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Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 1997
211: 239
DOI: 10.1243/0954406971521818

Dynamic analysis of a two-gyro Anschütz compass

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Abstract: In this paper, the dynamics of the two-gyro Anschütz compass in two cases, i.e. the earth spin velocity and zero vehicle velocity case, as well as the earth spin velocity and non-zero vehicle case, are studied. The detailed exact equations of motion of this compass are obtained by Lagrange's equations. The system is studied by the linear approximation method, and these equations are solved as eigenvalue problems. The stabilities of these motions are also discussed. The analytical stabilities for two cases from the linear approximation method are checked by the numerical solutions.

Keywords: gyrocompass, Anschütz compass, stability

NOTATION

C_1, C_2	angular momentum of rotor 1 and rotor 2
J_x, J_y, J_z	moments of inertia of the gyrosphere with respect to the x, y, z axes respectively (assume $J_x, J_y, J_z = J$)
J_{x1}, J_{y1}, J_{z1}	moments of inertia of rotor 1 with respect to the x_1, y_1, z_1 axes respectively
J_{x2}, J_{y2}, J_{z2}	moments of inertia of rotor 2 with respect to the x_2, y_2, z_2 axes respectively
k	stiffness of spring
K	heading angle of vehicle
l	distance between the origin O and the centre of gravity of the whole gyrocompass
l_0	unstretched length of spring
l_1	length of crank
l_2	length of spring when $\varphi = 0$
P	weight of the gyrocompass
R	radius of the earth
$\Re$	Rayleigh's dissipation function
v	velocity of vehicle
$\varepsilon_0(-\varepsilon_0)$	initial angles between the x axis and $x_2(x_1)$ axis
ξ	latitude angle at the position of the gyrocompass
φ_1, φ_2	spin angles of rotor 1 and rotor 2
ω_1, ω_2	angular velocity components of the earth spin with respect to the OX_0, OZ_0 axes, where $\omega_1 = \omega_e \cos \xi$, $\omega_2 = \omega_e \sin \xi$
ω_e	earth spin velocity
$\omega_x, \omega_y, \omega_z$	angular velocity components of the gyrosphere with respect to the x, y, z axes respectively
$\omega_{x1}, \omega_{y1}, \omega_{z1}$	angular velocity components of rotor 1 with respect to the x_1, y_1, z_1 axes respectively
$\omega_{x2}, \omega_{y2}, \omega_{z2}$	angular velocity components of rotor 2 with respect to the x_2, y_2, z_2 axes respectively

The MS was received on 31 January 1995 and was accepted for publication on 12 September 1996.

1 INTRODUCTION

The gyrocompass is used for the guidance of vehicles. Two kinds of gyrocompass are used, the single-rotor gyrocompass and the multirotor gyrocompass. In general, the multirotor gyrocompass is used to decrease the error of the single-rotor gyrocompass when the vehicle is wobbling.

In this paper, one kind of multirotor gyrocompass will be discussed, the two-gyro Anschütz compass. The main parts of this gyrocompass are two rotors and a gyrosphere. The rotor axles of these two gyros are set at an angle of ε_0, which is usually at 45° to the north/south line of the compass card, and, although provided with the freedom to rotate about the vertical axis, the casings are linked together so that their spinning angles make an angle equal with the meridian. The two rotors are carried in a frame which is closed by a sphere, known as the gyrosphere. The gyrosphere is placed inside the outer sphere, known as the liquid container. The inner diameter of the liquid container is slightly larger than the diameter of the gyrosphere. The space between the gyrosphere and the liquid container is filled with the supporting liquid. The density of the supporting liquid is slightly less than the density of the gyrosphere, and the gyrosphere will in fact sink gently to the bottom of the liquid container in order to ensure its vertical position. In 1956, Ischlinskij (**1**) found the equations of motion of this system by neglecting the angular momentum of the gyrosphere. In 1982, Kalinovich (**2**) discussed the dynamics for the damping system and showed the possibility of using the newtonometer to indicate a method for designing a circuit that ensures damping of two-rotor gyrocompass natural oscillations. In 1988, Koshlyakov (**3**) studied the dynamic behaviour of the Anschütz compass (**4**) in a vibrating base. Since the equations of motion derived for the two-gyro Anschütz compass are either inexact or linearized approximate equations and neglect the second-order derivative terms (**5**–**9**), the exact Lagrange equations of motion of this compass will first be established and checked by the computer software package 'MACSYMA', and then the differential equations of the two cases will be solved and analysed in order to predict the new results more accurately.

2 EQUATIONS OF MOTION OF THE TWO-GYRO ANSCHÜTZ COMPASS

In this section, the Lagrange equations are used to describe the motion of the gyrocompass system. From

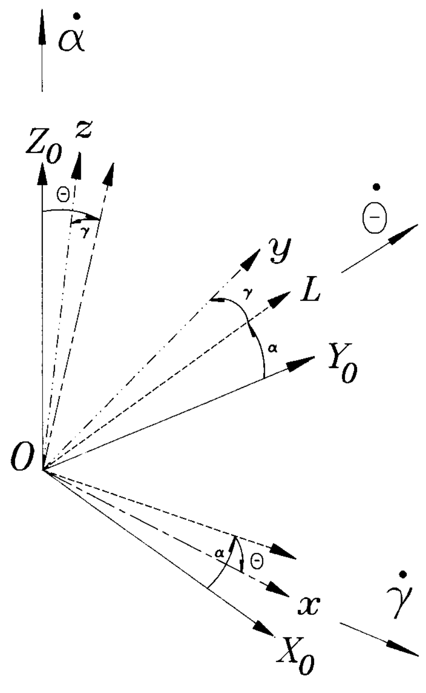

Fig. 1 The position of the gyrosphere relative to the geographic coordinte system

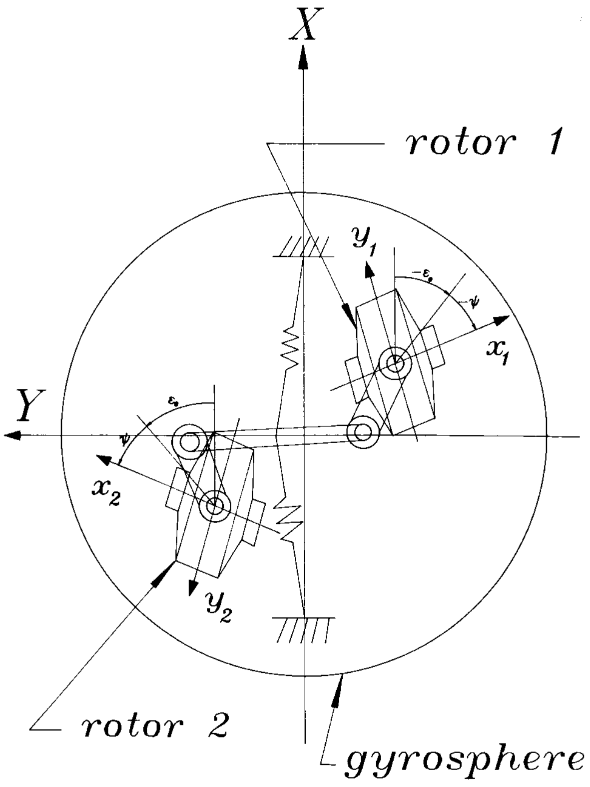

Fig. 2 Sketch of the gyrosphere

Figs 1 and 2, where the coordinate $X_0Y_0Z_0$ is the geographical coordinate system and the xyz frame is fixed with the gyrosphere, OX_0 points to the north, OY_0 points to the west, OZ_0 is the local vertical and the coordinate transformation is as given in Table 1, where θ, α and γ represent attitude angles of the gyrosphere coordinate relative to the geographic coordinate system. From these relations, the angular velocities of the gyrosphere of rotor 1 and of rotor 2 are obtained:

$$\omega_x = \dot{\gamma} + \left(\omega_1 + \frac{v \sin K}{R}\right) \cos\theta \cos\alpha - \left(\dot{\alpha} + \omega_2 + \frac{v \sin K}{R} \tan\xi\right) \sin\theta + \frac{v \cos K}{R} \cos\theta \sin\alpha$$

$$\omega_y = \dot{\theta}\cos\gamma + \left(\omega_1 + \frac{v \sin K}{R}\right) \times (-\cos\gamma \sin\alpha + \sin\gamma \sin\theta \cos\alpha) + \left(\dot{\alpha} + \omega_2 + \frac{v \sin K}{R} \tan\xi\right) \sin\gamma \cos\theta + \frac{v \cos K}{R} (\cos\gamma \cos\alpha + \sin\gamma \sin\theta \sin\alpha)$$

