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Embedding hamiltonian paths in hypercubes with a required vertex in a fixed position

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Abstract

Assume that n is a positive integer with $n \geqslant 2$. It is proved that between any two different vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l for any l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$. We expect such path $P_l(\mathbf{x}, \mathbf{y})$ can be further extended by including the vertices not in $P_l(\mathbf{x}, \mathbf{y})$ into a hamiltonian path from $\mathbf{x}$ to a fixed vertex $\mathbf{z}$ or a hamiltonian cycle. In this paper, we prove that for any two vertices $\mathbf{x}$ and $\mathbf{z}$ from different partite set of n-dimensional hypercube Q_n, for any vertex $\mathbf{y} \in V(Q_n) - \{\mathbf{x}, \mathbf{z}\}$, and for any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1 - h(\mathbf{y}, \mathbf{z})$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$, there exists a hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. Moreover, for any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n and for any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^{n-1}$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$, there exists a hamiltonian cycle $S(\mathbf{x}, \mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x}, \mathbf{y}) = l$.

Keywords: Interconnection networks

1. Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation, we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets V_0 and V_1 such that every edge joins V_0 with V_1. Two vertices u and v are *adjacent* if $(u, v) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possible $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$ where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* is a cycle of length $|V(G)|$. A *hamiltonian path* is a path of length $|V(G)| - 1$.

Interconnection networks play an important role in parallel computing/communication systems. The graph embedding problem is a central issue in evaluating a network. The graph embedding problem asked if the

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quest graph is a subgraph of a host graph, and an important benefit of the graph embeddings is that we can apply existing algorithm for guest graphs to host graphs. This problem has attracted a burst of studies in recent years. Cycle networks and path networks are suitable for designing simple algorithms with low communication costs. The cycle embedding problem, which deals with all possible length of the cycles in a given graph, is investigated in a lot of interconnection networks [2, 7,9,11,14]. The path embedding problem, which deals with all possible length of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks [3–6,11,13,14].

Let $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$ be an n-bit binary strings. The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of i such that $u_i = 1$. Let $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$ and $\mathbf{v} = v_n v_{n-1} \ldots v_2 v_1$ be two n-bit binary strings. The *Hamming distance* $h(\mathbf{u}, \mathbf{v})$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, denoted by Q_n, consists of all n-bit binary strings as its vertices and two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Thus, Q_n is a bipartite graph with bipartition $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$ and $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$. A vertex $\mathbf{u}$ of Q_n is white if $w(\mathbf{u})$ is odd, otherwise $\mathbf{u}$ is black. It is known that $d_{Q_n}(\mathbf{u}, \mathbf{v}) = h(\mathbf{u}, \mathbf{v})$. For $i = 0, 1$, let Q^i_{n-1} denote the subgraph of Q_n induced by $\{\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1 \mid u_n = i\}$. Obviously, Q^i_{n-1} is isomorphic to Q_{n-1}. For any vertex $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$, we use $\mathbf{u}^n$ to denote the vertex $\mathbf{v} = v_n v_{n-1} \ldots v_2 v_1$ with $u_i = v_i$ for $1 \leqslant i \leqslant n-1$ and $u_n = 1 - v_n$.

The hypercube Q_n is one of the most popular interconnection networks for parallel computer/communication system [10]. This is partly due to its attractive properties such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm.

For the path embedding problem on hypercube, Li et al. [11] proved that between any two different vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l for any l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$. Note that the requirement $2|(l - h(\mathbf{x}, \mathbf{y}))$ is needed because Q_n is a bipartite graph for every positive integer n. Moreover, the requirement $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1$ is needed because the distance between $\mathbf{x}$ and $\mathbf{y}$ in Q_n is $h(\mathbf{x}, \mathbf{y})$. Obviously, we expect such path $P_l(\mathbf{x}, \mathbf{y})$ can be further extended by including the vertices not in $P_l(\mathbf{x}, \mathbf{y})$ into a hamiltonian path from $\mathbf{x}$ to a fixed vertex $\mathbf{z}$ or a hamiltonian cycle. For this reason, we prove that for any two vertices $\mathbf{x}$ and $\mathbf{z}$ from different partite set of Q_n, for any vertex $\mathbf{y} \notin \{\mathbf{x}, \mathbf{z}\}$, and for any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1 - h(\mathbf{y}, \mathbf{z})$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$, there exists a hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. As a corollary, we prove that for any $n \geqslant 2$, for any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n and for any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^{n-1}$ and $2|(l - h(\mathbf{x}, \mathbf{y}))$ there exists a hamiltonian cycle $S(\mathbf{x}, \mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x}, \mathbf{y}) = l$.

