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ON USING NEWTON'S METHOD FOR COMPUTING THE NONCENTRALITY PARAMETER OF THE NONCENTRAL F DISTRIBUTION

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Key Words and Phrases: Newton's method; noncentral beta distribution; noncentral F distribution

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ABSTRACT

This article deals with how to enhance computational efficiency of Newton's method for computing the noncentrality parameter of the noncentral F distribution. Some relationship between recurrence formulas for computing the noncentral F distribution function and its derivative with respect to the noncentrality parameter is investigated. Use of the relationship discovered can combine the evaluations of the noncentral F distribution function and its derivative, and therefore can reduce the amount of computations during Newton's process. An efficient algorithm featuring this finding is provided in a step-by-step form.

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INTRODUCTION

Let $F(x; r_1, r_2, \lambda)$ denote the noncentral F distribution function with r_1, r_2 degrees of freedom and noncentrality parameter λ. Computing the noncentrality parameter $\hat{\lambda}$ such that $F(x; r_1, r_2, \hat{\lambda}) = p$ for given x (abscissa), r_1, r_2 (degrees of freedom), and cumulative probability p ($0 < p < F(x; r_1, r_2, 0)$) always relies on numerical root-finding methods (see, e.g., Narula and Weistroffer 1986; Guirguis 1990), in which an algorithm for evaluating $F(x; r_1, r_2, \lambda)$ is required.

It is well known that the noncentral F distribution function can be evaluated through the noncentral beta distribution function, which can be expressed as a series of central beta distribution functions with Poisson weights. More explicitly (see, e.g., Johnson and Kotz 1970b, p.192),

$$F(x; r_1, r_2, \lambda) = B(y; \frac{r_1}{2}, \frac{r_2}{2}, \lambda) = \sum_{i=0}^{\infty} u_i(\lambda)\, I_y(\frac{r_1}{2} + i, \frac{r_2}{2}), \qquad (1)$$

where B and I denote, respectively, the noncentral and central beta distribution function, $y = r_1 x / (r_1 x + r_2)$, and $u_i(\lambda) = (\lambda/2)^i \exp(-\lambda/2)/(i!)$. Norton (1983) gave a simple algorithm for evaluating (1). A bound on the error of truncating the series was proposed to control the computational accuracy. The corresponding FORTRAN program was provided by Narula and Weistroffer (1986). Lenth (1987) developed a recursive algorithm, where only one evaluation of the central beta distribution function is required instead of several, and gave a sharper truncation error bound. Posten (1993) extracted Lenth's algorithm and provided a step-by-step listing for easy programming in any desired computer language. Singh and Relyea (1992) proposed methods of computing (1) for integer degrees of freedom, which are based on exact expressions for $I_y(r_1/2, r_2/2)$ and Lenth's error bound. Ding (1994b) developed an algorithm for computing the noncentral beta distribution function based on an alternative series representation. No central beta routine is

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required. The accuracy can be effectively controlled by using the corresponding truncation error bound.

A simple approach for computing $\hat{\lambda}$ is to use a numerical root-finding method to solve the equation $F(x; r_1, r_2, \lambda) - p = 0$, where F may be evaluated by any of the algorithms mentioned above. Narula and Weistroffer (1986) used the quadratic interpolation procedure to computed $\hat{\lambda}$, and claimed that it is faster than the bisection method. Guirguis (1990) introduced a modification to Newton's method, which is robust to poor starting values. He also compared the performance of Newton's method (called L-Newton), modified Newton's method (called E-Newton), and quadratic interpolation for computing $\hat{\lambda}$, and concluded that L-Newton and E-Newton are both faster than quadratic interpolation, and E-Newton outperforms L-Newton when a good starting value is absent. Use of Newton's method, no matter with E-Newton or L-Newton, requires that we evaluate both $F(x; r_1, r_2, \lambda)$ and its derivative $F'(\lambda)$ with respect to λ. The purpose of this article is to investigate some relationship between recurrence formulas for evaluating F and F' so that their evaluations can be combined at each Newton's iteration. The amount of computations during Newton's process can then be greatly reduced. The idea follows that given in Ding (1994a) for computing the noncentrality parameter of the noncentral chi-squared distribution, but the analytical process is different. An algorithm for computing $\hat{\lambda}$ based on efficient E-Newton is provided in a step-by-step form.

