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THE PERFORMANCE OF PROCESS CAPABILITY INDEX C_S ON SKEWED DISTRIBUTIONS

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ABSTRACT

Wright (1995) considered a new process capability index C_s, which extends the most useful index to date for processes with two-sided specification limits, C_{pmk} proposed by Pearn, Kotz and Johnson (1992). The new index C_s not only takes into account the process variation as well as the location of the process mean relative to the specification limits, but also considers the asymmetry of the distribution by incorporating a penalty for skewness. Wright (1995) investigated an estimator of C_s and studied its bias and variance by simulation. The simulation study, however, was restricted to normal distributions where skewness is not present. In this paper, we extend Wright's simulation study to cover some skewed distributions including chi-square, lognormal, and Weibull distributions for some parameter values. The results show that the percentage bias of the estimator increases as the skewness coefficient $|\mu_3/\sigma^3|$ increases. Extensive simulation results, comparisons, and analysis are provided.

1. INTRODUCTION

Process capability indices (PCIs) have become widely used in the manufacturing industry to provide measures for process quality. Several basic indices including C_p, C_{pk}, and C_{pm} (Kane (1986), Chan, Cheng and Spiring (1988)) have been proposed to monitor the process potential and process performance. These indices are useful management tools, which provide numerical measures of a process characteristic standardized by the process target and specifications. Combining the three basic indices, Pearn, Kotz and Johnson (1992) developed the index C_{pmk}, which is considered to be the most useful index to date for processes with two-sided specification limits. This index, designed for normal and near-normal processes, is constructed as with all process capability indices, that the larger the index, the more capable the process. The index C_{pmk} is defined as the following:

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\},$$

where USL and LSL are the upper and lower specification limits, μ and σ are process mean and process standard deviation, and T is the target value.

The index C_{pmk} provides warnings of the increase of process variation and process departure (the deviation of process mean from its target), but provides no sensitivity to the changes in the shape of the distribution, particularly, the skewness. To detect the shape changes of the processes due to skewness, Wright (1995) considered a new process capability index C_s to extend C_{pmk}. The new index C_s not only takes into account the process variation as well as the departure of process mean from the target, but also the asymmetry of the distribution by incorporating a penalty for skewness. Utilizing the third central moment $\mu_3 = E(X - \mu)^3$ as a measure of skewness, the new index C_s is defined as the following:

$$C_s = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2 + |\mu_3/\sigma|}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2 + |\mu_3/\sigma|}}\right\},$$

with μ_3 divided by σ to ensure that the skewness term is expressed in the same units as the other terms in the denominator (Wright (1995)). Utilizing the identity $\min(x, y) = (x + y)/2 - |x - y|/2$, the index C_s can be rewritten as the following, where $d = (USL - LSL)/2$:

$$C_s = \frac{d - |\mu - T|}{3\sqrt{\sigma^2 + (\mu - T)^2 + |\mu_3/\sigma|}} .$$

2. ESTIMATION OF C_S

To estimate the index C_s, Wright (1995) considered a complicated estimator (defined in the following). We note that for normal samples, $E[\Sigma (x_i - T)^2/n] = \sigma^2 + (\mu - T)^2$, $E[(m_2)^{1/2}] = \{(n-1)/n\}^{1/2} c_4 \sigma$, and $E(m_3) = (n-1)(n-2)\mu_3/n^2$, where $m_r = (1/n)\Sigma (x_i - \bar{x})^r$ is the r-th sample central moment, and $c_4 = \{2/(n-1)\}^{1/2}\Gamma(n/2)\Gamma\{(n-1)/2\}^{-1}$ (see Wright (1995)).

$$\hat{C}_s = \frac{d - |\bar{X} - T|}{3\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - T)^2 + \left|\frac{n^2 m_3}{(n-1)(n-2)} \times \left(\frac{n}{n-1}\frac{m_2}{c_4^2}\right)^{-1/2}\right|}}$$

$$= \frac{d - |\bar{X} - T|}{3\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - T)^2 + \left|\frac{n c_4}{(n-1)^{1/2}(n-2)}\sum_{i=1}^{n}(X_i - \bar{X})^3 \times \left(\sum_{i=1}^{n}(X_i - \bar{X})^2\right)^{-1/2}\right|}} .$$

Obviously, if the third sample central moment is zero, then the estimator $\hat{C}_s$ defined above reduces to the estimator $\hat{C}_{pmk}$ considered by Pearn, Kotz and Johnson (1992) for the index C_{pmk}. The distribution of $\hat{C}_s$ is intractable even under normality assumption. Wright (1995) used a simulation technique to compute the expected value and variance of $\hat{C}_s$. The simulation study, however, was restricted to normal distributions where skewness is not present.

Before investigating the performance of the estimator $\hat{C}_s$ under nonnormal (skewed) samples, we repeat the calculation on the moments of $\hat{C}_s$ based on 15,000,000 random samples of size n from the uniform distribution, U(0, 1), which are generated by AS183 generator (Wichmann and Hill (1987)) with multiple seeds using IBM RISC/6000 work stations. Note that we have extended the sample size for the simulation to n = 500. Tables 1(a) and 1(b) display the expected values, variances, and the performance of $\hat{C}_s$ in terms of percentage bias, $\{E(\hat{C}_s) - C_S\}/C_S$, in normal samples for various values of d/σ, and $|(\mu - T)/\sigma|$. Our simulation results are almost identical to those presented in Wright (1995). In the next section, we extend Wright's simulation study on percentage bias of the estimator $\hat{C}_s$ to cover some skewed distributions including the chi-square

Table 1(a). Expected value, variance, and percentage bias of $\hat{C}_s$ for normal samples with n=10, 20, 30, 40, 50

$\lvert(\mu-T)/\sigma\rvert$		0.0		0.5		1.0		1.5		2.0		$\lvert(\mu-T)/\sigma\rvert$				
n	d/σ	E.V.	VAR	E.V.	VAR	E.V.	VAR	E.V.	VAR	E.V.	VAR	0.0	0.5	1.0	1.5	2.0
10	2	.5274	.0294	.4163	.0267	.2342	.0134	.0975	.0054	.0057	.0023	-20.9%	-6.9%	-.6%	5.5%	***%
	3	.8275	.0643	.6908	.0538	.4584	.0263	.2767	.0106	.1517	.0045	-17.3%	-7.3%	-2.8%	-.2%	1.8%
	4	1.1275	.1137	.9653	.0916	.6825	.0443	.4559	.0178	.2978	.0076	-15.4%	-7.5%	-3.5%	-1.4%	-.1%
	5	1.4275	.1775	1.2398	.1402	.9066	.0673	.6351	.0271	.4438	.0116	-14.4%	-7.6%	-3.8%	-1.9%	-.8%
	6	1.7275	.2557	1.5144	.1995	1.1307	.0953	.8144	.0385	.5898	.0165	-13.6%	-7.6%	-4.1%	-2.1%	-1.1%
20	2	.5448	.0143	.4133	.0132	.2279	.0061	.0932	.0025	.0029	.0011	-18.3%	-7.6%	-3.3%	.8%	***%
	3	.8433	.0309	.6860	.0259	.4506	.0120	.2717	.0049	.1485	.0021	-15.7%	-8.0%	-4.4%	-2.0%	-.4%
	4	1.1417	.0544	.9588	.0435	.6734	.0202	.4502	.0083	.2942	.0036	-14.4%	-8.1%	-4.8%	-2.6%	-1.3%
	5	1.4402	.0848	1.2316	.0661	.8962	.0307	.6286	.0126	.4398	.0055	-13.6%	-8.2%	-4.9%	-2.9%	-1.7%
	6	1.7387	.1220	1.5044	.0938	1.1189	.0435	.8071	.0178	.5855	.0078	-13.1%	-8.3%	-5.1%	-3.0%	-1.8%
30	2	.5575	.0098	.4142	.0089	.2268	.0040	.0921	.0016	.0019	.0007	-16.4%	-7.4%	-3.8%	-.4%	***%
	3	.8579	.0212	.6882	.0174	.4502	.0079	.2708	.0032	.1478	.0014	-14.2%	-7.7%	-4.5%	-2.4%	-.9%
	4	1.1582	.0372	.9622	.0292	.6735	.0133	.4496	.0054	.2936	.0024	-13.1%	-7.8%	-4.8%	-2.7%	-1.5%
	5	1.4585	.0580	1.2363	.0444	.8969	.0202	.6284	.0082	.4394	.0036	-12.5%	-7.9%	-4.9%	-2.9%	-1.7%
	6	1.7588	.0833	1.5103	.0630	1.1202	.0286	.8072	.0117	.5852	.0051	-12.1%	-7.9%	-4.9%	-3.0%	-1.9%
40	2	.5668	.0075	.4156	.0067	.2267	.0030	.0916	.0012	.0015	.0005	-15.0%	-7.1%	-3.8%	-.9%	***%
	3	.8690	.0163	.6911	.0132	.4508	.0059	.2708	.0024	.1475	.0010	-13.1%	-7.3%	-4.4%	-2.4%	-1.1%
	4	1.1713	.0286	.9665	.0221	.6749	.0099	.4499	.0040	.2935	.0018	-12.1%	-7.4%	-4.6%	-2.7%	-1.6%
	5	1.4735	.0444	1.2420	.0337	.8990	.0151	.6291	.0062	.4395	.0027	-11.6%	-7.4%	-4.6%	-2.8%	-1.7%
	6	1.7758	.0639	1.5174	.0477	1.1231	.0214	.8082	.0087	.5855	.0038	-11.2%	-7.5%	-4.7%	-2.9%	-1.8%
50	2	.5739	.0062	.4171	.0054	.2269	.0024	.0914	.0010	.0012	.0004	-13.9%	-6.7%	-3.7%	-1.1%	***%
	3	.8779	.0134	.6937	.0106	.4516	.0047	.2709	.0019	.1474	.0008	-12.2%	-6.9%	-4.2%	-2.3%	-1.1%
	4	1.1818	.0234	.9704	.0179	.6763	.0079	.4503	.0032	.2936	.0014	-11.4%	-7.0%	-4.4%	-2.6%	-1.5%
	5	1.4858	.0365	1.2471	.0272	.9010	.0121	.6298	.0049	.4398	.0021	-10.9%	-7.0%	-4.4%	-2.7%	-1.7%
	6	1.7897	.0524	1.5238	.0386	1.1258	.0171	.8093	.0070	.5860	.0030	-10.5%	-7.1%	-4.5%	-2.7%	-1.7%

