Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 2901 – 2909
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

All graphs with maximum degree three whose complements have 4-cycle decompositions

Chin-Mei Fu[a], Hung-Lin Fu[b], C.A. Rodger[c], Todd Smith[d]

[a] *Department of Mathematics, Tamkang University, Tamsui, Taipei Shien, Taiwan, ROC*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, ROC*
[c] *Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, AL 36849-5310, USA*
[d] *Department of Mathematics, University Central of Florida, P.O. Box 161364, Orlando, FL 32816-1364, USA*

Received 30 May 2003; received in revised form 23 November 2004; accepted 29 September 2006
Available online 4 September 2007
In honor of Jennie Seberry on the occasion of her 60th birthday

Abstract

Let $\mathscr{G}$ be the set that contains precisely the graphs on n vertices with maximum degree 3 for which there exists a 4-cycle system of their complement in K_n. In this paper $\mathscr{G}$ is completely characterized.

MSC: 05B30; 05C38

Keywords: 4-cycle; Specified leave; 3-regular; Graph decomposition

1. Introduction

A k-cycle system of a graph G is a partition of the edges of G into sets, each of which induces a cycle of length k. Over the past 30 years there has been considerable interest in the problem of finding k-cycle systems for various families of graphs. The classic result in this area is that of Sotteau [17], who found necessary and sufficient conditions for the existence of a k-cycle system of $K_{a,b}$, the complete bipartite graph. Probably the best result follows after years of activity in the literature (see for example [12,14]) from a recent pair of papers that completely settle the existence problem for k-cycle systems for complete graphs, and for complete graphs with the edges in a 1-factor removed [1,16]. More recently, attention has focused on fixing k and allowing G to vary in non-symmetric ways. For example, it is now known when it is possible to find k-cycle systems of G when G is formed from K_n by removing the edges of *any* tree whenever k is 4 or 6 (see [9,2], respectively), and by removing the edges of *any* 2-regular graph whenever k is 3 [8], 4 [10], 6 [3], or n [4,6,15]. In this paper we solve the existence problem for 4-cycle systems of G for *all* the myriad of graphs one can form from K_n by removing the edges of *any* subgraph of maximum degree 3.

This problem has also been of some statistical interest. Balanced sampling designs excluding contiguous units were first introduced by Hedayat et al. [11]. In terms of this paper, they were interested in a 1-dimensional problem that is equivalent to a 3-cycle decomposition of the complete graph with the edges in a 2-factor removed. Since then,

E-mail addresses: cmfu@mail.tku.edu.tw (C.-M. Fu), hlfu@math.nctu.edu.tw (H.-L. Fu), rodgec1@auburn.edu (C.A. Rodger), tbsmith@mail.ucf.edu (T. Smith).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.07.082

2-dimensional variants have been considered, where the graph removed from K_n is a 4-factor [5]. One can also define similar variants of the existence problem for neighbor designs, a statistical experimental design that is equivalent to a decomposition of the complete graph into cycles of a fixed length [13], corresponding to antigens being placed around the rim of petri dishes, surrounding an antiserum. Results in this paper relate directly to this generalization, allowing for the situation where we do not need to see how some pairs of antigens react together. Another version allows one antigen to be placed at the center of the petri dish, so the design corresponds to a decomposition of K_n into wheels [7].

A graph G is said to be a *Δ_3-graph* if $\Delta(G)\leqslant 3$. G is said to be *odd* if all vertices have odd degree. A graph G on n vertices is said to be *n-admissible* if:

(a) all vertices in $K_n - E(G)$ have even degree and
(b) 4 divides $\binom{n}{2} - |E(G)|$.

If the value of n is irrelevant then we can also refer to G as being simply *admissible*. Let $\mathscr{G}(n) = \{G|G$ is an odd Δ_3-graph of order n for which there exists a 4-cycle system of $K_n - E(G)\}$. Clearly if $G \in \mathscr{G}(n)$ then G is n-admissible. Let $\mathscr{G} = \bigcup_{n\geqslant 1}\mathscr{G}(n)$. In this paper we completely determine $\mathscr{G}$ (see Theorem 1), showing that there are exactly two non-admissible graphs that are not in $\mathscr{G}$.

