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Maximal sets of hamilton cycles in $K_{2p} - F$

H.L. Fu[a], S.L. Logan[b], C.A. Rodger[b]

[a]*Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hseuh Road, Hsinchu, Taiwan*
[b]*Department of Discrete and Statistical Sciences, Auburn University, 235 Allison, AL 36849, USA*

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Abstract

A set S of edge-disjoint hamilton cycles in a graph T is said to be maximal if the hamilton cycles in S form a subgraph of T such that $T - E(S)$ has no hamilton cycle. The spectrum of a graph T is the set of integers m such that T contains a maximal set of m edge-disjoint hamilton cycles. This spectrum has previously been determined for all complete graphs, all complete bipartite graphs, and many complete multipartite graphs. One of the outstanding problems is to find the spectrum for the graphs formed by removing the edges of a 1-factor, F, from a complete graph, K_{2p}.

In this paper we completely solve this problem, giving two substantially different proofs. One proof uses amalgamations, and is of interest in its own right because it is the first example of an amalgamation where vertices from different parts are amalgamated. The other is a neat direct proof.

Keywords: Hamilton; Maximal; Amalgamations

1. Introduction

A *hamilton cycle* in a graph T is a spanning cycle of T. If S is a set of edge-disjoint hamilton cycles in T and if $E(S)$ is the set of edges occurring in the hamilton cycles in S, then S is said to be *maximal* if $T - E(S)$ has no hamilton cycle.

In 1993, Hoffman et al. [5] showed that there exists a maximal set S of m edge-disjoint hamilton cycles in K_n if and only if $m \in \{\lfloor (n+3)/4 \rfloor, \lfloor (n+3)/4 \rfloor + 1, \ldots, \lfloor (n-1)/2 \rfloor\}$. Using amalgamation techniques, Bryant et al. [2] showed that there exists a maximal set S of m edge-disjoint hamilton cycles in the complete bipartite graph $K_{n,n}$ if and only if $n/4 < m \leqslant n/2$. Later, Daven et al. [3] extended the use of amalgamation techniques by nearly showing that for $n \geqslant 3$ and $p \geqslant 3$, there exists a maximal set S of m hamilton cycles in the complete multipartite graph K_n^p (p parts of size n) if and only if $\lceil (n(p-1))/4 \rceil \leqslant m \leqslant \lfloor (n(p-1))/2 \rfloor$, and $m > (n(p-1))/4$ if n is odd and $p \equiv 1 \pmod 4$; the case where the result is still in doubt is when n is odd and $m \leqslant ((n+1)(p-1)-2)/4$.

In these results, if $T = K_n$ or $T = K_{n,n}$, then in every case the set S of m hamilton cycles is maximal because $T - E(S)$ is disconnected. However if $T = K_n^p$, then the construction in [3] usually results in $T - E(S)$ being disconnected, but in some cases it has edge-connectivity 1.

So we can put the results in [2,3,5] to prove the following result, as stated in [3].

E-mail address: rodgec1@auburn.edu (C. Rodger).

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Theorem 1.1. *There exists a maximal set of m hamilton cycles in K_n^p (p parts of size n) if and only if*

1. $\lceil n(p-1)/4\rceil \leqslant m \leqslant \lfloor n(p-1)/2\rfloor$ *and*
2. $m > n(p-1)/4$ *if*
   (a) *n is odd and* $p \equiv 1 (\mathrm{mod}\ 4)$, *or*
   (b) $p = 2, n = 1$

except possibly for the two undecided cases when $n=2$, *and when* $n \geqslant 3$ *is odd, p is odd, and* $m \leqslant ((n+1)(p-1)-2)/4$.

In this paper, we extend this result by removing the possible exception when $n=2$ (see Theorems 2.1 and 3.1). Clearly this complete multipartite graph with p parts of size 2 is simply formed from the complete graph on $2p$ vertices by removing the edges in a 1-factor. So, with $T = K_{2p} - F$ where F is a 1-factor of K_{2p}, and with $\lceil (p-1)/2\rceil \leqslant m \leqslant p-1$, we find a set S of m hamilton cycles in T which is maximal since $T - E(S)$ is disconnected. To do so, we provide two substantially different proofs. One proof uses amalgamations, and is of interest in its own right because it is the first example of an amalgamation where vertices from different parts are amalgamated (see [1] for a survey of amalgamations, and see [6] for a related proof). The other is a neat direct proof.

Throughout the paper, loops will count two towards the degree of the incident vertex. If h is an edge-coloring of a graph G, then let $h_i(u, v)$ denote the number of edges colored i joining u and v in G. The subgraph of a graph G induced by the edges colored i is known as the ith color class, and is denoted by G_i. An edge-coloring is said to be *equitable* if $|d_{G_i}(v) - d_{G_j}(v)| \leqslant 1$ for all pairs of colors i, j and all vertices $v \in V(G)$. If $|h_i(u, v) - h_j(u, v)| \leqslant 1$ for all pairs of colors i, j and all pairs of vertices $u, v \in V(G)$ then the edge-coloring h is said to be *balanced*. It has been shown that for all $k \geqslant 1$ and for any bipartite multigraph B there exists a k-edge-coloring of B (that is, an edge-coloring using k colors altogether) that is both equitable and balanced [4]. Let $\mu(v, w)$ denote the number of edges joining vertices v and w in G, and let $\mu(v, C)$ denote the number of edges joining vertex v to vertices in some set C of vertices.