RELATIONSHIP BETWEEN RECURRENCE FORMULAS

Computing $\hat{\lambda}$ by using Newton's method requires the evaluations of $F(x; r_1, r_2, \lambda)$ and $F'(\lambda)$. It is easy to show that (see also Guirguis 1990)

$$F'(\lambda) = \frac{1}{2} \sum_{i=0}^{\infty} u_i(\lambda) \left[I_y\left(\frac{r_1}{2} + 1 + i, \frac{r_2}{2}\right) - I_y\left(\frac{r_1}{2} + i, \frac{r_2}{2}\right) \right].$$

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Using the equation (see, e.g., Abromowitz and Stegun 1965, p. 944)

$$I_y(a+1, b) = I_y(a, b) - \frac{\Gamma(a+b)}{\Gamma(a+1)\,\Gamma(b)}\, y^a (1-y)^b,$$

we have

$$F'(\lambda) = -\frac{1}{2}\sum_{i=0}^{\infty} u_i(\lambda)\frac{\Gamma((r_1+r_2)/2+i)}{\Gamma(r_1/2+i+1)\,\Gamma(r_2/2)}\, y^{r_1/2+i}(1-y)^{r_2/2}$$

$$= -\frac{1}{2}\sum_{i=0}^{\infty} u_i\, t_i, \tag{2}$$

where $u_i = u_i(\lambda)$, and $t_i = \frac{\Gamma((r_1+r_2)/2+i)}{\Gamma(r_1/2+i+1)\,\Gamma(r_2/2)}\, y^{r_1/2+i}(1-y)^{r_2/2}$. The terms of the series in (2) for F' are closely related to the corresponding ones of the series in (1) for F, which can be evaluated recursively as follows (see Posten 1993):

$$F(x; r_1, r_2, \lambda) = \sum_{i=0}^{\infty} u_i\, I_i, \tag{3}$$

where $I_i = I_y(\frac{r_1}{2}+i, \frac{r_2}{2})$, and

$$u_0 = \exp(-\lambda/2),$$

$$u_i = u_{i-1}\lambda/(2i),\ i \geq 1,$$

$$I_0 = I_y(\frac{r_1}{2}, \frac{r_2}{2}), \tag{4}$$

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$$I_i = I_{i-1} - t_{i-1}, \quad i \geq 1,$$

$$t_0 = \frac{\Gamma((r_1 + r_2)/2)}{\Gamma(r_1/2 + 1)\,\Gamma(r_2/2)}\, y^{r_1/2}(1 - y)^{r_2/2},$$

$$t_i = t_{i-1}\, y\, ((r_1 + r_2)/2 + i - 1) / (r_1/2 + i), \quad i \geq 1.$$

The terms of the series in (2) can be evaluated by taking advantage of the recursion given in (4).

For large values of noncentrality parameter, the lower terms of the series in (2) and (3) make little contribution to the values of the series, and underflow error propagation may occur in recursive computations. Thus, both of the summations should be started at a higher index. A bound for the lower truncation error $\sum_{i=0}^{m-1} u_i I_i$ (error in omitting the first m terms) of the series in (3) was given by Frick (1990) as follows:

$$\sum_{i=0}^{m-1} u_i I_i < \sum_{i=0}^{m-1} u_i I_0 < \sum_{i=0}^{m-1} u_i \leq \Phi((m - \lambda/2)\sqrt{2}/\sqrt{\lambda}), \tag{5}$$