Let $\overline{G}$ denote the complement of G. Let $G \vee H$ denote the graph formed from two vertex-disjoint graphs G and H by joining each vertex in G to each vertex in H with one edge. Let $G[A]$ denote the subgraph of G induced by A. It is easy to see that the edges of $K_{a,b}$ with vertex sets A and B can be partitioned into sets of size 4, each of which induces a 4-cycle, if and only if both a and b are even; we let $C(A, B)$ denote such a set of 4-cycles.

2. Some helpful results

The following result makes it easy to check if a given graph is admissible.

Lemma 1. *If G is an odd Δ_3-graph, then G is admissible if and only if the number of vertices of degree* 1 *is congruent to* $|V(G)|$ *modulo* 4.

Proof. Suppose G is admissible. By (a), let $|V(G)| = 4x + 2y$ with $y \in \{0, 1\}$. Let the number of vertices of degree 1 in G be z. Then $2|E(G)| = z + 3(4x + 2y - z) = 12x + 6y - 2z$. By (b), 4 divides $(4x + 2y)(4x + 2y - 1)/2 - (6x + 3y - z) = 4(2x^2 + 2xy - 2x - y) + 2y^2 - z$, so $z \equiv 2y$ (modulo 4) as required. Similarly, if $z \equiv 2y$ (modulo 4) then $z \equiv 2y^2$ (modulo 4), so reversing the above argument shows that (b) is true. Of course (a) is true since every graph that is odd must have an even number of vertices. □

It will be useful to define $\mathscr{S} = \{n|$ there exists a 4-cycle system of $K_n - E(G)$ for all admissible odd Δ_3-graphs G with $|V(G)| = n\}$. An odd Δ_3-graph of order n, G, is said to be *maximal* if there is no odd Δ_3-graph of order n, say G', with the property that G' can be formed from G by adding 4 new edges which induce a 4-cycle. Clearly the following is true.

Lemma 2. *If each maximal odd Δ_3-graph on n vertices is in $\mathscr{G}(n)$, then $n \in \mathscr{S}$.*

In particular, since the complement of any subgraph of G induced by four vertices of degree 1 *must* contain a 4-cycle, it follows that to prove that $n \in \mathscr{S}$ we need only consider the admissible odd Δ_3-graphs that have at most 2 vertices of degree 1. Therefore, throughout this paper, when we consider graphs with at least 10 vertices we assume that they have at most 2 vertices of degree 1 (since $8 \notin \mathscr{S}$, all graphs on 8 vertices need to be considered).

The main result is proved inductively. One common tool used to effect such a proof is to remove four vertices of degree 3 in G that induce a path, P. Such a subgraph can arise in several ways, depending on the neighborhood of P. In the following result we consider four such cases. Let $N(P)$ denote the set of vertices that are each adjacent to at least one vertex in P.

Lemma 3. *Suppose that G is an admissible odd Δ_3-graph on n vertices that contains an induced path $P = (a_1, a_2, a_3, a_4)$ with $d_G(a_i) = 3$ for $1 \leqslant i \leqslant 4$. Excluding those in P, let the neighbors of a_1 be b_1 and b_2, of a_2 be b_3, of a_3 be c_3, and* *of* a_4 *be* c_1 *and* c_2 (*possibly these vertices are not all distinct*). *Suppose that one of the following conditions is satisfied by* G:

(1) $b_i \neq c_j$ *for* $1 \leqslant i, j \leqslant 3$ *and*
  (a) $|\{b_1, b_2, b_3\}| = |\{c_1, c_2, c_3\}| = 3$ *and* $\{\{b_1, c_3\}, \{b_2, c_2\}, \{b_3, c_1\}\} \cap E(G) = \emptyset$ *or*
  (b) $b_2 = b_3, c_2 = c_3$, *and* $\{\{b_1, c_2\}, \{b_2, c_2\}, \{b_2, c_1\}\} \cap E(G) = \emptyset$ *or*
  (c) $b_2 = b_3, |\{c_1, c_2, c_3\}| = 3$ *and* $\{\{b_1, c_1\}, \{b_2, c_2\}, \{b_2, c_3\}\} \cap E(G) = \emptyset$;
(2) $b_1 = c_1, |\{b_1, b_2, b_3, c_1, c_2, c_3\}| = 5$, *and* $\{\{b_1, b_3\}, \{b_1, c_3\}, \{b_2, c_2\}\} \cap E(G) = \emptyset$.

If $n - 4 \in \mathcal{S}$ *then* $G \in \mathcal{G}$.

Proof. We consider the four cases in turn.

Case (1*a*): Suppose that $b_3 \notin \{b_1, b_2\}$ and $c_3 \notin \{c_1, c_2\}$. Form G_1 from $G - V(P)$ by adding the edges $\{b_1, c_3\}, \{b_2, c_2\}$, and $\{b_3, c_1\}$. Then $d_{G_1}(v) = d_G(v)$ for each $v \in V(G_1)$. So since $n - 4 \in \mathcal{S}$, by Lemma 1 there exists a 4-cycle system $(V(G) - V(P), C_1)$ of $K_{n-4} - E(G_1)$. Let $C_2 = C_1 \cup \{(a_1, a_3, b_1, c_3), (a_1, a_4, b_2, c_2), (a_2, a_4, b_3, c_1), (a_1, b_3, a_3, c_1), (a_2, b_1, a_4, c_3), (a_2, b_2, a_3, c_2)\} \cup C(V(P), V(G) - N(P))$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Case (1*b*): Suppose $b_2 = b_3$ and $c_2 = c_3$. Form a graph G_1 from $G - V(P)$ by adding the edges $\{b_2, c_2\}, \{b_1, c_2\}$, and $\{b_2, c_1\}$. Then, as before, there exists a 4-cycle system $(V(G) - V(P), C_1)$ of $K_{n-4} - E(G_1)$. Define $C_2 = C_1 \cup \{(a_2, b_1, a_3, c_1), (a_1, a_4, b_2, c_2), (a_1, a_3, b_2, c_1), (a_2, a_4, b_1, c_2)\} \cup C(V(P), V(G) - N(P))$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Case (1*c*): Suppose $b_2 = b_3$ and $c_3 \notin \{c_1, c_2\}$. Since $|V(G)|$ is even, G contains another vertex z. Form G_1 from $G - V(P)$ by adding the edges $\{b_1, c_1\}, \{b_2, c_2\}$, and $\{b_2, c_3\}$. Then, as before, let $(V(G) - V(P), C_1)$ be a 4-cycle system of $K_{n-4} - E(G_1)$. Let $C_2 = C_1 \cup \{(a_1, a_4, b_2, c_3), (a_1, a_3, b_2, c_2), (a_2, a_4, b_1, c_1), (a_2, b_1, a_3, c_2), (a_1, c_1, a_3, z), (a_2, c_3, a_4, z)\} \cup C(V(P), V(G) - (N(P) \cup \{z\}))$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Case 2: Suppose $b_1 = c_1$. Since $|V(G)|$ is even, G contains another vertex z. Form G_1 from $G - V(P)$ by adding the edges $\{b_1, b_3\}, \{b_1, c_3\}$, and $\{b_2, c_2\}$. As before, there exists a 4-cycle system $(V(G) - V(P), C_1)$ of $K_{n-4} - E(G_1)$. Define $C_2 = C_1 \cup \{(a_1, b_3, a_3, z), (a_1, c_3, a_2, a_4), (a_1, c_2, b_2, a_3), (a_2, b_2, a_4, z), (a_2, b_1, a_3, c_2), (a_4, b_3, b_1, c_3)\} \cup C(V(P), V(G) - (N(P) \cup \{z\}))$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$. $\square$

3. The main result

The following result handles the small cases, of which there are many! For example, if G is connected, cubic and has 12 vertices then it is one of the 85 such graphs—the proof handles these in a neat way. Lemma 2 is useful throughout this section.

