Discrete-time quasi-sliding-mode control for a class of nonlinear control systems

S.-D. Xu, Y.-W. Liang and S.-W. Chiou

A quasi-sliding-mode control law is proposed for a class of discrete-time nonlinear control systems concerning system stabilisation and chatter alleviation. An illustrative example is also given to demonstrate the use and benefits of the scheme.

Introduction: It is known that sliding mode control (SMC) schemes possess the benefits of fast response and low sensitivity to system parameter uncertainties and disturbances [1, 2]. Therefore, they have been widely applied to control a variety of systems [1–6]. On the other hand, many control systems have been mathematically formulated as a discrete-time version because of the popularity and scale of applications of digital computers in technology and industry. Thus, the study of SMC for discrete-time systems has recently attracted considerable attention [1, 3–6]. However, owing to a finite sampling frequency characteristic in discrete-time systems, the system states can only be expected to approach the selected sliding surface and remain around it, instead of remaining on the surface when the system undergoes external disturbances. Therefore, the so-called quasi-sliding-mode (QSM) concept was introduced and discussed in discrete-time systems [1, 4, 6]. A brief summary is given here. The system states are required to monotonically approach the sliding surface until they enter the vicinity of the surface, and they then remain inside [1]. The vicinity of the sliding surface is called a quasi-sliding-mode band (QSMB). Under this QSM definition, it is noted that the system states are not required to cross the sliding surface, as in the definition given by Gao *et al.* [4]. In this Letter, we employ the QSM concept to study the stabilisation for a class of discrete-time nonlinear control systems.

Main results: Consider the discrete-time nonlinear control systems:

$$\mathbf{x}_1[k+1] = \mathbf{f}_1(\mathbf{x}_1[k], \mathbf{x}_2[k]) \tag{1}$$

and

$$\mathbf{x}_2[k+1] = \mathbf{f}_2(\mathbf{x}_1[k], \mathbf{x}_2[k]) + G(\mathbf{x}_1[k], \mathbf{x}_2[k])\mathbf{u}[k] + \mathbf{d}[k] \tag{2}$$

Here, $\mathbf{x}_1 \in \Re^{n_1}$ and $\mathbf{x}_2 \in \Re^{n_2}$ are state vectors, $\mathbf{u} \in \Re^{n_2}$ denotes system input, $\mathbf{d}[k]$ contains possible uncertainties and/or disturbances, $\mathbf{f}_1$, $\mathbf{f}_2$ and G are three smooth functions with appropriate dimensions, and $\mathbf{f}_1(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\mathbf{f}_2(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. In this study, we say a vector $\mathbf{a} \geq \mathbf{0}$ if and only if $a_i \geq 0$ for each component a_i of $\boldsymbol{a}$. Below we impose three assumptions for System (1)–(2):

Assumption 1: $G(\cdot)$ is a non-singular matrix for all state vectors.

Assumption 2: There exists a function $\mathbf{x}_2[k] = \phi(\mathbf{x}_1[k])$, $\phi(\mathbf{0}) = \mathbf{0}$, such that the origin of the reduced order system is $\mathbf{x}_1[k+1] = \mathbf{f}_1(\mathbf{x}_1[k], \phi(\mathbf{x}_1[k]))$ is asymptotically stable (AS).

Assumption 3: $\mathbf{d}_l \leq \mathbf{d}[k] \leq \boldsymbol{d}_u$, where $\mathbf{d}_l$ and $\mathbf{d}_u$ are two constant vectors.

Owing to the aforementioned merits of SMC designs, we apply the QSM approach to the controller design. From Assumption 2, we select the sliding surface to be:

$$\mathbf{s}[k] = \mathbf{x}_2[k] - \phi(\mathbf{x}_1[k]) = \mathbf{0} \tag{3}$$

Clearly, the origin of the reduced order dynamics (i.e. set $\mathbf{s}[k] \equiv \mathbf{0}$) is AS by Assumption 2. To derive a suitable ϕ for a sliding surface, several approaches have been proposed. For instance, Kalman and Bertram [7] used the second method of Lyapunov, while Zheng *et al.* [5] adopted the linear matrix inequalities (LMI) technique.

