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Stability of an elastically connected two-body space station by the perturbation method

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Abstract: This investigation deals with the stability, in a circular orbit, of a flexible space station consisting of two inertially identical rigid end bodies connected together by an elastic structure. The earth-pointing motion and the rotation with arbitrary initial angular velocity perpendicular to the orbital plane are studied by using the multiple-scales technique. The first-order approximate analytical solution and the conditions of stability are obtained.

Keywords: stability, space station, perturbation method, multiple-scales technique

NOTATION

A, B, C	principal moments of inertia of end bodies
d_1	ratio of L with respect to R
$\overline{E}$	the earth
F^{c}_{ij}	component of structural contact force of body i in the j direction
F^{g}_{ij}	component of gravitational force of body i in the j direction
G	universal gravitational constant
L	undeformed distance between the mass centres of end bodies
m	mass of end body R_i
M	mass of earth $\overline{E}$
N	inertial reference frame
O	reference frame perpendicular to N
O_j	right-handed set of mutually perpendicular axes of P_*
p_j	elastic displacements of structure
$\tilde{p}$	dimensionless constant extension due to constant initial angular velocity ϖ
P_i	mass centre of end body R_i
P_*	mass centre of space station
R	radius of the circular path
R_i	inertial identical rigid bodies
$[\mathbf{S}]$	stiffness matrix of elastic connected structure
S_{kl}	element of the stiffness matrix $[\mathbf{S}]$ in row k and column l
T^{c}_{ij}	component of structural contact torque of body i in the j direction
T^{g}_{ij}	component of gravitational torque of body i in the j direction
T_n	different time-scales, $n = 0, 1, 2, 3, \ldots$
X^{i}_{j}	mutually perpendicular principal axes of inertia of R_i for P_i in the j direction

Greek symbols

$\alpha_1 \sim \alpha_{25}$	transformed parameters in the ϵ first-order differential equations
$\beta_1 \sim \beta_{22}$	transformed parameters in the ϵ second-order differential equations
$\gamma_0 \sim \gamma_8$	natural frequencies of the first-order differential equations
ϵ	small dimensionless parameter
θ_j	deformed angle of R_1 with respect to R_0
τ	dimensionless time
ψ_j	attitude angle of R_0 with respect to reference frame O
ω_j	component of angular velocity of R_0 in the inertial reference frame N
ϖ	initial angular velocity of axis P_*O_3
Ω	orbital angular velocity of the space station

Subscripts

i	identifies the end body, $i = 1, 2$
j	direction, $j = 1, 2, 3$

Superscripts

c	contact
g	gravitation

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1 INTRODUCTION

In recent years there has been a growing trend towards large, lightweight, flexible spacecraft. With the success of the space shuttle it may soon be possible to see satellites ranging from a few metres to several hundred metres in size. International Space Station Alpha is an example, whose lattice-like framework spans 155 m. It will take at least 18 separate flights of the space shuttle over a four year period to assemble the full station. For this type of satellite the influence of environmental forces will be very significant.

In the past papers have investigated the dynamics of the same model, an elastically connected two-body space station. In 1964 and 1965, Frueh and Miller (**1**, **2**) dealt with the effect of elastic deformations on the performance of manned space stations. Austin (**3**) analysed two axisymmetric rigid bodies connected in such a way as to permit only relative rotation about a common axis of symmetry. It was concluded that the effects of elasticity on gross rigid-body motion are of minor importance. For this model, in 1967 Robe and Kane (**4**), using rather simplified models of space stations, showed that the nature of the elastic connection can affect the stability of the vehicle. Specifically, certain vehicle configurations which are predicted to be stable when rigid must be classed as unstable when flexibility is taken into account. With a proper selection of vehicle parameters the instability can be avoided. Similar dynamic models have been used in studies of the stability in 1984 (**5**) and non-linear oscillations in 1988 (**6**).

In the present study the differential equations of motion, given by Robe and Kane (**4**), are applied. There are twelve differential equations and twelve variables. In reference (**4**), the effect of gravitational force was neglected. In this paper this effect will be included since the effects of gravitational force are not always negligible for the stability problem. It will be considered as a perturbation term and the perturbation method will be applied. The multiple-scales technique is used to obtain the first-order approximate analytical solution and the stability conditions.

The work that follows is divided into four sections. Section 2, entitled 'The first-order and second-order differential equations using the multiple-scales technique', contains a detailed description of the system model to be analysed, non-dimensionalization of the differential equations of motion and the equilibrium motion. In the final subsection of Section 2, using the multiple-scales technique, the first-order and second-order differential equations about the equilibrium motion are obtained. In the first subsection of Section 3, 'The dynamics of an earth-pointing motion', four sets of the first-order simultaneous differential equations obtained in the previous section are solved, and the first-order approximate analytical solution is obtained. In the next subsection, the secular terms in the second-order differential equations are eliminated to obtain equations for the coefficients of the first-order analytical solution. The final subsection of Section 3 gives a conclusion for the earth-pointing motion. In Section 4, 'Rotational dynamics with arbitrary angular velocity perpendicular to the orbital plane', the analytical solution of the rotation with arbitrary initial angular velocity of the motion is obtained and conclusions are given. The last section contains the conclusions of the full article.

2 THE FIRST-ORDER AND SECOND-ORDER DIFFERENTIAL EQUATIONS USING THE MULTIPLE-SCALES TECHNIQUE

The aim of this section is to obtain the first-order and second-order differential equations by the multiple-scales technique from the governing differential equations of the space station for use in subsequential analysis.

2.1 Description of the system model

The dynamics model is indicated in Fig. 1, where R_0 and R_1 represent inertially identical rigid bodies connected by an elastic structure that is light in comparison with the end bodies. N designates an inertial reference frame in which the earth $\overline{E}$ is fixed. Also fixed in N is an 'orbital plane', in which the mass centre of the space station, P_*, is presumed to move in a circular path. The radius of its path is R. With its origin at P_*, a right-handed set of mutually perpendicular axes O_1, O_2 and O_3 is oriented such that O_1 is the extension of the line passing through $\overline{E}$ and P_*, and O_3 is normal to the orbital plane. A reference frame in which these axes are fixed is designated O and this reference frame has a constant angular velocity of magnitude Ω in reference frame N. P_0 and P_1 are mass centres with respect to R_0, R_1. Finally, X^i_j (superscript $i = 1, 2$ represents end bodies, subscript $j = 1, 2, 3$ represents three directions) designate mutually perpendicular principal axes of inertia of R_i for P_i in the j direction. The orientation of the body

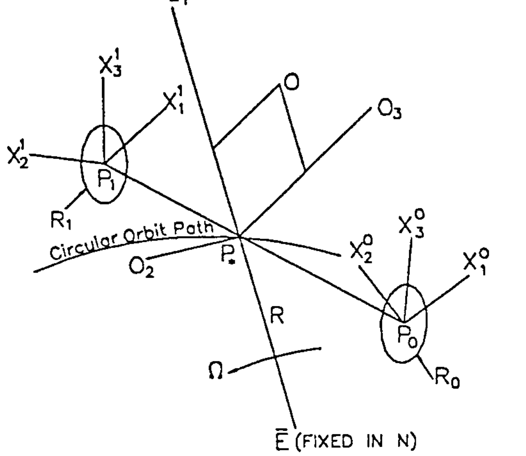

Fig. 1 Schematic representation of the space station in circular orbit

R_0 in reference frame O is described with attitude angles ψ_1, ψ_2 and ψ_3 (Fig. 2). Three successive right-handed rotations of amounts ψ_j are $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3$; the same sequence of rotations of R_1 with respect to R_0 is specified with deformed angles θ_1, θ_2 and θ_3 (Fig. 3), which have the same sequence of rotations $\theta_1 \rightarrow \theta_2 \rightarrow \theta_3$. It will be assumed that X_j^0 is parallel to X_j^1 and that X_2^0 and X_2^1 coincide when the structure connects R_0 and R_1 in the undeformed state. Finally, the elastic displacements of the structure p_1, p_2 and p_3 are presented in Fig. 4.

2.2 The non-dimensionalization of the differential equations of motion

In reference (**4**), the governing differential equations of motion are obtained in accordance with D'Alembert's principle, the resultant of all contact, inertia and gravitational forces acting on R_i, and the moments of these forces about P_i may be set equal to zero. As the analysis will be confined to deformations that are small in the usual sense of linear structural theory, all non-linear terms in p_j and θ_j may therefore be dropped. Thus, p_j and θ_j are restricted to small values, but ω_j and ψ_j are not limited in size.

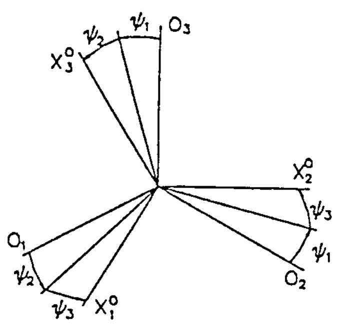

Fig. 2 Attitude angles between coordinate axes fixed in O and R_0

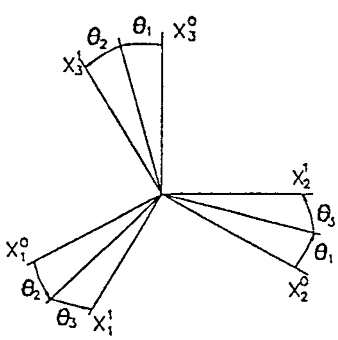

Fig. 3 Deformed angles between coordinate axes fixed in R_0 and R_1

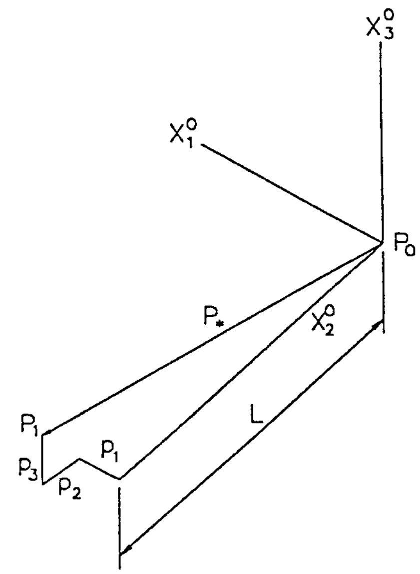

Fig. 4 Elastic displacements

To determine approximate analytical solutions of the problem by the perturbation method, first a process to keep certain elements, neglect some and approximate others must be involved. To accomplish this important step, the order of magnitude of the different elements of the system needs to be decided by comparing them with each other as well as with the basic elements of the system. Therefore, expressing the equations in dimensionless form brings out the important dimensionless parameters that govern the behaviour of the system. Consequently, dimensionless variables would always be introduced before attempting to make any approximations.