2. A proof using amalgamations

In this section we make use of the proof technique of amalgamations. The idea behind the method, informally speaking, is as follows. An *amalgamation* of a graph T is the graph U defined by a homomorphism $g : V(T) \to V(U)$. Each vertex u in U can be considered to "contain" $f(u) = |g^{-1}(u)|$ vertices of T; f is called the *amalgamation function* of (T, U). In the following proof, we begin with a graph U together with an *associated* function f that could conceivably be the amalgamation of $(T = K_{2p} - F, U)$. We then inductively prove that f is indeed this amalgamation function by disentangling each vertex u into $f(u)$ vertices one by one.

This proof is of particular interest since it is the first time amalgamations have been used where a vertex "contains" some, but not all, of the vertices from several parts of the corresponding complete multipartite graph; see [3] for an example where such an extension would be a great help. The more we know about disentangling, the simpler we can afford U to be, so the easier it is to construct U.

The following result is the focus of this paper.

Theorem 2.1. *There exists a maximal set S of m hamilton cycles in* $T = K_{2p} - F$ *if and only if* $\lceil (p-1)/2\rceil \leqslant m \leqslant p-1$, *where F is a 1-factor of* K_{2p}.

Proof. It is shown in [3] that $\lceil (p-1)/2\rceil \leqslant m \leqslant p-1$ is a necessary condition for S to exist, so we now prove the sufficiency.

Clearly we can assume that $p \geqslant 2$. For $1 \leqslant y \leqslant x \leqslant p$ we first define the set of m-edge-colored graphs $\mathcal{G}(x, y)$ in which each graph has vertex set $V \cup W$, where $V = \{v_1, v_2, \ldots, v_{x-1}, v_p\}$ and $W = \{w_1, w_2, \ldots, w_{y-1}, w_p\}$; so $x = |V|$ and $y = |W|$. Each graph in $\mathcal{G}(x, y)$ has the associated function f defined on the vertices by

$$f(u) = \begin{cases} p - x + 1 & \text{if } u = v_p, \\ p - y + 1 & \text{if } u = w_p, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

With this in mind, we define G with an m-edge-coloring to be a graph in $\mathscr{G}(x, y)$ if and only if it satisfies the following four defining properties.

(D1) The number of edges between vertices is

$$\mu(v_i, w_j) = \begin{cases} 0 & \text{if } 1 \leqslant i = j < y, \\ f(v_i)(f(w_j) - 1) & \text{if } i \geqslant y \text{ when } j = p, \\ f(v_i)f(w_j) & \text{otherwise, and} \end{cases}$$

$$\mu(u_1, u_2) \leqslant f(u_1)f(u_2) \quad \text{if } \{u_1, u_2\} \subseteq V \text{ or } \{u_1, u_2\} \subseteq W.$$

(D2) For any $u \in V(G)$, the number of loops on u is at most $\binom{f(u)}{2}$.
(D3) For any $u \in V(G)$, the degree of u in each color class is $2f(u)$.
(D4) Each color class is connected.

Now observe that if there exists a graph G in $\mathscr{G}(p, p)$, then by the definition of f it must be the case that $f(u) = 1$ for all vertices $u \in V(G)$. Therefore, using the properties (D1–D4) it is easily seen by (D1–D2) that G is a loopless simple graph with no edges between v_i and w_i for $1 \leqslant i \leqslant p$, so G is a subgraph of $K_{2p} - F$. Furthermore by (D1), G contains all edges $\{v_i, w_j\}$ where $i \neq j$, so the complement of G in $K_{2p} - F$ is disconnected. By (D3–D4) G has an edge-coloring in which each of the m color classes is 2-regular and connected, so each color class G_i is a hamilton cycle. So proving that $\mathscr{G}(p, p)$ is non-empty will prove Theorem 2.1.

To show that $\mathscr{G}(p, p)$ contains a graph, we proceed by induction. We first show there exists a graph in $\mathscr{G}(1, 1)$. We then complete the proof by showing that for $p \geqslant \alpha \geqslant \beta$ and $p - 1 \geqslant \beta$, if $\mathscr{G}(\alpha, \beta)$ contains a graph G, then we can use it to form a graph G' in $\mathscr{G}(\alpha, \beta + 1)$.

Constructing $G \in \mathscr{G}(1, 1)$: Let G be the multigraph with vertex set $V \cup W$ where $V = \{v_p\}$ and $W = \{w_p\}$, with associated amalgamation function satisfying $f(v_p) = f(w_p) = p$, defined as follows. As the definition proceeds we also check that $G \in \mathscr{G}(1, 1)$.

