Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3706–3710
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Maximal sets of Hamilton cycles in D_n

Liqun Pu[a], Hung-Lin Fu[b], Hao Shen[c]

[a] *Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsin Chu 30050, Taiwan*
[c] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China*

Received 4 December 2006; received in revised form 21 June 2007; accepted 5 July 2007
Available online 17 August 2007

Abstract

In this paper, we prove that there exists a maximal set of m directed Hamilton cycles in D_n if and only if $\lceil n/2\rceil \leqslant m \leqslant n-1$ for $n \geqslant 7$.

Keywords: Directed Hamilton cycles; Complete directed graph; Maximal sets

1. Introduction

A Hamilton cycle in a graph T is a spanning cycle of T. If S is a set of edge-disjoint Hamilton cycles in T and if $E(S)$ is the set of edges occurring in the Hamilton cycles in S, then S is said to be maximal if $T - E(S)$ has no Hamilton cycle.

In paper [5], Hoffman et al. showed that there exists a maximal set S of m edge-disjoint Hamilton cycles in the complete graph K_n if and only if $\lfloor (n+3)/4\rfloor \leqslant m \leqslant \lfloor (n-1)/2\rfloor$. In paper [1], Bryant et al. showed that there exists a maximal set S of m edge-disjoint Hamilton cycles in the complete bipartite graph $K_{n,n}$ if and only if $n/4 < m \leqslant n/2$. Later, Daven et al. [2] showed for $n \geqslant 3$ and $p \geqslant 3$, there exists a maximal set S of m Hamilton cycles in the complete multipartite graph K_n^p (p parts of size n) if and only if $\lceil n(p-1)/4\rceil \leqslant m \leqslant \lfloor n(p-1)/2\rfloor$, and $m > n(p-1)/4$ if n is odd and $p \equiv 1 \pmod 4$, except possibly if n is odd and $m \leqslant ((n+1)(p-1)-2)/4$. Recently, Fu et al. [3] extended the results in [2] and proved that if $\lceil (p-1)/2\rceil \leqslant m \leqslant p-1$, then there exists a maximal set S of m Hamilton cycles in $K_{2p} - F$, where F is a 1-factor of K_{2p}.

In this paper, we will extend these results to directed graphs and study maximal sets of Hamilton cycles in the complete directed graph D_n.

Throughout this paper, we will use the following notation and terminology. A digraph D is said to be r-regular if $\deg^+(v) = \deg^-(v) = r$ for each vertex v in D. Let D_n be a complete directed graph of order n (without loops); it is clear that D_n is $(n-1)$-regular. For disjoint sets A and B, define the directed complete bipartite graph $D_{A,B}$ as the

E-mail addresses: liqunpu@yahoo.com.cn, liqunpu@sina.com (L. Pu), hlfu@math.nctu.edu.tw (H. Fu), haoshen@sjtu.edu.cn (H. Shen).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.07.041

graph with vertices $V(D_{A,B}) = A \cup B$ and $E(D_{A,B}) = (A \times B) \cup (B \times A)$. Note that $|E(D_{A,B})| = 2mn$. If $|A| = m$ and $|B| = n$, then we often denote $D_{A,B}$ as $D_{m,n}$. Obviously, $D_{m+n} = D_m + D_{m,n} + D_n$.

Let $V(D_{m,n})$ be the vertex set of $D_{m,n}$ and $|V(D_{m,n})|$ be the number of vertices in $V(D_{m,n})$. Let $E(D_{m,n})$ be the edge set of $D_{m,n}$ and $|E(D_{m,n})|$ be the number of edges in $E(D_{m,n})$.

Let A_i be an ordered i-set defined on $\mathbb{Z}_t$ which can be expressed as slanted parentheses i.e., $A_i = (a_1, a_2, a_3, \ldots, a_{i-1}, a_i)$. Let A_i^* denote a path obtained from A_i, $A_i^* = \langle a_1, a_2, a_3, \ldots, a_{i-1}, a_i \rangle$. Furthermore, for $j \in \mathbb{Z}_t$, let $A_i + j = (a_1 + j, a_2 + j, a_3 + j, \ldots, a_{i-1} + j, a_i + j)$ and $A_i^* + j = \langle a_1 + j, a_2 + j, a_3 + j, \ldots, a_{i-1} + j, a_i + j \rangle$. Note that all the entries in $A_i + j$ and $A_i^* + j$ are done modulo t.

2. Maximal sets of Hamilton cycles in D_n

Before proving the main result, we need the following lemma.

Lemma 2.1. *For an integer n, $n \geqslant 7$, D_n has a Hamilton decomposition.*

Proof. D_n can be regarded as two K_n by orienting properly. When n is odd, K_n has a Hamilton decomposition [7]. When n is even, Tillson [8] gives the proof. □

The following lemma plays an important part to prove our main theorem.