Table 1(b). Expected value, variance, and percentage bias of $\hat{C}_s$ for normal samples with n=100, 200, 300, 400, 500

$\|(\mu-T)/\sigma\|$		0.0		0.5		1.0		1.5		2.0		$\|(\mu-T)/\sigma\|$				
n	d/σ	E.V.	VAR	E.V.	VAR	E.V.	VAR	E.V.	VAR	E.V.	VAR	0.0	0.5	1.0	1.5	2.0
100	2	.5945	.0033	.4224	.0028	.2281	.0012	.0912	.0005	.0007	.0002	-10.8%	-5.5%	-3.2%	-1.4%	***%
	3	.9041	.0071	.7032	.0055	.4550	.0024	.2718	.0009	.1474	.0004	-9.6%	-5.7%	-3.5%	-2.0%	-1.1%
	4	1.2136	.0125	.9841	.0092	.6819	.0040	.4523	.0016	.2942	.0007	-9.0%	-5.7%	-3.6%	-2.2%	-1.3%
	5	1.5231	.0194	1.2649	.0140	.9088	.0061	.6329	.0024	.4410	.0011	-8.6%	-5.7%	-3.6%	-2.2%	-1.4%
	6	1.8327	.0279	1.5457	.0199	1.1357	.0086	.8135	.0035	.5878	.0015	-8.4%	-5.7%	-3.6%	-2.2%	-1.4%
200	2	.6121	.0017	.4279	.0014	.2297	.0006	.0913	.0002	.0004	.0001	-8.2%	-4.3%	-2.5%	-1.2%	***%
	3	.9270	.0038	.7127	.0028	.4586	.0012	.2729	.0005	.1477	.0002	-7.3%	-4.4%	-2.7%	-1.6%	-.9%
	4	1.2419	.0066	.9975	.0047	.6876	.0020	.4546	.0008	.2951	.0003	-6.9%	-4.4%	-2.8%	-1.7%	-1.0%
	5	1.5568	.0102	1.2823	.0072	.9166	.0030	.6362	.0012	.4424	.0005	-6.6%	-4.4%	-2.8%	-1.7%	-1.1%
	6	1.8717	.0147	1.5671	.0102	1.1456	.0043	.8178	.0017	.5898	.0007	-6.4%	-4.4%	-2.8%	-1.7%	-1.1%
300	2	.6207	.0012	.4308	.0009	.2305	.0004	.0915	.0002	.0003	.0001	-6.9%	-3.7%	-2.2%	-1.0%	***%
	3	.9384	.0025	.7176	.0018	.4606	.0008	.2736	.0003	.1479	.0001	-6.2%	-3.7%	-2.3%	-1.4%	-.8%
	4	1.2560	.0044	1.0044	.0031	.6906	.0013	.4557	.0005	.2955	.0002	-5.8%	-3.7%	-2.3%	-1.4%	-.9%
	5	1.5736	.0068	1.2912	.0047	.9206	.0020	.6379	.0008	.4431	.0003	-5.6%	-3.8%	-2.4%	-1.4%	-.9%
	6	1.8913	.0098	1.5781	.0067	1.1506	.0028	.8200	.0011	.5908	.0005	-5.4%	-3.8%	-2.4%	-1.4%	-.9%
400	2	.6262	.0009	.4327	.0007	.2311	.0003	.0916	.0001	.0002	.0000	-6.1%	-3.2%	-2.0%	-.9%	***%
	3	.9456	.0019	.7208	.0013	.4618	.0005	.2740	.0002	.1480	.0001	-5.4%	-3.3%	-2.0%	-1.2%	-.7%
	4	1.2651	.0033	1.0090	.0023	.6925	.0009	.4565	.0004	.2958	.0002	-5.1%	-3.3%	-2.1%	-1.2%	-.8%
	5	1.5845	.0051	1.2971	.0035	.9232	.0014	.6390	.0006	.4436	.0002	-4.9%	-3.3%	-2.1%	-1.3%	-.8%
	6	1.9039	.0073	1.5853	.0049	1.1539	.0020	.8215	.0008	.5914	.0003	-4.8%	-3.3%	-2.1%	-1.3%	-.8%
500	2	.6301	.0007	.4341	.0005	.2316	.0002	.0916	.0001	.0002	.0000	-5.5%	-2.9%	-1.7%	-.9%	***%
	3	.9509	.0015	.7233	.0011	.4628	.0004	.2744	.0002	.1481	.0001	-4.9%	-3.0%	-1.8%	-1.1%	-.7%
	4	1.2718	.0025	1.0125	.0018	.6940	.0007	.4571	.0003	.2961	.0001	-4.6%	-3.0%	-1.9%	-1.1%	-.7%
	5	1.5926	.0040	1.3016	.0027	.9253	.0011	.6398	.0004	.4440	.0002	-4.4%	-3.0%	-1.9%	-1.1%	-.7%
	6	1.9134	.0057	1.5908	.0039	1.1565	.0016	.8226	.0006	.5920	.0003	-4.3%	-3.0%	-1.9%	-1.1%	-.7%

Table 2. Characteristics of the three distributions.

	$\chi^2(3)$	$\chi^2(4)$	$\chi^2(5)$	$\chi^2(6)$	$\chi^2(7)$
Mean	3.00	4.00	5.00	6.00	7.00
Variance	6.00	8.00	10.00	12.00	14.00
Skewness	1.63	1.41	1.26	1.15	1.07
	LN(0, 1)	LN(0, $\frac{1}{4}$)	LN(0, $\frac{1}{9}$)	LN(0, $\frac{1}{16}$)	LN(0, $\frac{1}{25}$)
Mean	1.65	1.13	1.06	1.03	1.02
Variance	4.67	0.36	0.13	0.07	0.04
Skewness	6.18	1.75	1.07	0.78	0.61
	W(1, $\frac{7}{5}$)	W(1, $\frac{8}{5}$)	W(1, $\frac{9}{5}$)	W(1, 2)	W(1, $\frac{11}{5}$)
Mean	0.91	0.90	0.89	0.89	0.89
Variance	0.44	0.33	0.26	0.22	0.18
Skewness	1.20	0.96	0.78	0.63	0.51

distribution $\chi^2(r)$, the lognormal distribution $\log N(\mu, \sigma^2)$, and the Weibull distribution $W(\alpha, \beta)$.

3. ESTIMATION OF C_S FOR SKEWED DISTRIBUTIONS

For skewed distributions, we consider the following three distributions: (a) chi-square distribution, $\chi^2(r)$, with probability density function $f(x) = \{\Gamma(r/2)\}^{-1} (1/2)^{r/2} (x)^{r/2-1} (e)^{-x/2}$, for $0 < x < \infty$, and degrees of freedom r = 3, 4, 5, 6, and 7; (b) lognormal distribution, $\log N(\mu, \sigma^2)$, with probability density function $f(x) = \{x (2\pi)^{1/2} \sigma\}^{-1} \exp\{-[\ln(x) - \mu]^2/(2\sigma^2)\}$, for $0 < x < \infty$, $-\infty < \mu < \infty$, and parameter values $\mu = 0$, and $\sigma = 1, 1/2, 1/3, 1/4$, and $1/5$; (c) Weibull distribution, $W(\alpha, \beta)$, with probability density function $f(x) = \{\beta (x)^{\beta-1} \alpha^{-\beta}\} \exp\{-(x/\alpha)^\beta\}$, for $0 \leq x < \infty$, $\alpha > 0$, $\beta > 0$, and parameters $\alpha = 1$, $\beta = 1.4, 1.6, 1.8, 2.0$, and 2.2.

The characteristics, including the means, the variances, and the skewness coefficients $|\mu_3/\sigma^3|$ of the three distributions are summarized in Table 2. We note that for chi-square distribution, $\chi^2(r)$, the skewness coefficient decreases as the value of the degrees of freedom increases. For lognormal distribution, logN(0, σ^2), the skewness coefficient decreases as the value of σ^2 decreases. For Weibull distribution, W(1, β), the skewness coefficient decreases as the value of β increases.