Table 1 The coordinate transformation

	OX_0	OY_0	OZ_0	OL
Ox	$\cos\theta\cos\alpha$	$\cos\theta\sin\alpha$	$-\sin\theta$	-0
Oy	$-\cos\gamma\sin\alpha+\sin\gamma\sin\theta\cos\alpha$	$\cos\gamma\cos\alpha+\sin\gamma\sin\theta\sin\alpha$	$\sin\gamma\cos\theta$	$\cos\gamma$
Oz	$\sin\alpha\sin\gamma+\cos\gamma\sin\theta\cos\alpha$	$-\sin\gamma\cos\alpha+\cos\gamma\sin\theta\sin\alpha$	$\cos\gamma\cos\theta$	$-\sin\gamma$

$$\omega_z = -\dot{\theta}\sin\gamma + \left(\omega_1 + \frac{v\sin K}{R}\right) \times (\sin\alpha\sin\gamma + \cos\gamma\sin\theta\cos\alpha) + \left(\dot{\alpha} + \omega_2 + \frac{v\sin K}{R}\tan\xi\right)\cos\gamma\cos\theta + \frac{v\cos K}{R}(\cos\gamma\sin\theta\sin\alpha - \sin\gamma\cos\alpha)$$

$$\omega_{x1} = \dot{\varphi}_1 + \omega_x\cos(\varepsilon_0+\varphi) - \omega_y\sin(\varepsilon_0+\varphi)$$

$$\omega_{y1} = \omega_x\sin(\varepsilon_0+\varphi) - \omega_y\cos(\varepsilon_0+\varphi)$$

$$\omega_{z1} = \omega_z - \dot{\varphi}$$

$$\omega_{x2} = \omega_x + \dot{\varphi}$$

$$\omega_{x2} = \dot{\phi}_2 + \omega_x\cos(\varepsilon_0+\varphi) + \omega_y\sin(\varepsilon_0+\varphi)$$

$$\omega_{y2} = -\omega_x\sin(\varepsilon_0+\varphi) + \omega_y\cos(\varepsilon_0+\varphi)$$

where ω_x, ω_y and ω_z represent the angular velocity components of the gyrosphere with respect to the x, y and z axes respectively, ω_{x1}, ω_{y1} and ω_{z1} represent the angular velocity components of rotor 1 with respect to the x_1, y_1 and z_1 axes respectively, ω_{x2}, ω_{y2} and ω_{z2} represent the angular velocity components of rotor 2 with respect to the x_2, y_2 and z_2 axes respectively, φ_1 and φ_2 represent spin angles of rotor 1 and rotor 2, K represents the heading angle of the vehicle, R represents the radius of the earth, v represents the velocity of the vehicles, $\varepsilon_0(-\varepsilon_0)$ represents the initial angle between the x axis and the $x_2(x_1)$ axis and ω_1 and ω_2 represent the angular velocity components of the earth spin with respect to the OX_0, OZ_0 axes, where $\omega_1 = \omega_e\cos\xi$, $\omega_2 = \omega_e\sin\xi$ and ω_e is the earth spin velocity. The kinetic energy of the system is

$$T_{total} = T_s + T_1 + T_2$$

where T_s is the kinetic energy of the gyrosphere, T_1 is the kinetic energy of rotor 1 and T_2 is the kinetic energy of rotor 2. The masses of linkage and casings are neglected. The total kinetic energy T_{total} is

$$T_{total} = \tfrac{1}{2}[J(\omega_x^2+\omega_y^2+\omega_z^2) + J_{x1}\omega_{x1}^2 + J_{y1}\omega_{y1}^2 + J_{z1}\omega_{z1}^2 + J_{x2}\omega_{x2}^2 + J_{y2}\omega_{y2}^2 + J_{z2}\omega_{z2}^2]$$

where J_y, J_y and J_z represent the moments of inertia of the gyrosphere with respect to the x, y and z axes respectively, J_{x1}, J_{y1} and J_{z1} represent the moment of inertia of rotor 1 with respect to the x_1, y_1 and z_1 axes respectively and J_{x2}, J_{y2} and J_{z2} represent the moments of inertia of rotor 2 with respect to the x_2, y_2 and z_2 axes respectively.

For the xyz coordinate, the centre of gravity of the system is $(0, 0, -l)$ and the weight of this system is P; the potential of the weight then becomes

$$V_1 = -Pl\cos\gamma\cos\theta$$

From the geometrical relation (Fig. 3), $\phi \approx \varphi$ when $\varphi \approx 0$. The potential energy of the spring is

$$V_2 = k[(l_1^2\varphi^2 + l_2^2)^{1/2} - l_0]^2$$

The potential energy of the inertial force field due to moving coordinate $X_0Y_0Z_0$ is

$$V_3 = \left(-\frac{\mathrm{d}vX_0}{\mathrm{d}t} - \omega vY_0\right) \times ml(\sin\alpha\sin\gamma + \cos\gamma\sin\theta\cos\alpha) + \left(-\frac{\mathrm{d}vY_0}{\mathrm{d}t} - \omega vX_0\right) \times ml(\cos\gamma\sin\theta\sin\alpha - \sin\gamma\cos\alpha) + m\frac{v^2}{R}l\cos y\cos\theta$$

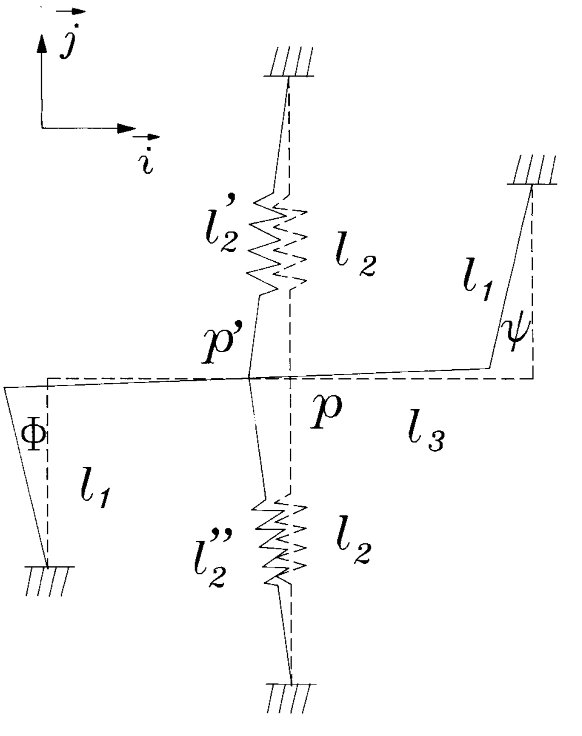

Fig. 3 Connecting cranks and spring

where v is the velocity of origin O with respect to the earth, vX_0, vY_0 are the components of velocity v along the $X_0 Y_0$ axes and ω is the component of the angular velocity of coordinate $X_0 Y_0 Z_0$ along the local normal to the earth. From Lagrange's equations, there are six equations of motion for φ_1, φ_2, θ, γ, α and φ (see the Appendix).