Lemma 2.2. *Let D be an r-regular directed graph on n vertices. If $r \geqslant \lceil n/2 \rceil$, then D contains a directed Hamilton cycle.*

In fact, this result is a special case of the following theorem which is obtained by Ghouila-Houri [4].

Theorem 2.1 (*Ghouila-Houri [4]*). *Let $\delta^+(D)$ and $\delta^-(D)$ denote the minimum outdegree and minimum indegree in D, respectively. If D is a strict digraph (without cycles of length 2), and $\min\{\delta^+(D), \delta^-(D)\} \geqslant |V(D)|/2$, then D is Hamiltonian.*

On the other hand, if D is regular strict digraph, then the condition for D to be Hamiltonian can be weaken slightly. The following result was obtained by Jackson.

Lemma 2.3 (*Jackson [6]*). *Every strict digraph of minimum in-degree and out-degree $k \geqslant 2$, on at most $2k+2$ vertices, is Hamiltonian.*

Now, we are ready to prove our main result. In order to construct maximal sets of Hamilton cycles. By Lemma 2.3, we conclude that the minimum number m in a maximal set of m directed Hamilton cycles in D_n is $\lceil n/2 \rceil$. Therefore, it suffices to construct a maximal set of m directed Hamilton cycles for each $\lceil n/2 \rceil \leqslant m \leqslant n-1$ for $n \geqslant 7$. In order to do that, the following lemma is essential. Mainly, we make an arrangement of directed differences in order that we can construct maximal sets of Hamilton cycles systematically.

Lemma 2.4. *For positive integers l and m where $m \geqslant 5$ and $1 \leqslant l \leqslant m-1$, there exists an ordered set $A_{m-l} = (a_1, a_2, \ldots, a_{m-l}) \subseteq Z_m$ such that $S = \{(a_{t+1} - a_t) \pmod m \mid t = 1, 2, \ldots, m-l-1\}$ is an $(m-l-1)$-subset of $\{1, 2, \ldots, m-1\}$ (all directed differences are distinct).*

Proof. We divide the proof into two cases.

Case 1: When m is even.

When $l = 1$, we delete the last element from the following ordered set $(0, 1, m-1, 2, m-2, 3, m-3, \ldots, m/2-2, m/2+2, m/2-1, m/2+1, m/2)$ to get A_{m-1}, i.e.,

$$A_{m-1} = \left(0, 1, m-1, 2, m-2, 3, m-3, \ldots, \frac{m}{2}-2, \frac{m}{2}+2, \frac{m}{2}-1, \frac{m}{2}+1\right).$$

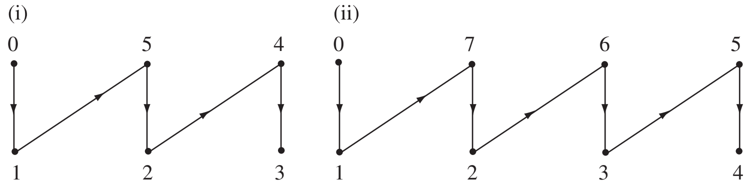

Fig. 1.

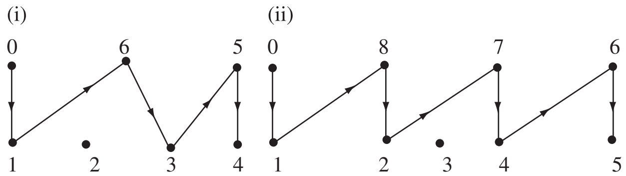

Fig. 2.

When $l = 2$, we delete the last element in A_{m-1} and get

$$A_{m-2} = \left(0, 1, m-1, 2, m-2, 3, m-3, \ldots, \frac{m}{2}-2, \frac{m}{2}+2, \frac{m}{2}-1\right).$$

We give examples in Fig. 1(i) ($m=6, l=1$) and Fig. 1(ii) ($m=8, l=1$).

Note that in Fig. 1(i), $A_5 = (0, 1, 5, 2, 4)$, in which 3 is deleted and in Fig. 1(ii), $A_7 = (0, 1, 7, 2, 6, 3, 5)$, in which 4 is deleted.

We generalize the above method and get

$$A_{m-l} = \begin{cases} \left(0, 1, m-1, 2, m-2, 3, m-3, 4, \ldots, \frac{m}{2} - \left(\frac{l}{2}+2\right), \frac{m}{2} \right. \\ \left. + \left(\frac{l}{2}+2\right), \frac{m}{2} - \left(\frac{l}{2}+1\right), \frac{m}{2} + \left(\frac{l}{2}+1\right), \frac{m}{2} - \frac{l}{2}\right) & \text{if } l \equiv 0 \pmod 2; \text{ and} \\ \left(0, 1, m-1, 2, m-2, 3, m-3, 4, \ldots, \frac{m}{2} - \left(\frac{l+1}{2}+2\right), \frac{m}{2} \right. \\ + \left(\frac{l+1}{2}+2\right), \frac{m}{2} - \left(\frac{l+1}{2}+1\right), \frac{m}{2} + \left(\frac{l+1}{2}+1\right), \frac{m}{2} \\ \left. -\frac{l+1}{2}, \frac{m}{2} + \frac{l+1}{2}\right) & \text{if } l \equiv 1 \pmod 2. \end{cases}$$

Case 2: When m is odd.