Proposition 1. *If* G *is a cubic graph with at most* 12 *vertices*, *then* $G \in \mathcal{G}$ *if and only if* $G \notin \{G_1, G_2\}$ (*see* Fig. 1).

Proof. The result is trivial if $n = 4$.

If $n = 6$ then G must have either 2 or 6 vertices of degree 1. So G is a 1-factor, or consists of the 2 components K_2 and K_4, or is induced by the edge set $\{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}$. In each case the required decomposition is easy to find.

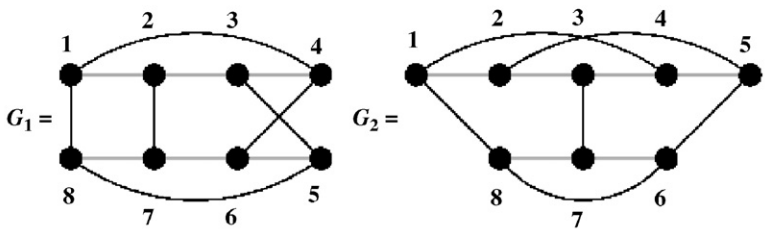

Fig. 1. The two exceptional cases.

Suppose that $n = 8$. Then G has 0, 4, or 8 vertices of degree 1. If G contains a component isomorphic to K_2 then the result follows by deleting this component, applying the solution when $n = 6$, then adding three 4-cycles between the six vertices and the replaced K_2. If G consists of two components isomorphic to K_4 then the result follows from a 4-cycle system of $K_{4,4}$. If G is connected and has four vertices of degree 1 then G has edge set $\{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{5, 7\}, \{6, 8\}\}$, or $\{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{5, 8\}\}$. In either case, G contains two vertices u and v of degree 1 whose neighbors w and x, respectively, are non-adjacent. Delete u and v and join w and x to form a graph on six vertices that by the case $n = 6$ has a 4-cycle system; to this system add the 4-cycle (u, v, w, x) and then the 4-cycles in a 4-cycle system of $K_{2,4}$. Finally we consider the case where G is connected and has 0 vertices of degree 1. All connected non-isomorphic cubic graphs on at most 12 vertices are listed on Gordon Royle's website: http://www.cs.uwa.edu.au/gordon/remote/cubics/index.html. There are five such graphs on eightvertices. It is easy to establish that three of them are in $\mathscr{G}(8)$ and the two depicted in Fig. 1 (graphs 2 and 5 in the Royle list) are not.

Suppose that $n=10$. As in the previous case, if G is disconnected then one component must be either K_2 or K_4, so the result follows by using previous cases unless one of the components is one of the graphs in Fig. 1. If it is G_1 then the 4-cycles in $\{(1, 7, 3, 6), (2, 5, 4, 8), (1, 3, 8, 9), (8, 6, 2, 10), (2, 4, 7, 9), (7, 5, 1, 10), (10, 3, 9, 4), (10, 5, 9, 6)\}$ provide the required system. If it is G_2, then the 4-cycles in $\{(1, 7, 2, 6), (3, 8, 5, 9), (3, 10, 4, 6), (1, 3, 5, 10), (2, 4, 9, 8), (4, 7, 10, 8), (2, 10, 6, 9), (1, 9, 7, 5)\}$ provide the required system.