In the following section, we introduce another interesting property, called 2RP-property, of hypercubes. We defer the proof of 2RP-property for hypercube in Section 3. Instead we prove that many interesting properties, including the aforementioned properties, of hypercube is a direct consequence of 2PR-property. In Section 4, we give a discussion on 2RP-property.

2. 2RP-property

Assume that n is any positive integer with $n \geqslant 2$. Let $\mathbf{u}$ and $\mathbf{x}$ be two distinct white vertices of Q_n and $\mathbf{v}$ and $\mathbf{y}$ be two distinct black vertices of Q_n. It is proved in [8] that there are two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans Q_n. We call such property the be the 2P property. The 2P property has been used in many applications of hypercubes [8,12]. Obviously, the lengths of P_1 and P_2 satisfy $l(P_1) + l(P_2) = 2^n - 2$. Yet, we can further require that the length of P_1, and hence the length of P_2, can be any odd integer such that $l(P_1) \geqslant h(\mathbf{u}, \mathbf{v})$ and $l(P_2) \geqslant h(\mathbf{x}, \mathbf{y})$. We call such property to be the 2RP property. More precisely, let $\mathbf{u}$ and $\mathbf{x}$ be two distinct white vertices of Q_n and $\mathbf{v}$ and $\mathbf{y}$ be two distinct black vertices of Q_n. Let l_1 and l_2 are odd integers with $l_1 \geqslant h(\mathbf{u}, \mathbf{v})$, $l_2 \geqslant h(\mathbf{x}, \mathbf{y})$, and $l_1 + l_2 = 2^n - 2$. Then there are two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_1) = l_1$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans Q_n. In next section, we will formally prove the following theorem.

Theorem 2.1. *Q_n satisfies the 2RP property if $n \geqslant 4$.*

Now, we make some remarks to illustrate that some interesting properties of hypercubes are consequences of Theorem 2.1.

Remark 1. The hamiltonian laceable property of hypercubes, proved in [15], states that there exists a hamiltonian path of Q_n joining any white vertex $\mathbf{u}$ to any black vertex $\mathbf{y}$. Now, we prove that Q_n is hamiltonian laceable by Theorem 2.1. Obviously, Q_n is hamiltonian laceable for $n = 1, 2, 3$. Since $n \geqslant 4$, we can choose a pair of adjacent vertices $\mathbf{v}$ and $\mathbf{x}$ such that $\mathbf{v}$ is a black vertex with $\mathbf{v} \neq \mathbf{y}$ and $\mathbf{x}$ be a white vertex with $\mathbf{x} \neq \mathbf{u}$.

By Theorem 2.1, there are two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining **u** to **v**, (2) P_2 is a path joining **x** to **y**, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}, P_2, \mathbf{y} \rangle$ forms a hamiltonian path joining **u** to **y**. Thus, Q_n is hamiltonian laceable.

Remark 2. The bipanconnected property of Q_n, proved in [11], stated that between any two different vertices **x** and **y** of Q_n there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l for any l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1$ and $2 | (l - h(\mathbf{x}, \mathbf{y}))$. Now, we prove that Q_n is bipanconnected by Theorem 2.1. Obviously, Q_n is bipanconnected for $n = 1, 2, 3$. Now, we consider $n \geqslant 4$. Without loss of generality, we assume that **x** is a white vertex.

Suppose that **y** is a black vertex. Thus, $h(\mathbf{x}, \mathbf{y})$ is odd. Let l be any odd integer with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1$. Suppose that $l = 2^n - 1$. By Remark 1, Q_n is hamiltonian laceable. Obviously, the hamiltonian path of Q_n joining **x** and **y** is of length $2^n - 1$. Suppose that $l < 2^n - 1$. Since $n \geqslant 4$, we can choose a pair of adjacent vertices **u** and **v** such that **u** is a white vertex with $\mathbf{u} \neq \mathbf{x}$ and **v** be a black vertex with $\mathbf{v} \neq \mathbf{y}$. Obviously, $h(\mathbf{u}, \mathbf{v}) = 1$. By Theorem 2.1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining **u** to **v** with $l(P_1) = 2^n - 2 - l$, (2) P_2 is a path joining **x** to **y** with $l(P_2) = l$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, P_2 is a path of length l joining **x** to **y**.