When $l = 1$, we can get A_{m-1} from $Z_m \backslash \{\lceil (m+1)/4 \rceil\}$.

$A_{m-1} = (0, 1, m-1, 2, m-2, 3, \ldots, \lceil (m+1)/4 \rceil - 1, m - \lceil (m+1)/4 \rceil + 1, \lceil (m+1)/4 \rceil + 1, m - \lceil (m+1)/4 \rceil, \lceil (m+1)/4 \rceil + 2, m - \lceil (m+1)/4 \rceil - 1, \ldots, (m+1)/2 + 2, (m+1)/2 - 1, (m+1)/2 + 1, (m+1)/2)$.

Similarly, we can get A_{m-2} by deleting the last element in A_{m-1}.

We give examples in Fig. 2(i) ($m=7, l=1$) and Fig. 2(ii) ($m=9, l=1$).

Note that in Fig. 2(i), $A_6 = (0, 1, 6, 3, 5, 4)$, in which 2 is deleted and in Fig. 2(ii), $A_8 = (0, 1, 8, 2, 7, 4, 6, 5)$, in which 3 is deleted.

We generalize the above method and get

Thus, we have

$$A_{m-l}=\begin{cases}\Big(0,1,m-1,2,m-2,3,m-3,4,\ldots,\left\lceil\frac{m+1}{4}\right\rceil-1,m\\ +1-\left\lceil\frac{m+1}{4}\right\rceil,\left\lceil\frac{m+1}{4}\right\rceil+1,m-\left\lceil\frac{m+1}{4}\right\rceil,\left\lceil\frac{m+1}{4}\right\rceil\\ +2,m-\left\lceil\frac{m+1}{4}\right\rceil-1,\ldots,\frac{m+1}{2}-\left(\left\lceil\frac{l}{2}\right\rceil+1\right),\frac{m+1}{2}\\ +\left(\left\lceil\frac{l}{2}\right\rceil+1\right),\frac{m+1}{2}-\left\lceil\frac{l}{2}\right\rceil,\frac{m+1}{2}+\left\lceil\frac{l}{2}\right\rceil\Big) & \text{if } l\equiv 0\ (\mathrm{mod}\ 2);\ \text{and}\\ \Big(0,1,m-1,2,m-2,3,m-3,4,\ldots,\left\lceil\frac{m+1}{4}\right\rceil-1,m+1\\ -\left\lceil\frac{m+1}{4}\right\rceil,\left\lceil\frac{m+1}{4}\right\rceil+1,m-\left\lceil\frac{m+1}{4}\right\rceil,\left\lceil\frac{m+1}{4}\right\rceil\\ +2,m-\left\lceil\frac{m+1}{4}\right\rceil-1,\ldots,\frac{m+1}{2}+\left\lceil\frac{l+1}{2}\right\rceil\\ +1,\frac{m+1}{2}-\left(\left\lceil\frac{l-1}{2}\right\rceil+1\right),\frac{m+1}{2}\\ +\left\lceil\frac{l+1}{2}\right\rceil,\frac{m+1}{2}-\left\lceil\frac{l-1}{2}\right\rceil\Big) & \text{if } l\equiv 1\ (\mathrm{mod}\ 2).\end{cases}$$

□

With the above preparations, we are now in a position to prove the main theorem of this paper.

Theorem 2.2. *For two positive integers $m, n, n \geqslant 7$, there exist maximal sets of m Hamilton cycles in D_n if and only if $\lceil n/2\rceil \leqslant m \leqslant n-1$.*

Proof. Suppose that $m < \lceil n/2\rceil$, and let S be a set of m edge-disjoint directed Hamilton cycles in D_n. Let $G(S)$ be the graph consisting of the union of the m Hamilton cycles in S. Then $G(S)^c$ is an r-regular directed graph with $r = n-m-1 \geqslant \lceil n/2\rceil$ except the case n is odd, $m=(n-1)/2$. So by Lemma 2.2, $G(S)^c$ contains a Hamilton cycle and S is not a maximal set. When n is odd, $m=(n-1)/2$, $G(S)^c$ is an $(n-1)/2$-regular directed graph of order n, by Lemma 2.3, $G(S)^c$ has at least one Hamilton cycle. Thus $m \geqslant \lceil n/2\rceil$.