In this mathematical modelling, θ_j and ψ_j are dimensionless variables and p_j and ω_j need to be considered. The elastic displacements p_j can be made dimensionless by using the undeformed distance between the mass centres of end bodies, L, as a characteristic distance, whereas the angular velocity of R_0, ω_j, can be made dimensionless by using the initial angular velocity of axis P_*O_3, $\overline{\omega}$, as a characteristic angular velocity. Then defining

$$\tau = \overline{\omega} t \tag{1}$$

and putting

$$p_1^* = \frac{p_1}{L}, \qquad p_2^* = \frac{p_2}{L}, \qquad p_3^* = \frac{p_3}{L} \tag{2}$$

$$\omega_1^* = \frac{\omega_1}{\overline{\omega}}, \qquad \omega_2^* = \frac{\omega_2}{\overline{\omega}}, \qquad \omega_3^* = \frac{\omega_3}{\overline{\omega}} \tag{3}$$

where the asterisks denote dimensionless quantities, and substituting these equations into the governing differential equations of motion (and for convenience neglecting the symbol *), the following equations are obtained:

$$\frac{d^2 p_1}{d\tau^2} = (\omega_2^2 + \omega_3^2)p_1 - (1 - k_3)\omega_1\omega_2(1 + p_2) - (1 + k_2)\omega_1\omega_3 p_3 + 2\omega_3 \frac{dp_2}{d\tau} - 2\omega_2 \frac{dp_3}{d\tau} + \frac{2}{mL\varpi^2}F_{11}^{c} - \frac{1}{C\varpi^2}(T_{13}^{c} - LF_{11}^{c}) + \frac{2}{mL\varpi^2}F_{11}^{g} - \frac{1}{B\varpi^2}T_{02}^{g}p_3 + \frac{1}{C\varpi^2}T_{03}^{g}(1 + p_2) \quad (4)$$

$$\frac{d^2 p_2}{d\tau^2} = (\omega_3^2 + \omega_1^2)(1 + p_2) - (1 - k_1)\omega_2\omega_3 p_3 - (1 + k_3)\omega_2\omega_1 p_1 + 2\omega_1 \frac{dp_3}{d\tau} - 2\omega_3 \frac{dp_1}{d\tau} + \frac{2}{mL\varpi^2}F_{12}^{c} + \frac{2}{mL\varpi^2}F_{12}^{g} - \frac{1}{C\varpi^2}T_{03}^{g}p_1 + \frac{1}{A\varpi^2}T_{01}^{g}p_3 \quad (5)$$

$$\frac{d^2 p_3}{d\tau^2} = (\omega_1^2 + \omega_2^2)p_3 - (1 - k_2)\omega_3\omega_1 p_1 - (1 + k_1)\omega_3\omega_2(1 + p_2) + 2\omega_2 \frac{dp_1}{d\tau} - 2\omega_1 \frac{dp_2}{d\tau} + \frac{2}{mL\varpi^2}F_{13}^{c} + \frac{1}{A\varpi^2}(T_{11}^{c} + LF_{13}^{c}) + \frac{2}{mL\varpi^2}F_{13}^{g} - \frac{1}{A\varpi^2}T_{01}^{g}(1 + p_2) + \frac{1}{B\varpi^2}T_{02}^{g}p_1 \quad (6)$$

$$\frac{d\omega_1}{d\tau} = k_1\omega_2\omega_3 - \frac{1}{A\varpi^2}(T_{11}^{c} + LF_{13}^{c} - T_{01}^{g}) \quad (7)$$

$$\frac{d\omega_2}{d\tau} = k_2\omega_3\omega_1 - \frac{1}{B\varpi^2}(T_{12}^{c} - T_{02}^{g}) \quad (8)$$

$$\frac{d\omega_3}{d\tau} = k_3\omega_1\omega_2 - \frac{1}{C\varpi^2}(T_{13}^{c} - LF_{11}^{c} - T_{03}^{g}) \quad (9)$$

$$\frac{d^2\theta_1}{d\tau^2} = k_1(\omega_3^2 - \omega_2^2)\theta_1 + (1 + k_1)\omega_3 \frac{d\theta_2}{d\tau} + (k_1 - 1)\omega_2 \frac{d\theta_3}{d\tau} + (k_1 + k_3)\omega_2\omega_1\theta_2 - (k_1 + k_2)\omega_1\omega_3\theta_3 + \frac{1}{A\varpi^2}(2T_{11}^{c} + LF_{13}^{c}) + \frac{1}{A\varpi^2}(T_{11}^{g} - T_{01}^{g}) - \frac{T_{02}^{g}}{B\varpi^2}\theta_3 + \frac{T_{03}^{g}}{C\varpi^2}\theta_2 \quad (10)$$

$$\frac{d^2\theta_2}{d\tau^2} = k_2(\omega_1^2 - \omega_3^2)\theta_2 + (1 + k_2)\omega_1 \frac{d\theta_3}{d\tau} + (k_2 - 1)\omega_3 \frac{d\theta_1}{d\tau} + (k_2 + k_1)\omega_2\omega_3\theta_3 - (k_2 + k_3)\omega_2\omega_1\theta_1 + \frac{1}{B\varpi^2}(2T_{12}^{c}) + \frac{1}{B\varpi^2}(T_{12}^{g} - T_{02}^{g}) - \frac{T_{03}^{g}}{C\varpi^2}\theta_1 + \frac{T_{01}^{g}}{A\varpi^2}\theta_3 \quad (11)$$

$$\frac{d^2\theta_3}{d\tau^2} = k_3(\omega_2^2 - \omega_1^2)\theta_3 + (1 + k_3)\omega_2 \frac{d\theta_1}{d\tau} + (k_3 - 1)\omega_1 \frac{d\theta_2}{d\tau} + (k_3 + k_2)\omega_1\omega_3\theta_1 - (k_3 + k_1)\omega_3\omega_2\theta_2 + \frac{1}{C\varpi^2}(2T_{13}^{c} - LF_{11}^{c}) + \frac{1}{C\varpi^2}(T_{13}^{g} - T_{03}^{g}) - \frac{T_{01}^{g}}{A\varpi^2}\theta_2 + \frac{T_{02}^{g}}{B\varpi^2}\theta_1 \quad (12)$$

$$\frac{d\psi_1}{d\tau} = \frac{1}{\cos\psi_2}(\omega_1\cos\psi_3 - \omega_2\sin\psi_3) + \frac{\Omega}{\varpi}\cos\psi_1\tan\psi_2 \quad (13)$$

$$\frac{d\psi_2}{d\tau} = \omega_1\sin\psi_3 + \omega_2\cos\psi_3 - \frac{\Omega}{\varpi}\sin\psi_1 \quad (14)$$

$$\frac{d\psi_3}{d\tau} = \omega_3 - \tan\psi_2(\omega_1\cos\psi_3 - \omega_2\sin\psi_3) - \frac{\Omega\cos\psi_1}{\varpi\cos\psi_2} \quad (15)$$

where

$$k_1 = \frac{B - C}{A}, \qquad k_2 = \frac{C - A}{B}, \qquad k_3 = \frac{A - B}{C} \quad (16)$$

and A, B and C denote the corresponding principal moments of inertia of end body R_i. It is assumed that they are identical for the two bodies.

F_{ij}^{c} is the component of contact forces and T_{ij}^{c} is the component of the contact torque, applied at P_i by the deformed connecting structure. The first subscript i identifies the end body and the second subscript j refers to the coordinate axis number. This convention will also be adopted for gravitational force and torque measure numbers introduced subsequently. This relationship is given as

$$\begin{bmatrix} F_{11}^{c} \\ F_{12}^{c} \\ F_{13}^{c} \\ T_{11}^{c} \\ T_{12}^{c} \\ T_{13}^{c} \end{bmatrix} = -\begin{bmatrix} S_{11} & 0 & 0 & 0 & 0 & S_{16} \\ 0 & S_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} & 0 & 0 \\ 0 & 0 & S_{43} & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ S_{61} & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} L_{p_1} \\ L_{p_2} \\ L_{p_3} \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad (17)$$

where S_{kl} is the element of the stiffness matrix [**S**] of the elastic connecting structure. In addition, the [**S**] satisfies the requirements

$$2S_{16} - LS_{11} = 2S_{61} - L_{11} = 0 \tag{18}$$

$$2S_{34} + LS_{33} = 2S_{43} + L_{33} = 0 \tag{19}$$

With a modest amount of structural symmetry, these requirements on [**S**] can easily be satisfied.

The approximate expressions of the component of the gravitational force, $F^{\rm g}_{ij}$, and torque, $T^{\rm g}_{ij}$, exerted on the body R_i by the earth $\overline{E}$ are

$$F^{\rm g}_{01} = -F^{\rm g}_{11} = \frac{3}{2}\left(\frac{GM}{R^3}\right) mL(\cos^2\psi_2 \sin\psi_3 \cos\psi_3) \tag{20}$$

$$F^{\rm g}_{02} = -F^{\rm g}_{12} = \frac{1}{2}\left(\frac{GM}{R^3}\right) mL(1 - 3\cos^2\psi_2 \sin^2\psi_3) \tag{21}$$

$$F^{\rm g}_{03} = -F^{\rm g}_{13} = \frac{3}{2}\left(\frac{GM}{R^3}\right) mL(\cos\psi_2 \sin\psi_2 \sin\psi_3) \tag{22}$$

$$T^{\rm g}_{01} = 3(B - C)\left(\frac{GM}{R^3}\right)\left[\left(1 - \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \times (\cos\psi_2 \sin\psi_2 \sin\psi_3) + \frac{L}{2R}\sin\psi_2\right] \tag{23}$$

$$T^{\rm g}_{02} = 3(C - A)\left(\frac{GM}{R^3}\right)\left[\left(1 - \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \times (-\cos\psi_2 \sin\psi_2 \cos\psi_3)\right] \tag{24}$$

$$T^{\rm g}_{03} = 3(A - B)\left(\frac{GM}{R^3}\right)\left[\left(1 - \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \times (\cos^2\psi_2 \cos\psi_3 \sin\psi_3) + \frac{L}{2R}\cos\psi_2 \cos\psi_3\right] \tag{25}$$

$$\begin{aligned} T^{\rm g}_{11} = 3(B - C)\left(\frac{GM}{R^3}\right)\bigg\{&\left(1 + \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \\ &\times [-\theta_1(\sin^2\psi_2 - \cos^2\psi_2 \sin^2\psi_3) \\ &+ \theta_2 \cos^2\psi_2 \sin\psi_3 \cos\psi_3 \\ &+ \theta_3 \cos\psi_2 \sin\psi_2 \cos\psi_3 \\ &+ \cos\psi_2 \sin\psi_2 \sin\psi_3] \\ &+ \frac{L}{2R}(-\theta_2 \cos\psi_2 \cos\psi_3 \\ &- 2\theta_1 \cos\psi_2 \sin\psi_3 - \sin\psi_2)\bigg\} \end{aligned} \tag{26}$$

$$\begin{aligned} T^{\rm g}_{12} = 3(C - A)\left(\frac{GM}{R^3}\right)\bigg\{&\left(1 + \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \\ &\times [\theta_2(\sin^2\psi_2 - \cos^2\psi_2 \cos^2\psi_3) \\ &+ \theta_3 \cos\psi_2 \sin\psi_2 \sin\psi_3 \\ &- \theta_1 \cos^2\psi_2 \sin\psi_3 \cos\psi_3 \\ &- \cos\psi_2 \sin\psi_2 \cos\psi_3] \\ &+ \frac{L}{2R}(\theta_1 \cos\psi_2 \cos\psi_3 - \theta_3 \sin\psi_2)\bigg\} \end{aligned} \tag{27}$$

$$\begin{aligned} T^{\rm g}_{13} = 3(A - B)\left(\frac{GM}{R^3}\right)\bigg\{&\left(1 + \frac{5L}{2R}\cos\psi_2 \sin\psi_3\right) \\ &\times [-\theta_3(\cos^2\psi_2 \sin^2\psi_3 - \cos^2\psi_2 \cos^2\psi_3) \\ &- \theta_1 \cos\psi_2 \sin\psi_2 \cos\psi_3 \\ &- \theta_2 \cos\psi_2 \sin\psi_2 \sin\psi_3 \\ &+ \cos^2\psi_2 \sin\psi_3 \cos\psi_3] \\ &+ \frac{L}{2R}(\theta_2 \sin\psi_2 + 2\theta_3 \cos\psi_2 \sin\psi_3 \\ &- \cos\psi_2 \cos\psi_3)\bigg\} \end{aligned} \tag{28}$$

In these equations, G is the universal gravitational constant and M and m are the masses of $\overline{E}$ and R_i respectively. Additionally, the component of angular velocity, ω_j, of R_0 in the inertial reference frame N with the attitude angle ψ_j have the following relationships:

$$\begin{aligned} \omega_1 = {} & \frac{\mathrm{d}\psi_1}{\mathrm{d}\tau}\cos\psi_2 \cos\psi_3 + \frac{\mathrm{d}\psi_2}{\mathrm{d}\tau}\sin\psi_3 \\ & + \frac{\Omega}{\varpi}(\sin\psi_1 \sin\psi_3 - \cos\psi_1 \sin\psi_2 \cos\psi_3) \end{aligned} \tag{29}$$

$$\begin{aligned} \omega_2 = {} & \frac{\mathrm{d}\psi_2}{\mathrm{d}\tau}\cos\psi_3 - \frac{\mathrm{d}\psi_1}{\mathrm{d}\tau}\cos\psi_3 \sin\psi_3 \\ & + \frac{\Omega}{\varpi}(\sin\psi_1 \cos\psi_3 + \cos\psi_1 \sin\psi_2 \sin\psi_3) \end{aligned} \tag{30}$$

$$\omega_3 = \frac{\mathrm{d}\psi_3}{\mathrm{d}\tau} + \frac{\mathrm{d}\psi_1}{\mathrm{d}\tau}\sin\psi_2 + \frac{\Omega}{\varpi}\cos\psi_1 \cos\psi_2 \tag{31}$$

The problem of the elastically connected two-body space station in circular orbit has now been reduced to a set of twelve differential equations, (4) to (15), in the twelve variables p_j, ω_j, θ_j, ψ_j. The equations are non-linear in ψ_j and ω_j but linear in p_j and θ_j; hence they are valid for large attitude motions accompanied by small elastic deformations.