To organise an appropriate QSM controller, we adopt the approach of Bartoszewicz [1] which was developed for linear systems. Suppose that $\mathbf{s}_d[k]$ has is a desired sliding variable trajectory. One candidate of $\mathbf{s}_d[k]$ has the following form [1]:

$$\mathbf{s}_d[k] = \frac{k^* - k}{k^*}\mathbf{s}[0] \quad \text{if} \quad k < k^*; \qquad \mathbf{s}_d[k] = \mathbf{0} \quad \text{if} \quad k \geq k^* \tag{4}$$

where k^* is a positive integer selected by the designer to make the system states reach the sliding surface in k^* steps; the choice depends on the desired convergence rate to the selected sliding surface and the maximum control magnitude a control system may provide. Once $\mathbf{s}_d[k]$ is determined, we choose the QSM controller to be:

$$\begin{aligned}\mathbf{u}[k] = {} & G(\mathbf{x}_1[k], \mathbf{x}_2[k])^{-1}\,[\phi(\mathbf{f}_1(\mathbf{x}_1[k], \mathbf{x}_2[k])) \\ & - \mathbf{f}_2(\mathbf{x}_1[k], \mathbf{x}_2[k]) - \mathbf{d}_0 + \mathbf{s}_d[k+1]]\end{aligned} \tag{5}$$

where $\mathbf{d}_0 = 1/2(\mathbf{d}_l + \mathbf{d}_u)$. Substituting (1), (2) and (5) into (3), we have $\|\mathbf{s}[k+1] - \mathbf{s}_d[k+1]\| = \|\mathbf{d}[k] - \mathbf{d}_0\| \leq \delta_d$, where $\delta_d = (1/2)\|\mathbf{d}_u - \mathbf{d}_l\|$ and $\|\cdot\|$ is a vector norm. Thus, the system states will approach the sliding surface and remain around there after k^* steps. The constant δ_d is the width of the QSMB. It is worth noting that the asymptotic stability performance can be achieved if $\mathbf{d}[k] \equiv \mathbf{0}$.

If the change rate of $\mathbf{d}[k]$ is limited by the relation $\|\mathbf{d}[k+1] - \mathbf{d}[k]\| \leq \Delta_d$, where Δ_d is a known constant and $\Delta_d < \delta_d$, then the control given by (5) can be modified as:

$$\begin{aligned}\mathbf{u}[k] = {} & G(\mathbf{x}_1[k], \mathbf{x}_2[k])^{-1}\Big[\phi(\mathbf{f}_1(\mathbf{x}_1[k], \mathbf{x}_2[k])) \\ & - \mathbf{f}_2(\mathbf{x}_1[k], \mathbf{x}_2[k]) - \mathbf{d}_0 + \mathbf{s}_d[k+1] - \sum_{i=0}^{k}(\mathbf{s}[i] - \mathbf{s}_d[i])\Big]\end{aligned} \tag{6}$$

We now show, by mathematical induction, that $\|\mathbf{s}[k+1] - \mathbf{s}_d[k+1]\| \leq \Delta_d$ for all k. From (1)–(3), the modified law (6) and the fact $\mathbf{s}[0] = \mathbf{s}_d[0]$, we have $\mathbf{s}[1] - \mathbf{s}_d[1] = \mathbf{d}[0] - \mathbf{d}_0$ and $\mathbf{s}[2] - \mathbf{s}_d[2] = \mathbf{d}[1] - \mathbf{d}_0 - (\mathbf{s}[1] - \mathbf{s}_d[1]) = \mathbf{d}[1] - \mathbf{d}[0]$. Suppose that $\mathbf{s}[i] - \mathbf{s}_d[i] = \mathbf{d}[i-1] - \mathbf{d}[i-2]$ for all $2 \leq i \leq k$. Then, $\mathbf{s}[k+1] - \mathbf{s}_d[k+1] = \mathbf{d}[k] - \mathbf{d}_0 - \sum_{i=0}^{k}(\mathbf{s}[i] - \mathbf{s}_d[i]) = \mathbf{d}[k] - \mathbf{d}_0 - (\mathbf{d}[0] - \mathbf{d}_0) - \sum_{i=2}^{k}(\mathbf{d}[i-1] - \mathbf{d}[i-2]) = \mathbf{d}[k] - \mathbf{d}[k-1]$. Thus, $\|\mathbf{s}[k+1] - \mathbf{s}_d[k+1]\| \leq \Delta_d$, as required. Since $\Delta_d < \delta_d$, it implies that this QSMB width is smaller than the former one.