If $n = 10$ and G is connected then proceed as follows. If G contains two vertices say 9 and 10 of degree 1 with $d_G(9, 10) \geqslant 4$ then delete 9 and 10 and join their neighbors u and v, respectively, to form G'. If $G' \neq G_1, G_2$ then adding the 4-cycle $(9, 10, u, v)$ to 4-cycle systems of $K_8 - E(G')$ and of $K_{2,6}$ (with bipartition $\{9, 10\}$ and $\{1, \dots, 8\} - \{u, v\}$) provides the result. If $G'=G_1$ then note that removing 8-cycle $(a_1, a_2, \dots, a_8)=(1, 3, 7, 5, 2, 4, 8, 6)$ from $K_8 - E(G')$ leaves two 4-cycles c_1 and c_2. Since for some i we have $u = a_{i+1}$ and $v = a_{i+3}, a_{i+4}$, or a_{i+5} (reducing the sum modulo 8), to c_1 and c_2 add the six 4-cycles in $\{(a_{i+5}, 9, a_{i+7}, 10), (9, a_{i+2}, a_{i+1}, a_{i+8}), (9, a_{i+4}, a_{i+5}, a_{i+6}), (10, a_{i+2}, a_{i+3}, a_{i+4}), (10, a_{i+6}, a_{i+7}, a_{i+8}), (9, 10, a_{i+1}, a_{i+3})\}$, $\{(a_{i+2}, 9, a_{i+8}, 10),(a_{i+3}, 9, a_{i+6}, 10)\ (9, a_{i+5}, a_{i+6}, a_{i+7}), (10, a_{i+1}, a_{i+8}, a_{i+7}),(a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}),(9, 10, a_{i+5}, a_{i+4})\}$, or $\{(a_{i+3}, 9, a_{i+7}, 10), (9, a_{i+2}, a_{i+1}, a_{i+8}), (9, a_{i+4}, a_{i+5}, a_{i+6}),(10, a_{i+2}, a_{i+3}, a_{i+4}), (10, a_{i+6}, a_{i+7}, a_{i+8}), (9, 10, a_{i+1}, a_{i+5})\}$, respectively, to form the required 4-cycle system. If $G' = G_2$ then this approach does not work. Fortunately, each of the two vertex-permutations $(1, 5)(2, 4)(3)(6, 8)(7)$ and $(1)(2)(3, 5)(4)(6, 7)(8)$ are automorphisms of G_2, so there exist automorphisms of G_2 acting transitively on the edges in $G_2[\{1, \dots, 5\}]$, on the edges in $\{\{1, 8\}, \{3, 7\}, \{5, 6\}\}$, and on the edges in $G_2[\{6, 7, 8\}]$. So we need only consider the three cases: where $(u, v) = (1, 2)$, where $(u, v) = (1, 8)$, and where $(u, v) = (6, 8)$. In these cases the result follows: from Lemma 2, Case 1(c) with $(a_1, a_2, a_3, a_4, c_1) = (8, 7, 3, 2, 5)$ when $(u, v) = (1, 2)$; by Lemma 2, Case 1(c) with $(a_1, a_2, a_3, a_4, c_1) = (8, 7, 3, 2, 5)$ when $(u, v) = (1, 8)$; and by the 4-cycles in $\{(9, 1, 6, 8), (9, 2, 6, 4), (9, 3, 1, 5), (9, 7, 1, 10), (10, 2, 4, 7), (10, 3, 8, 4), (10, 5, 3, 6), (2, 8, 5, 7)\}$ when $(u, v) = (6, 8)$.

If $n = 10$ and $d_G(9, 10) \leqslant 3$, then G is one of the graphs in Fig. 2. In each case except when G is G_3 or G_4, the vertices have been labeled so that the result follows from Lemma 2, case 1(a) or (c) (as indicated below the graph). If $G = G_3$ or G_4 then the 4-cycles in $\{(1, 2, 7, 8), (1, 4, 7, 6), (1, 5, 3, 7), (2, 4, 8, 5), (2, 6, 3, 8), (1, 9, 2, 10), (3, 9, 4, 10), (5, 9, 6, 10)\}$ or $\{(1, 2, 6, 5), (1, 4, 9, 6), (1, 7, 2, 8), (1, 9, 2, 10), (2, 4, 10, 5), (3, 6, 7, 9), (3, 5, 8, 10), (3, 7, 4, 8)\}$, respectively, provide the result.