Table 3 displays the results from the simulation for the chi-square distribution, $\chi^2(r)$, with degrees of freedom r = 3, 4, 5, 6, and 7. Table 4 displays the results from the simulation for the lognormal distribution, logN(μ, σ^2), with μ = 0, and σ = 1, 1/2, 1/3, 1/4, and 1/5. Table 5 displays the results from the simulation for the Weibull distribution, W(α, β), with α = 1, and β = 1.4, 1.6, 1.8, 2.0, and 2.2. The simulation was carried out for the following values, d/σ = 2, 3, 4, 5, and 6, and $|(\mu - T)/\sigma|$ = 0.0, 0.5, 1.0, 1.5, and 2.0. For simplicity of the presentation, the variance columns are omitted, only values of the percentages bias, $\{E(\hat{C}_s) - C_s\}/C_s$, are presented.

In Figures 1(a)-1(c), we plot the percentage bias versus skewness coefficient, $|\mu_3/\sigma^3|$, for the three distributions, with $d/\sigma = 3$, $|(\mu - T)/\sigma| = 0.5$, and n = 20, 30, 50. The figures show that for all three distributions, the percentage bias, $\{E(\hat{C}_s) - C_s\}/C_s$, increases as the skewness coefficient, $|\mu_3/\sigma^3|$, increases. From Tables 3, 4, and 5, we observed that this relationship remains intact for all values of d/σ, $|(\mu - T)/\sigma|$, and sample size n. In fact, Chen and Kotz (1996) have pointed out that the asymptotic behavior of the estimator $\hat{C}_s$ is highly sensitive to the skewness of the process distribution regardless of whether $\mu = T$ or $\mu \neq T$.

Chen and Kotz (1996) showed that $\hat{C}_s$ is a consistent and asymptotic unbiased estimator of C_s. But, they did not investigate the direction of bias. For normal distribution, the bias is negative except for some cases with small n regardless of whether $\mu = T$ or $\mu \neq T$ (see Wright (1995)). For skewed distributions, the direction of bias is quite different. Tables 3, 4, and 5 show that for the three distributions, $\chi^2(r)$, logN(0, σ^2), and W(1, β), the bias is positive for all n if the process is off-target ($\mu \neq T$). On the other hand, if the process is on-target ($\mu = T$), the bias tends to be positive for small n, and negative for large n.

In order to find the interpretations for such different behaviors of $\hat{C}_s$ between the normal distribution and skewed distributions, we consider the estimator, $\hat{\mu}_3/s$, of the term μ_3/σ in the denominator of C_s defined in Wright (1995) We perform the same simulation for $\hat{\mu}_3/s$ and calculate the percentage bias $\{E(\hat{\mu}_3/s) - \mu_3/\sigma\}/(\mu_3/\sigma)$. The results, which are displayed in Tables 6(a)-6(c), indicate that

Table 3. Percentage bias of $\hat{C}_s$ for chi-square distribution, $\chi^2(r)$, with r=3, 4, 5, 6, and 7.

$\|(\mu-T)/\sigma\|$		r=3					r=4					r=5				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
10	2	24.4%	24.7%	17.2%	13.9%	***%	19.4%	21.7%	15.7%	13.4%	***%	15.9%	19.6%	14.6%	13.1%	***%
	3	30.2%	25.9%	16.9%	11.4%	8.9%	24.9%	22.7%	15.1%	10.3%	8.3%	21.3%	20.4%	13.8%	9.6%	8.0%
	4	33.1%	26.4%	16.8%	10.9%	7.7%	27.7%	23.1%	14.9%	9.7%	7.0%	23.9%	20.7%	13.5%	8.9%	6.5%
	5	34.8%	26.7%	16.8%	10.7%	7.3%	29.3%	23.3%	14.7%	9.4%	6.6%	25.6%	20.9%	13.3%	8.6%	6.0%
	6	36.0%	26.9%	16.7%	10.5%	7.1%	30.4%	23.5%	14.7%	9.3%	6.4%	26.6%	21.0%	13.2%	8.4%	5.8%
20	2	13.4%	16.4%	11.0%	8.7%	***%	10.8%	14.7%	10.0%	8.3%	***%	9.0%	13.5%	9.3%	8.0%	***%
	3	17.1%	16.8%	10.8%	7.2%	5.6%	14.4%	14.9%	9.6%	6.5%	5.2%	12.6%	13.7%	8.8%	6.0%	4.9%
	4	18.9%	17.0%	10.8%	6.9%	4.9%	16.2%	15.0%	9.5%	6.1%	4.4%	14.3%	13.7%	8.7%	5.6%	4.0%
	5	20.0%	17.1%	10.7%	6.8%	4.6%	17.3%	15.1%	9.4%	6.0%	4.1%	15.4%	13.8%	8.6%	5.4%	3.8%
	6	20.8%	17.1%	10.7%	6.7%	4.5%	18.0%	15.1%	9.4%	5.9%	4.0%	16.1%	13.8%	8.5%	5.3%	3.6%
30	2	8.6%	12.3%	8.4%	6.9%	***%	6.8%	10.9%	7.5%	6.4%	***%	5.6%	10.1%	7.0%	6.1%	***%
	3	11.4%	12.4%	8.1%	5.5%	4.4%	9.6%	11.0%	7.1%	4.9%	3.9%	8.3%	10.1%	6.5%	4.5%	3.7%
	4	12.9%	12.5%	8.0%	5.2%	3.7%	11.0%	11.0%	7.0%	4.5%	3.3%	9.7%	10.1%	6.4%	4.2%	3.0%
	5	13.7%	12.5%	8.0%	5.1%	3.5%	11.8%	11.0%	7.0%	4.4%	3.1%	10.5%	10.1%	6.3%	4.0%	2.8%
	6	14.3%	12.5%	8.0%	5.0%	3.4%	12.3%	11.0%	6.9%	4.4%	3.0%	11.1%	10.1%	6.3%	3.9%	2.7%
40	2	6.0%	9.9%	6.9%	5.8%	***%	4.6%	8.7%	6.1%	5.3%	***%	3.6%	8.0%	5.6%	5.1%	***%
	3	8.4%	9.9%	6.6%	4.5%	3.7%	6.9%	8.7%	5.7%	4.0%	3.3%	6.0%	8.0%	5.2%	3.6%	3.1%
	4	9.6%	9.9%	6.5%	4.3%	3.1%	8.1%	8.7%	5.6%	3.7%	2.7%	7.1%	7.9%	5.1%	3.3%	2.5%
	5	10.3%	9.9%	6.4%	4.2%	2.9%	8.8%	8.7%	5.6%	3.6%	2.5%	7.8%	7.9%	5.0%	3.2%	2.3%
	6	10.8%	10.0%	6.4%	4.1%	2.8%	9.3%	8.7%	5.5%	3.5%	2.4%	8.3%	7.9%	5.0%	3.2%	2.2%
50	2	4.5%	8.3%	5.9%	5.1%	***%	3.2%	7.3%	5.2%	4.7%	***%	2.4%	6.6%	4.7%	4.4%	***%
	3	6.6%	8.3%	5.6%	3.9%	3.2%	5.3%	7.2%	4.8%	3.4%	2.9%	4.5%	6.6%	4.4%	3.1%	2.6%
	4	7.6%	8.3%	5.5%	3.7%	2.7%	6.3%	7.2%	4.7%	3.1%	2.3%	5.5%	6.5%	4.2%	2.8%	2.1%
	5	8.2%	8.3%	5.4%	3.5%	2.5%	6.9%	7.2%	4.6%	3.0%	2.2%	6.1%	6.5%	4.2%	2.7%	1.9%
	6	8.7%	8.3%	5.4%	3.5%	2.4%	7.4%	7.2%	4.6%	3.0%	2.1%	6.5%	6.5%	4.1%	2.6%	1.8%
100	2	1.2%	4.7%	3.7%	3.8%	***%	.6%	4.3%	3.4%	3.7%	***%	.1%	3.9%	3.1%	3.5%	***%
	3	2.6%	4.6%	3.3%	2.5%	2.3%	2.0%	4.1%	3.0%	2.3%	2.2%	1.5%	3.7%	2.7%	2.1%	2.0%
	4	3.3%	4.5%	3.2%	2.2%	1.8%	2.7%	4.0%	2.8%	2.0%	1.7%	2.2%	3.6%	2.5%	1.8%	1.5%
	5	3.7%	4.5%	3.1%	2.1%	1.6%	3.1%	4.0%	2.7%	1.9%	1.5%	2.6%	3.6%	2.4%	1.7%	1.3%
	6	4.0%	4.5%	3.0%	2.0%	1.5%	3.4%	4.0%	2.7%	1.8%	1.4%	2.9%	3.5%	2.4%	1.6%	1.2%
200	2	-.3%	2.8%	2.5%	3.0%	***%	-.5%	2.6%	2.4%	3.0%	***%	-.8%	2.3%	2.2%	2.9%	***%
	3	.7%	2.6%	2.0%	1.7%	1.8%	.5%	2.4%	1.9%	1.6%	1.7%	.1%	2.1%	1.7%	1.5%	1.7%
	4	1.2%	2.5%	1.9%	1.5%	1.3%	.9%	2.3%	1.7%	1.4%	1.2%	.6%	2.0%	1.5%	1.3%	1.1%
	5	1.5%	2.4%	1.8%	1.4%	1.1%	1.2%	2.2%	1.7%	1.3%	1.1%	.9%	1.9%	1.5%	1.1%	1.0%
	6	1.7%	2.4%	1.8%	1.3%	1.0%	1.4%	2.2%	1.6%	1.2%	1.0%	1.1%	1.9%	1.4%	1.1%	.9%
300	2	-.6%	2.1%	2.1%	2.9%	***%	-.8%	2.0%	2.0%	2.8%	***%	-1.0%	1.8%	1.9%	2.6%	***%
	3	.2%	1.9%	1.6%	1.5%	1.7%	.0%	1.8%	1.5%	1.4%	1.6%	-.2%	1.6%	1.4%	1.3%	1.5%
	4	.5%	1.8%	1.4%	1.2%	1.1%	.4%	1.7%	1.4%	1.2%	1.1%	.2%	1.5%	1.2%	1.0%	1.0%
	5	.8%	1.7%	1.4%	1.1%	1.0%	.6%	1.6%	1.3%	1.1%	.9%	.4%	1.4%	1.1%	.9%	.8%
	6	.9%	1.7%	1.3%	1.0%	.8%	.8%	1.6%	1.2%	1.0%	.8%	.6%	1.4%	1.1%	.9%	.8%
500	2	-.9%	1.6%	1.7%	2.6%	***%	-.9%	1.5%	1.7%	2.6%	***%	-1.1%	1.4%	1.6%	2.4%	***%
	3	-.3%	1.3%	1.2%	1.3%	1.5%	-.3%	1.3%	1.2%	1.2%	1.5%	-.5%	1.1%	1.1%	1.1%	1.4%
	4	.0%	1.2%	1.1%	1.0%	1.0%	.0%	1.2%	1.1%	1.0%	1.0%	-.2%	1.0%	.9%	.9%	.9%
	5	.2%	1.2%	1.0%	.9%	.8%	.2%	1.1%	1.0%	.9%	.8%	.0%	1.0%	.9%	.8%	.7%
	6	.4%	1.1%	.9%	.8%	.7%	.3%	1.1%	.9%	.8%	.7%	.1%	.9%	.8%	.7%	.7%