Example: Consider a trailer-truck kinematic model [8]:

$$\mathbf{x}_1[k+1] = \mathbf{x}_1[k] - \frac{\eta T}{L}\sin(\mathbf{x}_1[k]) + \frac{\eta T}{l}\tan(\mathbf{u}[k]) + \mathbf{d}[k] \tag{7}$$

$$\mathbf{x}_2[k+1] = \mathbf{x}_2[k] + \frac{\eta T}{L}\sin(\mathbf{x}_1[k]) \tag{8}$$

$$\mathbf{x}_3[k+1] = \mathbf{x}_3[k] + \eta T\cos(\mathbf{x}_1[k])\sin\left(\frac{\mathbf{x}_2[k] + \mathbf{x}_2[k+1]}{2}\right) \tag{9}$$

Here, the three states $\mathbf{x}_1[k]$, $\mathbf{x}_2[k]$ and $\mathbf{x}_3[k]$ are the angle difference between the trailer and the truck, the angle of the trailer, and the vertical position of the rear end of the trailer, respectively, $\mathbf{u}[k]$ and $\mathbf{d}[k]$ denote the steering angle and the possible disturbances, L is the length of the trailer, l is the length of the truck, T denotes the sampling period, and η is the constant speed of the backward movement. The geometry of the system can be found in Tanaka *et al.* [8]. By letting $\mathbf{x}_1[k] = (\mathbf{x}_2[k], \mathbf{x}_3[k])^T$, $\mathbf{x}_2[k] = \mathbf{x}_1[k]$ and $\mathbf{u}[k] = \tan(\mathbf{u}[k])$, System (7)–(9) can be put into the form of (1)–(2). The control objective is to realise the backward movement for the trailer-truck along the horizontal line $x_3 = 0$ without any forward movement; that is, to realise $\mathbf{x}_1 \to 0$, $\mathbf{x}_2 \to 0$ and $\mathbf{x}_3 \to 0$.

The parameters and initial states in this example are selected as follows: $L = 0.13$ m, $l = 0.087$ m, $\eta = -0.1$ m/s, $T = 0.5$ s, $\mathbf{x}_1[0] = (1.571,\ 1)^T$ and $\mathbf{x}_2[0] = 0$. The function ϕ given in (3) is set to be linear in the form of $\phi(\mathbf{x}_1[k]) = \mathbf{c}^T\,\mathbf{x}_1[k]$, where $\mathbf{c} = (1.3325,\ -3.9)^T$ is the vector such that the linearisation of the reduced-order model given in Assumption 2 has eigenvalues at 0.7 and 0.75. The desired sliding variable trajectory $\mathbf{s}_d[k]$ is taken in the form of (4) with $k^* = 10$. To demonstrate the robustness performance of the proposed schemes, we choose a slow varying disturbance, $\mathbf{d}[k] = 0.1\sin(0.1\,k)$.