2.3 The equilibrium motion

A popular type of motion is chosen to study. Intuitively, it seems possible that the elastic space station under consideration can have an initial spin about an axis parallel to X_3^0 and X_3^1 and that the resulting motion is then a steady initial spin of (constant) rate $\overline{\omega}$ accompanied by a constant elastic extension $\tilde{p}_2$. For this type of motion, there are the following parameter values:

$$p_1 = 0, \qquad p_2 = \frac{\tilde{p}_2}{L} = \tilde{p}_2, \qquad p_3 = 0 \tag{32}$$

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \omega_3 = \frac{\overline{\omega}}{\overline{\omega}} = 1 \tag{33}$$

$$\theta_1 = 0, \qquad \theta_2 = 0, \qquad \theta_3 = 0 \tag{34}$$

$$\psi_1 = 0, \qquad \psi_2 = 0, \qquad \psi_3 = \hat{\psi}_3 \tag{35}$$

The parameters of p_j, ω_j, θ_j satisfy the differential equations exactly, as shown in reference (**4**). The solution of attitude angle ψ_3 will now be found. Substituting equations (33) and (35) into equations (29) to (31) gives

$$\frac{\mathrm{d}\hat{\psi}_3}{\mathrm{d}\tau} = 1 - \frac{\Omega}{\overline{\omega}} \tag{36}$$

which in integral form is

$$\hat{\psi}_3 = \int \left(1 - \frac{\Omega}{\overline{\omega}}\right) \mathrm{d}\tau + c \tag{37}$$

In the circular path the orbital angular velocity Ω of mass centre P_* is constant, and the initial angular velocity $\overline{\omega}$ has a designated constant value. Then $\hat{\psi}_3$ becomes

$$\hat{\psi}_3 = \left(1 - \frac{\Omega}{\overline{\omega}}\right)\tau + c \tag{38}$$

From the initial condition $\hat{\psi}_3(0) = 0$, then $c = 0$. From the constant $d_2 = (1 - \Omega/\overline{\omega})$, it is found that ψ_3 is proportional to τ.

2.4 The first-order and second-order differential equations

Selecting $\epsilon = \frac{3}{2}(GM/R^3)(1/\overline{\omega}^2)$ as the small dimensionless parameter, if the perturbation method is applied the approximate analytical solution is obtained. The multiple-scales technique (**7**) introduces the different time-scales $T_0 = \tau$, $T_1 = \epsilon\tau$ and using the chain rule, the derivatives with respect to τ transform into

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = D_0 + \epsilon D_1 + \cdots \tag{39}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \cdots \tag{40}$$

where

$$D_0 = \frac{\partial}{\partial T_0}, \qquad D_1 = \frac{\partial}{\partial T_1} \tag{41}$$

An approximate analytical solution for equations (4) to (15) is sought in powers of ϵ in the form

$$p_1 = \epsilon p_{11} + \epsilon^2 p_{12} + \cdots, \qquad \omega_1 = \epsilon\omega_{11} + \epsilon^2\omega_{12} + \cdots \tag{42}$$

$$p_2 = \tilde{p}_2 + \epsilon p_{21} + \epsilon^2 p_{22} + \cdots, \qquad \omega_2 = \epsilon\omega_{21} + \epsilon^2\omega_{22} + \cdots \tag{43}$$

$$p_3 = \epsilon p_{31} + \epsilon^2 p_{32} + \cdots, \qquad \omega_3 = 1 + \epsilon\omega_{31} + \epsilon^2\omega_{32} + \cdots \tag{44}$$

$$\theta_1 = \epsilon\theta_{11} + \epsilon^2\theta_{12} + \cdots, \qquad \psi_1 = \epsilon\psi_{11} + \epsilon^2\psi_{12} + \cdots \tag{45}$$

$$\theta_2 = \epsilon\theta_{21} + \epsilon^2\theta_{22} + \cdots, \qquad \psi_2 = \epsilon\psi_{21} + \epsilon^2\psi_{22} + \cdots \tag{46}$$

$$\theta_3 = \epsilon\theta_{31} + \epsilon^2\theta_{32} + \cdots, \qquad \psi_3 = d_2\tau + \epsilon\psi_{31} + \epsilon^2\psi_{32} + \cdots \tag{47}$$

where p_{jm}, ω_{jm}, θ_{jm}, ψ_{jm} ($j = 1, 2, 3$ represents direction and $m = 1, 2, \ldots$ defines the orders of ϵ) are functions of $T_0, T_1, T_2 \ldots$. It is also assumed that

$$\sin(\epsilon\psi_{j1} + \epsilon^2\psi_{j2} + \cdots) = \epsilon\psi_{j1} + \epsilon^2\psi_{j2} + \cdots \tag{48}$$

$$\cos(\epsilon\psi_{j1} + \epsilon^2\psi_{j2} + \cdots) = 1 \tag{49}$$

Then

$$\sin\psi_3 = \sin d_2\tau + (\epsilon\psi_{31} + \epsilon^2\psi_{32})\cos d_2\tau + \epsilon^3 \cdots \tag{50}$$

$$\cos\psi_3 = \cos d_2\tau - (\epsilon\psi_{31} + \epsilon^2\psi_{32})\sin d_2\tau + \epsilon^3 \cdots \tag{51}$$

Equations (42) to (47) are substituted into equation (17) and equations (48) to (51) into equations (20) to (28) and then all the results are substituted into the governing equations (4) to (15). Since the equations are very lengthy, the expansions will not be listed here. After equating the coefficients of ϵ^0, ϵ^1 and ϵ^2 on both sides, the following are obtained:

1. The ϵ zero-order equations:

$$\tilde{p}_2 = \frac{1}{2S_{22}/m\overline{\omega}^2 - 1} \tag{52}$$

$$d_2 = 1 - \frac{\Omega}{\overline{\omega}} \tag{53}$$

where $\tilde{p}_2$ is a constant dimensionless extended length of an elastic structure during a constant spin rate $\overline{\omega}$. In a circular orbit d_2 is a constant also, as already mentioned. Certainly, the ϵ zero-order equations satisfy the conditions of an undisturbed state.

2. The ϵ first-order differential equations:

$$(D_0^2 + \alpha_1)p_{11} - 2D_0 p_{21} + \alpha_2\theta_{31} = [\alpha_3 \sin d_2\tau + \alpha_4(5\sin^2 d_2\tau - 1)]\cos d_2\tau \tag{54}$$

$$2D_0 p_{11} + (D_0^2 + \alpha_5)p_{21} + \alpha_6\omega_{31} = -2(\tfrac{1}{3} - \sin^2 d_2\tau) \tag{55}$$

$$(D_0^2 + \alpha_7)p_{31} + \alpha_8\omega_{21} + \alpha_9\theta_{11} = 0 \tag{56}$$

$$\alpha_{10}p_{31} + D_0\omega_{11} + \alpha_{11}\omega_{21} + \alpha_{12}\theta_{11} = 0 \tag{57}$$

$$\alpha_{13}\omega_{11} + D_0\omega_{21} + \alpha_{14}\theta_{21} = 0 \tag{58}$$

$$\alpha_{15}p_{11} + D_0\omega_{31} + \alpha_{16}\theta_{31} = [\alpha_{17}\sin d_2\tau + \alpha_{18}(5\sin^2 d_2\tau - 1)]\cos d_2\tau \tag{59}$$

$$(D_0^2 + \alpha_{19})\theta_{11} + \alpha_{20}D_0\theta_{21} = 0 \tag{60}$$

$$\alpha_{21}D_0\theta_{11} + (D_0^2 + \alpha_{22})\theta_{21} = 0 \tag{61}$$

$$(D_0^2 + \alpha_{23})\theta_{31} = \alpha_{24}(5\sin^2 d_2\tau - 1)\cos d_2\tau \tag{62}$$

$$\omega_{11}\cos d_2\tau - \omega_{21}\sin d_2\tau - D_0\psi_{11} + \alpha_{25}\psi_{21} = 0 \tag{63}$$

$$\omega_{11}\sin d_2\tau + \omega_{21}\cos d_2\tau - \alpha_{25}\psi_{11} - D_0\psi_{21} = 0 \tag{64}$$

$$\omega_{31} - D_0\psi_{31} = 0 \tag{65}$$

where

$$\alpha_1 = \frac{2}{m\overline{\omega}^2}S_{11} - \frac{L}{C\overline{\omega}^2}(S_{61} - LS_{11}) - 1$$

$$\alpha_2 = \frac{2}{mL\overline{\omega}^2}S_{16} - \frac{1}{C\overline{\omega}^2}(S_{66} - LS_{16})$$

$$\alpha_3 = 2k_3(1 + \tilde{p}_2) - 2$$

$$\alpha_4 = -k_3 d_1(1 + \tilde{p}_2)$$

$$\alpha_5 = \frac{2}{m\overline{\omega}^2}S_{22} - 1$$

$$\alpha_6 = -2(1 + \tilde{p}_2)$$

$$\alpha_7 = \frac{2}{m\overline{\omega}^2}S_{33} + \frac{L}{A\overline{\omega}^2}(S_{43} + LS_{33})$$

$$\alpha_8 = (1 + k_1)(1 + \tilde{p}_2)$$

$$\alpha_9 = \frac{2}{mL\overline{\omega}^2}S_{34} + \frac{1}{A\overline{\omega}^2}(S_{44} + LS_{34})$$

$$\alpha_{10} = -\frac{L}{A\overline{\omega}^2}(S_{43} + LS_{33})$$

$$\alpha_{11} = -k_1$$

$$\alpha_{12} = -\frac{1}{A\overline{\omega}^2}(S_{44} + LS_{34})$$

$$\alpha_{13} = -k_2$$

$$\alpha_{14} = -\frac{1}{B\overline{\omega}^2}S_{55}$$

$$\alpha_{15} = -\frac{L}{C\overline{\omega}^2}(S_{61} - LS_{11})$$

$$\alpha_{16} = -\frac{1}{C\overline{\omega}^2}(S_{66} - LS_{16})$$

$$\alpha_{17} = 2k_3$$

$$\alpha_{18} = -k_3 d_1$$

$$\alpha_{19} = \frac{1}{A\overline{\omega}^2}(2S_{44} + LS_{34}) - k_1$$

$$\alpha_{20} = -(1 + k_1)$$

$$\alpha_{21} = 1 - k_2$$

$$\alpha_{22} = \frac{2}{B\overline{\omega}^2}S_{55} + k_2$$

$$\alpha_{23} = \frac{1}{C\overline{\omega}^2}(2S_{66} - LS_{16})$$

$$\alpha_{24} = 2k_3 d_1$$

$$\alpha_{25} = \frac{\Omega}{\overline{\omega}}$$

3. The ϵ second-order differential equations:

$$(D_0^2+\alpha_1)p_{12}-2D_0p_{22}+\alpha_2\theta_{32}$$
$$=2((-D_0D_1+\omega_{31})p_{11}+\{D_1+\omega_{31}D_0$$
$$+\ [\beta_1\sin d_2\tau+\beta_2(5\sin^2 d_2\tau-1)]\cos d_2\tau\}p_{21}$$
$$+\ (\beta_3\omega_{11}-\omega_{21}D_0)p_{31}+\beta_4\omega_{11}\omega_{21}$$
$$+\ [\beta_5(\cos^2 d_2\tau-\sin^2 d_2\tau)$$
$$+\ \beta_6(5\sin^2 d_2\tau-10\cos^2 d_2\tau-1)\sin d_2\tau]\psi_{31}) \quad (66)$$