Suppose that **y** is a white vertex. Thus, $h(\mathbf{x}, \mathbf{y})$ is even. Let l be any even integer with $h(\mathbf{x}, \mathbf{y}) \leqslant l < 2^n - 1$. Since $n \geqslant 4$, we can choose two different neighbors **u** and **v** of **y** such that $h(\mathbf{x}, \mathbf{u}) = h(\mathbf{x}, \mathbf{y}) - 1$. By Theorem 2.1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining **x** to **u** with $l(P_1) = l - 1$, (2) P_2 is a path joining **y** to **v** with $l(P_2) = 2^n - l - 1$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{x}, P_1, \mathbf{u}, \mathbf{y} \rangle$ is a path of length l joining **x** to **y**.

Thus, Q_n is bipanconnected.

Remark 3. The edge-bipancyclic property of Q_n, proved in [11], stated that for any edge $e = (\mathbf{x}, \mathbf{y})$ and for any even integer with $4 \leqslant l \leqslant 2^n$ there exists a cycle of length l containing the edge e if $n \geqslant 2$. Again, we prove that Q_n is edge-bipancyclic by Theorem 2.1. Obviously, Q_n is edge-bipancyclic for $n = 2, 3$. Thus, we consider $n \geqslant 4$. Suppose that $l = 2^n$. By Remark 1, there exists a hamiltonian path P joining **x** to **y**. Obviously, $\langle \mathbf{x}, P, \mathbf{y}, \mathbf{x} \rangle$ forms a hamiltonian cycle of length 2^n containing the edge e. Suppose that $l < 2^n$. Since $n \geqslant 4$, we can choose a pair of adjacent vertices **u** and **v** such that **u** is a white vertex. By Theorem 2.1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining **u** to **v** with $l(P_1) = 2^n - l - 1$, (2) P_2 is a path joining **x** to **y** with $l(P_2) = l - 1$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{x}, P_2, \mathbf{y}, \mathbf{x} \rangle$ is a cycle of length l containing the edge e. Thus, Q_n is edge-bipancyclic for $n \geqslant 2$.

The following results are also consequence of Theorem 2.1.

Theorem 2.2. *Assume that n be any positive integer with $n \geqslant 2$. Let* **x** *and* **z** *be two vertices from different partite set of Q_n and* **y** *be a vertex of Q_n that is not in $\{\mathbf{x}, \mathbf{z}\}$. For any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^n - 1 - h(\mathbf{y}, \mathbf{z})$ and $2 | (l - h(\mathbf{x}, \mathbf{y}))$, there exists a hamiltonian path $R(\mathbf{x}, \mathbf{y}; \mathbf{z}, l)$ from* **x** *to* **z** *such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$.*

Proof. By brute force, we can check the theorem holds for $n = 2, 3$. Now, we consider $n \geqslant 4$. Without loss of generality, we assume that **x** is a white vertex and **z** is a black vertex.

Suppose that **y** is a black vertex. Obviously, $h(\mathbf{y}, \mathbf{z}) \geqslant 2$. There exists a neighbor **w** of **y** such that $\mathbf{w} \neq \mathbf{x}$ and $h(\mathbf{w}, \mathbf{z}) = h(\mathbf{y}, \mathbf{z}) - 1$. Obviously, **w** is a white vertex. By Theorem 2.1, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining **x** to **y** with $l(R_1) = l$, (2) R_2 is a path joining **w** to **z** with $l(R_2) = 2^n - l - 2$, and (3) $R_1 \cup R_2$ spans Q_n. We set R as $\langle \mathbf{x}, R_1, \mathbf{y}, \mathbf{w}, R_2, \mathbf{z} \rangle$. Obviously, R is the required hamiltonian path.