Now, suppose that $n = 12$ and G is connected. According to the Royle list, there are 85 possibilities for G. For example, the 4-cycles in $S_1= \{(1, 4, 2, 5), (3, 7, 4, 8), (5, 11, 6, 12), (1, 8, 6, 10), (4, 10, 9, 11), (2, 6, 3, 9), (2, 10, 3, 11), (1, 3, 5, 9), (2, 7, 5, 8), (1, 11, 8, 12), (4, 6, 7, 12), (7, 9, 12, 10)\}$ form a 4-cycle system of $K_{12} - G$ where G happens to be graph 15 in the list, containing the edges in $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{1, 7\}, \{2, 12\}, \{3, 12\}, \{4, 9\}, \{5, 10\}, \{6, 9\}, \{7, 8\}, \{7, 11\}, \{8, 9\}, \{8, 10\}, \{10, 11\}, \{11, 12\}\}$. This set S_1 of cycles has been carefully selected. Each of the first six cycles (a, b, c, d) listed have the property that the edges $\{a, c\}$ and $\{b, d\}$ are in $E(G)$. So we could replace this 4-cycle with (a, c, b, d) to form another 4-cycle system of $K_{12} - G_1$ for some other graph G_1, and then replace it with (a, c, d, b) to form another 4-cycle system of $K_{12} - G_2$ for some third graph G_2. So we can get 3^6 graphs by making all such switches. Of course there will be many repetitions, so we test for isomorphism using NAUTY, but hopefully most of the 85 graphs can be obtained in this way. In fact, doing this produces 4-cycle systems for all such graphs except for those numbered 1, 2, 3, 4, 5, 6, 9, 11, 12, 16, 17, 20, 23, 32, 39, 46, 56, and 67 in the Royle list. Similarly one can switch cycles using the first 4 and the 7th cycle in S_1 to get graphs 11, 17, and 56. If we start with the set of four-cycles $S_2=\{(5, 9, 6, 12), (7, 10, 8, 11), (2, 4, 7, 5), (1, 8, 2, 9), (1, 3, 5, 11), (1, 5, 10, 6),$