$$2D_0p_{12}+(D_0^2+\alpha_5)p_{22}+\alpha_6\omega_{32}$$
$$=2(\{(-D_1-\omega_{31}D_0)$$
$$-\ [\beta_1\sin d_2\tau+\beta_2(5\sin^2 d_2\tau-1)]\cos d_2\tau\}p_{11}$$
$$+\ (\omega_{31}-D_0D_1)p_{21}+(\omega_{11}D_0+\beta_7\omega_{21})p_{31}$$
$$+\ \beta_8(\omega_{11}^2+\omega_{31}^2)+2\sin d_2\tau\cos d_2\tau\psi_{31}) \quad (67)$$

$$(D_0^2+\alpha_7)p_{32}+\alpha_8\omega_{22}+\alpha_9\theta_{12}$$
$$=2\{(\omega_{21}D_0+\beta_9\omega_{11})p_{11}$$
$$+\ (\beta_{10}\omega_{21}-\omega_{11}D_0)p_{21}-D_0D_1p_{31}+\beta_{11}\omega_{31}\omega_{21}$$
$$+\ [\beta_{12}\sin d_2\tau+\beta_{13}(5\sin^2 d_2\tau-1)]\psi_{21}\} \quad (68)$$

$$\alpha_{10}p_{32}+D_0\omega_{12}+\alpha_{11}\omega_{22}+\alpha_{12}\theta_{12}$$
$$=-D_1\omega_{11}+\beta_{14}\omega_{21}\omega_{31}$$
$$+\ 2[\beta_{14}\sin d_2\tau+\beta_{15}(5\sin^2 d_2\tau-1)]\psi_{21} \quad (69)$$

$$\alpha_{13}\omega_{12}+D_0\omega_{22}+\alpha_{14}\theta_{22}$$
$$=-D_1\omega_{21}+\beta_{16}\omega_{11}\omega_{31}$$
$$-2(\beta_{16}-5\beta_{17}\sin d_2\tau)\cos d_2\tau\psi_{21} \quad (70)$$

$$\alpha_{15}p_{12}+D_0\omega_{32}+\alpha_{16}\theta_{32}$$
$$=-D_1\omega_{31}+\beta_1\omega_{11}\omega_{21}+\ 2[\beta_1(\cos^2 d_2\tau-\sin^2 d_2\tau)$$
$$-\beta_2(5\sin^2 d_2\tau-10\cos^2 d_2\tau-1)\sin d_2\tau]\psi_{31} \quad (71)$$

$$(D_0^2+\alpha_{19})\theta_{12}+\alpha_{20}D_0\theta_{22}$$
$$=2(\{-D_0D_1+\beta_{14}\omega_{31}$$
$$+\ [\beta_{14}\sin d_2\tau-\beta_{15}(5\sin^2 d_2\tau-2)]\sin d_2\tau\}\theta_{11}$$
$$+\ \{-B_{10}(\omega_{31}D_0+D_1)$$
$$+\ [(\beta_{14}+\beta_1)\sin d_2\tau$$
$$-\ (\beta_{15}-\beta_2)(5\sin^2 d_2\tau-1)]\cos d_2\tau\}\theta_{21}$$
$$+\ (\beta_7\omega_{21}D_0+\beta_{18}\omega_{11})\theta_{31}-2\beta_{15}(5\sin^2 d_2\tau-1)\psi_{21}) \quad (72)$$

$$\alpha_{21}D_0\theta_{12}+(D_0^2+\alpha_{22})\theta_{22}$$
$$=2(\{\beta_9(D_1+\omega_{31}D_0)$$
$$-\ [(\beta_{16}+\beta_1)\sin d_2\tau+(\beta_{17}+\beta_2)$$
$$\times\ (5\sin^2 d_2\tau-1)]\cos d_2\tau\}\theta_{11}$$
$$+\ [-D_0D_1-\beta_{16}\omega_{31}-(\beta_{16}+5\beta_{17}\sin d_2\tau)\cos^2 d_2\tau]\theta_{21}$$
$$-\ (\beta_{18}\omega_{21}+\beta_3\omega_{11}D_0)\theta_{31}-2\beta_{17}(5\sin d_2\tau\cos d_2\tau)\psi_{21}) \quad (73)$$

$$(D_0^2+\alpha_{23})\theta_{32}$$
$$=2((\beta_{19}\omega_{21}D_0+\beta_{21}\omega_{11})\theta_{11}$$
$$+\ (\beta_{20}\omega_{11}D_0+\beta_{22}\omega_{21})\theta_{21}+\{[\beta_1(\cos^2 d_2\tau-\sin^2 d_2\tau)$$
$$+\ \beta_2(5\sin^2 d_2\tau-5\cos^2 d_2\tau-2)\sin d_2\tau]-D_0D_1\}\theta_{31}$$
$$+2\beta_2(5\sin^2 d_2\tau-10\cos^2 d_2\tau-1)\sin d_2\tau\psi_{31}) \quad (74)$$

$$\omega_{12}\cos d_2\tau-\omega_{22}\sin d_2\tau-D_0\psi_{12}+\alpha_{25}\psi_{22}$$
$$=\ D_1\psi_{11}+(\omega_{11}\sin d_2\tau+\omega_{21}\cos d_2\tau)\psi_{31} \quad (75)$$

$$\omega_{12}\sin d_2\tau+\omega_{22}\cos d_2\tau-\alpha_{25}\psi_{12}-D_0\psi_{22}$$
$$=\ D_1\psi_{21}+(\omega_{21}\sin d_2\tau-\omega_{11}\cos d_2\tau)\psi_{31} \quad (76)$$

$$\omega_{32}-D_0\psi_{32}=D_1\psi_{31}+(\omega_{11}\cos d_2\tau-\omega_{21}\sin d_2\tau)\psi_{21} \quad (77)$$

where

$$\beta_1=k_3$$

$$\beta_2=-\tfrac{1}{2}k_3d_1$$

$$\beta_3=-\tfrac{1}{2}(1+k_2)$$

$$\beta_4=-\tfrac{1}{2}(1-k_3)(1+\tilde{p}_2)$$

$$\beta_5 = k_3(1+\tilde{p}_2) - 1$$

$$\beta_6 = \tfrac{1}{2}k_3 d_1(1+\tilde{p}_2)$$

$$\beta_7 = -\tfrac{1}{2}(1-k_1)$$

$$\beta_8 = \tfrac{1}{2}(1+\tilde{p}_2)$$

$$\beta_9 = -\tfrac{1}{2}(1-k_2)$$

$$\beta_{10} = -\tfrac{1}{2}(1+k_1)$$

$$\beta_{11} = -\tfrac{1}{2}(1+k_1)(1+\tilde{p}_2)$$

$$\beta_{12} = -k_1(1+\tilde{p}_2) - 1$$

$$\beta_{13} = \tfrac{1}{2}k_1 d_1(1+\tilde{p}_2)$$

$$\beta_{14} = k_1$$

$$\beta_{15} = -\tfrac{1}{2}k_1 d_1$$

$$\beta_{16} = k_2$$

$$\beta_{17} = \tfrac{1}{2}k_2 d_1$$

$$\beta_{18} = -\tfrac{1}{2}(k_1+k_2)$$

$$\beta_{19} = \tfrac{1}{2}(1+k_3)$$

$$\beta_{20} = -\tfrac{1}{2}(1-k_3)$$

$$\beta_{21} = \tfrac{1}{2}(k_3+k_2)$$

$$\beta_{22} = -\tfrac{1}{2}(k_3+k_1)$$

The first-order and second-order differential equations are the foundation of the two sequential sections.

3 THE DYNAMICS OF AN EARTH-POINTING MOTION

An 'earth-pointing' motion, which corresponds to an equilibrium solution of the set of differential equations, is accomplished when the assembly moves such that the rigid bodies are at rest in the orbital reference frame O and such that the connecting structure remains coincident with the line joining the centre of the earth $\overline{E}$ and the mass centre of the space station P_*. Such motion is of benefit in space communicational and navigational systems. It is also well suited for carrying Earth observation equipment.

3.1 The first-order approximate analytical solution

For this case of motion, the initial angular velocity $\overline{\omega}$ equals the orbital angular rate Ω. Then $d_2 = 1 - \Omega/\overline{\omega} = 0$, which gives $\sin d_2\tau = 0$, $\cos d_2\tau = 1$ and $\alpha_{25} = 1$. The ϵ first-order differential equations (54) to (65) become

$$(D_0^2+\alpha_1)p_{11} - 2D_0 p_{21} + \alpha_2\theta_{31} = -\alpha_4 \tag{78}$$

$$2D_0 p_{11} + (D_0^2+\alpha_5)p_{21} + \alpha_6\omega_{31} = -\tfrac{2}{3} \tag{79}$$

$$(D_0^2+\alpha_7)p_{31} + \alpha_8\omega_{21} + \alpha_9\theta_{11} = 0 \tag{80}$$

$$\alpha_{10}p_{31} + D_0\omega_{11} + \alpha_{11}\omega_{21} + \alpha_{12}\theta_{11} = 0 \tag{81}$$

$$\alpha_{13}\omega_{11} + D_0\omega_{21} + \alpha_{14}\theta_{21} = 0 \tag{82}$$

$$\alpha_{15}p_{11} + D_0\omega_{31} + \alpha_{16}\theta_{31} = -\alpha_{18} \tag{83}$$

$$(D_0^2+\alpha_{19})\theta_{11} + \alpha_{20}D_0\theta_{21} = 0 \tag{84}$$

$$\alpha_{21}D_0\theta_{11} + (D_0^2+\alpha_{22})\theta_{21} = 0 \tag{85}$$

$$(D_0^2+\alpha_{23})\theta_{31} = -\alpha_{24} \tag{86}$$

$$\omega_{11} - D_0\psi_{11} + \psi_{21} = 0 \tag{87}$$

$$\omega_{21} - \psi_{11} - D_0\psi_{21} = 0 \tag{88}$$

$$\omega_{31} - D_0\psi_{31} = 0 \tag{89}$$

On the right-hand sides of the above equations, α_4, α_{18} and α_{24} are constant values and have no effect on the stability of the system.