Suppose that **y** is a white vertex. Obviously, $h(\mathbf{x}, \mathbf{y}) \geqslant 2$. There exists a neighbor **w** of **y** such that $\mathbf{w} \neq \mathbf{z}$ and $h(\mathbf{w}, \mathbf{x}) = h(\mathbf{y}, \mathbf{x}) - 1$. Obviously, **w** is a black vertex. By Theorem 2.1, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining **x** to **w** with $l(R_1) = l - 1$, (2) R_2 is a path joining **y** to **z** with $l(R_2) = 2^n - l - 1$, and (3) $R_1 \cup R_2$ spans Q_n. We set R as $\langle \mathbf{x}, R_1, \mathbf{w}, \mathbf{y}, R_2, \mathbf{z} \rangle$. Obviously, R is the required hamiltonian path. □

Corollary 2.1. *Assume that n is a positive integer with $n \geqslant 2$. Let* **x** *and* **y** *be any two different vertices of Q_n. For any integer l with $h(\mathbf{x}, \mathbf{y}) \leqslant l \leqslant 2^{n-1}$ there exists a hamiltonian cycle $S(\mathbf{x}, \mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x}, \mathbf{y}) = l$ and $2 | (l - h(\mathbf{x}, \mathbf{y}))$.*

Proof. Let **z** be a neighbor of **x** such that $\mathbf{z} \neq \mathbf{y}$. By Theorem 2.2, there exits a hamiltonian path R joining **x** to **z** such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. We set S as $\langle \mathbf{x}, R, \mathbf{z}, \mathbf{x} \rangle$. Obviously, S forms the required hamiltonian cycle. □

3. Proof of Theorem 2.1

Now, we prove Theorem 2.1. By brute force, we can check the theorem holds for $n = 4$. Assume the theorem holds for any Q_k with $4 \leqslant k < n$. Without loss of generality, we can assume that $l_1 \geqslant l_2$. Thus, $l_2 \leqslant 2^{n-1} - 1$. Since Q_n is edge symmetric, we can assume that $\mathbf{u} \in V(Q_{n-1}^0)$ and $\mathbf{x} \in V(Q_{n-1}^1)$. We have the following cases.

Case 1. $\mathbf{v} \in V(Q_{n-1}^0)$ and $\mathbf{y} \in V(Q_{n-1}^1)$. Suppose that $l_2 < 2^{n-1} - 1$. By Remark 1, there exists a hamiltonian path R of Q_{n-1}^0 joining $\mathbf{u}$ and $\mathbf{v}$. Since the length of R is $2^{n-1} - 1$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{q}, R_2, \mathbf{v} \rangle$ for some black vertex $\mathbf{p}$ with $\mathbf{p}^n \neq \mathbf{x}$ and some white vertex $\mathbf{q}$ with $\mathbf{q}^n \neq \mathbf{y}$. Obviously, $h(\mathbf{p}^n, \mathbf{q}^n) = 1$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^n$ to $\mathbf{q}^n$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^n, S_1, \mathbf{q}^n, \mathbf{q}, R_2, \mathbf{v} \rangle$ and set P_2 as S_2. Obviously, P_1 and P_2 are the required paths. See Fig. 1(a) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. By Remark 1, there exists a hamiltonian path P_1 of Q_{n-1}^0 joining $\mathbf{u}$ and $\mathbf{v}$ and there exists a hamiltonian path P_2 of Q_{n-1}^1 joining $\mathbf{x}$ and $\mathbf{y}$. Obviously, P_1 and P_2 are the required paths. See Fig. 1(b) for illustration.

Case 2. $\{\mathbf{v}, \mathbf{y}\} \subset V(Q_{n-1}^1)$. Suppose that $l_2 < 2^{n-1} - 1$. We choose a neighbor $\mathbf{p}$ of $\mathbf{v}$ such that $\mathbf{p} \neq \mathbf{x}$. Obviously, $\mathbf{p}$ is a white vertex. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. By Remark 1, there exists a hamiltonian path R of Q_{n-1}^0 joining $\mathbf{u}$ and $\mathbf{p}^n$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^n, \mathbf{p}, S_1, \mathbf{v} \rangle$ and we set P_2 as S_2. Obviously, P_1 and P_2 are the required paths. See Fig. 1(c) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. Again, we choose a neighbor $\mathbf{p}$ of $\mathbf{v}$ such that $\mathbf{p} \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. Obviously, we can write S_2 as $\langle \mathbf{x}, S_2^1, \mathbf{r}, \mathbf{s}, S_2^2, \mathbf{y} \rangle$ for some black vertex $\mathbf{r}$ with $\mathbf{r}^n \neq \mathbf{u}$. Again by induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^n$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{r}^n$ to $\mathbf{s}^n$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q_{n-1}^0. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}^n, \mathbf{p}, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, S_2^1, \mathbf{r}, \mathbf{r}^n, \mathbf{s}^n, \mathbf{s}, S_2^2, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. See Fig. 1(d) for illustration.