3 NON-ZERO EARTH SPIN VELOCITY

In this section, the non-linear terms of the governing equations are neglected but the non-zero earth spin velocity and non-zero vehicle velocity cases are considered. Assuming that the vehicle velocity is constant, the approximate equations can be obtained and the solution found.

3.1 Non-zero vehicle velocity case

3.1.1 The undamped motion

From the equation of motion, it can be assumed that $J_{x1}\omega_{x1} = C_1 = J_{x2}\omega_{x2} = C_2$, $J_{y1} = J_{y2}$, $J_{y1} = J_{z1}$. The equations then become

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{C}][\dot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = [\mathbf{Q}] \tag{1}$$

where

$$[\mathbf{M}] = \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} 0 & C_{12} & C_{13} & C_{14} \\ -C_{12} & 0 & C_{23} & C_{24} \\ -C_{13} & -C_{23} & 0 & 0 \\ -C_{14} & -C_{24} & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix}$$

$$[\mathbf{Q}] = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

$$[\mathbf{X}] = \begin{bmatrix} \theta \\ \gamma \\ \alpha \\ \varphi \end{bmatrix}$$

where

$$M_{11} = J + 2J_{y1}\cos^2\varepsilon_0$$
$$M_{22} = J + 2J_{y1}\sin^2\varepsilon_0$$
$$M_{33} = J + 2J_{y1}$$
$$M_{44} = 2J_{y1}$$
$$C_{12} = J\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right)$$
$$C_{14} = -4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\frac{v\cos K}{R}$$
$$C_{23} = (J + 4J_{y1}\sin^2\varepsilon_0)\frac{v\cos K}{R}$$
$$C_{13} = 2C_1\cos\varepsilon_0 - (J + 4J_{y1}\cos^2\varepsilon_0)\left(\omega_1 + \frac{v\sin K}{R}\right)$$
$$C_{24} = -2C_1\sin\varepsilon_0 + 4J_{y1}\cos\varepsilon_0\left(\omega_1 + \frac{v\sin K}{R}\right)$$
$$K_{11} = lp + 2C_1\cos\varepsilon_0\left(\omega_1 + \frac{v\sin K}{R}\right) + 2J_{y1}\cos^2\varepsilon_0\left[\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right)^2 - \left(\omega_1 + \frac{v\sin K}{R}\right)^2\right] - m\frac{v^2}{R}$$
$$K_{22} = lp + 2J_{y1}\sin^2\varepsilon_0\left[\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right)^2 - \left(\frac{v\cos K}{R}\right)^2\right] - m\frac{v^2}{R}$$
$$K_{12} = 2J_{y1}\sin^2\varepsilon_0\frac{v\cos K}{R}\left(\omega_1 + \frac{v\sin K}{R}\right)$$
$$K_{23} = -2J_{y1}\sin^2\varepsilon_0\left(\omega + \frac{v\sin K}{R}\right) \times \left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right) + m\omega vX_0$$
$$K_{13} = -2J_{y1}\cos^2\varepsilon_0\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right)\frac{v\cos K}{R} - m\omega vY_0$$
$$K_{24} = 4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right)\frac{v\cos K}{R}$$
$$K_{14} = -2C_1\sin\varepsilon_0\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right) + 4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\left(\omega_2 + \frac{v\sin K}{R}\tan\xi\right) \times \left(\omega_1 + \frac{v\sin K}{R}\right) \tag{2}$$

$$K_{34}=\left[-8J_{y1}\cos\varepsilon_0\sin\varepsilon_0\left(\omega_1+\frac{v\sin K}{R}\right)+2C_1\sin\varepsilon_0\right]\frac{v\cos K}{R}$$

$$K_{33}=2C_1\cos\varepsilon_0\left(\omega_1+\frac{v\sin K}{R}\right)+2J_{y1}\left[\left(\omega_1+\frac{v\sin K}{R}\right)^2-\left(\frac{v\cos K}{R}\right)^2\right]\times(\sin^2\varepsilon_0-\cos^2\varepsilon_0)$$

$$K_{44}=2kl_1^2\left(1-\frac{l_0}{l_2}\right)+2C_1\cos\varepsilon_0\left(\omega_1+\frac{v\sin K}{R}\right)+2J_{y1}(\cos^2\varepsilon_0-\sin^2\varepsilon_0)\times\left[\left(\frac{v\cos K}{R}\right)^2-\left(\omega_1+\frac{v\sin K}{R}\right)^2\right]$$

$$Q_1=2C_1\sin\varepsilon_0\left(\omega_2+\frac{v\sin K}{R}\tan\xi\right)-2J_{y1}\cos^2\varepsilon_0\left(\omega_1+\frac{v\sin K}{R}\right)\times\left(\omega_2+\frac{v\sin K}{R}\tan\xi\right)$$

$$Q_2=-2J_{y1}\sin^2\varepsilon_0\left(\omega_2+\frac{v\sin K}{R}\tan\xi\right)\frac{v\cos K}{R}$$

$$Q_3=-2C_1\cos\varepsilon_0\frac{v\cos K}{R}-2J_{y1}(\sin^2\varepsilon_0-\cos^2\varepsilon_0)\times\left(\frac{v\cos K}{R}\right)\left(\omega_1+\frac{v\sin K}{R}\right)$$

$$Q_4=2C_1\sin\varepsilon_0\left(\omega_1+\frac{v\sin K}{R}\right)-2J_{y1}\cos\varepsilon_0\sin\varepsilon_0\times\left[\left(\omega_1+\frac{v\sin K}{R}\right)^2-\left(\frac{v\cos K}{R}\right)^2\right]$$

By normalization, the differential equations can be represented in the following form:

$$[\mathbf{A}][\dot{\mathbf{Y}}]+[\mathbf{B}][\mathbf{Y}]=[\mathbf{Q}'] \quad (3)$$

where

$$[\mathbf{A}]=\begin{bmatrix}[\mathbf{0}] & [\mathbf{M}]\\ [\mathbf{M}] & [\mathbf{C}]\end{bmatrix}$$

$$[\mathbf{B}]=\begin{bmatrix}-[\mathbf{M}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{K}]\end{bmatrix}$$

$$[\mathbf{Q}']=\begin{bmatrix}[\mathbf{0}]\\ [\mathbf{Q}]\end{bmatrix}$$

$$[\mathbf{Y}]=\begin{bmatrix}[\dot{\mathbf{Y}}]\\ [\mathbf{X}]\end{bmatrix}$$

Let $\theta=a_1\,\mathrm{e}^{-\lambda t}$, $\gamma=a_2\,\mathrm{e}^{-\lambda t}$, $\alpha=a_3\,\mathrm{e}^{-\lambda t}$, $\varphi=a_4\,\mathrm{e}^{-\lambda t}$. Then for the homogeneous solution of these differential equations, the following characteristic equation is given:

$$M_{11}M_{22}M_{33}M_{44}\lambda^8+\cdots=0 \quad (4)$$

If

$$\lambda^2=\Lambda,\qquad \omega_e=\varepsilon$$

$$\omega_1+\frac{v\sin K}{R}=a\varepsilon,\qquad \omega_2+\frac{v\sin K}{R}\tan\xi=b\varepsilon$$

$$\frac{v\cos K}{R}=c\varepsilon,\quad -m\omega vX_0=d\varepsilon$$

$$-m\omega vY_0=e\varepsilon,\qquad -m\frac{v^2}{R}=f\varepsilon$$

$$\Lambda=\Lambda_0+\varepsilon\Lambda_1+\cdots$$

and a, b, c, d, e, f are constants, by the perturbation method, the following equations are obtained:

$$\begin{aligned}\varepsilon^0:&\quad M_{11}M_{22}M_{33}M_{44}\Lambda_0^4+\cdots=0\\ \varepsilon^1:&\quad 4M_{11}M_{22}M_{33}M_{44}\Lambda_0^3\Lambda_1+\cdots=0\end{aligned} \quad (5)$$

where

$$\begin{aligned}&G_{12}=J,\qquad G_{23}=4J_{y1}\sin^2\varepsilon_0+J\\ &H_{44}=2kl_1^2\left(1-\frac{l_0}{l_2}\right),\qquad G_{14}=-4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\\ &F_{13}=2C_1\cos\varepsilon_0,\qquad G_{13}=-(4J_{y1}\cos^2\varepsilon_0+J)\\ &F_{14}=-2C_1\sin\varepsilon_0,\qquad G_{24}=4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\\ &H_{11}=H_{22}=lp,\qquad I_{11}=2C_1\cos\varepsilon_0=I_{33}=I_{44}\\ &I_{33}=I_{22}=I_{23}=1,\qquad I_{14}=2C_1\sin\varepsilon_0,\\ &I_{34}=2C_1\sin\varepsilon_0\end{aligned} \quad (6)$$