Join v_p to w_p with $p(p-1)$ edges. So clearly, G satisfies (D1). Next, notice that $p(2m-p+1)/2 = (2mp - p(p-1))/2$ is an integer since $p(p-1)$ is even. Also $p(2m - p + 1)/2 \geqslant (p(2\lceil (p-1)/2 \rceil - (p-1)))/2$ since $m \geqslant \lceil (p-1)/2 \rceil$. So if p is even then $p(2m - p + 1)/2 \geqslant p(2(p/2) - (p-1))/2 = p/2$, which is non-negative since p is positive, and if p is odd then $p(2m - p + 1) \geqslant p(2((p-1)/2) - (p-1))/2 = 0$; so $p(2m - p + 1)/2$ is non-negative. Therefore we can place $p(2m - p + 1)/2$ loops on each of v_p and w_p; so then each vertex has degree $2mp$. To verify that (D2) is satisfied, we should check that $p(2m - p + 1)/2 \leqslant \binom{p}{2}$. Clearly $p(2m - p + 1)/2 = (2mp - p(p-1))/2 \leqslant p(p-1) - p(p-1)/2 = p(p-1)/2$, since we are given that $m \leqslant p - 1$. Therefore (D2) is satisfied.

To define the m-edge-coloring h with colors in $\{1, \ldots, m\}$, first arbitrarily name the edges joining v_p to w_p with $e_1, \ldots, e_{p(p-1)}$. For $1 \leqslant i \leqslant p(p-1)$ color the edge e_i with the color congruent to $\lceil i/2 \rceil \pmod m$. Since $p(p-1)$ is even, it follows that:

(i) $h_i(v_p, w_p)$ is even for $1 \leqslant i \leqslant m$, and
(ii) $|h_i(v_p, w_p) - h_j(v_p, w_p)| \leqslant 2$ for $i, j \in \mathbb{Z}_m$.

So $h_i(v_p, w_p)$ is either $p(p-1)/m$ if this number happens to be an even integer, or is one of the two even integers closest to $(p(p-1))/m$ otherwise. Next, since we are given that $m \geqslant \lceil (p-1)/2 \rceil$, it follows that $m \geqslant (p-1)/2$, so $2p \geqslant p(p-1)/m$. Therefore, being an integer, $2p$ is at least the smallest even number greater than or equal to $p(p-1)/m$, so is at least $h_i(v_p, w_p)$. This implies that $2p - h_i(v_p, w_p)$ is non-negative and even, so for $1 \leqslant i \leqslant m$ we can color $(2p - h_i(v_p, w_p))/2$ loops with color i on v_p and also on w_p. Then each of v_p and w_p has degree $2p$ in G_i, so (D3) is satisfied. Also $\sum_i (2p - h_i(v_p, w_p))/2 = (2mp - p(p-1))/2 = p(2m - p + 1)/2$, so each loop has been colored.

Since $p - 1 \geqslant m$, $p(p-1)/2m \geqslant p/2 \geqslant 1$. Now observe that for $1 \leqslant i \leqslant m$, G_i is connected, since $h_i(v_p, w_p) \geqslant 2\lfloor (p(p-1)/2m \rfloor \geqslant 2$. Therefore (D4) is satisfied. Because G satisfies (D1–D4), $G \in \mathscr{G}(1, 1)$.

Constructing $G' \in \mathscr{G}(\alpha, \beta + 1)$ *from* $G \in \mathscr{G}(\alpha, \beta)$: Let $G \in \mathscr{G}(\alpha, \beta)$. We can assume that there exists a vertex $u \in V(G)$ such that $f(u) \geqslant 2$, for otherwise $G \in \mathscr{G}(p, p)$ and we are finished. Without loss of generality, we can assume $f(w_p) \geqslant 2$. Recall we are also assuming that $p \geqslant \alpha \geqslant \beta$, and $p - 1 \geqslant \beta$.

Let B be the bipartite multigraph with bipartition $\{C, Z\}$ where $C=\{c_1,\dots,c_m\}\cup\{d\}$ and $Z=(V(G)-\{w_p\})\cup\{l\}$ formed as follows:

(i) for each edge colored k joining $z\in Z$ to w_p in G, let B contain an edge joining c_k to z;
(ii) for each loop colored k on w_p in G join c_k to l in B with *two* edges; and
(iii) for $\beta\leqslant i\leqslant\alpha-1$ and for $i=p$ join v_i to d with $f(v_i)$ edges.

(The edges in B incident with vertex d "represent" edges $\{v_i, w_i\}$ for $\beta\leqslant i\leqslant\alpha-1$, which are edges that, of course, are *not* in $K_{2p}-F$.)

By (D3), w_p has degree $2f(w_p)$ in each color class in G, so by (i) and (ii) it immediately follows that $d_B(c_i)=2f(w_p)$ for $1\leqslant i\leqslant m$. Also, $d_B(d)=(\alpha-\beta)+(p-\alpha+1)=p-\beta+1=f(w_p)$ by the definition of f, $d_B(l)\leqslant 2\binom{f(w_p)}{2}$ by (D2), and by (D1) $d_B(z)\leqslant f(z)f(w_p)$ for each $z\in V(Z)$, with equality holding if $z\in\{v_1,\dots,v_{\alpha-1},v_p\}$.