Case 3. $\mathbf{y} \in V(Q_{n-1}^0)$ and $\mathbf{v} \in V(Q_{n-1}^1)$. Suppose that $l_2 = 1$. Obviously, $\mathbf{x} = \mathbf{y}^n$. Let $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q_{n-1}^0 such that $\mathbf{y}^n \neq \mathbf{v}$ and let $\mathbf{q}$ be a neighbor of $\mathbf{p}$ in Q_{n-1}^0 such that $\mathbf{p} \neq \mathbf{y}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{q}$ and $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ and $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q_{n-1}^0. Obviously, $\mathbf{p}^n$ is a black vertex and $\mathbf{q}^n$ is a white vertex. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{q}^n$ to $\mathbf{v}$ with $l(S_1) = 2^{n-1} - 3$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{q}, \mathbf{p}, \mathbf{p}^n, \mathbf{q}^n, S_1, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, \mathbf{y} \rangle$. Obvi-

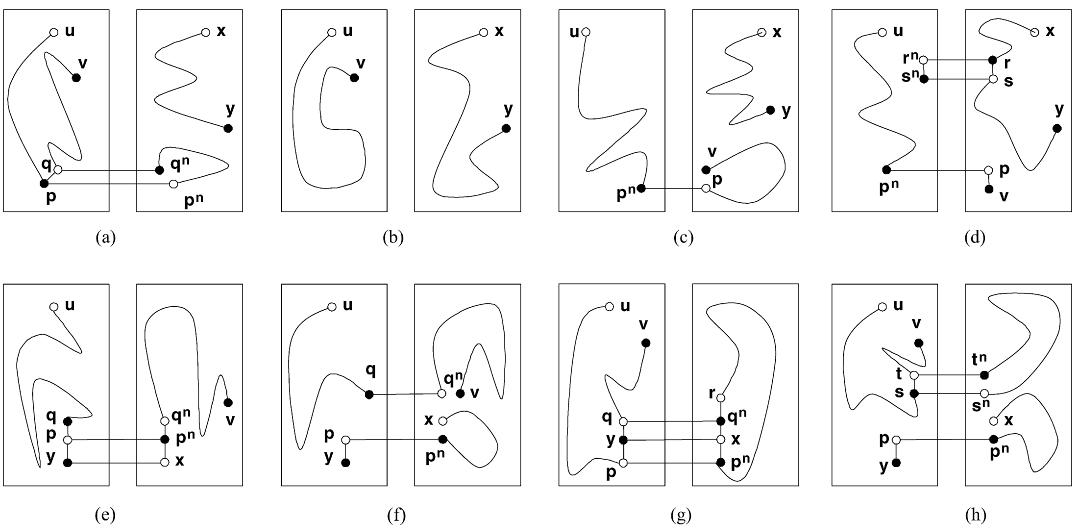

Fig. 1. Illustration for Theorem 2.1.

ously, P_1 and P_2 are the required paths. See Fig. 1(e) for illustration.

Suppose that $l_2 \geqslant 3$. We set $\mathbf{p}$ be a neighbor in Q_{n-1}^0 of $\mathbf{y}$ with $\mathbf{p} \neq \mathbf{u}$ if $h(\mathbf{x}, \mathbf{y}) = 1$ and set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q_{n-1}^0 with $\mathbf{p} \neq \mathbf{u}$ and $h(\mathbf{p}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) - 1$ if $h(\mathbf{x}, \mathbf{y}) \geqslant 3$. Let $\mathbf{q}$ be a neighbor $\mathbf{v}^n$ in Q_{n-1}^0 such that $\mathbf{q} \neq \mathbf{y}$ and $\mathbf{q}^n \neq \mathbf{x}$. Thus, $h(\mathbf{q}^n, \mathbf{v}) = 1$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{q}$ to $\mathbf{y}$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q_{n-1}^0. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{q}^n$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1} + 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = 1_2 - 2$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{q}, \mathbf{q}^n, S_1, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, S_2, \mathbf{p}^n, \mathbf{p}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. See Fig. 1(f) for illustration.