Using the computer software package 'MACSYMA', the solution of equations (5) is obtained:

$$\begin{aligned}&\Lambda_{10}=0\\ &\Lambda_{30}=-\frac{F_{13}^2+H_{11}M_{33}}{M_{11}M_{33}}\\ &\Lambda_{50}=\frac{\sqrt{(H_{11}^2M_{44}^2-2H_{11}H_{44}M_{22}M_{44}+2F_{24}^2H_{11}M_{44}+H_{44}^2M_{22}^2+2F_{24}^2H_{44}M_{44}+F_{24}^4)}}{2M_{22}M_{44}}-\frac{F_{24}^2+H_{44}M_{22}+H_{11}M_{22}}{2M_{22}M_{44}}\\ &\Lambda_{70}=-\frac{\sqrt{(H_{11}^2M_{44}^2-2H_{11}H_{44}M_{22}M_{44}+2F_{24}^2H_{11}M_{44}+H_{44}^2M_{22}^2+2F_{24}^2H_{44}M_{44}+F_{24}^4)}}{2M_{22}M_{44}}-\frac{F_{24}^2+H_{44}M_{22}+H_{11}M_{22}}{2M_{22}M_{44}}\end{aligned} \quad (7)$$

Substituting $\Lambda_{10} = 0$ into equations (5) gives

$$\Lambda_{11} = -\frac{aH_{11}I_{33}}{H_{11}M_{33} + F_{13}^2}$$

Because Λ_{30}, Λ_{50}, Λ_{70}, Λ_{11} are all negative and $\Lambda = \pm\sqrt{\lambda}$, the eigenvalues of these governing equations are pure imaginary. The motion is stable.

Substituting the eigenvalue into equation (1) gives the following homogeneous solution:

$$\begin{aligned}
\theta &= a_1\chi_{11}\,\mathrm{e}^{-\lambda_1 t} + a_2\chi_{13}\,\mathrm{e}^{-\lambda_3 t} + a_3\chi_{15}\,\mathrm{e}^{-\lambda_5 t} \\
&\quad + a_4\chi_{17}\,\mathrm{e}^{-\lambda_7 t} + \text{c.c.} \\
\gamma &= a_1\chi_{21}\,\mathrm{e}^{\lambda_1 t} + a_2\chi_{23}\,\mathrm{e}^{-\lambda_3 t} + a_3\chi_{25}\,\mathrm{e}^{-\lambda_5 t} \\
&\quad + a_4\chi_{27}\,\mathrm{e}^{-\lambda_7 t} + \text{c.c.} \\
\alpha &= a_1\chi_{31}\,\mathrm{e}^{\lambda_1 t} + a_2\chi_{33}\,\mathrm{e}^{-\lambda_3 t} + a_3\chi_{35}\,\mathrm{e}^{-\lambda_5 t} \\
&\quad + a_4\chi_{37}\,\mathrm{e}^{-\lambda_7 t} + \text{c.c.} \\
\varphi &= a_1\,\mathrm{e}^{\lambda_1 t} + a_2\,\mathrm{e}^{-\lambda_3 t} + a_3\,\mathrm{e}^{-\lambda_5 t} + a_4\,\mathrm{e}^{-\lambda_7 t} + \text{c.c.}
\end{aligned} \tag{8}$$

where a_i is constant, $i = 1$–4, and

$$\chi_{1i} = -\frac{M_{22}M_{33}M_{44}\lambda_i^6 + \cdots}{M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$

$$\chi_{1i} = \frac{C_{12}M_{33}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$

$$\chi_{1i} = \frac{C_{13}M_{22}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$

The particular solution of equation (1) is

$$\begin{aligned}
\theta &= -\frac{K_{12}K_{23}K_{34}Q_4 + \cdots}{K_{11}K_{22}K_{33}K_{44} + \cdots} \\
\gamma &= \frac{K_{11}K_{23}K_{34}Q_4 + \cdots}{K_{11}K_{22}K_{33}K_{44} + \cdots} \\
\alpha &= \frac{K_{11}K_{22}K_{34}Q_4 + \cdots}{K_{11}K_{22}K_{33}K_{44} + \cdots} \\
\varphi &= \frac{K_{11}K_{22}K_{33}Q_4 + \cdots}{K_{11}K_{22}K_{33}K_{44} + \cdots}
\end{aligned} \tag{9}$$

The complete solution is the sum of the homogeneous solution and the particular solution.

3.1.2 The damped motion

In this section, the effect of damping will be studied. Let Rayleigh's dissipation function $\Re$ be

$$\Re = \frac{C_{11}\dot{\theta}^2 + C_{22}\dot{\gamma}^2 + C_{33}\dot{\alpha}^2 + C_{44}\dot{\varphi}^2}{2}$$

Then the equations of motion become

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{C}][\dot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = [\mathbf{Q}] \tag{10}$$

where

$$[\mathbf{M}] = \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ -C_{12} & C_{22} & C_{23} & C_{24} \\ -C_{13} & -C_{23} & C_{33} & 0 \\ -C_{14} & -C_{24} & 0 & C_{44} \end{bmatrix}$$

$$[\mathbf{K}] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix}$$

$$[\mathbf{Q}] = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

$$[\mathbf{X}] = \begin{bmatrix} \theta \\ \gamma \\ \alpha \\ \varphi \end{bmatrix}$$

where M_{11}, M_{22}, M_{33}, M_{44}, C_{12}, C_{14}, C_{23}, C_{13}, C_{24}, K_{11}, K_{22}, K_{12}, K_{23}, K_{13}, K_{24}, K_{14}, K_{34}, K_{33}, K_{44}, Q_1, Q_2, Q_3, Q_4 are the same as equation (2). By normalization, the differential equations can be represented as equation (3). Then for the homogeneous solution of these differential equations, the characteristic equation is given as equation (4). If

$$\omega_{\mathrm{e}} = \varepsilon, \qquad \omega_1 + \frac{v \sin K}{R} = a\varepsilon$$

$$\omega_2 + \frac{v \sin K}{R} \tan \xi = b\varepsilon, \qquad \frac{v \cos K}{R} = c\varepsilon$$

$$-m\omega v X_0 = d\varepsilon, \qquad -m\omega v Y_0 = e\varepsilon$$

$$m\frac{v^2}{R} = f\varepsilon, \qquad \lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$$

and a, b, c, d, e, f are constants, by the perturbation method, the following equations are obtained:

$$\begin{aligned}
\varepsilon^0&: \quad M_{11}M_{22}M_{33}M_{44}\lambda_0^8 + \cdots = 0 \\
\varepsilon^1&: \quad 8M_{11}M_{22}M_{33}M_{44}\lambda_0^7\lambda_1 + \cdots = 0
\end{aligned} \tag{11}$$

where G_{12}, G_{23}, G_{14}, G_{13}, G_{24}, F_{13}, F_{24}, H_{11}, H_{22}, H_{44},

I_{11}, I_{33}, I_{44}, I_{13}, I_{22}, I_{23}, I_{14}, I_{34} are the same as in equations (6).