Equations (78) to (89) are divided into four sets of simultaneous equations, and it is convenient to express the general solutions in a complex form. First solving equations (84) to (86), from equation (86), the following equation is obtained:

$$\theta_{31} = K_{31}\,\mathrm{e}^{\mathrm{i}\gamma_1 T_0} + \text{c.c.} \tag{90}$$

where

$$\gamma_1 = \sqrt{\alpha_{23}} \tag{91}$$

and c.c. stands for the complex conjugate of the preceding terms. This notation serves to display long expressions efficiently. For a first-order expansion, K_{31} is considered as a function of T_1 only, and is complex [K_{31} will be obtained in equation (127)]. The first subscript j of K_{jn} represents direction and the second subscript n refers to γ_n. This notation will be used sequentially. Next, solving equations (84) and (85), the solution is assumed to be

$$\theta_{11} = c_1\,\mathrm{e}^{\mathrm{i}\omega_n T_0}, \qquad \theta_{21} = c_2\,\mathrm{e}^{\mathrm{i}\omega_n T_0} \tag{92}$$

Substituting equation (92) into equations (84) and (85), the coefficients of $\exp(\mathrm{i}\omega_n T_0)$ equal zero as follows:

$$\begin{bmatrix} -\omega_n^2+\alpha_{19} & \mathrm{i}\omega_n\alpha_{20} \\ \mathrm{i}\omega_n\alpha_{21} & -\omega_n^2+\alpha_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \tag{93}$$

Since c_1 and c_2 have a non-zero solution, the equation above must satisfy

$$\det\begin{bmatrix} -\omega_n^2+\alpha_{19} & \mathrm{i}\omega_n\alpha_{20} \\ \mathrm{i}\omega_n\alpha_{21} & -\omega_n^2+\alpha_{22}\end{bmatrix}=0 \tag{94}$$

or

$$\omega_n^4+[\alpha_{20}\alpha_{21}-(\alpha_{19}+\alpha_{22})]\omega_n^2+\alpha_{19}\alpha_{22}=0 \tag{95}$$

The roots are

$$\omega_n=\pm\sqrt{\left[\frac{-\delta_1\pm\sqrt{(\delta_1^2-4\delta_2)}}{2}\right]} \tag{96}$$

where the parameters are transformed as

$$\delta_1=\alpha_{20}\alpha_{21}-(\alpha_{19}+\alpha_{22}),\qquad \delta_2=\alpha_{19}\alpha_{22} \tag{97}$$

and set as

$$\gamma_2=\sqrt{\left[\frac{\delta_1+\sqrt{(\delta_1^2-4\delta_2)}}{2}\right]}\qquad \gamma_3=\sqrt{\left[\frac{\delta_1-\sqrt{(\delta_1^2-4\delta_2)}}{2}\right]} \tag{98}$$

The solution will be

$$\theta_{11}=K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+\mathrm{c.c.} \tag{99}$$

$$\theta_{21}=K_{22}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+K_{23}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+\mathrm{c.c.} \tag{100}$$

Substituting equations (99) and (100) into equation (84) gives

$$K_{22}=\mathrm{i}\left(\frac{\alpha_{19}-\gamma_2^2}{\alpha_{20}\gamma_2}\right)K_{12}=\mathrm{i}O_{22}K_{12} \tag{101}$$

$$K_{23}=\mathrm{i}\left(\frac{\alpha_{19}-\gamma_3^2}{\alpha_{20}\gamma_3}\right)K_{13}=\mathrm{i}O_{23}K_{13} \tag{102}$$

Then the solution of θ_{21} is

$$\theta_{21}=\mathrm{i}O_{22}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+\mathrm{i}O_{23}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+\mathrm{c.c.} \tag{103}$$

where

$$O_{22}=\frac{\alpha_{19}-\gamma_2^2}{\alpha_{20}\gamma_2},\qquad O_{23}=\frac{\alpha_{19}-\gamma_3^2}{\alpha_{20}\gamma_3}$$

By the same procedure, solving the other sets of equations gives

$$p_{11}=M_{11}K_{31}\,\mathrm{e}^{\mathrm{i}\gamma_1T_0}+I_{17}\,\mathrm{e}^{\mathrm{i}\gamma_7T_0}+I_{18}\,\mathrm{e}^{\mathrm{i}\gamma_8T_0}+\mathrm{c.c.} \tag{104}$$

$$p_{21}=\mathrm{i}M_{21}K_{31}\,\mathrm{e}^{\mathrm{i}\gamma_1T_0}+\mathrm{i}M_{27}I_{17}\,\mathrm{e}^{\mathrm{i}\gamma_7T_0}+\mathrm{i}M_{28}I_{18}\,\mathrm{e}^{\mathrm{i}\gamma_8T_0}+\mathrm{c.c.} \tag{105}$$

$$p_{31}=M_{32}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+M_{33}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+I_{34}\,\mathrm{e}^{\mathrm{i}\gamma_4T_0}+I_{35}\,\mathrm{e}^{\mathrm{i}\gamma_5T_0}+\mathrm{c.c.} \tag{106}$$

$$\omega_{11}=\mathrm{i}N_{12}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+\mathrm{i}N_{13}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+\mathrm{i}N_{14}I_{34}\,\mathrm{e}^{\mathrm{i}\gamma_4T_0}+\mathrm{i}N_{15}I_{35}\,\mathrm{e}^{\mathrm{i}\gamma_5T_0}+J_{16}\,\mathrm{e}^{\mathrm{i}\gamma_6T_0}+\mathrm{c.c.} \tag{107}$$

$$\omega_{21}=N_{22}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+N_{23}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+N_{24}I_{34}\,\mathrm{e}^{\mathrm{i}\gamma_4T_0}+N_{25}I_{35}\,\mathrm{e}^{\mathrm{i}\gamma_5T_0}+\mathrm{i}N_{26}J_{16}\,\mathrm{e}^{\mathrm{i}\gamma_6T_0}+\mathrm{c.c.} \tag{108}$$

$$\omega_{31}=\mathrm{i}N_{31}K_{31}\,\mathrm{e}^{\mathrm{i}\gamma_1T_0}+\mathrm{i}N_{37}I_{17}\,\mathrm{e}^{\mathrm{i}\gamma_7T_0}+\mathrm{i}N_{38}I_{18}\,\mathrm{e}^{\mathrm{i}\gamma_8T_0}+\mathrm{c.c.} \tag{109}$$

$$\psi_{11}=P_{12}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+P_{13}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+P_{14}I_{24}\,\mathrm{e}^{\mathrm{i}\gamma_4T_0}+P_{15}I_{25}{}^{\mathrm{i}\gamma_5T_0}+\mathrm{i}P_{16}J_{16}\,\mathrm{e}^{\mathrm{i}\gamma_6T_0}+\mathrm{i}P_{19}L_{19}\,\mathrm{e}^{\mathrm{i}T_0}+\mathrm{c.c.} \tag{110}$$

$$\psi_{21}=\mathrm{i}P_{22}K_{12}\,\mathrm{e}^{\mathrm{i}\gamma_2T_0}+\mathrm{i}P_{23}K_{13}\,\mathrm{e}^{\mathrm{i}\gamma_3T_0}+\mathrm{i}P_{24}I_{24}\,\mathrm{e}^{\mathrm{i}\gamma_4T_0}+\mathrm{i}P_{25}I_{25}\,\mathrm{e}^{\mathrm{i}\gamma_5T_0}+\mathrm{i}P_{26}J_{16}\,\mathrm{e}^{\mathrm{i}\gamma_6T_0}+L_{29}\,\mathrm{e}^{\mathrm{i}T_0}+\mathrm{c.c.} \tag{111}$$

$$\psi_{31}=P_{31}K_{31}\,\mathrm{e}^{\mathrm{i}\gamma_1T_0}+P_{37}I_{17}\,\mathrm{e}^{\mathrm{i}\gamma_7T_0}+P_{38}I_{18}\,\mathrm{e}^{\mathrm{i}\gamma_8T_0}+\mathrm{c.c.} \tag{112}$$

where

$$\delta_3=\alpha_7-\alpha_{11}\alpha_{13},\qquad \delta_4=\alpha_8\alpha_{10}\alpha_{13}-\alpha_7\alpha_{11}\alpha_{13}$$

$$\delta_5=\alpha_9\alpha_{11}\alpha_{13}-\alpha_8\alpha_{12}\alpha_{13},\qquad \delta_6=\alpha_8\alpha_{14}$$

$$\delta_7=\alpha_1+\alpha_5+4,\qquad \delta_8=\alpha_1\alpha_5-2\alpha_6\alpha_{15}$$

$$\delta_9=2\alpha_6\alpha_{16}-\alpha_2\alpha_5$$

$$\gamma_{4,5}=\sqrt{\left[\frac{\delta_3\pm\sqrt{(\delta_3^2-4\delta_4)}}{2}\right]},\qquad \gamma_6=\sqrt{(-\alpha_{11}\alpha_{13})}$$

$$\gamma_{7,8}=\sqrt{\left[\frac{\delta_7\pm\sqrt{(\delta_7^2-4\delta_8)}}{2}\right]}$$

$$M_{11}=\frac{\alpha_2\gamma_1^2+\delta_9}{\gamma_1^4-\delta_7\gamma_1^2+\delta_8}$$

$$M_{21}=\frac{(\gamma_1^2-\alpha_1)M_{11}-\alpha_2}{2\gamma_1}$$

$$M_{27} = \frac{\gamma_7^2 - \alpha_1}{2\gamma_7}$$

$$M_{28} = \frac{\gamma_8^2 - \alpha_1}{2\gamma_8}$$

$$M_{32} = \frac{\alpha_9\gamma_2^2 + \delta_5 - \delta_6\gamma_2 O_{22}}{\gamma_2^4 - \delta_3\gamma_2^2 + \delta_4}$$

$$M_{33} = \frac{\alpha_9\gamma_3^2 + \delta_5 - \delta_6\gamma_3 O_{23}}{\gamma_3^4 - \delta_3\gamma_3^2 + \delta_4}$$

$$N_{12} = \frac{\alpha_{10}\gamma_2 M_{32} + \alpha_{12}\gamma_2 - \alpha_{11}\alpha_{14} O_{22}}{\gamma_2^2 + \alpha_{11}\alpha_{13}}$$

$$N_{13} = \frac{\alpha_{10}\gamma_3 M_{33} + \alpha_{12}\gamma_3 - \alpha_{11}\alpha_{14} O_{23}}{\gamma_3^2 + \alpha_{11}\alpha_{13}}$$

$$N_{14} = \frac{\alpha_{10}\gamma_4}{\gamma_4^2 + \alpha_{11}\alpha_{13}}$$

$$N_{15} = \frac{\alpha_{10}\gamma_5}{\gamma_5^2 + \alpha_{11}\alpha_{13}}$$

$$N_{22} = \frac{\gamma_2 N_{12} - \alpha_{10} M_{32} - \alpha_{12}}{\alpha_{11}}$$

$$N_{23} = \frac{\gamma_3 N_{13} - \alpha_{10} M_{33} - \alpha_{12}}{\alpha_{11}}$$

$$N_{24} = \frac{\gamma_4 N_{14} - \alpha_{10}}{\alpha_{11}}$$

$$N_{25} = \frac{\gamma_5 N_{15} - \alpha_{10}}{\alpha_{11}}$$

$$N_{26} = \frac{-\gamma_6}{\alpha_{11}}$$

$$N_{31} = \frac{\alpha_{15} M_{11} + \alpha_{16}}{\gamma_1}$$

$$N_{37} = \frac{\alpha_{15}}{\gamma_7}, \quad N_{38} = \frac{\alpha_{15}}{\gamma_8}$$

$$P_{12} = N_{22} + \gamma_2 P_{22}$$

$$P_{13} = N_{23} + \gamma_3 P_{23}$$

$$P_{14} = N_{24} + \gamma_4 P_{24}$$

$$P_{15} = N_{25} + \gamma_5 P_{25}$$

$$P_{16} = N_{26} - \gamma_6 P_{26}$$

$$P_{19} = -1$$

$$P_{22} = \frac{\gamma_2 N_{22} - N_{12}}{1 - \gamma_2^2}$$

$$P_{23} = \frac{\gamma_3 N_{23} - N_{13}}{1 - \gamma_3^2}$$

$$P_{24} = \frac{\gamma_4 N_{24} - N_{14}}{1 - \gamma_4^2}$$

$$P_{25} = \frac{\gamma_5 N_{25} - N_{15}}{1 - \gamma_5^2}$$

$$P_{26} = \frac{-(\gamma_6 N_{26} + 1)}{1 - \gamma_6^2}$$

$$P_{31} = \frac{N_{31}}{\gamma_1}, \quad P_{37} = \frac{N_{37}}{\gamma_7}, \quad P_{38} = \frac{N_{38}}{\gamma_8}$$

The first-order approximate analytical solution, equations (90), (99), (103), (104) to (112), can be collected and summarized in Table 1. The coefficients of the same column are related to each other by the generators of coefficients, K_{31}, K_{12}, K_{13}, I_{34}, I_{35}, J_{16}, I_{17}, I_{18}, L_{29}. They will be determined in the next section.