Now we shall show when $\lceil n/2\rceil \leqslant m \leqslant n-2$, we can construct maximal sets of m Hamilton cycles in D_n.

When integer $l>0$, let $l=n-m-1$. Otherwise let integer $l=0$. By Lemma 2.4, we can choose ordered set A_{m-l} and b_i $(1\leqslant i\leqslant l)$ from Z_m and thus Z_m can be partitioned into A_{m-l} and everything else not in A_{m-l}. That is, $Z_m = A_{m-l} \cup \{b_1, b_2, \ldots, b_l\}$ where $A_{m-l} = (a_1, a_2, \ldots, a_{m-l})$ and $A_{m-l} \cap \{b_1, b_2, \ldots, b_l\} = \emptyset$.

Since $D_n = D_{n-m} + D_{m,n-m} + D_m$, let $V(D_{m,n-m}) = \{i_0 | i \in Z_{n-m}\} \cup \{i | i \in Z_m\}$. Further, $\{i | i \in Z_m\}$ can be partitioned as $A_{m-l} \cup \{b_1, b_2, \ldots, b_l\}$. Now, we obtain a set S of directed Hamilton cycles with $S = \{(0_0, (b_1 + j), 1_0, (b_2+j), \ldots, (n-m-2)_0, (b_{n-m-1}+j), (n-m-1)_0, (A^*_{m-l}+j)) | j \in Z_m\}$ from $D_m + D_{m,n-m}$. Note that the entries of $\{b_t + j | 1 \leqslant t \leqslant n-m-1\}$ and $A^*_{m-l}+j$ are done modulo m. Obviously, $E(D_{m,n-m}) \subseteq E(S)$ and $D_n - E(S)$ is disconnected. Thus S is a maximal set of Hamilton cycles in D_n when $\lceil n/2\rceil \leqslant m \leqslant n-2$.

As for $m=n-1$, D_n can be decomposed into directed Hamilton cycles by Lemma 2.1. Thus, we conclude the proof. □

Example 1. When $n=8$, $4\leqslant m\leqslant 7$, we give a construction of maximal sets of Hamilton cycles for $m \in \{4,5,6\}$ and the case $m=7$ can be settled by applying Lemma 2.1:

$m=4$, the set is $\{(0_0, j, 1_0, 1+j, 2_0, 2+j, 3_0, 3+j) | j \in Z_4\}$.

$m=5$, the set is $\{(0_0, j, j+1, j+4, 1_0, 2+j, 2_0, 3+j) | j \in Z_5\}$.

$m=6$, the set is $\{(0_0, j, j+1, j+5, j+2, j+4, 1_0, 3+j) | j \in Z_6\}$.

Example 2. When $n = 9$, $5 \leqslant m \leqslant 8$, we give a construction of maximal sets of Hamilton cycles for $m \in \{5, 6, 7\}$ and the case $m = 8$ can be settled by applying Lemma 2.1:

$$m = 5, \text{ the set is } \{(0_0, j, j+1, 1_0, j+4, 2_0, 3+j, 3_0, 2+j) | j \in Z_5\}.$$

$$m = 6, \text{ the set is } \{(0_0, j, j+1, j+5, j+2, 1_0, 3+j, 2_0, 4+j) | j \in Z_6\}.$$

$$m = 7, \text{ the set is } \{(0_0, j, j+1, j+6, j+3, j+5, j+4, 1_0, 2+j) | j \in Z_7\}.$$

Acknowledgements

The authors wish to thank the referees for their valuable comments and suggestions, and in particular, one of the referees for his/her effort to provide assistance in rewriting the paper.

References

[1] D.E. Bryant, S. El-Zanati, C.A. Rodger, Maximal sets of Hamilton cycles in $K_{n,n}$, J. Graph Theory 33 (2000) 25–31.
[2] M. Daven, J.A. MacDougall, C.A. Rodger, Maximal sets of Hamilton cycles in complete multipartite graphs, J. Graph Theory 43 (1) (2003) 49–66.
[3] H.L. Fu, S.L. Logan, C.A. Rodger, Maximal sets of Hamilton cycles in $K_{2p} - F$, Discrete Math., to appear.
[4] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit Hamiltonien, C. R. Acad. Sci. Paris 251 (1960) 495–497.
[5] D.G. Hoffman, C.A. Rodger, A. Rosa, Maximal sets of 2-factors and Hamilton cycles, J. Combin. Theory (B) 57 (1993) 69–76.
[6] B. Jackson, Long paths and cycles in oriented graphs, J. Graph Theory 5 (1981) 145–157.
[7] E. Lucas, Recreations Mathématiques, vol. 2, Gauthiers-Villars, Paris, 1892.
[8] T.W. Tillson, A Hamiltonian decomposition of K^*_{2m}, $2m \geqslant 8$, J. Combin. Theory (B) 29 (1980) 68–74.