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Pooling spaces associated with finite geometry

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Abstract

Motivated by the works of Ngo and Du [H. Ngo, D. Du, A survey on combinatorial group testing algorithms with applications to *DNA* library screening, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 55 (2000) 171–182], the notion of pooling spaces was introduced [T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Mathematics 282 (2004) 163–169] for a systematic way of constructing pooling designs; note that geometric lattices are among pooling spaces. This paper attempts to draw possible connections from finite geometry and distance regular graphs to pooling spaces: including the projective spaces, the affine spaces, the attenuated spaces, and a few families of geometric lattices associated with the orbits of subspaces under finite classical groups, and associated with d-bounded distance-regular graphs.

1. Introduction

A mechanism has been considered so that each binary vector of length n with weight (at most) d will be associated with a binary vector of length t, and the minimum (Hamming) distance among the associated vectors is as large as possible, where d is small comparing with n. More precisely, let $\mathcal{P} = \{x \mid x \in Z_2^n \text{ with weight at most } d\} \subseteq Z_2^n$, we are looking for a matrix M of order $t \times n$, such that the minimum distance of the set $\{\overline{Mx} \mid x \in \mathcal{P}\} \subseteq Z_2^t$ is as large as possible.

$$\begin{aligned} x \in \mathcal{P} \subseteq Z_2^n \text{ (message)} &\to r(x) = \overline{Mx} \in Z_2^t \text{ (encoded message)} \\ &\to r(x) + e \in Z_2^t \text{ (reported message)} \\ &\to x \text{ (decoded message)}. \end{aligned}$$

If the columns of M are identified with the set of n items to be tested in pools (for an unknown positive subset of size at most d), and the rows of M are the characteristic vectors of those pools

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(i.e., subsets of items) to be tested, the binary vector x corresponds to a (unknown) positive set to be identified, the associated vector $\overline{Mx}$ is called the ideal outcome vector, and indeed the outcome vector $\overline{Mx} + e$ is received with a possible error e occurring during the conducting of experiments. The condition posed over M must guarantee that the correspondence between x and $r(x)$ is one to one for identifying purposes; and moreover the requirement over the minimum distance of the set $\{\overline{Mx} \mid x \in \mathcal{P}\} \subseteq Z_2^t$ is for error-correcting purposes.

Similar to the situation of classical error-correcting codes, this model provides a mechanism for non-adaptive group testing purpose if $\overline{Mx}$ is specified appropriately, see Section 2, and the matrix M is therefore called a *pooling design* under this consideration. The notion of traditional group testing can be traced back to around 1941 for blood testing purpose; the items to be tested nowadays have been transformed to *DNA* segments, refer to [2,15] for an overview of up-to-date results on combinatorial group testing algorithms along with its applications to *DNA* library screening.

The incidence matrices of the system $\left(\binom{[n]}{d}, \binom{[n]}{k}; \subseteq\right)$ and of its q-analogue $\left(\begin{bmatrix} V \\ d \end{bmatrix}_q, \begin{bmatrix} V \\ k \end{bmatrix}_q; \subseteq\right)$ were studied extensively by Macula [13,14] and by Ngo and Du [16] respectively for the disjunct property, see also [4,5], where $[n] = \{1, 2, \ldots, n\}$, $\binom{[n]}{i}$ is the family of all i-element subsets of $[n]$, $V = F_q^n$ is a vector space of dimension n over the finite field F_q, and $\begin{bmatrix} V \\ i \end{bmatrix}_q$ is the family of all i-dimensional subspaces of V. Note that both $\binom{[n]}{d}$ and $\begin{bmatrix} V \\ d \end{bmatrix}_q$ are levels of some well known partially ordered sets and the vertices sets of some distance regular graphs as well. Ngo and Du [15] therefore asked for possible generalizations of the Boolean algebra B_n of power sets of $\{1, 2, \ldots, n\}$; and also for conditions over some two levels of lattices for disjunct purposes; usually some regularity constraints must also be added for avoiding vagueness and for the ease of analysis. It would be better if some information about the capacity of error correcting of the matrix being constructed can be derived from the lattices themselves. Inspired by the remarks made by Ngo and Du, a comprehensive treatment of constructions of d-disjunct matrices in terms of ranked partially ordered sets was considered by Huang and Weng [8], leading to the introduction of pooling spaces over ranked partially order sets.