3.2 The secular terms

In seeking a solution in the form of equations (42) to (47), practical considerations dictate that the form be limited to the first few terms. This can produce an unbounded solution owing to the appearance in the solution of terms that grow indefinitely with time, these terms being fre-

Table 1 The components of the first-order approximate analytical solution

	$e^{i\gamma_1 T_0}$	$e^{i\gamma_2 T_0}$	$e^{i\gamma_3 T_0}$	$e^{i\gamma_4 T_0}$	$e^{i\gamma_5 T_0}$	$e^{i\gamma_6 T_0}$	$e^{i\gamma_7 T_0}$	$e^{i\gamma_8 T_0}$	e^{iT_0}
p_{11}	$M_{11}K_{31}$						I_{17}	I_{18}	
p_{21}	$iM_{21}K_{31}$						$iM_{27}I_{17}$	$iM_{28}I_{18}$	
p_{31}		$M_{32}K_{12}$	$M_{33}K_{13}$	I_{34}	I_{35}				
ω_{11}		$iN_{12}K_{12}$	$iN_{13}K_{13}$	$iN_{14}I_{34}$	$iN_{15}I_{35}$	J_{16}			
ω_{21}		$N_{22}K_{12}$	$N_{23}K_{13}$	$N_{24}I_{34}$	$N_{25}I_{35}$	$iN_{26}J_{16}$			
ω_{31}	$iN_{31}K_{31}$						$iN_{37}I_{17}$	$iN_{38}I_{18}$	
θ_{11}		K_{12}	K_{13}						
θ_{21}		$iO_{22}K_{12}$	$iO_{23}K_{13}$						
θ_{31}	K_{31}								
ψ_{11}		$P_{12}K_{12}$	$P_{13}K_{13}$	$P_{14}I_{34}$	$P_{15}I_{35}$	$iP_{16}J_{16}$			$iP_{19}L_{29}$
ψ_{21}		$iP_{22}K_{12}$	$iP_{23}K_{13}$	$iP_{24}I_{34}$	$iP_{25}I_{35}$	$P_{26}J_{16}$			L_{29}
ψ_{31}	$P_{31}K_{31}$						$P_{37}I_{17}$	$P_{38}I_{18}$	

quently referred to as ‘secular’ terms. In the present study, the first-order approximate analytical solution is sought. There is no intention to find the second-order resonance phenomenon, but the coefficients of the first-order approximate analytical solution can be found by vanishing the coefficients of the secular terms. The condition for the elimination of secular terms from p_{j2}, ω_{j2}, θ_{j2}, ψ_{j2} demands that each of the coefficients of $\exp(i\gamma_1 T_0) \sim \exp(iT_0)$ vanish independently.

As in the previous section, in the circular orbital earth-pointing motion the d_2 vanished and $\sin d_2\tau = 0$, $\cos d_2\tau = 1$ and $\alpha_{25} = 1$ are obtained. By substituting in equations (66) to (77) and comparing the left-hand sides of equations (66) to (77) with equations (54) to (65), it is found that they have the same coefficients. Therefore, on the right-hand sides the terms of second degree of variables will not produce resonance, and can be dropped. Equations (66) to (77) become

$$(D_0^2 + \alpha_1)p_{12} - 2D_0 p_{22} + \alpha_2\theta_{32} = 2[-D_0D_1 p_{11} + (D_1 - \beta_2)p_{21} + \beta_5\psi_{31}] \tag{113}$$

$$2D_0 p_{12} + (D_0^2 + \alpha_5)p_{22} + \alpha_6\omega_{32} = -2[(D_1 - \beta_2)p_{11} + D_0D_1 p_{21}] \tag{114}$$

$$(D_0^2 + \alpha_7)p_{32} + \alpha_8\omega_{22} + \alpha_9\theta_{12} = -2(D_0D_1 p_{31} + \beta_{13}\psi_{21}) \tag{115}$$

$$\alpha_{10}p_{32} + D_0\omega_{12} + \alpha_{11}\omega_{22} + \alpha_{12}\theta_{12} = -D_1\omega_{11} - \beta_{15}\psi_{21} \tag{116}$$

$$\alpha_{13}\omega_{12} + D_0\omega_{22} + \alpha_{14}\theta_{22} = -D_1\omega_{21} - 2\beta_{16}\psi_{21} \tag{117}$$

$$\alpha_{15}p_{12} + D_0\omega_{32} + \alpha_{16}\theta_{32} = -D_1\omega_{31} + 2\beta_1\psi_{31} \tag{118}$$

$$(D_0^2 + \alpha_{19})\theta_{12} + \alpha_{20}D_0\theta_{22} = -2\{D_0D_1\theta_{11} + [\beta_{10}D_1 - (\beta_{15} - \beta_2)]\theta_{21} - 2\beta_{15}\psi_{21}\} \tag{119}$$

$$\alpha_{21}D_0\theta_{12} + (D_0^2 + \alpha_{22})\theta_{22} = 2[(\beta_9 D_1 + \beta_{17} + \beta_2)\theta_{11} - (D_0D_1 + \beta_{16})\theta_{21}] \tag{120}$$

$$(D_0^2 + \alpha_{23})\theta_{32} = 2(-D_0D_1 + \beta_1)\theta_{31} \tag{121}$$

$$\omega_{12} - D_0\psi_{12} + \psi_{22} = D_1\psi_{11} \tag{122}$$

$$\omega_{22} - \psi_{12} - D_0\psi_{22} = D_1\psi_{21} \tag{123}$$

$$\omega_{32} - D_0\psi_{32} = D_1\psi_{31} \tag{124}$$

Substituting the first-order approximate analytical solution of Table 1 into equations (113) to (124) and considering the same sets of simultaneous equations with the procedure of solving the first-order differential equations, to solve the first set, equation (121) gives

$$(D_0^2 + \alpha_{23})\theta_{32} = 2(-i\gamma_1 D_1 + \beta_1)K_{31}\, e^{i\gamma_1 T_0} \tag{125}$$

The condition for the elimination of secular terms from θ_{32} demands that the coefficient of $\exp(i\gamma_1 T_0)$ vanishes, which yields

$$\left[D_1 + i\left(\frac{\beta_1}{\gamma_1}\right)\right]K_{31} = 0 \tag{126}$$

Therefore

$$K_{31} = a_1\, e^{-i(\beta_1/\gamma_1)T_1} = a_1\, e^{-iv_1 T_1} \tag{127}$$

where

$$v_1 = \frac{\beta_1}{\gamma_1} \tag{128}$$

and a_1 is a complex constant because K_{31} is a function of T_1 only and depends on the initial condition. Note that v_1 is real. The other two equations (119) and (120), are

$$\begin{aligned}(D_0^2 + \alpha_{19})\theta_{12} + \alpha_{20}D_0\theta_{22} &= 2\{-i(\gamma_2 + \beta_{10}O_{22})D_1 \\ &\quad + i[(\beta_{15} - \beta_2)O_{22} + 2\beta_{15}P_{22}]\}K_{12}\, e^{i\gamma_2 T_0} \\ &\quad + 2\{-i(\gamma_3 + \beta_{10}O_{23})D_1 \\ &\quad + i[(\beta_{15} - \beta_2)O_{23} + 2\beta_{15}P_{23}]\}K_{13}\, e^{i\gamma_3 T_0} + \cdots\end{aligned}$$

$$\begin{aligned}\alpha_{21}D_0\theta_{12} + (D_0^2 + \alpha_{22})\theta_{22} &= 2\{(\beta_9 + \gamma_2 O_{22})D_1 \\ &\quad + [(\beta_{17} + \beta_2) - i\beta_{16}O_{22}]\}K_{12}\, e^{i\gamma_2 T_0} \\ &\quad + 2\{(\beta_9 + \gamma_3 O_{23})D_1 \\ &\quad + [(\beta_{17} + \beta_2) - i\beta_{16}O_{23}]\}K_{13}\, e^{i\gamma_3 T_0} + \cdots\end{aligned}$$

By the superposition method, the term of $\exp(i\gamma_2 T_0)$ is first considered:

$$\begin{aligned}(D_0^2 + \alpha_{19})\theta_{12} + \alpha_{20}D_0\theta_{22} &= 2\{-i(\gamma_2 + \beta_{10}O_{22})D_1 \\ &\quad + i[(\beta_{15} - \beta_2)O_{22} + 2\beta_{15}P_{22}]\}K_{12}\, e^{i\gamma_2 T_0}\end{aligned} \tag{129}$$

$$\alpha_{21} D_0 \theta_{12} + (D_0^2 + \alpha_{22})\theta_{22} = 2\{(\beta_9 + \gamma_2 O_{22})D_1 + [(\beta_{17} + \beta_2) - i\beta_{16} O_{22}]\} K_{12}\, e^{i\gamma_2 T_0} \quad (130)$$

Let the solution be

$$\theta_{12} = P(T_1)\, e^{i\gamma_2 T_0}, \qquad \theta_{22} = Q(T_1)\, e^{i\gamma_2 T_0} \quad (131)$$

Substituting equations (131) into equations (129) and (130), the coefficients of the left-hand sides satisfy

$$\begin{vmatrix} -\gamma_2^2 + \alpha_{19} & i\alpha_{20}\gamma_2 \\ i\alpha_{21}\gamma_2 & -\gamma_2^2 + \alpha_{22} \end{vmatrix} = 0 \quad (132)$$

The solution is found from the solvability condition (**7**) using the following condition:

$$\begin{vmatrix} -\gamma_2^2 + \alpha_{19} & 2\{-i(\gamma_2 + \beta_{10} O_{22})D_1 + i[(\beta_{15} - \beta_2)O_{22} + 2\beta_{15} P_{22}]\} K_{12} \\ i\alpha_{21}\gamma_2 & 2\{(\beta_9 + \gamma_2 O_{22})D_1 + [(\beta_{17} + \beta_2) - i\beta_{16} O_{22}]\} K_{12} \end{vmatrix} = 0 \quad (133)$$

Then

$$K_{12} = a_2\, e^{-(u_2 + iv_2)T_1} \quad (134)$$

where u_2 and v_2 are real are follows:

$$u_2 = \frac{(\alpha_{19} - \gamma_2^2)(\beta_{17} + \beta_2) + \alpha_{21}\gamma_2[(\beta_{15} - \beta_2)O_{22} + 2\beta_{15} P_{22}]}{(\alpha_{19} - \gamma_2^2)(\beta_9 + \gamma_2 O_{22}) - \alpha_{21}\gamma_2(\gamma_2 + \beta_{10} O_{22})} \quad (135)$$

$$v_2 = \frac{-(\alpha_{19} - \gamma_2^2)\beta_{16} O_{22}}{(\alpha_{19} - \gamma_2^2)(\beta_9 + \gamma_2 O_{22}) - \alpha_{21}\gamma_2(\gamma_2 + \beta_{10} O_{22})} \quad (136)$$