Give B an equitable and balanced $f(w_p)$-edge-coloring with colors in $\{1,2,\dots,f(w_p)\}$. We can assume that the edge $\{d, v_\beta\}$ is colored 1 if $\beta<\alpha$, and an edge $\{d, v_p\}$ is colored 1 if $\beta=\alpha$.

At this point, it would be easy to use the edges colored 1 in B to determine which edges and loops to choose in G in order to detach one of their ends from w_p and then reattach to a new vertex w_β instead; in so doing (D1–D3) would be satisfied by the new graph, G'. Details of this are unnecessary, since unfortunately G' may not satisfy (D4) if such an approach were to be used. To address the connectivity issue, note that the only way a color class G'_i can be disconnected in G' is that there is exists a component H in G_i-w_p that is joined to w_p with exactly 2 edges (the number of such edges is necessarily even, since each vertex in H has even degree in G_i), and both these edges are selected to be detached from w_p and joined to w_β in G'. To avoid selecting such pairs of "disconnecting edges", we can ensure that at most one edge from each such pair is chosen. This is accomplished by focusing on B', the subgraph of B induced by the edges colored 1 and 2. Each vertex c_i has degree 4 in B', so we then form B'' by splitting c_i into two vertices c_i and c'_i in such a way that disconnecting pairs of edges in G_i correspond to adjacent edges in B'' (assuming the corresponding pair of edges are even in B'—they may receive colors other than 1 or 2 in B). Now an equitable 2-edge-coloring of B'' ensures that at most one of the edges in B'' corresponding to a disconnecting pair is colored 1. As we will see, it then turns out that the edges in B''_1 (that is, the edges colored 1 in B'') can be used to form G'.

More formally, we consider the subgraph B' of B induced by the edges colored 1 and 2 (these colors exist since $f(w_p)\geqslant 2$). B' has the same bipartition of the vertices as does B. Now form a third bipartite graph B'' with $V(B')=V(B'')\cup\{c'_i|i\in\{1,\dots,m\}\}$ as follows. For $1\leqslant i\leqslant m$, consider the *four* (since c_i is incident with exactly $d_B(c_i)/f(w_p)=2$ edges of each color in B) edges incident with vertex c_i in B'. If c_i is incident with two edges that correspond to the unique pair of edges in G that join w_p to the same component of $G_i-\{w_p\}$, (clearly each edge is in at most one such pair) or if there are at least two edges joining c_i to l in B', then detach exactly two of them from c_i and join them to c'_i instead. Otherwise detach any pair of edges from c_i in B' and join them to c'_i instead to form B''. So $d_{B''}(c_i)=d_{B''}(c'_i)=2$.

Give B'' an equitable 2-edge-coloring with colors 1 and 2; again we can assume that the edge $\{d, v_\beta\}$ is colored 1 if $\beta<\alpha$ and an edge $\{d, v_p\}$ is colored 1 if $\beta=\alpha$. Note that the pairing process and the balanced edge-coloring ensures that

$$\mu_{B''_1}(l,\{c_i,c'_i\})\leqslant\frac{\mu_B(l,c_i)}{2}, \tag{1}$$

so $\mu_{B''_1}(l,\{c_i,c'_i\})$ is at most the number of loops colored i on w_p in G.

We now construct a graph G' from G as follows. For each edge in B''_1 joining a vertex c_i or c'_i to a vertex $z\in Z-\{l\}$, replace the corresponding edge in G_i joining z to w_p with an edge colored i joining z to a new vertex w_β. Also for each edge in B''_1 joining a vertex c_i or c'_i to the vertex l, replace one loop on w_p colored i with an edge colored i joining w_p to w_β. (Recall each loop corresponds to two edges in B; but by (1) this step is possible.) So B''_1 corresponds to a subset of edges in G which are detached from w_p and joined to w_β instead to result in a new graph G'.

Now we must show $G'\in\mathscr{G}(\alpha,\beta+1)$ by verifying that it satisfies the properties (D1–D4). Clearly we need only concern ourselves with the edges incident with the new vertex w_β and the modified vertex w_p. Note that $|V|=\alpha$ and $|W|=\beta+1$ in G'. Let f' be the associated function of the graph G' defined by $f'(w_\beta)=1$, $f'(w_p)=f(w_p)-1$, and $f'(u)=f(u)$ for all vertices $u\in V(G')-\{w_\beta, w_p\}$.