It was also pointed out by Ngo and Du [15] that this is a young and interesting field with deep connections to coding theory and design theory; the theory of *association schemes*, in particular *distance regular graphs* [1], should play an important role in improving the performance of pooling designs. Based on the notion of pooling spaces, this paper attempts to draw possible connections from finite geometry and association schemes, in particular distance regular graphs to pooling spaces as much as possible. The condition of d^e-disjunct and its variations are given in Section 2, followed by a decoding algorithm; the pooling spaces and their capability of error-correcting are given in Section 3. We note that geometric lattices are among pooling spaces, and a few families of geometric lattices associated with the orbits of subspaces under finite classical groups by Wan and Huo [18,19], and associated with d-bounded distance-regular graphs of diameter d by Gao et al. [6] respectively, therefore they provide some interesting families of pooling spaces, see Section 4.

2. d^e-disjunct matrices and their decodings

For a binary matrix M of order $t \times n$, let $\{C_1, C_1, \ldots, C_n\}$ be the family of subsets of $[t] = \{1, 2, \ldots, t\}$ with the corresponding columns of M as their characteristic vectors; while

$\{T_1, T_2, \ldots, T_t\}$ be the family of subsets of $[n] = \{1, 2, \ldots, n\}$ with the corresponding rows of M as their characteristic vectors. These two set systems $([t], \{C_i\}_{1\le i\le n})$ and $([n], \{T_j\}_{1\le j\le t})$ are called a *dual pair of set systems*, or dual pair in short, with respect to the binary matrix M. To identify a positive subset P, unknown at the beginning, of the items $[n]$, the pools $T_1, T_2, \ldots, T_t$ over the items $[n]$ are arranged in advance; an ideal outcome vector $z_P = (z_1, z_2, \ldots, z_t)^t$ will be reported after these t tests were performed simultaneously, where $z_j = 1$ if and only if $T_j \cap P$ is nonempty. If $x = (x_1, x_1, \ldots, x_n)^t$, and $Mx = (y_1, y_1, \ldots, y_t)^t$ is defined as the usual product over the integers, we define $\overline{Mx} = (\bar{y_1}, \bar{y_2}, \ldots, \bar{y_t})^t$ where $\bar{y_i} = 1$ if $y_i \ge 1$ and $\bar{y_i} = 0$ if $y_i = 0$. Note that $\overline{Mx}$ is equivalent to the Boolean sum of those columns corresponding to the nonzero entries of x, and thus $\overline{Mx}$ corresponds to the union $\bigcup_{i:x_i=1} C_i$ of those columns with $x_i = 1$. For $P \subseteq [n]$, $M(P)$ is defined to be the Boolean sum of the columns vectors corresponding to elements in P.

For the purpose of group testing, the following models have been considered in the literature. A matrix M, and also the corresponding family $C = \{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}$, is called *$d$-separable* if $M(P_1) \ne M(P_2)$ for distinct $P_1, P_2 \subseteq [n]$ with $|P_1|, |P_2| \le d$, and called *d-disjunct* if $C_i \nsubseteq M(P)$ whenever $i \notin P$. The notion of d-disjunct matrices was first introduced by Kautz and Singleton [12] in 1964. Note that the condition of d-disjunct is stricter than that of d-separable, and hence it provides more information for identifying purposes. The one to one correspondence between subsets P and its outcome vector provides a starting point for the purpose of pooling designs.

(1) If $\{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}$ is d-separable, the dual family $\{T_1, T_2, \ldots, T_t\} \subseteq 2^{[n]}$ satisfies the condition that for each vector $(x_i) \in \mathbb{Z}_2^t$, there exists $P \subseteq [n]$ with $|P| \le d$ such that $|P \cap T_i| = 0$ if and only if $x_i = 0$ for $i \le t$.
(2) If $\{C_1, C_2, \ldots, C_n\} \subseteq 2^{[t]}$ is d-disjunct and $P \subseteq [n]$ with $|P| \le d$, then the dual family $\{T_1, T_2, \ldots, T_t\} \subseteq 2^{[n]}$ satisfies the condition that $\bigcup_{j \notin M(P)} T_j = [n] - P$, and vice versa.