The solution of equations (11) can be found using the computer software package 'MACSYMA'. The solution is as follows:

$$\lambda_0 = 0$$
$$\lambda_1 = \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots$$
$$\lambda_2 = \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots$$
$$\lambda_3 = \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots$$
$$\lambda_4 = \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots \qquad (12)$$
$$\lambda_5 = \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots$$
$$\lambda_6 = \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots$$
$$\lambda_7 = \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots$$

Substituting the eigenvalue into equation (10), the homogeneous solution obtained is

$$\begin{aligned}
\theta &= a_1\chi_{11}\,e^{-\lambda_1 t} + a_2\chi_{12}\,e^{-\lambda_2 t} + a_3\chi_{13}\,e^{-\lambda_3 t} \\
&\quad + a_4\chi_{14}\,e^{-\lambda_4 t} + a_5\chi_{15}\,e^{-\lambda_5 t} + a_6\chi_{16}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{17}\,e^{-\lambda_7 t} + a_8\chi_{18}\,e^{-\lambda_8 t} \\
\gamma &= a_1\chi_{21}\,e^{-\lambda_1 t} + a_2\chi_{22}\,e^{-\lambda_2 t} + a_3\chi_{23}\,e^{-\lambda_3 t} \\
&\quad + a_4\chi_{24}\,e^{-\lambda_4 t} + a_5\chi_{25}\,e^{-\lambda_5 t} + a_6\chi_{26}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{27}\,e^{-\lambda_7 t} + a_8\chi_{28}\,e^{-\lambda_8 t} \\
\alpha &= a_1\chi_{31}\,e^{-\lambda_1 t} + a_2\chi_{32}\,e^{-\lambda_2 t} + a_3\chi_{33}\,e^{-\lambda_3 t} \qquad (13) \\
&\quad + a_4\chi_{34}\,e^{-\lambda_4 t} + a_5\chi_{35}\,e^{-\lambda_5 t} + a_6\chi_{36}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{37}\,e^{-\lambda_7 t} + a_8\chi_{38}\,e^{-\lambda_8 t} \\
\varphi &= a_1\,e^{-\lambda_1 t} + a_2\,e^{-\lambda_2 t} + a_3\,e^{-\lambda_3 t} + a_4\,e^{-\lambda_4 t} + a_5\,e^{-\lambda_5 t} \\
&\quad + a_6\,e^{-\lambda_6 t} + a_7\,e^{-\lambda_7 t} + a_8\,e^{-\lambda_8 t}
\end{aligned}$$

where a_i is constant, $i = 1$–8, and

$$\chi_{1i} = \frac{M_{11}M_{22}M_{33}M_{44}\lambda_i^6 + \cdots}{-M_{11}M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$
$$\chi_{1i} = \frac{-M_{11}C_{12}M_{33}M_{44}\lambda_i^5 + \cdots}{-M_{11}M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$
$$\chi_{1i} = \frac{-C_{13}M_{11}M_{22}M_{44}\lambda_i^5 + \cdots}{-M_{11}M_{22}M_{33}C_{14}\lambda_i^5 + \cdots}$$

If the numerical constants of Section 3.1.3 are substituted into the solution of equations (11), the following results are obtained:

$$\begin{aligned}
&\lambda_{01} = 0, \quad \lambda_{02} = 0.0017, \quad \lambda_{03} = 0.0172 \\
&\lambda_{04} = 0.0196, \quad \lambda_{05,06} = 1557.71 \pm 2239.37\mathrm{i} \qquad (14) \\
&\lambda_{07,08} = 1573.85 \pm 2494.49\mathrm{i}, \quad \varepsilon\lambda_{11} = 9.34 \times 10^{-5}
\end{aligned}$$

Because all of the main parts of eigenvalues, $\lambda_{02}, \ldots, \lambda_{08}$, $\varepsilon\lambda_{11}$, have a positive real part, the motion is asymptotically stable.

The particular solution of equation (10) is the same as equation (9). The complete solution is the sum of the homogeneous solution and the particular solution.

3.1.3 *Numerical examples*

From reference (**5**), the constants of elements of the gyrosphere are given as:

$$J = 10.89 \text{ gw cm s}^2, \quad J_{y1} = 40.13 \text{ gw cm s}^2$$
$$2C_1 \cos\varepsilon_0 = 1.585 \times 10^5 \text{ gw cm s}$$
$$lp = 6.675 \times 10^3 \text{ gw cm}$$
$$2kl_1^2\left(1 - \frac{l_0}{l_2}\right) = 140 \text{ gw cm}, \quad \varepsilon_0 = 45^\circ$$
$$C_{11} = C_{22} = C_{33} = C_{44} = 1 \times 10^5 \text{ gw cm s}$$

and

$$\xi = 30^\circ \text{ N}, \quad v = 0.67 \text{ m/s}, \quad K = 90^\circ$$

The following numerical equations are obtained:

(a) Undamped system:

$$1.9 \times 10^7\lambda^8 + 2.19 \times 10^{14}\lambda^6 + 6.31 \times 10^{20}\lambda^4 + 2.68 \times 10^{16}\lambda^2 + 6.7 \times 10^{10} = 0 \qquad (15)$$

The solution of the above equation using the computer software package 'MATHEMATICA' is

$$\begin{aligned}
&\lambda_{1,2} = \pm 0.0016\mathrm{i}, \quad \lambda_{3,4} = \pm 2387.46\mathrm{i} \\
&\lambda_{5,6} = \pm 0.0062\mathrm{i}, \quad \lambda_{7,8} = \pm 2412.88\mathrm{i}
\end{aligned} \qquad (16)$$

(b) Damped system:

$$\begin{aligned}
&1.9 \times 10^7\lambda^8 - 1.19 \times 10^{11}\lambda^7 + 4.93 \times 10^{14}\lambda^6 \\
&- 9.6 \times 10^{17}\lambda^5 + 1.23 \times 10^{21}\lambda^4 - 4.75 \times 10^{19}\lambda^3 \\
&+ 4.95 \times 10^{17}\lambda^2 - 7.17 \times 10^{14}\lambda + 6.7 \times 10^{10} = 0
\end{aligned} \qquad (17)$$

The solution of the above equation using the computer software package 'MATHEMATICA' is

$$\begin{aligned}
&\lambda_1 = 0.0001, \quad \lambda_2 = 0.00159, \quad \lambda_3 = 0.0173 \\
&\lambda_4 = 0.0195, \quad \lambda_{5,6} = 1557.71 \pm 2239.37\mathrm{i} \qquad (18) \\
&\lambda_{7,8} = 1573.85 \pm 2429.49\mathrm{i}
\end{aligned}$$

From these results, the undamped motion is stable and the damped motion is asymptotically stable.

If $K = 45^\circ$ is taken and other assumptions remain unchanged, the following numerical equations are obtained:

(a) Undamped system:

$$1.9 \times 10^7\lambda^8 + 2.19 + 10^{14}\lambda^6 + 6.31 \times 10^{20}\lambda^4 + 2.68 \times 10^{16}\lambda^2 + 6.7 \times 10^{10} = 0 \quad (19)$$

(b) Damped system:

$$1.9 \times 10^7\lambda^8 - 1.19 \times 10^{11}\lambda^7 + 4.93 \times 10^{14}\lambda^6 - 9.6 \times 10^{17}\lambda^5 + 1.23 \times 10^{21}\lambda^4 - 4.75 \times 10^{19}\lambda^3 + 4.95 \times 10^{17}\lambda^2 - 7.17 \times 10^{14}\lambda + 6.7 \times 10^{10} = 0 \quad (20)$$

These two equations are the same as the equations for the case $K = 90°$. Therefore the solutions are the same.

3.2 Zero vehicle velocity case

Since θ, α, γ, φ are usually small, the non-linear terms of the governing equations are neglected and the vehicle velocity is taken to be zero. Using these assumptions, the approximate equations can be obtained and the solution found.