Table 3. (continued) Percentage bias of $\hat{C}_S$ for chi-square distribution, $\chi^2(r)$, with r=3, 4, 5, 6, and 7.

$\|(\mu-T)/\sigma\|$		r=6					r=7				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
10	2	13.2%	17.9%	13.7%	12.8%	***%	11.1%	16.6%	13.0%	12.5%	***%
	3	18.5%	18.6%	12.7%	9.1%	7.6%	16.3%	17.2%	11.9%	8.6%	7.4%
	4	21.1%	18.9%	12.4%	8.3%	6.1%	18.9%	17.4%	11.5%	7.8%	5.8%
	5	22.7%	19.1%	12.2%	8.0%	5.6%	20.4%	17.6%	11.4%	7.4%	5.3%
	6	23.7%	19.2%	12.1%	7.8%	5.4%	21.4%	17.7%	11.3%	7.3%	5.1%
20	2	7.7%	12.6%	8.8%	7.8%	***%	6.6%	11.8%	8.4%	7.6%	***%
	3	11.2%	12.7%	8.2%	5.7%	4.7%	10.0%	11.9%	7.8%	5.4%	4.5%
	4	12.9%	12.8%	8.1%	5.2%	3.8%	11.8%	11.9%	7.6%	4.9%	3.6%
	5	14.0%	12.8%	8.0%	5.1%	3.5%	12.8%	12.0%	7.5%	4.7%	3.3%
	6	14.6%	12.8%	7.9%	5.0%	3.4%	13.5%	12.0%	7.4%	4.6%	3.2%
30	2	4.7%	9.4%	6.6%	5.9%	***%	4.0%	8.9%	6.3%	5.7%	***%
	3	7.4%	9.4%	6.1%	4.2%	3.5%	6.7%	8.9%	5.8%	4.0%	3.4%
	4	8.8%	9.4%	5.9%	3.9%	2.9%	8.1%	8.9%	5.6%	3.6%	2.7%
	5	9.6%	9.4%	5.9%	3.7%	2.6%	8.9%	8.9%	5.5%	3.5%	2.5%
	6	10.2%	9.4%	5.8%	3.6%	2.5%	9.4%	8.9%	5.5%	3.4%	2.4%
40	2	3.0%	7.5%	5.3%	4.9%	***%	2.5%	7.1%	5.0%	4.7%	***%
	3	5.3%	7.4%	4.9%	3.4%	2.9%	4.8%	7.0%	4.6%	3.2%	2.8%
	4	6.4%	7.4%	4.7%	3.1%	2.3%	5.9%	7.0%	4.4%	2.9%	2.2%
	5	7.1%	7.4%	4.7%	3.0%	2.1%	6.6%	7.0%	4.4%	2.8%	2.0%
	6	7.6%	7.4%	4.6%	2.9%	2.0%	7.1%	7.0%	4.3%	2.7%	1.9%
50	2	1.9%	6.2%	4.4%	4.3%	***%	1.5%	5.9%	4.2%	4.1%	***%
	3	3.9%	6.1%	4.0%	2.9%	2.6%	3.5%	5.8%	3.8%	2.7%	2.4%
	4	5.0%	6.1%	3.9%	2.6%	2.0%	4.5%	5.7%	3.7%	2.4%	1.9%
	5	5.6%	6.1%	3.9%	2.5%	1.8%	5.1%	5.7%	3.6%	2.3%	1.7%
	6	6.0%	6.1%	3.8%	2.4%	1.7%	5.5%	5.7%	3.6%	2.3%	1.6%
100	2	-.2%	3.6%	2.9%	3.4%	***%	-.4%	3.4%	2.8%	3.2%	***%
	3	1.2%	3.4%	2.5%	2.0%	2.0%	.9%	3.2%	2.4%	1.9%	1.9%
	4	1.9%	3.3%	2.3%	1.7%	1.4%	1.6%	3.1%	2.2%	1.6%	1.4%
	5	2.3%	3.3%	2.2%	1.6%	1.2%	2.0%	3.1%	2.1%	1.5%	1.2%
	6	2.6%	3.3%	2.2%	1.5%	1.1%	2.3%	3.1%	2.1%	1.4%	1.1%
200	2	-1.0%	2.1%	2.0%	2.8%	***%	-1.2%	2.0%	1.9%	2.6%	***%
	3	-.1%	1.9%	1.6%	1.4%	1.6%	-.2%	1.8%	1.5%	1.3%	1.5%
	4	.4%	1.8%	1.4%	1.2%	1.1%	.3%	1.7%	1.3%	1.1%	1.0%
	5	.7%	1.8%	1.3%	1.0%	.9%	.6%	1.6%	1.2%	1.0%	.9%
	6	.9%	1.7%	1.3%	1.0%	.8%	.8%	1.6%	1.2%	.9%	.8%
300	2	-1.2%	1.6%	1.7%	2.5%	***%	-1.3%	1.5%	1.6%	2.5%	***%
	3	-.4%	1.4%	1.3%	1.2%	1.4%	-.5%	1.3%	1.2%	1.2%	1.4%
	4	.0%	1.3%	1.1%	1.0%	1.0%	-.1%	1.2%	1.0%	.9%	.9%
	5	.2%	1.3%	1.0%	.9%	.8%	.1%	1.2%	.9%	.8%	.7%
	6	.4%	1.2%	1.0%	.8%	.7%	.3%	1.1%	.9%	.7%	.7%
500	2	-1.2%	1.2%	1.5%	2.4%	***%	-1.3%	1.1%	1.4%	2.2%	***%
	3	-.6%	1.0%	1.0%	1.1%	1.3%	-.7%	.9%	.9%	1.0%	1.3%
	4	-.3%	.9%	.8%	.8%	.8%	-.4%	.8%	.8%	.8%	.8%
	5	-.1%	.8%	.8%	.7%	.7%	-.2%	.8%	.7%	.7%	.7%
	6	.0%	.8%	.7%	.7%	.6%	-.1%	.7%	.7%	.6%	.6%

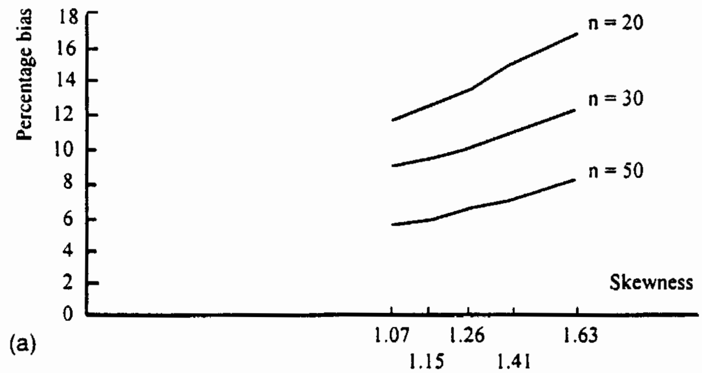

Figure 1(a). Percentage bias plot for $\chi^2(r)$ distribution, $r = 3, 4, 5, 6, 7$, $d/\sigma = 3$, $|(\mu - T)/\sigma| = 0.5$, $n = 20, 30, 50$.