The notion of d-disjunct matrices has been generalized to d^e-disjunct matrices [14], (s, l)-superimposed codes and designs [3], $(s, l)^e$-generalized cover free families [17] over the past four decades. All these structures can be used in combinatorial group testing algorithms applicable to *DNA* library screening. The idea used above can be generalized for decoding algorithms for pooling designs based on d^e-disjunct matrices.

Definition 2.1. A binary matrix M of order $t \times n$ is called d^e-disjunct if $|C_j - \bigcup_{i\in P} C_i| \ge e + 1$ for any d-element subset P of $[n]$ and any $j \in [n] - P$.

A decoding algorithm for pooling designs based on d^e-disjunct matrices is behind the following theorem, where χ_U is the characteristic vector of the set U:

Theorem 2.1 ([7]). *Let M be a d^e-disjunct matrix of order $t \times n$, $P \subseteq [n]$ with $|P| \le d$ and $U \subseteq [t]$, and let $T = \{j \mid |C_j - U| \le \lfloor \frac{e}{2} \rfloor\}$, then*

(a) *if $d_H(M(P), \chi_U) \le \lfloor \frac{e}{2} \rfloor$, then $T = P$.*
(b) *if $d_H(M(P), \chi_U) \le e$ and $|T| \le d$, then $M(P) = \chi_U$ if and only if $M(T) = \chi_U$.*

Definition 2.2 ([17]). Let s, l and e be positive integers, a set system $(X, \mathcal{F})$ with $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ is called an *$(s, l; e)$-cover-free-family* provided that $|\bigcap_{j\in S} C_j - \bigcup_{i\in L} C_i| \ge e$ for disjoint $S, L \subseteq [n]$ with $|S| = s$ and $|L| = l$.

Less formally, it says that the intersection of any s blocks contains at least e elements not in the union of l other blocks. A decoding algorithm for pooling designs based on $(s, l)^e$-disjunct matrices was considered in [9]. It was shown that $(s, l)^e$-disjunct matrices also provide a class of pooling designs over *complexes*, called *setwise group testing*.

A connected regular graph Γ of diameter d is called *distance-regular* if the parameters

$$c_i(x, y) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i - 1\}|,$$
$$a_i(x, y) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i\}|,$$
$$b_i(x, y) = |\{z \mid d(x, z) = 1 \text{ and } d(z, y) = i + 1\}|$$

are constants c_i, a_i, b_i respectively for each i with $0 \le i \le d$, independent of the vertices x and y at distance i chosen. For example, the collinearity graphs of the system $\left(\binom{[n]}{d}, \binom{[n]}{d-1}; \subseteq\right)$ and of its q-analogue $\left(\begin{bmatrix} V \\ d \end{bmatrix}_q, \begin{bmatrix} V \\ d-1 \end{bmatrix}_q; \subseteq\right)$ are the distance-regular *Johnson graphs* $J(n, d)$ and the *Grassmann graphs* $J_q(n, d)$ respectively, see [1] for the details. A few families of d^e-disjunct matrices defined over the incidence matrices associated with Johnson graphs and Grassmann graphs were considered in [10]. A comprehensive treatment about the containment as the above examples of pooling designs can also be found in [4,5].

3. Pooling spaces and the capacity of error-correcting

Inspired by the remarks made by Ngo and Du [15], a comprehensive treatment of constructions of d^e-disjunct matrices in terms of ranked partially ordered sets was given by Huang and Weng [8], leading to the introduction of pooling spaces over ranked partially ordered sets. The notion of pooling spaces together with their capability of error-correcting is included in this section, followed by a few families of examples.

Let $P = (X, \le)$ be a finite *partially ordered set* (or *poset* in short) with the least element 0. An *atom* in P is an element in P that covers 0; let A_P be the set of all atoms in P. Moreover, let $P_0 = \{0\}$, $P_1 = A_P$ and let P_i be the set of all rank i elements of P for $i \ge 2$. A ranked poset P is called *atomic* whenever each element $x \in P$ is the least upper bound of the set $[0, x] \cap P_1$.

Definition 3.1. A *pooling space* is a finite ranked partially ordered set $P = (X, \le)$ such that $w^+ = \{y \ge w \mid y \in P\}$ is atomic for each $w \in P$.

A *geometric lattice* is an upper semi-modular atomic lattice. The projective geometry $PG(n, q)$ and the affine geometry $AG(n, q)$ are typical examples of *geometric lattices*. Note also that the difference between geometric lattices and combinatorial geometries is similar to that between incidence structures and families of subsets of a set.