Numerical simulations are given in Figs. 1 and 2. Among these, three different control laws are adopted. Two of them are the QSM controller (5) (labelled by QSMC1) and the modified QSM controller (6) (labelled by QSMC2), while the other is an existing fuzzy scheme [8] (labelled by fuzzy). It is observed from Figs. 1*a*–*c* that the states by the three schemes exhibit oscillation because of the effect of disturbance; however, the amplitudes of the oscillation by the two QSM schemes are much smaller than those of the fuzzy design. This demonstrates the robustness characteristic of the QSM designs. Fig. 2 shows the time response of the sliding variables and their magnified scale by the two QSM schemes. Both of the sliding variables are seen to approach the selected sliding surface within 10 steps, as desired. By direct inspection, the QSMB widths of the QSMC1 and the QSMC2 schemes are $\delta_d \simeq 0.2$ and $\Delta_d = |d[k] - d[k-1]| \simeq 0.01$, respectively, which agree with the theoretical results. In addition, the proposed QSM schemes do not require the system states to cross the sliding surface at each step (i.e. each sampling instant) when the states are within the QSMB. Therefore, the chattering phenomenon can be greatly alleviated when comparing to the QSM definition given by Gao *et al.* [4]. By direct calculation based on Fig. 1*d*, we have $(\|u\|_\infty)_{QSMC2} = 0.3946 < (\|u\|_\infty)_{QSMC1} = 0.3981 < (\|u\|_\infty)_{Fuzzy} = 0.4771$ and $(\|u\|_2)_{QSMC1} = 2.0157 < (\|u\|_2)_{QSMC2} = 2.0424 < (\|u\|_2)_{fuzzy} = 2.9723$, where $\|u\|_\infty := \max_k |u[k]|$ and $\|u\|_2 := \sqrt{\sum_k u^2[k]}$. From this example, it is concluded that the proposed QSM schemes are not only more robust than the existing fuzzy controller, but are also able to alleviate chatter without creating an extra control burden.

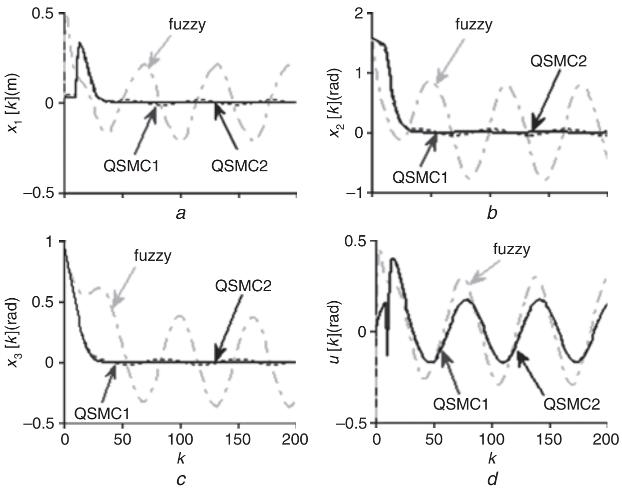

Fig. 1 *Time histories*

a–c Time history of system states
d Time history of control by the three schemes

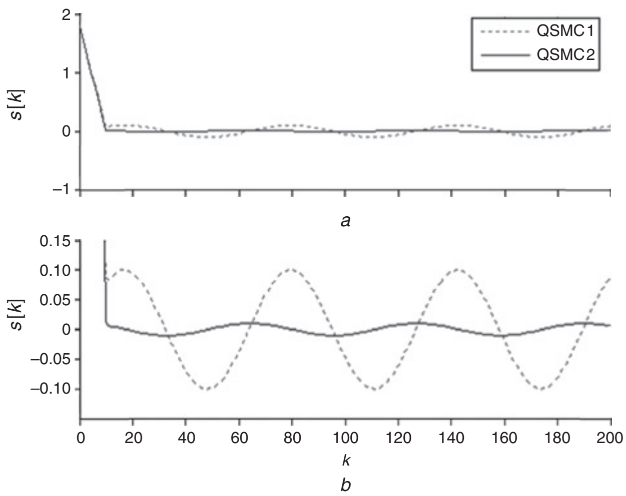

Fig. 2 *Time histories of sliding variables*

a Time history of sliding variables
b Time history of sliding variables on magnified scale by the two QSM schemes

Conclusions: A quasi-sliding-mode control scheme for system stabilisation is proposed for a class of discrete-time nonlinear control systems. It has been shown that the scheme not only achieves the stabilisation performance, but it also alleviates chatter without creating an extra control burden.

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