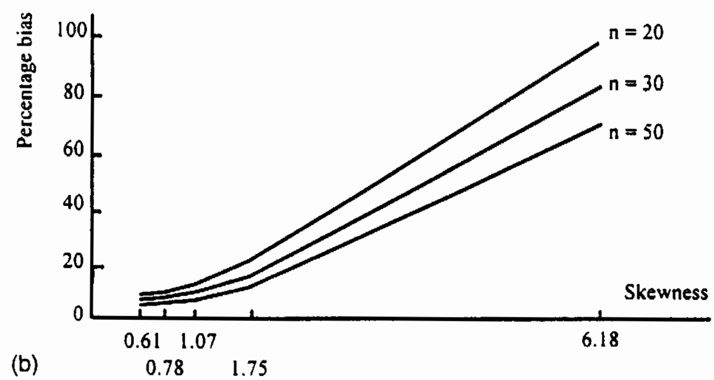

Figure 1(b). Percentage bias plot for $\log N(0, \sigma^2)$ distribution, $\sigma = 1, 1/2, 1/3, 1/4, 1/5$, $d/\sigma = 3$, $|(\mu - T)/\sigma| = 0.5$, $n = 20, 30, 50$.

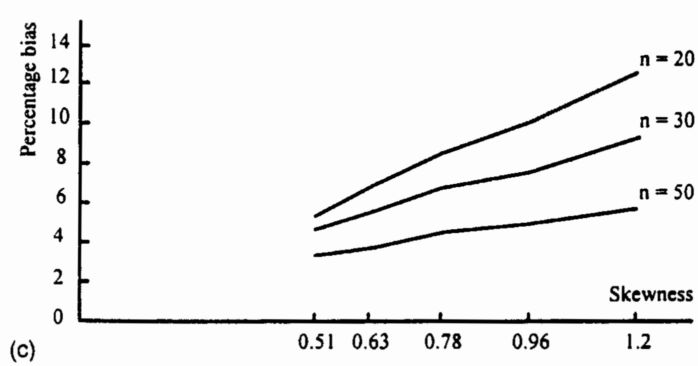

Figure 1(c). Percentage bias plot for $W(1, \beta)$ distribution, $\beta = 1.4, 1.6, 1.8, 2.0, 2.2$, $d/\sigma = 3$, $|(\mu - T)/\sigma| = 0.5$, $n = 20, 30, 50$.

Table 4. Percentage bias of $\hat{C}_S$ for lognormal distribution, logN$(0,\sigma^2)$, with σ=1, 1/2, 1/3, 1/4, and 1/5.

$\vert(\mu\text{-}T)/\sigma\vert$		σ=1					σ=1/2					σ=1/3				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
	2	172.9%	117.3%	72.9%	43.8%	****%	34.0%	31.2%	20.3%	13.9%	****%	14.0%	18.3%	13.4%	11.5%	***%
	3	183.6%	122.4%	77.5%	50.7%	34.1%	39.9%	32.7%	20.6%	13.1%	9.2%	19.2%	19.1%	12.6%	8.6%	6.8%
10	4	188.9%	124.6%	79.1%	52.1%	35.8%	42.9%	33.4%	20.8%	13.0%	8.7%	21.8%	19.4%	12.4%	8.0%	5.7%
	5	192.1%	125.8%	79.8%	52.7%	36.4%	44.7%	33.7%	20.8%	12.9%	8.5%	23.4%	19.6%	12.3%	7.8%	5.3%
	6	194.3%	126.6%	80.3%	53.0%	36.7%	45.9%	34.0%	20.9%	12.9%	8.4%	24.4%	19.7%	12.2%	7.6%	5.1%
	2	120.7%	93.0%	61.0%	37.7%	****%	22.6%	22.5%	14.0%	8.8%	****%	9.9%	13.7%	8.9%	6.7%	***%
	3	127.2%	96.1%	64.3%	43.2%	29.5%	26.4%	23.3%	14.4%	8.9%	6.0%	13.4%	14.0%	8.7%	5.5%	4.1%
20	4	130.4%	97.4%	65.4%	44.3%	30.9%	28.4%	23.6%	14.6%	9.0%	5.8%	15.1%	14.1%	8.6%	5.3%	3.6%
	5	132.4%	98.2%	65.9%	44.8%	31.3%	29.5%	23.8%	14.7%	9.0%	5.8%	16.2%	14.2%	8.6%	5.2%	3.4%
	6	133.7%	98.7%	66.2%	45.0%	31.6%	30.3%	23.9%	14.7%	9.0%	5.8%	16.9%	14.2%	8.5%	5.1%	3.3%
	2	97.3%	79.9%	54.3%	34.4%	****%	16.7%	17.7%	11.0%	6.7%	****%	7.3%	10.8%	6.8%	4.9%	***%
	3	102.1%	82.2%	56.9%	39.0%	26.9%	19.7%	18.2%	11.4%	7.0%	4.6%	10.1%	11.0%	6.7%	4.2%	3.0%
30	4	104.5%	83.1%	57.7%	39.8%	28.0%	21.2%	18.4%	11.5%	7.1%	4.6%	11.4%	11.0%	6.7%	4.0%	2.7%
	5	105.9%	83.7%	58.2%	40.2%	28.4%	22.1%	18.5%	11.6%	7.1%	4.5%	12.3%	11.1%	6.6%	4.0%	2.6%
	6	106.9%	84.0%	58.4%	40.4%	28.6%	22.7%	18.6%	11.6%	7.1%	4.5%	12.8%	11.1%	6.6%	3.9%	2.5%
	2	83.7%	71.4%	49.7%	32.1%	****%	13.3%	14.7%	9.2%	5.6%	****%	5.6%	8.8%	5.5%	3.9%	***%
	3	87.6%	73.2%	51.9%	36.0%	25.0%	15.8%	15.1%	9.5%	5.9%	3.8%	7.9%	9.0%	5.5%	3.4%	2.4%
40	4	89.5%	74.0%	52.6%	36.8%	26.1%	17.1%	15.3%	9.6%	5.9%	3.8%	9.1%	9.0%	5.5%	3.3%	2.1%
	5	90.7%	74.4%	52.9%	37.1%	26.4%	17.8%	15.4%	9.7%	6.0%	3.8%	9.8%	9.1%	5.4%	3.2%	2.1%
	6	91.5%	74.7%	53.2%	37.3%	26.6%	18.3%	15.4%	9.7%	6.0%	3.8%	10.3%	9.1%	5.4%	3.2%	2.0%
	2	74.6%	65.2%	46.3%	30.3%	****%	11.1%	12.7%	7.9%	4.8%	****%	4.4%	7.5%	4.6%	3.2%	***%
	3	77.9%	66.7%	48.1%	33.7%	23.6%	13.3%	13.0%	8.3%	5.1%	3.3%	6.5%	7.6%	4.6%	2.8%	2.0%
50	4	79.5%	67.3%	48.7%	34.4%	24.6%	14.4%	13.1%	8.4%	5.1%	3.3%	7.5%	7.7%	4.6%	2.8%	1.8%
	5	80.5%	67.7%	49.1%	34.7%	24.8%	15.0%	13.2%	8.4%	5.2%	3.3%	8.1%	7.7%	4.6%	2.7%	1.7%
	6	81.2%	67.9%	49.2%	34.9%	25.0%	15.4%	13.3%	8.4%	5.2%	3.3%	8.5%	7.7%	4.6%	2.7%	1.7%
	2	52.5%	48.6%	36.4%	25.0%	****%	6.0%	7.8%	5.0%	3.0%	***%	1.7%	4.3%	2.7%	1.9%	***%
	3	54.5%	49.5%	37.5%	27.2%	19.4%	7.5%	8.0%	5.2%	3.2%	2.0%	3.1%	4.4%	2.7%	1.6%	1.1%
100	4	55.6%	49.8%	37.9%	27.6%	20.0%	8.2%	8.1%	5.3%	3.3%	2.1%	3.8%	4.4%	2.7%	1.6%	1.0%
	5	56.2%	50.0%	38.1%	27.7%	20.2%	8.7%	8.2%	5.3%	3.3%	2.1%	4.2%	4.4%	2.6%	1.6%	1.0%
	6	56.6%	50.1%	38.2%	27.9%	20.3%	9.0%	8.2%	5.3%	3.3%	2.1%	4.5%	4.4%	2.7%	1.6%	1.0%
	2	37.3%	35.8%	28.0%	20.0%	****%	3.1%	4.8%	3.1%	2.0%	***%	.3%	2.4%	1.5%	1.0%	***%
	3	38.6%	36.2%	28.6%	21.3%	15.6%	4.1%	4.8%	3.2%	2.0%	1.2%	1.3%	2.4%	1.5%	.9%	.7%
200	4	39.3%	36.4%	28.8%	21.5%	15.9%	4.6%	4.9%	3.2%	2.0%	1.3%	1.8%	2.4%	1.5%	.9%	.6%
	5	39.7%	36.6%	28.9%	21.6%	16.0%	4.9%	4.9%	3.2%	2.0%	1.3%	2.0%	2.4%	1.5%	.9%	.6%
	6	39.9%	36.6%	29.0%	21.7%	16.1%	5.1%	4.9%	3.2%	2.0%	1.3%	2.2%	2.4%	1.5%	.9%	.5%
	2	30.7%	29.8%	23.8%	17.4%	****%	2.1%	3.5%	2.4%	1.4%	***%	-.1%	1.7%	1.1%	.7%	***%
	3	31.7%	30.2%	24.2%	18.3%	13.5%	2.9%	3.6%	2.4%	1.5%	.9%	.7%	1.7%	1.1%	.7%	.5%
300	4	32.2%	30.3%	24.4%	18.5%	13.8%	3.3%	3.6%	2.4%	1.5%	1.0%	1.1%	1.7%	1.1%	.6%	.4%
	5	32.5%	30.4%	24.5%	18.6%	13.8%	3.5%	3.6%	2.4%	1.5%	1.0%	1.3%	1.7%	1.1%	.6%	.4%
	6	32.7%	30.4%	24.5%	18.6%	13.9%	3.7%	3.6%	2.4%	1.5%	1.0%	1.5%	1.7%	1.1%	.6%	.4%
	2	24.0%	23.6%	19.2%	14.4%	****%	1.1%	2.4%	1.6%	1.0%	***%	-.4%	1.1%	.7%	.5%	***%
	3	24.7%	23.8%	19.5%	14.9%	11.2%	1.8%	2.4%	1.6%	1.1%	.7%	.2%	1.1%	.7%	.5%	.4%
500	4	25.1%	23.9%	19.6%	15.1%	11.4%	2.1%	2.4%	1.6%	1.1%	.7%	.5%	1.1%	.7%	.4%	.3%
	5	25.3%	23.9%	19.6%	15.1%	11.4%	2.2%	2.4%	1.7%	1.0%	.7%	.7%	1.1%	.7%	.4%	.3%
	6	25.5%	24.0%	19.7%	15.2%	11.4%	2.4%	2.5%	1.7%	1.1%	.7%	.8%	1.1%	.7%	.4%	.3%