Theorem 3.1 (*[11]*).

(1) *A ranked semi-lattice such that each interval is atomic is a pooling space.*
(2) *A geometric lattice is a pooling space.*

Some d^e-disjunct matrices can be associated with pooling spaces naturally as shown in the following theorem; moreover, some information of the capacity of error-correcting of the pooling designs based on such pooling spaces is also included.

Theorem 3.2 (*[8]*). *For a pooling space $P = (X, \leq)$ with rank $D \geq 1$, and integers $1 \leq d \leq l \leq D$, the binary incidence matrix $M = M(l, D)$ of the incidence structure $(P_l, P_D, \leq)$ is d^e-disjunct, where*

$$e + 1 = \min |\cup([y, x] \cap P_l)|,$$

the minimum is taken over all pairs (x, T) with $T \subseteq P_D$, $|T| \leq d$ and $x \in P_D - T$; the union is taken over all $y \in [0, x] \cap P_d$ such that $y \not\leq z$ for all $z \in T$.

The parameter e in the above theorem seems complicated; however, the number $|[y, x] \cap P_l|$ is a constant in the known examples. The *truncation* of a pooling space is again a pooling space, i.e., if P is a pooling space with rank D, then so is $\bigcup_{i=0}^{k} P_i$ with rank k for $0 \leq k \leq D$. Hence we can choose any k with $0 \leq k \leq D$, and replace P_D by P_k in the construction of M. The binary incidence matrices of the system $\left(\binom{[n]}{d}, \binom{[n]}{k}; \subseteq\right)$ and of its q-analogue $\left(\begin{bmatrix} F_q^n \\ d \end{bmatrix}_q, \begin{bmatrix} F_q^n \\ k \end{bmatrix}_q; \subseteq\right)$ have been studied extensively [4,5,13,14,16].

Theorem 3.3. *Let $P = (X, \leq)$ be a pooling space of rank D such that each interval of rank i in P is isomorphic to the projective geometry $PG(i-1, q)$. For $1 \leq d < k \leq D$ with $k - d \geq 2$, the incidence matrix $M(d, k)$ of the incidence structure $(P_d, P_k; \leq)$ is s^e-disjunct for $1 \leq s \leq \frac{q(q^{k-1}-1)}{q^{k-d}-1}$ and*

$$e = q^{k-d} \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q - (s-1)q^{k-d-1} \begin{bmatrix} k-2 \\ d-1 \end{bmatrix}_q - 1.$$

Example 3.1 (*[5,13] The Boolean Algebra*). As mentioned before, for $d < k < n$, the incidence matrix $J(n, d, k)$ of the incidence structure $\left(\binom{[n]}{d}, \binom{[n]}{k}; \subseteq\right)$ is d-disjunct; more precisely,

(1) [5,13] for $1 \leq s \leq d \leq k \leq n$, the matrix $J(n, d, k)$ is s^e-disjunct where $e = \binom{k-s}{k-d} - 1$;
(2) [14] for $1 \leq d \leq k \leq n$ and $k - d \geq 2$, let $\mathcal{K}$ be a family of k-subsets of $[n]$ such that the Hamming distance between any pair of k-sets in $\mathcal{K}$ is at least $2r$, then the incidence matrix of the system $\left(\binom{[n]}{d}, \mathcal{K}\right)$ is $d^{\alpha_d - 1}$-disjunct where $\alpha_d = \min(r^d, k - d)$.

Example 3.2 (*[5,8] The Hamming Spaces $H(n, q)$*). Let $X = \{(x_1, x_2, \ldots, x_n) \mid x_i \in F\}$ where $F = \{0, 1, \ldots, q\}$, we define $x \leq y$ if $x_i = 0$ or $x_i = y_i$ otherwise for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in X. The rank of $x = (x_1, x_2, \ldots, x_n)$ is the weight of x, and $|P_i| = \binom{n}{i} q^i$. Let $1 \leq s \leq d \leq k \leq n$, then the incidence matrix of the incidence structure $(P_d, P_k; \leq)$ is s^e-disjunct where $e = \binom{k-s}{k-d} - 1$.