For $1 \leqslant i \leqslant \beta - 1$, we have that $\mu_{G'}(v_i, w_\beta) = d_{B_1''}(v_i) = d_B(v_i)/f(w_p) = f(v_i) = f'(v_i)f'(w_\beta)$ since $f'(w_\beta) = 1$. Similarly, for $1 \leqslant i \leqslant \beta - 1$, since $\mu_{G'}(v_i, w_\beta) = f(v_i)$ edges joining v_i to w_p in G are detached in forming G', $\mu_{G'}(v_i, w_p) = \mu_G(v_i, w_p) - f(v_i) = f(v_i)f(w_p) - f(v_i) = f(v_i)(f(w_p) - 1) = f'(v_i)f'(w_p)$. In the same manner we see that for $1 \leqslant i \leqslant \beta - 1$, $\mu_{G'}(w_i, w_\beta) = d_{B_1''}(w_i) \leqslant \lceil d_B(w_i)/f(w_p)\rceil \leqslant f(w_i) = f'(w_i)f'(w_\beta)$, and $\mu_{G'}(w_\beta, w_p) \leqslant d_{B_1''}(l) = \lceil d_B(l)/f(w_p)\rceil \leqslant \lceil f(w_p)(f(w_p) - 1)/f(w_p)\rceil = f(w_p) - 1 = f'(w_\beta)f'(w_p)$. Next, if $\beta < \alpha$ then $\mu_{G'}(v_\beta, w_\beta) = 0$ since the edge incident with d in B_1'' is $\{d, v_\beta\}$, the only edge incident with v_β in B_1''. This also means that when $\beta < \alpha$ we have: $\mu_{G'}(v_\beta, w_p) = \mu_{G'}(v_\beta, w_\beta) + \mu_{G'}(v_\beta, w_p) = \mu_G(v_\beta, w_p) = f(v_\beta)(f(w_p) - 1) = f'(v_\beta)f'(w_p)$; $\mu_{G'}(v_p, w_\beta) = d_{B_1''}(v_p) = d_B(v_p)/f(w_p) = f(v_p) = f'(v_p)f'(w_\beta)$; and $\mu_{G'}(v_p, w_p) = \mu_G(v_p, w_p) - f(v_p) = f(v_p)(f(w_p) - 1) - f(v_p) = f'(v_p)(f'(w_p) - 1)$. However, if $\beta = \alpha$, then the edge incident with d in B_1'' is $\{d, v_p\}$, in which case: $\mu_{G'}(v_p, w_\beta) = d_{B_1''}(v_p) - 1 = f(v_p) - 1 = (f'(v_p) - 1)f'(w_\beta)$; and since $\beta = \alpha$ implies that $f(v_p) = f(w_p)$, we have that $\mu_{G'}(v_p, w_p) = \mu_G(v_p, w_p) - (f(v_p) - 1) = f(v_p)(f(w_p) - 1) - (f(v_p) - 1) = f(w_p)(f(v_p) - 1) - (f(v_p) - 1) = (f(w_p) - 1)(f(v_p) - 1) = f'(w_p)(f(v_p) - 1)$. These values of $\mu_{G'}(v_p, w_\beta)$ and $\mu_{G'}(v_p, w_p)$ seem to be reversed in the roles of V and W when compared to (D1), which is in fact the case because this is the one and only situation in which G' has $|W| > |V|$. Finally, for $\beta < i \leqslant \alpha - 1$, since $d_{B_1''}(v_i) = d_B(v_i)/f(w_p) = f(v_i)$ edges joining v_i to w_p in G are detached in forming G', $\mu_{G'}(v_i, w_\beta) = f(v_i) = f'(v_i)f'(w_\beta)$, and $\mu_{G'}(v_i, w_p) = \mu_G(v_i, w_p) - f(v_i) = f(v_i)(f(w_p) - 1) - f(v_i) = f(v_i)(f(w_p) - 2) = f'(v_i)(f'(w_p) - 1)$. So (D1) is satisfied.

Clearly w_β is incident with no loops. By (D2), the number of loops on w_p in G is $\binom{f(w_p)}{2} - \gamma$, where γ is some non-negative integer. Since $d_B(l) = (f(w_p)(f(w_p) - 1)) - 2\gamma$, and therefore $d_{B_1''}(l) \geqslant (f(w_p) - 1) - \lceil (2\gamma/f(w_p))\rceil$, the number of loops on w_p in G'

$$
\begin{aligned}
&= \left(\binom{f(w_p)}{2} - \gamma\right) - d_{B_1''}(l) \\
&\leqslant \frac{f(w_p)(f(w_p) - 1)}{2} - \gamma - \left((f(w_p) - 1) - \left\lceil \frac{2\gamma}{f(w_p)} \right\rceil\right) \\
&= \frac{(f(w_p) - 1)(f(w_p) - 2)}{2} - \left(\gamma - \left\lceil \frac{2\gamma}{f(w_p)} \right\rceil\right) \\
&\leqslant \frac{f'(w_p)(f'(w_p) - 1)}{2} - \left(\gamma - \left\lceil \frac{2\gamma}{2} \right\rceil\right) \\
&= \binom{f'(w_p)}{2}.
\end{aligned}
$$

Thus (D2) is satisfied.