3.2.1 The undamped motion

From the equation of motion, $J_{x1}\omega_{x1} = C_1 = J_{x2}\omega_{x2} = C_2$, $J_{y1} = J_{y2}$, $J_{y1} = J_{z1}$ are assumed and the equations become

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{C}][\dot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = [\mathbf{Q}] \quad (21)$$

where

$$[\mathbf{M}] = \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} 0 & C_{12} & C_{13} & 0 \\ -C_{12} & 0 & 0 & C_{24} \\ -C_{13} & 0 & 0 & 0 \\ 0 & -C_{24} & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}] = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} \\ 0 & K_{22} & K_{23} & 0 \\ 0 & K_{23} & K_{33} & 0 \\ K_{14} & 0 & 0 & K_{44} \end{bmatrix}$$

$$[\mathbf{Q}] = \begin{bmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{bmatrix}$$

$$[\mathbf{X}] = \begin{bmatrix} \theta \\ \gamma \\ \alpha \\ \varphi \end{bmatrix}$$

where

$$\begin{aligned}
M_{11} &= J + 2J_{y1}\cos^2\varepsilon_0 \\
M_{22} &= J + 2J_{y1}\sin^2\varepsilon_0 \\
M_{33} &= J + 2J_{y1} \\
M_{44} &= 2J_{y1} \\
C_{12} &= J\omega_2 \\
C_{13} &= 2C_1\cos\varepsilon_0 - (J + 4J_{y1}\cos^2\varepsilon_0)\omega_1 \\
C_{24} &= -2C_1\sin\varepsilon_0 + 4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\omega_1 \\
K_{11} &= lp + 2C_1\cos\varepsilon_0\omega_1 + 2J_{y1}\cos^2\varepsilon_0(\omega_2^2 - \omega_1^2) \\
K_{22} &= lp + 2J_{y1}\sin^2\varepsilon_0\omega_2^2 \\
K_{23} &= -2J_{y1}\sin^2\varepsilon_0\omega_1\omega_2 \\
K_{14} &= -2C_1\sin\varepsilon_0\omega_2 + 4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\omega_2\omega_1 \\
K_{33} &= 2C_1\cos\varepsilon_0\omega_1 + 2J_{y1}\omega_1^2(\sin^2\varepsilon_0 - \cos^2\varepsilon_0) \\
K_{44} &= 2kl_1^2\left(1 - \frac{l_0}{l_2}\right) + 2C_1\cos\varepsilon_0\omega_1 \\
&\quad - 2J_{y1}(\cos^2\varepsilon_0 - \sin^2\varepsilon_0)\omega_1^2 \\
Q_1 &= 2C_1\sin\varepsilon_0\omega_2 - 2J_{y1}\sin^2\varepsilon_0\omega_1\omega_2 \\
Q_4 &= 2C_1\sin\varepsilon_0\omega_1 - 2J_{y1}\cos\varepsilon_0\sin\varepsilon_0\omega_1^2
\end{aligned} \quad (22)$$

By normalization, the differential equations can be represented as equation (3) in Section 3.1.1. The characteristic equation is obtained as equation (4) in Section 3.1.1. If $\lambda^2 = \Lambda$, $\omega_e = \varepsilon$, $\Lambda = \Lambda_0 + \varepsilon\Lambda_i + \cdots$, by the perturbation method, the following equations are obtained:

$$\begin{aligned}
\varepsilon^0:&\quad M_{11}M_{22}M_{33}M_{44}\Lambda_0^4 + \cdots = 0 \\
\varepsilon^1:&\quad 4M_{11}M_{22}M_{33}M_{44}\Lambda_0^3\Lambda_1 + \cdots = 0
\end{aligned} \quad (23)$$

where

$$\begin{aligned}
G_{12} &= J\sin\xi, \qquad F_{13} = 2C_1\cos\varepsilon_0 \\
H_{44} &= 2kl_1^2\left(1 - \frac{l_0}{l_2}\right) \\
G_{13} &= -(4J_{y1}\cos^2\varepsilon_0 + J)\cos\xi \\
F_{24} &= -2C_1\sin\varepsilon_0 \\
G_{24} &= 4J_{y1}\cos\varepsilon_0\sin\varepsilon_0\cos\xi \\
H_{11} &= H_{22} = lp, \qquad I_{11} = 2C_1\cos\varepsilon_0\cos\xi = I_{33} = I_{44} \\
I_{14} &= 2C_1\cos\varepsilon_0\sin\xi
\end{aligned} \quad (24)$$

Using the computer software package 'MACSYMA', the solution of equations (23) is obtained:

$$\Lambda_{10} = 0$$

$$\Lambda_{30} = -\frac{F_{13}^2 + H_{11}M_{33}}{M_{11}M_{33}}$$

$$\Lambda_{50} = \frac{\sqrt{(H_{11}^2M_{44}^2 - 2H_{11}H_{44}M_{22}M_{44} + 2F_{24}^2H_{11}M_{44} + H_{44}^2M_{22}^2 + 2F_{24}^2H_{44}M_{22} + F_{24}^4)}}{2M_{22}M_{44}} - \frac{F_{24}^2 + H_{44}M_{22} + H_{11}M_{22}}{2M_{22}M_{44}} \quad (25)$$

$$\Lambda_{70} = -\frac{\sqrt{(H_{11}^2M_{44}^2 - 2H_{11}H_{44}M_{22}M_{44} + 2F_{24}^2H_{11}M_{44} + H_{44}^2M_{22}^2 + 2F_{24}^2H_{44}M_{22} + F_{24}^4)}}{2M_{22}M_{44}} - \frac{F_{24}^2 + H_{44}M_{22} + H_{11}M_{22}}{2M_{22}M_{44}}$$

Substituting $\Lambda_{10} = 0$ into equations (23) gives

$$\Lambda_{11} = -\frac{aH_{11}I_{11}}{H_{11}M_{33} + F_{13}^2}$$

Because Λ_{30}, Λ_{50}, Λ_{70}, Λ_{11} are all negative and $\Lambda = \pm\sqrt{\lambda}$, the eigenvalues of these governing equations are pure imaginary. The motion is stable.

Substituting the eigenvalue into equation (21), the following homogeneous solution can be obtained:

$$\theta = a_1\chi_{11}\,e^{\lambda_1 t} + a_2\chi_{13}\,e^{-\lambda_3 t} + a_3\chi_{15}\,e^{-\lambda_5 t} + a_4\chi_{17}\,e^{-\lambda_7 t} + \text{c.c.}$$

$$\gamma = a_1\chi_{21}\,e^{\lambda_1 t} + a_2\chi_{23}\,e^{-\lambda_3 t} + a_3\chi_{25}\,e^{-\lambda_5 t} + a_4\chi_{27}\,e^{-\lambda_7 t} + \text{c.c.}$$

$$\alpha = a_1\chi_{31}\,e^{\lambda_1 t} + a_2\chi_{33}\,e^{-\lambda_3 t} + a_3\chi_{35}\,e^{-\lambda_5 t} + a_4\chi_{37}\,e^{-\lambda_7 t} + \text{c.c.}$$

$$\varphi = a_1\,e^{\lambda_1 t} + a_2\,e^{-\lambda_3 t} + a_3\,e^{-\lambda_5 t} + a_4\,e^{-\lambda_7 t} + \text{c.c.}$$

where a_i is constant, $i = 1$–4, and

$$\chi_{1i} = -\frac{M_{22}M_{33}M_{44}\lambda_i^6 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^4 + \cdots}$$

$$\chi_{1i} = \frac{C_{12}M_{33}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^5 + \cdots}$$

$$\chi_{1i} = \frac{C_{13}M_{22}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^5 + \cdots}$$

The convergence of the above solution by the perturbation method can be certificated by numerical results. The particular solution of equation (21) is

$$\theta = \frac{Q_1K_{44} - Q_4K_{14}}{K_{11}K_{44} - K_{14}^2}, \qquad \varphi = \frac{Q_4K_{11} - Q_1K_{14}}{K_{11}K_{44} - K_{14}^2} \quad (26)$$

The complete solution is the sum of the homogeneous solution and the particular solution.