Example 3.3 (*[4,5] The Projective Geometry*). For $1 \leq d < k \leq n$ with $k - d \geq 2$, the incidence matrix $J_q(n, d, k)$ of the incidence structure $\left(\begin{bmatrix} F_q^n \\ d \end{bmatrix}_q, \begin{bmatrix} F_q^n \\ k \end{bmatrix}_q; \subseteq\right)$ is s^e-disjunct for $1 \leq s \leq \frac{q(q^{k-1}-1)}{q^{k-d}-1}$, and

$$e = q^{k-d} \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q - (s-1)q^{k-d-1} \begin{bmatrix} k-2 \\ d-1 \end{bmatrix}_q - 1.$$

Example 3.4 (*[11] The Affine Geometry*). Let V be the n-dimensional vector space over the finite field F_q, and $\mathcal{P}$ be the family of all affine subspaces of V and the empty set $\emptyset$ ordered by inclusion, then $\mathcal{P}$ is a geometric lattice. Let P_{i+1} be the family consisting of all affine i-subspaces of $V = F_q^n$. The incidence matrix of the incidence structure (P_{r+1}, P_{k+1}) of order $q^{n-r}\begin{bmatrix} n \\ r \end{bmatrix}_q \times q^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q$ is s^e-disjunct for any $1 \le s < \frac{q(q^k-1)}{q^{k-r}-1}$ and $e = q^{k-r}\begin{bmatrix} k \\ r \end{bmatrix}_q - sq^{k-r-1}\begin{bmatrix} k-1 \\ r \end{bmatrix}_q - 1$.

We next consider a substructure of $\left(\begin{bmatrix} F_q^m \\ r \end{bmatrix}_q, \begin{bmatrix} F_q^m \\ k \end{bmatrix}_q; \subseteq\right)$ which carries the structure of attenuated spaces and the bilinear forms graphs $H_q(d, n)$ as well.

Example 3.5 (*[8] The Attenuated Spaces*). Let V be the vector space of dimension $n + d$ over F_q and $W = \langle w_1, w_2, \ldots, w_n \rangle \subseteq V$ a subspace of dimension $n\,(\ge d)$. Let further $\{w_1, w_2, \ldots, w_n; u_1, u_2, \ldots, u_d\}$ be a basis for V, and $U = \langle u_1, u_2, \ldots, u_d \rangle$. Let $P_i = \{A \mid A \in \begin{bmatrix} V \\ i \end{bmatrix}_q$ and $\dim(A \cap U) = 0\}$, then each $A \in P_d$ corresponds to a unique matrix M_A of order $d \times n$ in the following way: Let $A = \langle u_1 + v_1, u_2 + v_2, \ldots, u_d + v_d \rangle$ for unique choices of $v_1, v_2, \ldots, v_d \in W$, then $M_A = [a_{ij}]_{d\times n}$ where $v_i = \sum_{j=1}^{n} a_{ij} w_j$ for $1 \le i \le d$. If A, B correspond to M_A, M_B respectively as given, then $d - \dim(A \cap B) = \operatorname{rank}(M_A - M_B)$. Moreover, each interval of rank i is isomorphic to the projective geometry $PG(i-1, q)$. As a consequence of Theorem 3.3, for integers $1 \le r < k \le d$ with $k - r \ge 2$, the incidence matrix of the system $(P_r, P_k; \subseteq)$ associated with the attenuated space of rank d is s^e-disjunct for $1 \le s \le \frac{q(q^{k-1}-1)}{q^{k-r}-1}$ and

$$e = q^{k-r}\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q - (s-1)q^{k-r-1}\begin{bmatrix} k-2 \\ r-1 \end{bmatrix}_q - 1.$$

Example 3.6 (*[20] The Hermitian Forms Space* Her(q, d)). Let P be the set of all *weak geodesic subgraphs* of the *Hermitian forms graph* of diameter d, ordered by reversed inclusion. Then P is a ranked semi-lattice with atomic intervals, and with $|P_i| = \begin{bmatrix} d \\ i \end{bmatrix}_{q^2} q^{i(2d-i)}$ $(0 \le i \le d)$.

4. More examples of pooling spaces

Some families of geometric lattices associated with subspaces of d-bounded distance-regular graphs of diameter d, and associated with finite geometry and classical groups were given by Gao et al. [6], and by Huo and Wan [18] respectively. All of them turn out to be families of pooling spaces as shown in Section 3, and will be summarized in this section.