(continued)

Table 4. (continued) Percentage bias of $\hat{C}_s$ for lognormal distribution, $\log N(0,\sigma^2)$, with σ=1, 1/2, 1/3, 1/4, and 1/5.

$\|(\mu-T)/\sigma\|$		σ=1/4					σ=1/5				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
10	2	5.5%	12.6%	10.3%	10.4%	***%	.4%	9.1%	8.5%	9.8%	***%
	3	10.3%	13.0%	9.2%	6.7%	5.9%	5.0%	9.3%	7.1%	5.5%	5.2%
	4	12.7%	13.1%	8.8%	5.9%	4.5%	7.3%	9.4%	6.6%	4.7%	3.7%
	5	14.1%	13.2%	8.6%	5.6%	4.0%	8.7%	9.4%	6.4%	4.3%	3.2%
	6	15.1%	13.3%	8.5%	5.4%	3.8%	9.6%	9.5%	6.2%	4.1%	2.9%
20	2	4.2%	9.8%	6.8%	6.0%	***%	.7%	7.3%	5.5%	5.4%	***%
	3	7.5%	9.9%	6.3%	4.2%	3.4%	3.9%	7.2%	4.8%	3.4%	3.0%
	4	9.2%	9.9%	6.1%	3.9%	2.8%	5.5%	7.2%	4.6%	3.0%	2.2%
	5	10.2%	9.9%	6.0%	3.7%	2.5%	6.4%	7.2%	4.5%	2.8%	2.0%
	6	10.9%	9.9%	6.0%	3.7%	2.4%	7.1%	7.2%	4.4%	2.7%	1.8%
30	2	3.3%	8.0%	5.3%	4.4%	***%	.7%	6.1%	4.3%	4.0%	***%
	3	6.0%	8.0%	5.0%	3.2%	2.5%	3.3%	6.1%	3.9%	2.6%	2.2%
	4	7.3%	8.0%	4.9%	3.0%	2.1%	4.6%	6.1%	3.7%	2.4%	1.7%
	5	8.1%	8.0%	4.8%	2.9%	1.9%	5.4%	6.0%	3.6%	2.2%	1.5%
	6	8.6%	8.0%	4.8%	2.8%	1.8%	5.9%	6.0%	3.6%	2.2%	1.4%
40	2	2.6%	6.7%	4.3%	3.4%	***%	.6%	5.3%	3.6%	3.2%	***%
	3	4.8%	6.7%	4.1%	2.6%	2.0%	2.8%	5.3%	3.3%	2.2%	1.8%
	4	6.0%	6.7%	4.0%	2.5%	1.7%	3.9%	5.2%	3.2%	2.0%	1.4%
	5	6.7%	6.7%	4.0%	2.4%	1.6%	4.6%	5.2%	3.1%	1.9%	1.3%
	6	7.1%	6.8%	4.0%	2.3%	1.5%	5.1%	5.2%	3.1%	1.8%	1.2%
50	2	2.0%	5.7%	3.7%	2.8%	***%	.4%	4.6%	3.1%	2.7%	***%
	3	4.0%	5.8%	3.5%	2.2%	1.7%	2.4%	4.6%	2.8%	1.9%	1.5%
	4	5.0%	5.8%	3.4%	2.1%	1.4%	3.4%	4.6%	2.8%	1.7%	1.2%
	5	5.6%	5.8%	3.4%	2.0%	1.3%	4.0%	4.6%	2.7%	1.6%	1.1%
	6	6.0%	5.8%	3.4%	2.0%	1.3%	4.4%	4.6%	2.7%	1.6%	1.0%
100	2	.4%	3.3%	2.1%	1.6%	***%	-.3%	2.8%	1.8%	1.6%	**%
	3	1.7%	3.3%	2.0%	1.2%	1.0%	1.1%	2.8%	1.7%	1.1%	.9%
	4	2.4%	3.3%	2.0%	1.2%	.8%	1.8%	2.8%	1.7%	1.0%	.7%
	5	2.9%	3.3%	2.0%	1.2%	.7%	2.2%	2.8%	1.6%	1.0%	.7%
	6	3.1%	3.3%	1.9%	1.1%	.7%	2.4%	2.8%	1.6%	1.0%	.6%
200	2	-.5%	1.8%	1.2%	.9%	***%	-.8%	1.6%	1.0%	.8%	**%
	3	.5%	1.8%	1.1%	.7%	.5%	.1%	1.5%	.9%	.6%	.5%
	4	1.0%	1.8%	1.1%	.6%	.5%	.6%	1.5%	.9%	.6%	.4%
	5	1.2%	1.8%	1.1%	.6%	.4%	.9%	1.5%	.9%	.5%	.4%
	6	1.4%	1.8%	1.1%	.6%	.4%	1.1%	1.5%	.9%	.5%	.4%
300	2	-.7%	1.3%	.8%	.7%	***%	-.9%	1.1%	.7%	.6%	**%
	3	.1%	1.2%	.8%	.5%	.4%	-.2%	1.1%	.7%	.5%	.4%
	4	.5%	1.2%	.8%	.5%	.3%	.2%	1.1%	.6%	.4%	.3%
	5	.7%	1.2%	.7%	.5%	.3%	.5%	1.0%	.6%	.4%	.3%
	6	.9%	1.2%	.7%	.4%	.3%	.6%	1.0%	.6%	.4%	.2%
500	2	-.8%	.8%	.6%	.4%	***%	-1.0%	.7%	.5%	.5%	**%
	3	-.2%	.8%	.5%	.4%	.3%	-.4%	.7%	.4%	.3%	.2%
	4	.1%	.8%	.5%	.3%	.2%	-.0%	.7%	.4%	.3%	.2%
	5	.3%	.8%	.5%	.3%	.2%	.1%	.7%	.4%	.3%	.2%
	6	.4%	.8%	.5%	.3%	.2%	.3%	.7%	.4%	.2%	.2%

Table 5. Percentage bias of $\hat{C}_S$ for Weibull distribution, W(1,β), with β=1.4, 1.6, 1.8, 2.0, and 2.2.