where Φ denotes the standard normal distribution function. For a specified accuracy α, the starting index m is determined by $m = [\max(\lambda/2 + U_\alpha\sqrt{\lambda/2}\ , 0)]$, where U_α is the (100α)th percentage point of the standard normal distribution and [.] denotes 'largest integer less than or equal to '. Setting $U_\alpha = -5$ would be sufficient since $\Phi(-5) < 3 \times 10^{-7}$. The starting index m determined above also applies for computing the series in (2) because $\sum_{i=0}^{m-1} u_i t_i$ is dominated by $\sum_{i=0}^{m-1} u_i I_i$. This can be seen by noting that $I_i = \sum_{k=i}^{\infty} t_k$ (see equation (4) in Ding 1994b), and therefore $t_i < I_i$ for $i \geq 0$. The upper truncation errors, $\sum_{i=n}^{\infty} u_i I_i$ and $\sum_{i=n}^{\infty} u_i t_i$ (

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errors in truncating the summations at $i = n$ $(n > m)$), are both bounded above by $1 - \sum_{i=m}^{n-1} u_i$ since

$$\sum_{i=n}^{\infty} u_i t_i < \sum_{i=n}^{\infty} u_i I_i < \sum_{i=n}^{\infty} u_i = 1 - \sum_{i=0}^{n-1} u_i \leq 1 - \sum_{i=m}^{n-1} u_i. \qquad (6)$$

To approximate the series in (2) and (3), sum up the terms from $i = m$ until the error bound given in (6) is less than a predetermined accuracy ε (e.g., 10^{-6}).

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COMPUTING THE NONCENTRALITY PARAMETER

Newton's method is an efficient approach for computing $\hat{\lambda}$. The process is to repeat the iteration

$$\lambda_{j+1} = \lambda_j - \frac{F(x; r_1, r_2, \lambda_j) - p}{F'(\lambda_j)}, \quad j = 0, 1, 2, \ldots, \qquad (7)$$

until $| \lambda_{j+1} - \lambda_j | \leq \delta \lambda_{j+1}$, where δ denotes a convergence criterion (e.g., 10^{-4}). However, when we lack a good starting value, convergence may not be achieved. Guirguis (1990) proposed a modification, called E-Newton, to deal with this problem. It is given by

$$\lambda_{j+1} = \lambda_j - \frac{F(x; r_1, r_2, \lambda_j)}{F'(\lambda_j)} \log\left(\frac{p}{F(x; r_1, r_2, \lambda_j)}\right), \quad j = 0, 1, 2, \ldots$$

No matter using E-Newton or the regular Newton in (7) (called L-Newton by Guirguis), both F and F' need to be evaluated at each iteration. Since the factors u_i and t_i, computed through (4), are needed for both F and F', and the bounds in (5) and (6) for the lower and upper truncation errors are commonly used for

computing F and F', the evaluations of F and F' should be combined, rather than separated, to avoid redundant computations. In addition, the factors t_i and I_i do not depend upon λ, and hence their values can be repeatedly used for computing new iterates. Before Newton's iteration is performed, t_i and I_i are computed recursively and saved until $I_i \leq \varepsilon$. The efficiency of Newton's method can be greatly enhanced by the treatments mentioned above. Note that the evaluations of F and F' should be precise enough to ensure the accuracy of Newton's solution.

To carry out Newton's iteration, we need a starting value λ_0. Cox and Reid (1987) proposed a simple formula for approximating the percentage point of the noncentral F distribution, which can be used to obtain λ_0. The formula implies that

$$\lambda_0 = r_1(x / F_{r_1, r_2}(p) - 1),$$

where $F_{r_1, r_2}(p)$ is the $(100p)$th percentage point of the F distribution with r_1 and r_2 degrees of freedom. Since λ_0 is just a guess, a rough estimate of $F_{r_1, r_2}(p)$ would be sufficient, and could be obtained by $\chi^2_{r_1}(p)/r_1$, where $\chi^2_{r_1}(p)$ denote the $(100p)$th percentage point of the χ^2 distribution with r_1 degrees of freedom. An approximation for $\chi^2_{r_1}(p)$ is given by (see, e.g., Johnson and Kotz 1970a, p.176)

$$\chi^2_{r_1}(p) \approx (U_p + (2r_1 - 1)^{1/2})^2 / 2,$$

where U_p could be approximated by Hastings' formula (see, e.g., Abramowitz and Stegun 1965, p.933). In case the starting value λ_0 is nonpositive, set $\lambda_0 = \delta$. If the new iterate λ_{j+1} is nonpositive, set $\lambda_{j+1} = \lambda_j/2$.