4.1. Some d-bounded distance-regular graphs

An induced subgraph Δ of Γ of diameter d is called *strongly closed* if $\{x \mid (d(u, x), d(x, v)) = (1, i-1) \text{ or } (1, i)\} \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$ at distance i for each $i \le d$. It is obvious that strongly closed subgraphs are connected and $d_\Gamma(x, y) = d_\Delta(x, y)$ for all $x, y \in \Delta$. A *subspace* of Γ is a regular subgraph induced by a strongly closed subset. For subspaces Δ_1, Δ_2 of Γ, the join $\Delta_1 + \Delta_2$ of Δ_1 and Δ_2 is the smallest subspace containing $\Delta_1 \cup \Delta_2$. A distance-regular graph Γ with diameter d is called *d-bounded* if every strongly closed subgraph of Γ is regular, and any two vertices $x, y \in V(\Gamma)$ are contained in a common strongly closed subgraph of diameter $d(x, y)$. For a d-bounded distance-regular graph Γ with diameter $d \geq 3$, let

(1) $P(x)$ be the set of strongly closed subgraphs containing the vertex $x \in V(\Gamma)$,
(2) $P(x, i) = \{\Delta \mid \Delta \in P(x)$ with diameter $i\}$, and
(3) $L(x, i)$ be the set of the intersection of elements in $P(x, i)$, with the convention that $\Gamma \in L(x, i)$ for $i \in [d-1]$. It is called the set generated by the intersection of elements in $P(x, i)$.

The lattices $(L(x, i), \subseteq)$, and $(L(x, i), \supseteq)$ were studied in [6] for d-bounded distance-regular graphs with diameter d at least 3. It was proved that $(L(x, i), \subseteq)$ and $(L(x, i), \supseteq)$ are both finite atomic lattices, and conditions for them being geometric lattices were given.

Example 4.1 (*[6]*). Let Γ be a d-bounded distance-regular graph with diameter d at least 3. For each vertex x and each $i \in [d-1]$,

(1) $(L(x, i), \subseteq)$ is a finite geometric lattice.
(2) $(L(x, i), \supseteq)$ is a finite geometric lattice if and only if $i = 1$ or $i = d-1$ and

$$d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) = d(\Delta_1) + d(\Delta_2), \quad \forall \Delta_1,\ \Delta_2 \in P(x).$$

4.2. Classical polar spaces

Example 4.2. Let V be a vector space with a given non-degenerate form over the field F_q, a subspace of $V$is called isotropic whenever the form vanishes completely on that subspace. It is known that all the maximal isotropic subspaces have the same dimension, denoted by D. Let $\mathcal{F}$ be the family of all isotropic subspaces of V, $A \leq B$ whenever A is a subspace of B for $A, B \in \mathcal{F}$; moreover $\text{rank}(A) = \dim(A)$.

Name	dim(V)	Form	$\lvert P_i \rvert$
$B_D(q)$	$2D+1$	Quadratic	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$C_D(q)$	$2D$	Alternating	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$D_D(q)$	$2D$	Quadratic (with index D)	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k})$
${}^2D_{D+1}(q)$	$2D+2$	Quadratic (with index D)	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=-1}^{i-2}(1+q^{D-k})$
${}^2A_{2D}(r)$	$2D+1$	Hermitian ($q = r^2$)	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k+\frac{1}{2}})$
${}^2A_{2D-1}(r)$	$2D$	Hermitian ($q = r^2$)	$\begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k+\frac{1}{2}})$

Example 4.3. Let V be a vector space with a given degenerate form over the finite field F_q, a subspace of V is called isotropic if the form vanishes completely on that space. It is known that all the maximal isotropic subspaces intersecting trivially with $V^{\perp}$ have the same dimension, denoted by D. Let $\dim(V^{\perp}) = l$.

(1) Let $\mathcal{F}$ be the family of all isotropic subspaces of V intersecting trivially with $V^{\perp}$, $A \leq B$ whenever A is a subspace of B for $A, B \in \mathcal{F}$; moreover $\text{rank}(A) = \dim(A)$.