|(μ-T)/σ|		β=1.4					β=1.6					β=1.8				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
10	2	14.3%	18.0%	13.1%	11.3%	***%	8.3%	14.3%	11.4%	11.0%	***%	3.7%	11.4%	10.0%	10.8%	***%
	3	19.6%	18.8%	12.6%	8.6%	6.9%	13.3%	14.9%	10.5%	7.5%	6.4%	8.5%	11.8%	8.8%	6.7%	6.0%
	4	22.2%	19.2%	12.4%	8.1%	5.8%	15.8%	15.2%	10.1%	6.8%	5.1%	10.9%	12.0%	8.4%	5.9%	4.5%
	5	23.8%	19.4%	12.3%	7.8%	5.4%	17.3%	15.3%	10.0%	6.5%	4.6%	12.4%	12.1%	8.2%	5.5%	4.1%
	6	24.9%	19.6%	12.2%	7.7%	5.2%	18.3%	15.4%	9.9%	6.4%	4.4%	13.3%	12.1%	8.0%	5.3%	3.8%
20	2	7.9%	12.1%	8.0%	6.2%	***%	4.7%	10.2%	7.1%	6.1%	***%	2.3%	8.7%	6.4%	6.1%	***%
	3	11.4%	12.5%	7.9%	5.1%	3.8%	8.1%	10.3%	6.7%	4.5%	3.6%	5.6%	8.7%	5.8%	4.1%	3.4%
	4	13.1%	12.6%	7.8%	4.9%	3.3%	9.8%	10.3%	6.5%	4.2%	3.0%	7.2%	8.7%	5.6%	3.7%	2.7%
	5	14.2%	12.7%	7.8%	4.8%	3.2%	10.8%	10.4%	6.4%	4.0%	2.8%	8.2%	8.7%	5.5%	3.5%	2.5%
	6	14.9%	12.7%	7.7%	4.7%	3.1%	11.5%	10.4%	6.4%	4.0%	2.6%	8.9%	8.7%	5.4%	3.4%	2.3%
30	2	4.8%	8.9%	5.8%	4.3%	***%	2.7%	7.5%	5.1%	4.3%	***%	1.2%	6.7%	4.7%	4.3%	***%
	3	7.5%	9.1%	5.7%	3.7%	2.7%	5.4%	7.6%	4.8%	3.2%	2.5%	3.8%	6.6%	4.3%	3.0%	2.4%
	4	8.9%	9.2%	5.6%	3.5%	2.4%	6.7%	7.6%	4.7%	3.0%	2.1%	5.1%	6.6%	4.2%	2.7%	1.9%
	5	9.7%	9.2%	5.6%	3.4%	2.3%	7.5%	7.7%	4.7%	2.9%	1.9%	5.9%	6.6%	4.1%	2.6%	1.8%
	6	10.3%	9.3%	5.6%	3.4%	2.2%	8.0%	7.7%	4.7%	2.9%	1.9%	6.4%	6.6%	4.1%	2.5%	1.7%
40	2	3.1%	7.0%	4.5%	3.3%	***%	1.5%	5.9%	4.0%	3.3%	***%	.5%	5.3%	3.7%	3.3%	***%
	3	5.4%	7.2%	4.5%	2.9%	2.1%	3.8%	6.0%	3.8%	2.5%	1.9%	2.7%	5.3%	3.4%	2.3%	1.9%
	4	6.6%	7.2%	4.5%	2.7%	1.9%	4.9%	6.0%	3.7%	2.3%	1.6%	3.8%	5.3%	3.3%	2.1%	1.5%
	5	7.3%	7.3%	4.4%	2.7%	1.8%	5.6%	6.0%	3.7%	2.3%	1.5%	4.5%	5.3%	3.3%	2.1%	1.4%
	6	7.8%	7.3%	4.4%	2.7%	1.7%	6.0%	6.0%	3.7%	2.2%	1.5%	4.9%	5.3%	3.2%	2.0%	1.3%
50	2	2.1%	5.8%	3.7%	2.8%	***%	.7%	4.9%	3.2%	2.6%	***%	-.1%	4.4%	3.0%	2.7%	***%
	3	4.2%	5.9%	3.7%	2.4%	1.7%	2.7%	4.9%	3.1%	2.0%	1.6%	1.9%	4.4%	2.8%	1.9%	1.6%
	4	5.2%	5.9%	3.7%	2.3%	1.5%	3.7%	4.9%	3.0%	1.9%	1.3%	2.9%	4.4%	2.7%	1.8%	1.3%
	5	5.8%	6.0%	3.7%	2.2%	1.5%	4.3%	4.9%	3.0%	1.8%	1.2%	3.5%	4.4%	2.7%	1.7%	1.1%
	6	6.2%	6.0%	3.7%	2.2%	1.4%	4.7%	4.9%	3.0%	1.8%	1.2%	3.9%	4.4%	2.7%	1.6%	1.1%
100	2	.2%	3.1%	2.0%	1.5%	**%	-.7%	2.5%	1.7%	1.3%	**%	-1.1%	2.3%	1.6%	1.4%	**%
	3	1.5%	3.2%	2.0%	1.3%	.9%	.7%	2.6%	1.6%	1.0%	.9%	.3%	2.3%	1.5%	1.0%	.8%
	4	2.2%	3.2%	2.0%	1.2%	.8%	1.4%	2.6%	1.6%	1.0%	.7%	1.0%	2.3%	1.4%	.9%	.7%
	5	2.6%	3.2%	2.0%	1.2%	.8%	1.8%	2.6%	1.6%	1.0%	.6%	1.4%	2.3%	1.4%	.9%	.6%
	6	2.9%	3.2%	2.0%	1.2%	.8%	2.1%	2.6%	1.6%	.9%	.6%	1.7%	2.3%	1.4%	.9%	.6%
200	2	-.6%	1.7%	1.1%	.9%	**%	-1.2%	1.3%	.9%	.7%	**%	-1.3%	1.2%	.8%	.8%	**%
	3	.3%	1.7%	1.1%	.7%	.5%	-.2%	1.3%	.8%	.5%	.4%	-.4%	1.2%	.8%	.5%	.5%
	4	.8%	1.7%	1.1%	.7%	.4%	.3%	1.3%	.8%	.5%	.3%	.1%	1.2%	.8%	.5%	.4%
	5	1.1%	1.7%	1.1%	.7%	.4%	.6%	1.3%	.8%	.5%	.3%	.4%	1.2%	.7%	.5%	.3%
	6	1.3%	1.7%	1.1%	.7%	.4%	.8%	1.3%	.8%	.5%	.3%	.6%	1.2%	.7%	.5%	.3%
300	2	-.8%	1.1%	.8%	.6%	**%	-1.2%	.9%	.6%	.6%	**%	-1.3%	.8%	.6%	.6%	**%
	3	-.0%	1.2%	.8%	.5%	.4%	-.4%	.9%	.5%	.4%	.3%	-.5%	.8%	.5%	.4%	.3%
	4	.4%	1.2%	.7%	.5%	.3%	-.0%	.9%	.5%	.3%	.2%	-.1%	.8%	.5%	.3%	.3%
	5	.6%	1.2%	.7%	.5%	.3%	.2%	.9%	.5%	.3%	.2%	.1%	.8%	.5%	.3%	.2%
	6	.7%	1.2%	.7%	.4%	.3%	.4%	.9%	.5%	.3%	.2%	.3%	.8%	.5%	.3%	.2%
500	2	-.9%	.7%	.5%	.4%	**%	-1.2%	.5%	.4%	.3%	**%	-1.2%	.5%	.4%	.5%	**%
	3	-.3%	.7%	.5%	.3%	.2%	-.6%	.5%	.3%	.2%	.2%	-.6%	.5%	.3%	.3%	.3%
	4	.0%	.7%	.5%	.3%	.2%	-.3%	.5%	.3%	.2%	.2%	-.3%	.5%	.3%	.2%	.2%
	5	.2%	.7%	.5%	.3%	.2%	-.1%	.5%	.3%	.2%	.1%	-.1%	.5%	.3%	.2%	.2%
	6	.3%	.7%	.5%	.3%	.2%	.1%	.5%	.3%	.2%	.1%	.0%	.5%	.3%	.2%	.1%

(continued)

Table 5. (continued) Percentage bias of $\hat{C}_s$ for Weibull distribution, W(1,β), with β=1.4, 1.6, 1.8, 2.0, and 2.2.