Secondly, the term $\exp(i\gamma_3 T_0)$ is used to obtain

$$K_{13} = a_3\, e^{-(u_3 + iv_3)T_1} \quad (137)$$

where u_3 and v_3 are real also:

$$u_3 = \frac{(\alpha_{19} - \gamma_3^2)(\beta_{17} + \beta_2) + \alpha_{21}\gamma_3[(\beta_{15} - \beta_2)O_{23} + 2\beta_{15} P_{23}]}{(\alpha_{19} - \gamma_3^2)(\beta_9 + \gamma_3 O_{23}) - \alpha_{21}\gamma_3(\gamma_3 + \beta_{10} O_{23})} \quad (138)$$

$$v_3 = \frac{-(\alpha_{19} - \gamma_3^2)\beta_{16} O_{23}}{(\alpha_{19} - \gamma_3^2)(\beta_9 + \gamma_3 O_{23}) - \alpha_{21}\gamma_3(\gamma_3 + \beta_{10} O_{23})} \quad (139)$$

Similarly, when solving the other sets of equations,

$$I_{17} = a_7\, e^{-(u_7 + iv_7)T_1}, \qquad I_{18} = a_8\, e^{-(u_8 + iv_8)T_1} \quad (140)$$

$$I_{34} = a_4\, e^{-(u_4 + iv_4)T_1}, \qquad I_{35} = a_5\, e^{-(u_5 + iv_5)T_1} \quad (141)$$

$$J_{16} = a_6\, e^{-(u_6 + iv_6)T_1} \quad (142)$$

where

$$u_4 = \frac{(2\beta_{13}\gamma_4^2 - \alpha_8\alpha_{13}\beta_{15} + 2\alpha_{11}\alpha_{13}\beta_{13})P_{24}}{2\gamma_4^3 - \alpha_8\alpha_{13} N_{14} + \alpha_8\gamma_4 N_{24} + 2\alpha_{11}\alpha_{13}\gamma_4} \quad (143)$$

$$v_4 = \frac{2\alpha_8\beta_{16}\gamma_4 P_{24}}{2\gamma_4^3 - \alpha_8\alpha_{13} N_{14} + \alpha_8\gamma_4 N_{24} + 2\alpha_{11}\alpha_{13}\gamma_4} \quad (144)$$

$$u_5 = \frac{(2\beta_{13}\gamma_5^2 - \alpha_8\alpha_{13}\beta_{15} + 2\alpha_{11}\alpha_{13}\beta_{13})P_{25}}{2\gamma_5^3 - \alpha_8\alpha_{13} N_{15} + \alpha_8\gamma_5 N_{25} + 2\alpha_{11}\alpha_{13}\gamma_5} \quad (145)$$

$$v_5 = \frac{2\alpha_8\beta_{16}\gamma_5 P_{25}}{2\gamma_5^3 - \alpha_8\alpha_{13} N_{15} + \alpha_8\gamma_5 N_{25} + 2\alpha_{11}\alpha_{13}\gamma_5} \quad (146)$$

$$u_6 = \frac{-(2\beta_{13}\gamma_4^2 - \alpha_8\alpha_{13}\beta_{15} + 2\alpha_{11}\alpha_{13}\beta_{13})P_{26}}{\alpha_8\alpha_{13} + \alpha_8\gamma_4 N_{26}} \quad (147)$$

$$v_6 = \frac{-2\alpha_8\beta_{16}\gamma_4 P_{26}}{\alpha_8\alpha_{13} + \alpha_8\gamma_4 N_{26}} \quad (148)$$

$$u_7 = \frac{\beta_2[(\alpha_5 - \gamma_7^2)M_{27} + 2\gamma_7]}{(\gamma_7^2 - \alpha_5)(M_{27} - \gamma_7) - \alpha_6 N_{37} - 2\gamma_7(\gamma_7 M_{27} - 1)} \quad (149)$$

$$v_7 = \frac{[\beta_5(\alpha_5 - \gamma_7^2) - 2\alpha_6\beta_1]P_{37}}{(\gamma_7^2 - \alpha_5)(M_{27} - \gamma_7) - \alpha_6 N_{37} - 2\gamma_7(\gamma_7 M_{27} - 1)} \quad (150)$$

$$u_8 = \frac{\beta_2[(\alpha_5 - \gamma_8^2)M_{28} + 2\gamma_8]}{(\gamma_8^2 - \alpha_5)(M_{28} - \gamma_8) - \alpha_6 N_{38} - 2\gamma_8(\gamma_8 M_{28} - 1)} \quad (151)$$

$$v_8 = \frac{[\beta_5(\alpha_5 - \gamma_8^2) - 2\alpha_6\beta_1]P_{38}}{(\gamma_8^2 - \alpha_5)(M_{28} - \gamma_8) - \alpha_6 N_{38} - 2\gamma_8(\gamma_8 M_{28} - 1)} \quad (152)$$

3.3 Conclusions

Substituting (127), (134), (137), (140) to (142) into Table 1, the first-order approximate analytical solution becomes

$$p_{11} = a_1 M_{11}\, e^{i(\gamma_1 T_0 - v_1 T_1)} + a_7\, e^{-u_7 T_1 + i(\gamma_7 T_0 - v_7 T_1)} + a_8\, e^{-u_8 T_1 + i(\gamma_8 T_0 - v_8 T_1)} + \text{c.c.} \quad (153)$$

$$p_{21} = ia_1 M_{22}\, e^{i(\gamma_1 T_0 - v_1 T_1)} + ia_7 M_{27}\, e^{-u_7 T_1 + i(\gamma_7 T_0 - v_7 T_1)} + ia_8 M_{28}\, e^{-u_8 T_1 + i(\gamma_8 T_0 - v_8 T_1)} + \text{c.c.} \quad (154)$$

$$p_{31} = a_2 M_{32}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + a_3 M_{33}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + a_4\,\mathrm{e}^{-u_4T_1+\mathrm{i}(\gamma_4T_0-v_4T_1)} + a_5\,\mathrm{e}^{-u_5T_1+\mathrm{i}(\gamma_5T_0-v_5T_1)} + \text{c.c.} \quad (155)$$

$$\omega_{11} = \mathrm{i}a_2 N_{12}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + \mathrm{i}a_3 N_{13}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + \mathrm{i}a_4 N_{14}\,\mathrm{e}^{-u_4T_1+\mathrm{i}(\gamma_4T_0-v_4T_1)} + \mathrm{i}a_5 N_{15}\,\mathrm{e}^{-u_5T_1+\mathrm{i}(\gamma_5T_0-v_5T_1)} + a_6\,\mathrm{e}^{-u_6T_1+\mathrm{i}(\gamma_6T_0-v_6T_1)} + \text{c.c.} \quad (156)$$

$$\omega_{21} = a_2 N_{22}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + a_3 N_{23}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + a_4 N_{24}\,\mathrm{e}^{-u_4T_1+\mathrm{i}(\gamma_4T_0-v_4T_1)} + a_5 N_{25}\,\mathrm{e}^{-u_5T_1+\mathrm{i}(\gamma_5T_0-v_5T_1)} + \mathrm{i}a_6 N_{26}\,\mathrm{e}^{-u_6T_1+\mathrm{i}(\gamma_6T_0-v_6T_1)} + \text{c.c.} \quad (157)$$

$$\omega_{31} = \mathrm{i}a_1 N_{31}\,\mathrm{e}^{\mathrm{i}(\gamma_1T_0-v_1T_1)} + \mathrm{i}a_7 N_{37}\,\mathrm{e}^{-u_7T_1+\mathrm{i}(\gamma_7T_0-v_7T_1)} + \mathrm{i}a_8 N_{38}\,\mathrm{e}^{-u_8T_1+\mathrm{i}(\gamma_8T_0-v_8T_1)} + \text{c.c.} \quad (158)$$

$$\theta_{11} = a_2\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + a_3\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + \text{c.c.} \quad (159)$$

$$\theta_{21} = \mathrm{i}a_2 O_{22}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + \mathrm{i}a_3 O_{23}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + \text{c.c.} \quad (160)$$

$$\theta_{31} = a_1\,\mathrm{e}^{\mathrm{i}(\gamma_1T_0-v_1T_1)} + \text{c.c.} \quad (161)$$

$$\psi_{11} = a_2 P_{12}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + a_3 P_{13}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + a_4 P_{14}\,\mathrm{e}^{-u_4T_1+\mathrm{i}(\gamma_4T_0-v_4T_1)} + a_5 P_{15}\,\mathrm{e}^{-u_5T_1+\mathrm{i}(\gamma_5T_0-v_5T_1)} + \mathrm{i}a_6 P_{16}\,\mathrm{e}^{-u_6T_1+\mathrm{i}(\gamma_6T_0-v_6T_1)} + \mathrm{i}a_9 P_{19}\,\mathrm{e}^{\mathrm{i}T_0} + \text{c.c.} \quad (162)$$

$$\psi_{21} = \mathrm{i}a_2 P_{22}\,\mathrm{e}^{-u_2T_1+\mathrm{i}(\gamma_2T_0-v_2T_1)} + \mathrm{i}a_3 P_{23}\,\mathrm{e}^{-u_3T_1+\mathrm{i}(\gamma_3T_0-v_3T_1)} + \mathrm{i}a_4 P_{24}\,\mathrm{e}^{-u_4T_1+\mathrm{i}(\gamma_4T_0-v_4T_1)} + \mathrm{i}a_5 P_{25}\,\mathrm{e}^{-u_5T_1+\mathrm{i}(\gamma_5T_0-v_5T_1)} + a_6 P_{26}\,\mathrm{e}^{-u_6T_1+\mathrm{i}(\gamma_6T_0-v_6T_1)} + a_9\,\mathrm{e}^{\mathrm{i}T_0} + \text{c.c.} \quad (163)$$

$$\psi_{31} = a_1 P_{32}\,\mathrm{e}^{+\mathrm{i}(\gamma_1T_0-v_1T_1)} + a_7 P_{37}\,\mathrm{e}^{-u_7T_1+\mathrm{i}(\gamma_7T_0-v_7T_1)} + a_8 P_{38}\,\mathrm{e}^{-u_8T_1+\mathrm{i}(\gamma_8T_0-v_8T_1)} + \text{c.c.} \quad (164)$$

Except for θ_{31}, the exponents of the complex form solution, equations (153) to (164), contain real parts. For stability, the coefficients of real parts must be non-positive. Thus, for the earth-pointing motion the conditions of stability are

$$u_s \geqslant 0, \qquad s = 2, 3, 4, 5, 6, 7, 8 \quad (165)$$

The values of the transformed parameters (u_s), listed in (135), (138), (143), (145), (147), (149) and (151), in equations (165) originate from the dimensions of the space station and the stiffness of structure. They are independent of gravitational forces and torques.

4 ROTATIONAL DYNAMICS WITH ARBITRARY ANGULAR VELOCITY PERPENDICULAR TO THE ORBITAL PLANE

To provide suitable living conditions in space, the rotation of the space station is intended to generate an artificial gravitational environment. For this reason, the dynamics of the arbitrary spinning angular velocity $\overline{\omega}$ perpendicular to the orbital plane is also studied.