Name	dim(V)	Form	$\lvert P_i \rvert$
$B_{D,l}(q)$	$2D+l+1$	Quadratic	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$C_{D,l}(q)$	$2D+l$	Alternating	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$D_{D,l}(q)$	$2D+l$	Quadratic (with index D)	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k})$
${}^2D_{D+1,l}(q)$	$2D+l+2$	Quadratic (with index D)	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=-1}^{i-2}(1+q^{D-k})$
${}^2A_{2D,l}(r)$	$2D+l+1$	Hermitian ($q=r^2$)	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k+\frac{1}{2}})$
${}^2A_{2D-1,l}(r)$	$2D+l$	Hermitian ($q=r^2$)	$q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k+\frac{1}{2}})$

(2) Let $\mathcal{F}$ be the family of all cosets of isotropic subspaces of V intersecting trivially with $V^{\perp}$ and the empty set $\emptyset$, $A \leq B$ whenever A is an affine subspace of B for $A, B \in \mathcal{F}$ or $A = \emptyset$; moreover rank$(A) = \dim(A) + 1$.

Name	dim(V)	Form	$\lvert P_i \rvert$
$B_{D,l}(q)$	$2D+l+1$	Quadratic	$q^{2D+l+1-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$C_{D,l}(q)$	$2D+l$	Alternating	$q^{2D+l-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k})$
$D_{D,l}(q)$	$2D+l$	Quadratic (with index D)	$q^{2D+l-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k})$
${}^2D_{D+1,l}(q)$	$2D+l+2$	Quadratic (with index D)	$q^{2D+l+2-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-2}(1+q^{D+1-k})$
${}^2A_{2D,l}(r)$	$2D+l+1$	Hermitian ($q=r^2$)	$q^{2D+l+1-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=0}^{i-1}(1+q^{D-k+\frac{1}{2}})$
${}^2A_{2D-1,l}(r)$	$2D+l$	Hermitian ($q=r^2$)	$q^{2D+l-i} \cdot q^{il} \cdot \begin{bmatrix} D \\ i \end{bmatrix}_q \cdot \prod_{k=1}^{i}(1+q^{D-k+\frac{1}{2}})$

4.3. Some lattices generated by transitive sets of subspaces

Let $V = F_q^n$ be the n-dimensional vector space over the finite field F_q of q elements, and let G_n be one of the classical group of degree n over F_q. The family of all subspaces of V is partitioned into orbits under the action of the group G_n. Let M be any nontrivial orbit of subspaces under G_n, and let $L(M)$ be the set of subspaces generated by M, i.e., the family consists of V and of all subspaces of V which are the intersection of those subspaces in M. Then both $(L(M), \subseteq)$ and $(L(M), \supseteq)$ are lattices. The geometricity of those lattices were classified by Wan and Huo [18,19] recently.

For the case $G_n = GL_n(F_q)$, the set $L(m, n)$ of subspaces generated by the orbit $M(m, n)$ consisting of all m-dimensional subspaces consists of V and all of its subspaces of dimension at most m.

Example 4.4 (*[18,19]*). The lattices $(L(1, n), \supseteq)$, $(L(n-1, n), \supseteq)$ and $(L(m, n), \subseteq)$ with $1 \leq m \leq n-1$ are geometric lattices.

For the symplectic case, G_n is the *symplectic group* $Sp_{2\nu}(F_q)$ of degree $n = 2\nu$ over F_q consisting of all $2\nu \times 2\nu$ matrices T over F_q satisfying $TK^tT = K$ where $K = \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}$. An m-dimensional subspace P is of *type* (m, s) if PK^tP is of rank $2s$, it is known that subspaces with type (m, s) exist if and only if $2s \le m \le \nu + s$, and that the set $M(m, s; 2\nu)$ of subspaces with the same type (m, s) forms an orbit. The set $L(m, s; 2\nu)$ of intersections of subspaces with type (m, s) consists of V and all subspaces of type (m_1, s_1) with $m - m_1 \ge s - s_1 \ge 0$.

Example 4.5 (*[18,19]*). $(L(1, 0; 2\nu), \subseteq)$, $(L(1, 0; 2\nu), \supseteq)$, $(L(2\nu - 1, \nu - 1; 2\nu), \subseteq)$, and $(L(2\nu - 1, \nu - 1; 2\nu), \supseteq)$ are geometric lattices.

Similar results hold for the unitary case, the orthogonal case and the pseudo-symplectic case; refer to [18,19] for more details.

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