|(μ-T)/σ|		β=2.0					β=2.2				
n	d/σ	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
10	2	-.3%	8.8%	8.7%	10.4%	***%	-3.7%	6.4%	7.5%	9.9%	***%
	3	4.3%	9.0%	7.2%	5.9%	5.6%	.7%	6.5%	5.8%	5.1%	5.2%
	4	6.6%	9.1%	6.7%	4.9%	4.0%	2.9%	6.5%	5.3%	4.1%	3.5%
	5	8.0%	9.1%	6.5%	4.6%	3.5%	4.3%	6.5%	5.0%	3.7%	2.9%
	6	8.9%	9.1%	6.3%	4.4%	3.2%	5.2%	6.5%	4.8%	3.5%	2.7%
20	2	-.1%	7.1%	5.7%	5.9%	***%	-2.3%	5.5%	4.8%	5.6%	***%
	3	3.2%	7.0%	4.9%	3.6%	3.2%	.9%	5.4%	4.0%	3.2%	3.0%
	4	4.8%	7.0%	4.7%	3.2%	2.4%	2.4%	5.3%	3.7%	2.7%	2.1%
	5	5.7%	6.9%	4.5%	3.0%	2.1%	3.4%	5.3%	3.6%	2.4%	1.8%
	6	6.4%	6.9%	4.4%	2.9%	2.0%	4.0%	5.2%	3.5%	2.3%	1.7%
30	2	-.3%	5.7%	4.3%	4.2%	***%	-1.7%	4.7%	3.8%	4.1%	***%
	3	2.3%	5.6%	3.8%	2.7%	2.4%	.8%	4.6%	3.2%	2.4%	2.2%
	4	3.6%	5.6%	3.6%	2.4%	1.8%	2.1%	4.5%	3.0%	2.1%	1.6%
	5	4.4%	5.5%	3.5%	2.3%	1.6%	2.8%	4.5%	2.9%	1.9%	1.4%
	6	4.9%	5.5%	3.5%	2.2%	1.5%	3.3%	4.4%	2.9%	1.9%	1.3%
40	2	-.6%	4.7%	3.4%	3.3%	***%	-1.6%	4.1%	3.1%	3.3%	***%
	3	1.7%	4.6%	3.0%	2.2%	1.9%	.6%	3.9%	2.7%	2.0%	1.8%
	4	2.8%	4.6%	2.9%	1.9%	1.4%	1.7%	3.9%	2.6%	1.7%	1.3%
	5	3.4%	4.6%	2.9%	1.8%	1.3%	2.4%	3.9%	2.5%	1.6%	1.1%
	6	3.9%	4.5%	2.8%	1.8%	1.2%	2.8%	3.8%	2.4%	1.6%	1.1%
50	2	-.8%	4.0%	2.9%	2.7%	***%	-1.5%	3.5%	2.7%	2.7%	***%
	3	1.2%	3.9%	2.6%	1.8%	1.5%	.4%	3.4%	2.3%	1.7%	1.5%
	4	2.1%	3.8%	2.4%	1.6%	1.2%	1.4%	3.4%	2.2%	1.5%	1.1%
	5	2.7%	3.8%	2.4%	1.5%	1.1%	2.0%	3.3%	2.1%	1.4%	1.0%
	6	3.1%	3.8%	2.4%	1.5%	1.0%	2.4%	3.3%	2.1%	1.3%	.9%
100	2	-1.4%	2.1%	1.5%	1.4%	***%	-1.6%	2.1%	1.5%	1.6%	**%
	3	-.0%	2.1%	1.4%	1.0%	.8%	-.2%	2.0%	1.3%	.9%	.8%
	4	.6%	2.1%	1.3%	.9%	.6%	.5%	2.0%	1.3%	.8%	.6%
	5	1.1%	2.1%	1.3%	.8%	.6%	.9%	2.0%	1.2%	.8%	.6%
	6	1.3%	2.0%	1.3%	.8%	.5%	1.1%	1.9%	1.2%	.8%	.5%
200	2	-1.5%	1.1%	.8%	.8%	***%	-1.6%	1.1%	.8%	.9%	**%
	3	-.6%	1.1%	.7%	.5%	.4%	-.6%	1.1%	.7%	.5%	.5%
	4	-.1%	1.0%	.7%	.5%	.3%	-.1%	1.0%	.7%	.5%	.3%
	5	.2%	1.0%	.7%	.4%	.3%	.1%	1.0%	.7%	.4%	.3%
	6	.4%	1.0%	.7%	.4%	.3%	.3%	1.0%	.6%	.4%	.3%
300	2	-1.4%	.7%	.6%	.6%	***%	-1.5%	.7%	.6%	.6%	**%
	3	-.7%	.7%	.5%	.3%	.3%	-.7%	.7%	.5%	.4%	.3%
	4	-.3%	.7%	.4%	.3%	.2%	-.3%	.7%	.5%	.3%	.2%
	5	-.0%	.7%	.4%	.3%	.2%	-.1%	.7%	.5%	.3%	.2%
	6	.1%	.7%	.4%	.3%	.2%	.1%	.7%	.4%	.3%	.2%
500	2	-1.3%	.4%	.4%	.5%	***%	-1.3%	.5%	.4%	.4%	**%
	3	-.7%	.4%	.3%	.2%	.2%	-.7%	.5%	.3%	.3%	.2%
	4	-.4%	.4%	.3%	.2%	.2%	-.4%	.4%	.3%	.2%	.2%
	5	-.2%	.4%	.3%	.2%	.1%	-.2%	.4%	.3%	.2%	.2%
	6	-.1%	.4%	.3%	.2%	.1%	-.1%	.4%	.3%	.2%	.1%

Table 6(a). Percentages bias of $\widehat{\mu}_3/s$ for $\chi^2(r)$.

n	$\chi^2(3)$	$\chi^2(4)$	$\chi^2(5)$	$\chi^2(6)$	$\chi^2(7)$
10	- 24.9%	- 22.3%	- 20.1%	- 18.0%	- 16.0%
20	- 15.7%	- 14.2%	- 13.2%	- 12.3%	- 11.5%
30	- 11.5%	- 10.4%	- 9.6%	- 9.0%	- 8.5%
40	- 9.2%	- 8.2%	- 7.6%	- 7.1%	- 6.8%
50	- 7.7%	- 6.8%	- 6.3%	- 5.9%	- 5.6%
100	- 4.5%	- 3.9%	- 3.5%	- 3.2%	- 3.0%
200	- 2.8%	- 2.3%	- 2.0%	- 1.7%	- 1.6%
300	- 1.9%	- 1.5%	- 1.2%	- 1.0%	- 0.9%
500	- 1.7%	- 1.3%	- 1.0%	- 0.9%	- 0.7%

Table 6(b). Percentages bias of $\widehat{\mu}_3/s$ for $\log N(0, \sigma^2)$.

n	LN(0, 1)	LN(0, $\frac{1}{4}$)	LN(0, $\frac{1}{9}$)	LN(0, $\frac{1}{16}$)	LN(0, $\frac{1}{25}$)
10	- 65.8%	- 31.5%	- 18.1%	- 5.9%	8.1%
20	- 56.7%	- 22.2%	- 14.0%	- 7.5%	0.0%
30	- 51.3%	- 17.3%	- 11.1%	- 7.2%	- 2.7%
40	- 47.5%	- 14.4%	- 9.2%	- 6.6%	- 3.4%
50	- 44.7%	- 12.3%	- 7.7%	- 5.9%	- 3.8%
100	- 36.3%	- 7.6%	- 4.5%	- 3.4%	- 3.0%
200	- 29.2%	- 4.7%	- 2.5%	- 1.9%	- 1.9%
300	- 25.8%	- 3.6%	- 1.9%	- 1.4%	- 1.1%
500	- 22.3%	- 2.7%	- 1.3%	- 1.0%	- 0.7%

Table 6(c). Percentages bias of $\widehat{\mu}_3/s$ for $W(1, \beta)$.

n	W(1, $\frac{7}{5}$)	W(1, $\frac{8}{5}$)	W(1, $\frac{9}{5}$)	W(1, 2)	W(1, $\frac{11}{5}$)
10	- 19.0%	- 13.5%	- 7.1%	2.5%	15.9%
20	- 12.4%	- 9.9%	- 7.6%	- 3.4%	3.0%
30	- 8.9%	- 7.3%	- 6.4%	- 4.1%	- 0.4%
40	- 7.0%	- 5.7%	- 5.3%	- 3.9%	- 1.8%
50	- 5.7%	- 4.7%	- 4.5%	- 3.6%	- 2.3%
100	- 3.0%	- 2.3%	- 2.5%	- 2.0%	- 2.0%
200	- 1.6%	- 1.0%	- 1.3%	- 1.0%	- 1.1%
300	- 1.1%	- 0.6%	- 0.9%	- 0.6%	- 0.8%
500	- 0.7%	- 0.3%	- 0.6%	- 0.3%	- 0.4%

for all three skewed distributions, the bias of $\hat{\mu}_3/s$ is negative. The under-estimate of $\hat{\mu}_3/s$ for the term μ_3/σ results in a reduction for the value of the denominator of $\hat{C}_s$. Consequently, $\hat{C}_s$ over-estimates C_s, and the bias becomes positive.

4. CONCLUSIONS

Wright (1995) considered a new process capability index C_s, which takes into account the process variation, the location of the process mean relative to the specification limits, and the asymmetry of the distribution. Wright (1995) investigated an estimator of C_s and studied its bias and variance by simulation for normal distributions where skewness is not present. In this paper, we extend Wright's simulation study to cover some skewed distributions including chi-square, lognormal, and Weibull distributions.

The result show that for all three skewed distributions, the percentage bias, $\{E(\hat{C}_s) - C_s\}/C_s$, increases as the skewness coefficient, $|\mu_3/\sigma^3|$, increases. For the normal distribution, the bias is negative except for some cases with small n regardless of whether $\mu = T$ or $\mu \neq T$. For skewed distributions, the bias is positive for all n if the process is off-target ($\mu \neq T$). On the other hand, if the process is on-target ($\mu = T$), the bias tends to be positive for small n, and negative for large n. Although the index C_s is sensitive to skewed distributions, and has some interesting properties over C_{pmk}, but the estimator $\hat{C}_s$ proposed by Wright (1995) is highly unstable in the presence of skewness. In fact, we demonstrated that the percentage bias, $\{E(\hat{C}_s) - C_s\}/C_s$, increases as the skewness coefficient $|\mu_3/\sigma^3|$ increases for the three typical skewed distributions we investigated. Evidently, for the index C_s to be acceptable by the practitioners a more stable estimator is needed.

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