4.1 The first-order approximate analytical solution

Under this condition,

$$\sin d_2\tau = \frac{\mathrm{e}^{\mathrm{i}d_2T_0} - \mathrm{e}^{-\mathrm{i}d_2T_0}}{2i}, \qquad \cos d_2\tau = \frac{\mathrm{e}^{\mathrm{i}d_2T_0} + \mathrm{e}^{-\mathrm{i}d_2T_0}}{2} \quad (166)$$

Substituting equations (166) into equations (54) to (65), by the same procedure to solve them, the particular solution is

$$p_{11\,p} = I_{1(3d_2)}\,\mathrm{e}^{3\mathrm{i}d_2T_0} + I_{1(2d_2)}\,\mathrm{e}^{2\mathrm{i}d_2T_0} + I_{1(d_2)}\,\mathrm{e}^{\mathrm{i}d_2T_0} + \text{c.c.} \quad (167)$$

$$p_{21p} = I_{2(3d_2)}\,e^{3id_2T_0} + I_{2(2d_2)}\,e^{2id_2T_0} + I_{2(d_2)}\,e^{id_2T_0} + \text{c.c.} \quad (168)$$

$$\omega_{31p} = J_{3(3d_2)}\,e^{3id_2T_0} + J_{3(2d_2)}\,e^{2id_2T_0} + J_{3(d_2)}\,e^{id_2T_0} + \text{c.c.} \quad (169)$$

$$\theta_{31p} = K_{3(3d_2)}\,e^{3id_2T_0} + K_{3(d_2)}\,e^{id_2T_0} + \text{c.c.} \quad (170)$$

$$\begin{aligned}\psi_{11p} = {} & L_{1211}\,e^{i(\gamma_2+d_2)T_0} + L_{1212}\,e^{i(\gamma_2-d_2)T_0} \\ & + L_{1311}\,e^{i(\gamma_3+d_2)T_0} + L_{1312}\,e^{i(\gamma_3-d_2)T_0} \\ & + L_{1411}\,e^{i(\gamma_4+d_2)T_0} + L_{1412}\,e^{i(\gamma_4-d_2)T_0} \\ & + L_{1511}\,e^{i(\gamma_5+d_2)T_0} + L_{1512}\,e^{i(\gamma_5-d_2)T_0} \\ & + L_{1611}\,e^{i(\gamma_6+d_2)T_0} + L_{1612}\,e^{i(\gamma_6-d_2)T_0} + \text{c.c.}\end{aligned} \quad (171)$$

$$\begin{aligned}\psi_{21p} = {} & L_{2211}\,e^{i(\gamma_2+d_2)T_0} + L_{2212}\,e^{i(\gamma_2-d_2)T_0} \\ & + L_{2311}\,e^{i(\gamma_3+d_2)T_0} + L_{2312}\,e^{i(\gamma_3-d_2)T_0} \\ & + L_{2411}\,e^{i(\gamma_4+d_2)T_0} + L_{2412}\,e^{i(\gamma_4-d_2)T_0} \\ & + L_{2511}\,e^{i(\gamma_5+d_2)T_0} + L_{2512}\,e^{i(\gamma_5-d_2)T_0} \\ & + L_{2611}\,e^{i(\gamma_6+d_2)T_0} + L_{2612}\,e^{i(\gamma_6-d_2)T_0} + \text{c.c.}\end{aligned} \quad (172)$$

$$\psi_{31p} = L_{3(3d_2)}\,e^{3id_2T_0} + L_{3(2d_2)}\,e^{2id_2T_0} + L_{3(d_2)}\,e^{id_2T_0} + \text{c.c.} \quad (173)$$

where

$$I_{1(3d_2)} = \frac{(9d_2^2-\alpha_5)(\alpha_2K_{3(3d_2)}+\frac{5}{8}\alpha_4)+\alpha_6(2\alpha_{16}K_{3(3d_2)}+\frac{5}{4}\alpha_{18})}{81d_2^4-9\delta_7d_2^2+\delta_8} \quad (174)$$

$$I_{1(2d_2)} = \frac{i[\alpha_3(d_2^2-\frac{1}{4}\alpha_5)-2d_2+\frac{1}{2}\alpha_6\alpha_{17}]}{16d_2^4-4\delta_7d_2^2+\delta_8} \quad (175)$$

$$I_{1(d_2)} = \frac{(d_2^2-\alpha_5)(\alpha_2K_{3(d_2)}-\frac{1}{8}\alpha_4)+\alpha_6(2\alpha_{16}K_{3(d_2)}-\frac{1}{4}\alpha_{18})}{d_2^4-\delta_7d_2^2+\delta_8} \quad (176)$$

$$I_{2(3d_2)} = \frac{i(9d_2^2-\alpha_1-\alpha_2K_{3(3d_2)}-\frac{5}{8}\alpha_4)}{6d_2} \quad (177)$$

$$I_{2(2d_2)} = \frac{i(4d_2^2-\alpha_1)+\frac{1}{4}\alpha_3}{4d_2} \quad (178)$$

$$I_{2(d_2)} = \frac{i(d_2^2-\alpha_1-\alpha_2K_{3(d_2)}+\frac{1}{8}\alpha_4)}{2d_2} \quad (179)$$

$$J_{3(3d_2)} = \frac{i(\alpha_{15}I_{1(3d_2)}+\alpha_{16}K_{3(3d_2)}+\frac{5}{8}\alpha_{18})}{3d_2} \quad (180)$$

$$J_{3(2d_2)} = \frac{i(\alpha_{15}I_{1(2d_2)})-\frac{1}{4}\alpha_{17}}{2d_2} \quad (181)$$

$$J_{3(d_2)} = \frac{i(\alpha_{15}I_{1(d_2)}+\alpha_{16}K_{3(d_2)}-\frac{1}{8}\alpha_{18})}{d_2} \quad (182)$$

$$K_{3(3d_2)} = \frac{-5\alpha_{24}}{8(\alpha_{23}-9d_2^2)} \quad (183)$$

$$K_{3(d_2)} = \frac{\alpha_{24}}{8(\alpha_{23}-d_2^2)} \quad (184)$$

$$L_{1211} = \frac{(N_{22}+N_{12})K_{12}-2i(\gamma_2+d_2)L_{2211}}{2\alpha_{25}} \quad (185)$$

$$L_{1212} = \frac{(N_{22}-N_{12})K_{12}-2i(\gamma_2-d_2)L_{2212}}{2\alpha_{25}} \quad (186)$$

$$L_{1311} = \frac{(N_{23}+N_{13})K_{13}-2i(\gamma_3+d_2)L_{2311}}{2\alpha_{25}} \quad (187)$$

$$L_{1312} = \frac{(N_{23}-N_{13})K_{13}-2i(\gamma_3-d_2)L_{2312}}{2\alpha_{25}} \quad (188)$$

$$L_{1411} = \frac{(N_{24}+N_{14})I_{34}-2i(\gamma_4+d_2)L_{2411}}{2\alpha_{25}} \quad (189)$$

$$L_{1412} = \frac{(N_{24}-N_{14})I_{34}-2i(\gamma_4-d_2)L_{2412}}{2\alpha_{25}} \quad (190)$$

$$L_{1511} = \frac{(N_{25}+N_{15})I_{35}-2i(\gamma_5+d_2)L_{2511}}{2\alpha_{25}} \quad (191)$$

$$L_{1512} = \frac{(N_{25}-N_{15})I_{35}-2i(\gamma_5-d_2)L_{2512}}{2\alpha_{25}} \quad (192)$$

$$L_{1611} = \frac{-i[(1-N_{26})J_{16}+2(\gamma_6+d_2)L_{2611}]}{2\alpha_{25}} \quad (193)$$

$$L_{1612} = \frac{i[(1+N_{26})J_{16}-2(\gamma_6-d_2)L_{2612}]}{2\alpha_{25}} \quad (194)$$

$$L_{2211} = \frac{i\{[\gamma_2N_{22}-(\alpha_{25}-d_2)N_{12}] + [\gamma_2N_{12}-(\alpha_{25}-d_2)N_{22}]\}K_{12}}{2[\alpha_{25}^2-(\gamma_2+d_2)^2]} \quad (195)$$

$$L_{2212} = \frac{i\{[\gamma_2N_{22}-(\alpha_{25}-d_2)N_{12}] - [\gamma_2N_{12}-(\alpha_{25}-d_2)N_{22}]\}K_{12}}{2[\alpha_{25}^2-(\gamma_2-d_2)^2]} \quad (196)$$

$$L_{2311} = \frac{\mathrm{i}\{[\gamma_3 N_{23} - (\alpha_{25} - d_2)N_{13}] + [\gamma_3 N_{13} - (\alpha_{25} - d_2)N_{23}]\}K_{13}}{2[\alpha_{25}^2 - (\gamma_3 + d_2)^2]} \quad (197)$$

$$L_{2312} = \frac{\mathrm{i}\{[\gamma_3 N_{23} - (\alpha_{25} - d_2)N_{13}] - [\gamma_3 N_{13} - (\alpha_{25} - d_2)N_{23}]\}K_{13}}{2[\alpha_{25}^2 - (\gamma_3 - d_2)^2]} \quad (198)$$

$$L_{2411} = \frac{\mathrm{i}\{[\gamma_4 N_{24} - (\alpha_{25} - d_2)N_{14}] + [\gamma_4 N_{14} - (\alpha_{25} - d_2)N_{24}]\}I_{34}}{2[\alpha_{25}^2 - (\gamma_4 + d_2)^2]} \quad (199)$$

$$L_{2412} = \frac{\mathrm{i}\{[\gamma_4 N_{24} - (\alpha_{25} - d_2)N_{14}] - [\gamma_4 N_{14} - (\alpha_{25} - d_2)N_{24}]\}I_{34}}{2[\alpha_{25}^2 - (\gamma_4 - d_2)^2]} \quad (200)$$

$$L_{2511} = \frac{\mathrm{i}\{[\gamma_5 N_{25} - (\alpha_{25} - d_2)N_{15}] + [\gamma_5 N_{15} - (\alpha_{25} - d_2)N_{25}]\}I_{35}}{2[\alpha_{25}^2 - (\gamma_5 + d_2)^2]} \quad (201)$$

$$L_{2512} = \frac{\mathrm{i}\{[\gamma_5 N_{25} - (\alpha_{25} - d_2)N_{15}] - [\gamma_5 N_{15} - (\alpha_{25} - d_2)N_{25}]\}I_{35}}{2[\alpha_{25}^2 - (\gamma_5 - d_2)^2]} \quad (202)$$

$$L_{2611} = \frac{\{-[\gamma_6 N_{26} + (\alpha_{25} - d_2)] + [\gamma_6 + (\alpha_{25} - d_2)N_{26}]\}J_{16}}{2[\alpha_{25}^2 - (\gamma_6 + d_2)^2]} \quad (203)$$

$$L_{2612} = \frac{\{-[\gamma_6 N_{26} + (\alpha_{25} - d_2)] - [\gamma_6 + (\alpha_{25} - d_2)N_{26}]\}J_{16}}{2[\alpha_{25}^2 - (\gamma_6 - d_2)^2]} \quad (204)$$

$$L_{3(3d_2)} = \frac{-\mathrm{i}J_{3(3d_2)}}{3d_2} \quad (205)$$

$$L_{3(2d_2)} = \frac{-\mathrm{i}J_{3(2d_2)}}{2d_2} \quad (206)$$

$$L_{3(d_2)} = \frac{-\mathrm{i}J_{3(d_2)}}{d_2} \quad (207)$$

4.2 Conclusions

For the stability of arbitrary angular velocity perpendicular to the orbital plane, the criteria (165) obtained in the previous section must be satisfied, and the frequencies of a particular solution can not be equal to the natural frequencies of the harmonic solutions in order to avoid resonance. Thus, the following stability conditions are obtained:

$$d_2 \neq \frac{\gamma_q}{\hbar}, \qquad q = 1, 7, 8, \quad \hbar = 1, 2, 3 \quad (208)$$

$$(\gamma_p \pm d_2) \neq \gamma_r, \alpha_{25}, \qquad p, r = 2, 3, 4, 5, 6 \quad (209)$$

and

$$u_s \geqslant 0, \qquad s = 2, 3, 4, 5, 6, 7, 8 \quad (210)$$

5 CONCLUSIONS

As a result of the present work, the following points can be made in conclusion:

1. In earth-pointing motion, the gravitational terms do not affect the stability with regard to the first-order approximate analytical solution.
2. According to the stability criteria (165), the stability of earth-pointing motion depends on the configuration of the space station and the elastic properties of the connecting structure.
3. For the motion with an arbitrary initial spinning rate perpendicular to the orbital plane, the gravitational forces and torques will limit the spinning rate. Therefore, besides the stability criteria (165), the stable spinning rate has some limitations, equations (208) and (209).

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