Since c_i is incident with two edges of each color in B and $d_{B_1''}(c_i) = d_{B_1''}(c_i') = 1$, these two edges of color 1 incident with c_i and c_i' correspond to the two edges of G_1' incident with w_β. So clearly, w_β is incident with $2f(w_\beta) = 2$ edges in each color class. Likewise, since two edges of each color class are removed from w_p, $d_{G_i'}(w_p) = 2f(w_p) - 2 = 2(f(w_p) - 1) = 2f'(w_p)$. Thus (D3) is satisfied.

Notice that since each vertex in G_i has even degree, it follows that each edge-cut has even size. Therefore, since exactly two edges colored i are detached from w_p and reattached to w_β, the only way that G_i' could be disconnected would be if there exists a component of $G_i - w_p$ that is joined to w_p in G_i by exactly two edges, and those two edges are the ones detached from w_p and joined to w_β in forming G'. Clearly the choice of edges incident with c_i' in B'' prevents this. So by the construction of B'' we have ensured that G' remains connected, thus verifying D4.

Since (D1–D4) are satisfied, there exists a graph $G' \in \mathscr{G}(\alpha, \beta + 1)$. So the result now follows. $\square$

3. A direct construction

In this section we provide a direct construction for finding a maximal set S of m edge-disjoint hamilton cycles in the graph $T = K_{2p} - F$. We ensure that each set is indeed maximal by showing that $T - E(S)$ is a disconnected graph, and therefore contains no hamilton cycle.

Let $T = K_{2p} - F$ with vertex set $V \cup W$ where $V = \{v_0, v_1, \ldots, v_{p-1}\}$ and $W = \{w_0, w_1, \ldots, w_{p-1}\}$; and with $F = \{\{v_j, w_j\} | 0 \leqslant j \leqslant p-1\}$. We will define a maximal set S of m hamilton cycles by considering various cases in turn.

Let $\mathbb{Z}_s = \{0, 1, \ldots s-1\}$. The hamilton cycles in S are built from hamilton cycles or hamilton paths in smaller graphs in which the vertices are either elements of $\mathbb{Z}_s$, $\mathbb{Z}_s \cup \{\infty\}$ or $\mathbb{Z}_s \cup \{\infty_1, \infty_2\}$ for some s. If $H = (z_0, \ldots, z_y)$ is such a path or cycle then let $H_s(i) = (z'_0, \ldots, z'_y)$ where if z_j is in $\mathbb{Z}_s$ then $z'_j = z_j + i \pmod{s}$, and if $z_j = \infty, \infty_1$ or ∞_2 then $z'_j = z_j$.

Theorem 3.1. *There exists a maximal set S of m hamilton cycles in $T = K_{2p} - F$ if and only if $\lceil (p-1)/2 \rceil \leqslant m \leqslant p-1$, where F is a 1-factor of K_{2p}.*

Proof. Again, we observe that it is shown in [3] that $\lceil (p-1)/2 \rceil \leqslant m \leqslant p-1$ is a necessary condition for S to exist, so we now prove the sufficiency.

Case 1: p is odd. The classic hamilton cycle decomposition of K_p on the vertex set $\mathbb{Z}_{p-1} \cup \{\infty\}$ is formed by $\{H_{p-1}(i) | 0 \leqslant i \leqslant (p-3)/2\}$ where H is the hamilton cycle $(z_0, \ldots, z_{p-1})$ defined by

$$z_j = (-1)^j \lceil j/2 \rceil + 1 \pmod{p-1} \quad \text{for } 0 \leqslant j \leqslant (p-3)/2,$$

$$z_{p-1-j} = z_j + (p-1)/2 \pmod{p-1} \quad \text{for } 0 \leqslant j \leqslant (p-3)/2, \text{ and}$$

$$z_{(p-1)/2} = \infty.$$

Let $H'[i]$ be a copy of $H_{p-1}(i)$ formed by renaming the vertex ∞ with $p-1$. Notice that since H contains the edges $\{0, 1\}$ and $\{0, 2\}$, it follows that for $0 \leqslant i \leqslant (p-3)/2$

$$H'[i] \text{ contains both the edges } \{i, i+1\} \text{ and } \{i, i+2\}. \tag{2}$$

Recall that T has bipartition $V = \{v_0, \ldots, v_{p-1}\}$ and $W = \{w_o, \ldots, w_{p-1}\}$. For $0 \leqslant i \leqslant (p-3)/2$, we can use $H'[i]$ to define a hamilton cycle $\mathscr{H}[i]$ in T as follows. For each edge $\{k, l\} \in E(H'[i])$, let $\mathscr{H}[i]$ contain the edges $\{v_k, w_l\}$ and $\{v_l, w_k\}$. If $m = (p-1)/2$, the smallest possible value, then the set $S = \{\mathscr{H}[0], \ldots, \mathscr{H}[(p-3)/2]\}$ forms the desired maximal set of hamilton cycles since each edge in T joining a vertex in V to a vertex in W occurs in one of these hamilton cycles, thus $T - E(S)$ is disconnected.