ALGORITHM

In this section, we provide an algorithm, based on E-Newton with the

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properties mentioned above, for computing the noncentrality parameter of the noncentral F distribution, assuming that accurate routines for computing (the logarithm of) the gamma function $\Gamma(.)$ (see, e.g., Macleod 1989; Pike and Hill 1966) and the central beta distribution function (see, e.g., Majunder and Bhattacharjee 1985) are available. The algorithm, named NCPF, is presented in a step-by-step form. It can be easily converted to a computer program in any desired language. The algorithm involves two user-specified stopping parameters, ε and δ. The input of the algorithm includes r_1, r_2 (degrees of freedom), x (the abscissa), and p (the cumulative probability). Note that there will be no solution if $p > F(x; r_1, r_2, 0)$.

Algorithm NCPF:

Step 1. Set $i \leftarrow 1$, $y \leftarrow r_1 x / (r_1 x + r_2)$, $t_0 \leftarrow \dfrac{\Gamma((r_1 + r_2)/2)}{\Gamma(r_1/2 + 1)\,\Gamma(r_2/2)}\, y^{r_1/2}(1 - y)^{r_2/2}$, and $I_0 \leftarrow I_y(\dfrac{r_1}{2}, \dfrac{r_2}{2})$.

Step 2. Do *Steps 2.1-2.2* until $I_i \le \varepsilon$.

2.1 Compute $I_i \leftarrow I_{i-1} - t_{i-1}$, and $t_i \leftarrow t_{i-1}\, y\, ((r_1 + r_2)/2 + i - 1) / (r_1/2 + i)$.

2.2 Set $i \leftarrow i + 1$.

Step 3. Set $N \leftarrow i - 1$. Then obtain an initial guess λ_0 and set $\lambda \leftarrow \lambda_0$.

Step 4. Do *Steps 4.1-4.7* until $|DIFF| \le \delta\lambda$.

4.1 Set $m \leftarrow [\max(\lambda/2 - 5\sqrt{\lambda/2},\ 0)]$.

4.2 Set $u \leftarrow \exp(m \ln(\lambda/2) - \lambda/2 - \ln \Gamma(m+1))$, $v \leftarrow u$, $CDF \leftarrow u\, I_m$, and $DER \leftarrow u\, t_m$.

4.3 For $i = m+1, m+2, \ldots, N$ do *Steps 4.3.1-4.3.2*.

4.3.1 Set $u \leftarrow u\lambda / (2i)$, $v \leftarrow v + u$, $CDF \leftarrow CDF + u\, t_i$, and $DER \leftarrow DER + u\, I_i$.

4.3.2 If $1 - v \le \varepsilon$, go to *Step 4.6*.

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4.4 Set $i \leftarrow N + 1$, $t^* \leftarrow t_N$, and $I^* \leftarrow I_N$.

4.5 Do Steps *4.5.1*-*4.5.2* until $1 - v \leq \varepsilon$.

4.5.1 Compute $I^* \leftarrow I^* \cdot t^*$, $t^* \leftarrow t^* y\,((r_1 + r_2)/2 + i - 1)/(r_1/2 + i)$, $u \leftarrow u\lambda/(2i)$, $v \leftarrow v + u$, $CDF \leftarrow CDF + u\,t^*$, and $DER \leftarrow DER + u\,I^*$.

4.5.2 Set $i \leftarrow i + 1$.

4.6 Set $DIFF \leftarrow 2\,CDF \ln(p/CDF)/DER$.

4.7 If $\lambda + DIFF \leq 0$, set $\lambda \leftarrow \lambda/2$; otherwise set $\lambda \leftarrow \lambda + DIFF$.

Step 5. Stop with $\hat{\lambda} \leftarrow \lambda$.

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