Case 4. $\{\mathbf{v}, \mathbf{y}\} \subset V(Q_{n-1}^0)$. Suppose that $l_2 = 1$. Obviously, $\mathbf{y} = \mathbf{x}^n$. By Remark 1, there exist a hamiltonian path R of Q_{n-1}^0 joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, R can be written as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{y}, \mathbf{q}, R_2, \mathbf{v} \rangle$. Note that $\mathbf{u} = \mathbf{p}$ if $l(R_1) = 0$. Obviously, $\mathbf{p}$ and $\mathbf{q}$ are white vertices. Thus, $\mathbf{p}^n$ and $\mathbf{q}^n$ are black vertices. Let $\mathbf{r}$ be a neighbor of $\mathbf{q}^n$ in Q_{n-1}^1 such that $\mathbf{r} \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^n$ to $\mathbf{r}$ with $l(S_1) = 2^{n-1} - 3$, (2) S_2 is a path joining $\mathbf{q}^n$ to $\mathbf{x}$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^n, S_1, \mathbf{r}, \mathbf{q}^n, \mathbf{q}, R_2, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. See Fig. 1(g) for illustration.

Suppose that $l_2 \geqslant 3$. We set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q_{n-1}^0 with $\mathbf{p} \neq \mathbf{u}$ if $h(\mathbf{x}, \mathbf{y}) = 1$ and set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q_{n-1}^0 with $\mathbf{p} \neq \mathbf{u}$ and $h(\mathbf{p}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) - 1$ if $h(\mathbf{x}, \mathbf{y}) \geqslant 3$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q_{n-1}^0. Obviously, we can write R_1 as $\langle \mathbf{u}, R_1^1, \mathbf{s}, \mathbf{t}, R_1^2, \mathbf{v} \rangle$ for some black vertex $\mathbf{s}$ such that $\mathbf{s}^n \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{t}^n$ with $l(S_1) = l_1 - 2^{n-1} - 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = l_2 - 2$, and (3) $S_1 \cup S_2$ spans Q_{n-1}^1. We set P_1 as $\langle \mathbf{u}, R_1^1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{t}^n, \mathbf{t}, R_1^2, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, S_2, \mathbf{p}^n, \mathbf{p}, \mathbf{y} \rangle$. Obviously, P_1 and P_2 are the required paths. See Fig. 1(h) for illustration.

4. Discussion

Since there are four vertices in Q_2, it is easy to check that Q_2 satisfies the 2RP-property. However, the 2RP-property does not hold for Q_3. Let $\mathbf{u} = 000$, $\mathbf{v} = 111$, $\mathbf{x} = 011$, and $\mathbf{y} = 001$. By brute force, we can check that we cannot find two disjoint paths P_1 and P_2 such that P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_1) = 3$ and P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_2) = 3$.

By changing the roles of the vertices in bipartite sets in Theorem 2.1, we have the following theorem. The proof is similar to the proof of Theorem 2.1.

Theorem 4.1. *Assume that n is a positive integer with $n \geqslant 4$. Let $\mathbf{u}$ and $\mathbf{x}$ be two distinct white vertices of Q_n and $\mathbf{v}$ and $\mathbf{y}$ be two distinct black vertices of Q_n. Let l_1 and l_2 be even integers with $l_1 \geqslant h(\mathbf{u}, \mathbf{x})$, $l_2 \geqslant h(\mathbf{v}, \mathbf{y})$, and $l_1 + l_2 = 2^n - 2$ except for the case $\{l_1, l_2\} = \{2, 2^n - 4\}$ with $\{\mathbf{u}, \mathbf{x}, \mathbf{v}, \mathbf{y}\}$ inducing a cycle of length 4. There exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{x}$ and $l(P_1) = l_1$, (2) P_2 is a path joining $\mathbf{v}$ to $\mathbf{y}$ and $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans Q_n.*

Suppose that $n = 3$. Let $\mathbf{u} = 000$, $\mathbf{v} = 011$, $\mathbf{x} = 100$, and $\mathbf{y} = 111$. We can check that there are no two disjoint paths P_1 and P_2 such that P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ and P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ such that $P_1 \cup P_2$ spans Q_3. Again, the above theorem does not hold for $n = 3$.

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