If $m > (p-1)/2$ then for $0 \leqslant i \leqslant m - ((p+1)/2)$, $\mathscr{H}[i]$ is slightly altered and $\mathscr{H}[i + ((p-1)/2)]$ is formed using two of the edges in $\mathscr{H}[i]$ as follows. Begin by letting $\mathscr{H}[i + ((p-1)/2)]$ be the disconnected graph containing the edges $\{v_k, v_l\}$ and $\{w_k, w_l\}$ for each edge $\{k, l\} \in E(H'[i])$. Next, by (2) $\mathscr{H}[i]$ contains the pair of edges $\{v_i, w_{i+2}\}$ and $\{v_{i+1}, w_i\}$, and $\mathscr{H}[i + ((p-1)/2)]$ contains the pair of edges $\{v_i, v_{i+1}\}$ and $\{w_i, w_{i+2}\}$; so we can now interchange the pair of edges between $\mathscr{H}[i]$ and $\mathscr{H}[i + ((p-1)/2)]$. It is easy to check that since p is odd, this results in two hamilton cycles (see Fig. 1). Notice that the formation of each hamilton cycle $\mathscr{H}[i + ((p-1)/2)]$ requires an alteration in the original definition of each hamilton cycle $\mathscr{H}[i]$. So the set $S = \{\mathscr{H}[0], \ldots, \mathscr{H}[m-1]\}$ is a maximal set of m hamilton cycles, since clearly all edges joining vertices in V to vertices in W occur in $E(S)$.

Case 2: p is even. The classic hamilton path decomposition of K_p on the vertex set $\mathbb{Z}_p$ is formed by $\{P_p(i) | 0 \leqslant i \leqslant (p-2)/2\}$ where P is the hamilton path $(z_0, \ldots, z_{p-1})$ defined by

$$z_j = (-1)^j \lceil j/2 \rceil + 1 \pmod{p} \quad \text{for } 0 \leqslant j \leqslant p-1.$$

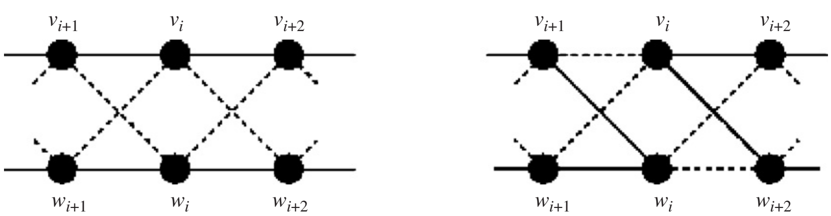

Fig. 1. The edge-interchange in Case 1.

Notice that since P contains the edges $\{0, 1\}$ and $\{0, 2\}$, it follows that for $0 \leqslant i \leqslant (p-2)/2$

$$P_p(i) \text{ contains the edges } \{i, i+1\} \text{ and } \{i, i+2\}. \tag{3}$$

The classic hamilton cycle decomposition of $K_p - F'$ on the vertex set $\mathbb{Z}_{p-2} \cup \{\infty_1, \infty_2\}$ with $F' = \{\{j, ((p-2)/2) + j\}, \{\infty_1, \infty_2\} | j \in \mathbb{Z}_{(p-2)/2}\}$ is formed by $\{H_{p-2}(i) | 0 \leqslant i \leqslant (p-4)/2\}$ where H is the hamilton cycle $(z_0, \ldots, z_{p-1})$ defined by

$$z_j = (-1)^{j+1} \lceil j/2 \rceil (\text{mod } p-2) \quad \text{for } 0 \leqslant j \leqslant (p-4)/2,$$

$$z_{p-2-j} = z_j + (p-2)/2 \quad \text{for } 0 \leqslant j \leqslant (p-4)/2,$$

$$z_{p-1} = \infty_2, \text{ and}$$

$$z_{(p-2)/2} = \infty_1.$$

Notice that since H contains the edge $\{0, 1\}$, it follows that for $0 \leqslant i \leqslant (p-4)/2$

$$H_{p-2}(i) \text{ contains the edge } \{i, i+1\} \text{ for } 0 \leqslant i \leqslant (p-4)/2. \tag{4}$$

For $0 \leqslant i \leqslant (p-4)/2$, let $H'[i]$ be a copy of $H_{p-2}(i)$ on the vertex set $\mathbb{Z}_p$ formed by renaming each vertex $z \in \mathbb{Z}_{p-2} \cup \{\infty_1, \infty_2\}$ with $f(z)$, where

$$f(j) = j \quad \text{for } 0 \leqslant j \leqslant (p-4)/2,$$

$$f((p-2)/2 + j) = f(j) + p/2 \quad \text{for } 0 \leqslant j \leqslant (p-4)/2,$$

$$f(\infty_1) = p-1, \text{ and}$$

$$f(\infty_2) = (p-2)/2.$$

It is crucial to notice that if $\{z_1, z_2\} \in F'$, then $f(z_1) = f(z_2) + p/2$ (mod p). Therefore this renaming produces a hamilton decomposition of $K_p - F''$ where $F'' = \{\{j, j + p/2\} | j \in \mathbb{Z}_{p/2}\}$, in which by (4):