3.2.2 *The damped motion*

In this section, the effect of damping will be studied. Let Rayleigh's dissipation function $\Re$ be

$$\Re = \frac{C_{11}\dot{\theta}^2 + C_{22}\dot{\gamma}^2 + C_{33}\dot{\alpha}^2 + C_{44}\dot{\varphi}^2}{2}$$

Then the equations of motion become

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{C}][\dot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = [\mathbf{Q}] \quad (27)$$

where

$$[\mathbf{M}] = \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ -C_{12} & C_{22} & 0 & C_{24} \\ -C_{13} & 0 & C_{33} & 0 \\ 0 & -C_{24} & 0 & C_{44} \end{bmatrix}$$

$$[\mathbf{K}] = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} \\ 0 & K_{22} & K_{23} & 0 \\ 0 & K_{23} & K_{33} & 0 \\ K_{14} & 0 & 0 & K_{44} \end{bmatrix}$$

$$[\mathbf{Q}] = \begin{bmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{bmatrix}$$

$$[\mathbf{X}] = \begin{bmatrix} \theta \\ \gamma \\ \alpha \\ \varphi \end{bmatrix}$$

where M_{11}, M_{22}, M_{33}, M_{44}, C_{12}, C_{13}, C_{24}, K_{11}, K_{22}, K_{23}, K_{14}, K_{33}, K_{44}, Q_1, Q_4 are the same as equations (22). By normalization, the differential equations can be represented as equation (3). Then for the homogeneous solution of these differential equations, through the characteristic equation (4), if $\omega_e = \varepsilon$, $\lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$, by the perturbation method, the following equations are obtained:

$$\begin{aligned} \varepsilon^0: &\quad M_{11}M_{22}M_{33}M_{44}\lambda_0^8 + \cdots = 0 \\ \varepsilon^1: &\quad 8M_{11}M_{22}M_{33}M_{44}\lambda_0^7\lambda_1 + \cdots = 0 \end{aligned} \quad (28)$$

where M_{11}, M_{22}, M_{33}, M_{44}, G_{12}, G_{13}, G_{24}, F_{13}, F_{24}, H_{11}, H_{44}, I_{11}, I_{44} are the same as equations (24).

The solution of equations (28) can be found by using the computer software package 'MACSYMA'. The solution is as follows:

$$\begin{aligned}
\lambda_0 &= 0 \\
\lambda_1 &= \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots \\
\lambda_2 &= \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots \\
\lambda_3 &= \frac{C_{11}M_{33} + C_{33}M_{11}}{3M_{11}M_{33}} + \cdots \\
\lambda_4 &= \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots \\
\lambda_5 &= \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots \\
\lambda_6 &= \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots \\
\lambda_7 &= \frac{C_{22}M_{44} + C_{44}M_{22}}{4M_{22}M_{44}} + \cdots
\end{aligned} \tag{29}$$

Substituting the eigenvalue into equation (27), the following homogeneous solution can be obtained:

$$\begin{aligned}
\theta &= a_1\chi_{11}\,e^{-\lambda_1 t} + a_2\chi_{12}\,e^{-\lambda_2 t} + a_3\chi_{13}\,e^{-\lambda_3 t} \\
&\quad + a_4\chi_{14}\,e^{-\lambda_4 t} + a_5\chi_{15}\,e^{-\lambda_5 t} + a_6\chi_{16}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{17}\,e^{-\lambda_7 t} + a_8\chi_{18}\,e^{-\lambda_8 t} \\
\gamma &= a_1\chi_{21}\,e^{-\lambda_1 t} + a_2\chi_{22}\,e^{-\lambda_2 t} + a_3\chi_{23}\,e^{-\lambda_3 t} \\
&\quad + a_4\chi_{24}\,e^{-\lambda_4 t} + a_5\chi_{25}\,e^{-\lambda_5 t} + a_6\chi_{26}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{27}\,e^{-\lambda_7 t} + a_8\chi_{28}\,e^{-\lambda_8 t} \\
\alpha &= a_1\chi_{31}\,e^{-\lambda_1 t} + a_2\chi_{32}\,e^{-\lambda_2 t} + a_3\chi_{33}\,e^{-\lambda_3 t} \\
&\quad + a_4\chi_{34}\,e^{-\lambda_4 t} + a_5\chi_{35}\,e^{-\lambda_5 t} + a_6\chi_{36}\,e^{-\lambda_6 t} \\
&\quad + a_7\chi_{37}\,e^{-\lambda_7 t} + a_8\chi_{38}\,e^{-\lambda_8 t} \\
\varphi &= a_1\,e^{-\lambda_1 t} + a_2\,e^{-\lambda_2 t} + a_3\,e^{-\lambda_3 t} + a_4\,e^{-\lambda_4 t} + a_5\,e^{-\lambda_5 t} \\
&\quad + a_6\,e^{-\lambda_6 t} + a_7\,e^{-\lambda_7 t} + a_8\,e^{-\lambda_8 t}
\end{aligned} \tag{30}$$

where a_i is constant, $i = 1$–8, and

$$\chi_{1i} = -\frac{M_{22}M_{33}M_{44}\lambda_i^6 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^4 + \cdots}$$

$$\chi_{1i} = \frac{C_{12}M_{33}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^4 + \cdots}$$

$$\chi_{1i} = \frac{C_{13}M_{22}M_{44}\lambda_i^5 + \cdots}{M_{22}M_{33}K_{14}\lambda_i^4 + \cdots}$$

If the numerical constants of Section 3.1.3 are substituted into the solution of equations (28), the following results are obtained:

$$\begin{aligned}
&\lambda_{01} = 0, \quad \lambda_{02} = 0.0017, \quad \lambda_{03} = 0.017 \\
&\lambda_{04} = 0.019, \quad \lambda_{05,06} = 1434.71 \pm 2243.09\mathrm{i} \\
&\lambda_{07,08} = 700.63 \pm 2494.49\mathrm{i}, \quad \varepsilon\lambda_{11} = 9.34 \times 10^{-5}
\end{aligned} \tag{31}$$

Because all of the main parts of eigenvalues, $\lambda_{02}, \ldots, \lambda_{08}$, $\varepsilon\lambda_{11}$, have a positive real part, the motion is asymptotically stable.

The particular solution of equation (27) is the same as equation (26). The complete solution is the sum of the homogeneous solution and the particular solution.

3.2.3 *Numerical examples*

Taking the assumptions of Section 3.1.3 and $\xi = 30°$ N, the following numerical equations are obtained:

(a) Undamped system:

$$3.62 \times 10^{14}\lambda^8 + 4.18 \times 10^{21}\lambda^6 + 1.2 \times 10^{28}\lambda^4 + 5.11 \times 10^{23}\lambda^2 + 1.27 \times 10^{18} = 0 \tag{32}$$

The solution of the above equation using the computer software package 'MATHEMATICA' is

$$\begin{aligned}
&\lambda_{1,2} = \pm 0.0016\mathrm{i}, \quad \lambda_{3,4} = \pm 2490.45\mathrm{i} \\
&\lambda_{5,6} = \pm 0.0063\mathrm{i}, \quad \lambda_{7,8} = \pm 2311.85\mathrm{i}
\end{aligned} \tag{33}$$

(b) Damped system:

$$\begin{aligned}
&3.62 \times 10^{14}\lambda^8 - 2.27 \times 10^{18}\lambda^7 + 9.4 \times 10^{21}\lambda^6 \\
&\quad - 1.82 \times 10^{25}\lambda^5 + 2.34 \times 10^{28}\lambda^4 - 9.05 \times 10^{26}\lambda^3 \\
&\quad + 9.43 \times 10^{24}\lambda^2 - 1.36 \times 10^{22}\lambda + 1.27 \times 10^{18} = 0
\end{aligned} \tag{34}$$

The solution of the above equation using the computer software package 'MATHEMATICA' is

$$\begin{aligned}
&\lambda_1 = 0.0001, \quad \lambda_2 = 0.0016, \quad \lambda_3 = 0.017 \\
&\lambda_4 = 0.019, \quad \lambda_{5,6} = 1434.71 \pm 2243.09\mathrm{i} \\
&\lambda_{7,8} = 1700.63 \pm 2429.99\mathrm{i}
\end{aligned} \tag{35}$$

From these results, the undamped motion is stable and the damped motion is asymptotically stable.

