim_{\Gamma\to\infty}\Pi(\Gamma,0) = (c + \frac{c_2}{\theta})(p-d)$, we do not need to consider Case (d) on the boundary as $\Gamma \to \infty$.

Appendix E. For the instantaneous replenishment case, the value of $B(\Gamma_0)$

From $\Gamma_0 = \frac{1}{\theta}\ln\left[1 + (\frac{p}{d})\frac{c_4(1-B)dm\theta}{(c\theta+c_2)(p-d)-c_4(1-B)dm\theta}\right]$ and $z = \frac{c_4(1-B)\theta}{c\theta+c_2}$, when $p \to \infty$, we derive that $\lim_{p\to\infty}\Gamma_0 = \frac{1}{\theta}\ln(1+z)$ and $\lim_{p\to\infty}p\beta(\Gamma_0) = \frac{dz}{\theta}$.

Hence, when $p \to \infty$, it follows that

$$\begin{aligned}\lim_{p\to\infty}B(\Gamma_0) &= c_4(1-B)d\Gamma_0 - c_1 - \left(c + \frac{c_2}{\theta}\right)\left[\frac{d}{\theta}(\exp^{\Gamma_0\theta} - 1) - d\Gamma_0\right]\\ &= \frac{(c\theta + c_2)z}{\theta}\frac{d}{\theta}\ln(1+z) - c_1 - \frac{(c\theta + c_2)}{\theta}\frac{dz}{\theta} + \frac{(c\theta + c_2)}{\theta}\frac{d}{\theta}\ln(1+z)\\ &= \frac{d(c\theta + c_2)}{\theta^2}[(1+z)\ln(1+z) - z] - c_1.\end{aligned}$$

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