$$H'[i] \text{ contains the edge } \{i, i+1\} \text{ for } 0 \leqslant i < (p-4)/2$$

$$\text{and the edge } \{i, i+2\} \quad \text{for } i = (p-4)/2. \tag{5}$$

Let $H''[i]$ be another copy of $H_{p-2}(i)$ on the vertex set $\mathbb{Z}_p$, this time formed by renaming each vertex $z \in \mathbb{Z}_{p-2} \cup \{\infty_1, \infty_2\}$ with $f(z)$, where

$$f(j) = 2j \quad \text{for } 0 \leqslant j \leqslant \lceil p/4 \rceil - 1,$$

$$f((p-4)/2 - j) = 3 + 2j \quad \text{for } 0 \leqslant j \leqslant \lfloor p/4 \rfloor - 2,$$

$$f((p-2)/2 + j) = f(j) + (p/2) \quad \text{for } 0 \leqslant j \leqslant (p-4)/2,$$

$$f(\infty_1) = (p+2)/2, \text{ and}$$

$$f(\infty_2) = 1.$$

Again, it is crucial to notice that if $\{z_1, z_2\} \in F'$ then $f(z_1) = f(z_2) + p/2$ (mod p), so this renaming also produces a hamilton decomposition of $K_p - F''$, but this time it satisfies

$$H''[i] \text{ contains the edge } \{2i, 2i+2\} \text{ for } 0 \leqslant i \leqslant \lceil p/4 \rceil - 2,$$

$$H''[(p-4)/2) - i] \text{ contains the edge } \{2i+1, 2i+3\} \text{ for } 0 \leqslant i \leqslant \lfloor p/4 \rfloor - 2, \text{ and}$$

$$H''[i] \text{ contains the edge } \{i, i+1\} \text{ for } i = \lceil p/4 \rceil - 1. \tag{6}$$

Let $H'' = \{H''[i] \mid 0 \leqslant i \leqslant (p-4)/2\}$. We now match each $H'[i]$ with a hamilton cycle $M(H'[i])$ in H'' in such a way that

$$\text{one of the edges } \{i, i+1\} \text{ and } \{i, i+2\} \text{ occurs in } H'[i]$$
$$\text{and the other occurs in a hamilton cycle in } M(H'[i]). \tag{7}$$

In view of (5) and (6) we can achieve (7) by matching $H'[i]$ with $H''[i/2]$ if i is even and matching $H'[i]$ with $H''[(p-i-3)/2]$ if i is odd.

We are now ready to define the hamilton cycles in $T = K_{2p} - F$. For $0 \leqslant i \leqslant (p-2)/2$, define $\mathscr{H}[i]$ to be the hamilton cycle containing the edges $\{v_{i+1}, v_{i+((p+2)/2)}\}$ and $\{w_{i+1}, w_{i+((p+2)/2)}\}$, as well as the edges $\{v_k, w_l\}$ and $\{v_l, w_k\}$ for each edge $\{k, l\} \in E(P_p(i))$. It is easy to check that $\mathscr{H}[i]$ is a hamilton cycle. So if $m = p/2$, the smallest possible value of m, then the set $S = \{\mathscr{H}[0], \ldots, \mathscr{H}[(p-2)/2]\}$ forms the desired maximal set of hamilton cycles since each edge joining a vertex in V to a vertex in W occurs in one of these hamilton cycles, thus $T - E(S)$ is disconnected.

If $m > p/2$ then for $0 \leqslant i \leqslant m - ((p+2)/2)$, $\mathscr{H}[i+(p/2)]$ is formed by using two of the edges in $\mathscr{H}[i]$ as follows. Begin by letting $\mathscr{H}[i+(p/2)]$ contain the edges $\{v_k, v_l\}$ for each edge $\{k, l\} \in E(H'[i])$ and $\{w_k, w_l\}$ for each edge $\{k, l\} \in E(M(H'[i]))$. Now, by (3), $\mathscr{H}[i]$ contains the pair of edges $\{v_i, w_{i+2}\}$ and $\{v_{i+1}, w_i\}$ and the pair of edges $\{v_i, w_{i+1}\}$ and $\{v_{i+2}, w_i\}$, whereas by (7) $\mathscr{H}[i+(p/2)]$ either contains the pair of edges $\{v_i, v_{i+1}\}$ and $\{w_i, w_{i+2}\}$ or contains the pair of edges $\{v_i, v_{i+2}\}$ and $\{w_i, w_{i+1}\}$. In either case we can interchange the pair of edges in $\mathscr{H}[i]$ with the pair in $\mathscr{H}[i+(p/2)]$ so that both are still 2-regular. Then it is easy to check that both the resulting updated graphs $\mathscr{H}[i]$ and $\mathscr{H}[i+(p/2)]$ are hamilton cycles since p is even. So the set $S = \{\mathscr{H}[0], \ldots, \mathscr{H}[m-1]\}$ is a maximal set of m hamilton cycles, since clearly all the edges joining vertices in V to vertices in W occur in $E(S)$. $\square$

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