4 CONCLUSIONS

The detailed exact Lagrange equations of motion of the two-gyro Anschütz compass are obtained. The earth spin velocity and zero vehicle case and the non-zero earth spin velocity and non-zero velocity case are studied using the linear approximation method.

All solutions of undamped motion in the two cases are periodic functions of time, which means that these motions are stable. Rayleigh's dissipation function is added to the equations of motion and then the eigenvalues, i.e. the solutions of characteristic equations of damped motion for the two cases, are obtained. All of the exponential parts decrease with time, and these motions are asymptotically stable by the numerical results of these eigenvalues.

From the numerical examples, the characteristic equations are seen to be slightly different from each other when the vehicle velocity and heading angle parameters vary. The effect of these parameters is rather small.

ACKNOWLEDGEMENT

This research is supported by the National Science Council, Republic of China, under Grant NSC 84-2212-E-009-027.

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APPENDIX

In this Appendix, the Lagrange equations are used to describe the equations of motion of the two-gyro Anschütz compass and obtain six equations of motion as follows:

1. *The equation of motion for* φ_1:

$$J_{x1}\omega_{x1} = C_1 \tag{36}$$

2. *The equation of motion for* φ_2:

$$J_{x2}\omega_{x2} = C_2 \tag{37}$$

3. *The equation of motion for* θ:

$$\begin{aligned} J\Bigg\{\cos\gamma\bigg[\ddot{\theta}\cos\gamma - \dot{\theta}\sin\gamma\dot{\gamma} + \left(\omega_1 + \frac{v\sin K}{R}\right) \\ \times(\sin\gamma\dot{\gamma}\sin\alpha - \cos\gamma\cos\alpha\dot{\alpha} + \cos\gamma\sin\theta\cos\alpha\dot{\gamma} + \sin\gamma\cos\theta\cos\alpha\dot{\theta} - \sin\gamma\sin\theta\sin\alpha\dot{\alpha})\bigg] \\ + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R} + \frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)(-\cos\gamma\sin\alpha + \sin\gamma\sin\theta\cos\alpha) \\ + \left[\ddot{\alpha} + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R} + \frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\tan\xi\right]\sin\gamma\cos\theta \\ + \left(\dot{\alpha} + \omega_2 + \frac{v\sin K}{R}\tan\xi\right)(\cos\gamma\cos\theta\dot{\gamma} - \sin\gamma\sin\theta\dot{\theta})\Bigg\} + \cdots = 0 \end{aligned} \tag{38}$$

4. *The equation of motion for* γ:

$$J\left\{\ddot{\gamma}+\left(\omega_1+\frac{v\sin K}{R}\right)(-\sin\theta\dot{\theta}\cos\alpha-\cos\theta\sin\alpha\sin\alpha\dot{\alpha})+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\cos\theta\cos\alpha\right.$$
$$-\left[\ddot{\alpha}+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\tan\xi\right]\sin\theta-\left(\dot{\alpha}+\omega_2+\frac{v\sin K}{R}\tan\xi\right)\cos\theta\dot{\theta}$$
$$\left.+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\cos\theta\sin\alpha+\frac{v\cos K}{R}(\cos\alpha\cos\theta\dot{\alpha}-\sin\theta\sin\alpha\dot{\theta})\right\}+\cdots=0 \tag{39}$$

5. *The equation of motion for* α:

$$-J\left(\cos\theta\dot{\theta}\left(\dot{\gamma}+\omega_1+\frac{v\sin K}{R}\right)\cos\theta\cos\alpha-\left(\dot{\alpha}+\omega_2+\frac{v\sin K}{R}\tan\xi\right)\sin\theta+\frac{v\cos K}{R}\cos\theta\sin\alpha\right.$$
$$+\sin\theta\left\{\ddot{\gamma}+\left(\omega_1+\frac{v\sin K}{R}\right)(-\sin\theta\dot{\theta}\cos\alpha-\cos\theta\sin\alpha\dot{\alpha})+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\sin\theta\sin\alpha\right.$$
$$-\left[\ddot{\alpha}+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)\tan\xi\right]\sin\theta-\left(\dot{\alpha}+\omega_2+\frac{v\sin K}{R}\tan\xi\right)\cos\theta\dot{\theta}$$
$$\left.\left.+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\cos K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\sin K}{R}\right)\cos\theta\sin\alpha+\frac{v\cos K}{R}(\cos\alpha\cos\theta\dot{\alpha}-\sin\theta\sin\alpha\dot{\theta})\right\}\right)$$
$$+J\left\{\sin\gamma\cos\theta\left[\ddot{\theta}\cos\gamma-\dot{\theta}\sin\gamma\dot{\gamma}+\left(\omega_2+\frac{v\sin K}{R}\right)(\sin\gamma\dot{\gamma}\sin\alpha\right.\right.$$
$$\left.-\cos\gamma\cos\alpha\dot{\alpha}+\cos\gamma\sin\theta\cos\alpha\dot{\theta}+\sin\gamma\cos\alpha\dot{\theta}-\sin\gamma\sin\theta\sin\alpha\dot{\alpha})\right]$$
$$\left.+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)(-\cos\gamma\sin\alpha+\sin\gamma\sin\theta\cos\alpha)\right\}+\cdots=0 \tag{40}$$

6. *The equation of motion for* φ:

$$-J_{z1}\left[-\ddot{\theta}\sin\gamma-\dot{\theta}\cos\gamma\dot{\gamma}+\left(\omega_2+\frac{v\sin K}{R}\right)\right.$$
$$\times(\cos\alpha\sin\gamma\dot{\alpha}+\sin\alpha\cos\gamma\dot{\gamma}-\sin\gamma\sin\theta\cos\alpha\dot{\gamma}+\cos\gamma\cos\theta\cos\alpha\dot{\theta}-\cos\gamma\sin\theta\sin\alpha\dot{\theta})$$
$$+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\sin K}{R}+\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\cos K}{R}\right)(\sin\alpha\sin\gamma+\cos\gamma\sin\theta\cos\alpha)$$
$$+\left(\dot{\alpha}+\omega_1+\frac{v\sin K}{R}\tan\xi\right)(-\sin\gamma\cos\theta\dot{\gamma}-\cos\gamma\sin\theta\dot{\theta})$$
$$-\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\cos K}{R}-\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\sin K}{R}\right)\cos\gamma\sin\theta+\left(\frac{v\cos K}{R}\right)\cos\gamma\cos\theta$$
$$+\frac{v\sin K}{R}(-\sin\gamma\sin\theta\sin\alpha\dot{\gamma}+\cos\gamma\cos\theta\sin\alpha\dot{\theta}+\cos\gamma\sin\theta\cos\alpha\dot{\alpha}-\cos\gamma\cos\alpha\dot{\gamma}+\sin\gamma\sin\alpha\dot{\alpha})$$
$$\left.+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\cos K}{R}-\frac{\mathrm{d}K}{\mathrm{d}t}\frac{v\sin K}{R}\right)(\cos\gamma\sin\theta\sin\alpha-\sin\gamma\cos\alpha)-\ddot{\varphi}\right]+\cdots=0 \tag{41}$$