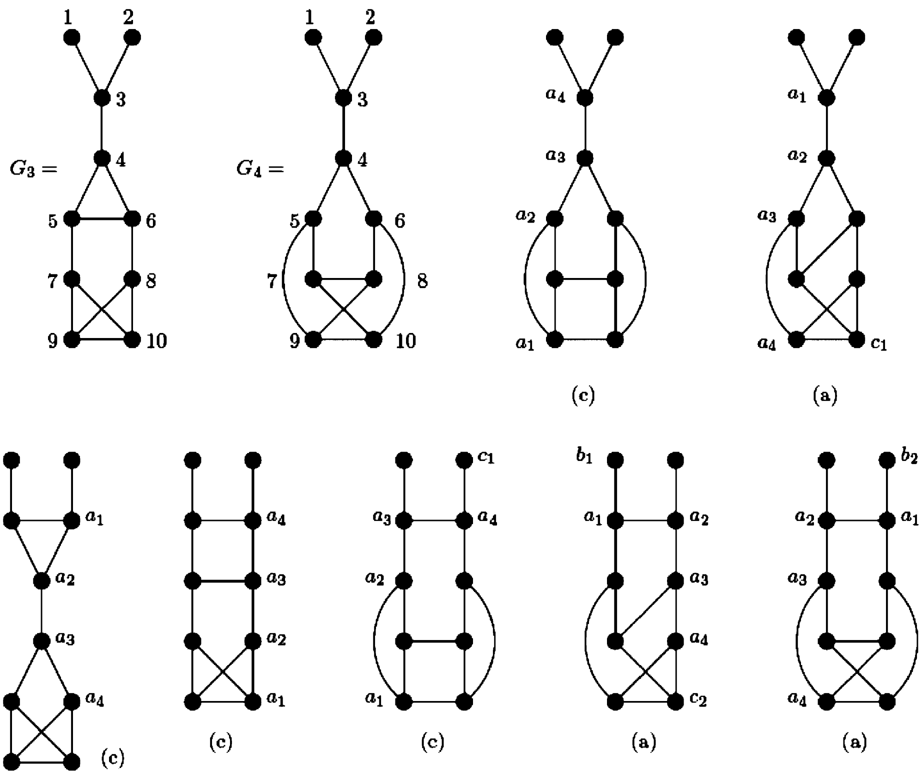

Fig. 2. Graphs on 10 vertices.

(1, 7, 12, 10), (2, 6, 8, 12), (2, 10, 4, 11), (3, 6, 4, 8), (3, 7, 9, 11), (3, 9, 4, 12)} then all edges occur in a 4-cycle except for those in {1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}, {10, 11}, {11, 12}, {12, 1}, {1, 4}, {2, 7}, {3, 10}, {5, 8}, {6, 11}, {9, 12}; these edges induce graph 5 in the Royle list. In S_2 the first four cycles can be switched to produce up to 3^4 possible graphs for which there exists a 4-cycle system of the complement in K_{12}. Among these, graphs 1, 2, 3, 4, 5 and 12 in the Royle list appear. Next let $S_3 = \{(2, 10, 3, 11), (5, 11, 6, 12), (4, 7, 10, 8), (1, 3, 12, 9), (1, 7, 2, 8), (1, 4, 12, 10), (1, 5, 9, 11), (2, 4, 6, 9), (2, 6, 8, 12), (3, 5, 10, 6), (3, 7, 5, 8), (4, 9, 7, 11)\}$, thus avoiding the edges in {1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}, {10, 11}, {11, 12}, {12, 1}, {1, 6}, {2, 5}, {3, 9}, {4, 10}, {7, 12}, {8, 11} which induce graph 6. The first four cycles can be switched, thereby producing graphs 6, 9, 16, 20, 23, and 39 in the Royle list. We also get graph 67 by replacing the first, fourth and fifth cycles in S_3 with (2, 3, 11, 10), (1, 12, 9, 3), and (1, 2, 7, 8). The edges in the 4-cycles in {(1, 3, 8, 9), (1, 6, 4, 12), (1, 7, 3, 11), (1, 8, 2, 10), (2, 3, 10, 7), (2, 6, 5, 11), (2, 9, 3, 12), (4, 5, 12, 11), (4, 7, 5, 8), (4, 9, 5, 10), (6, 7, 12, 10), (6, 8, 11, 9)} avoid the edges in graph 32. Finally, the edges in the 4-cycles in {(1, 3, 8, 4), (1, 5, 10, 7), (1, 8, 12, 10), (1, 11, 2, 12), (2, 4, 7, 9), (2, 5, 8, 10), (2, 6, 3, 7), (3, 5, 7, 11), (3, 9, 5, 12), (4, 6, 8, 11), (4, 9, 6, 10), (6, 11, 9, 12)} avoid the edges in graph 46.

It remains to suppose that $n = 12$ and G is disconnected. Then G has 0, 4, 8, or 12 vertices of degree 1. First, if G has a component K_2, then the result follows by deleting this component, using the solution when $n = 10$ and adding a 4-cycle system of $K_{2,10}$. Second, if G has a component of order 4, then it must be K_4. If the rest of G is $G' \neq G_1, G_2$ (see Fig. 1) then this can be handled by combining 4-cycle systems of G' and $K_{4,8}$. Otherwise, G' is either G_1 or G_2. Observe that both $\overline{G}_1$ and $\overline{G}_2$, can be decomposed into the two Hamiltonian cycles in: $\{(a_1, a_2, \ldots, a_8), (b_1, b_2, \ldots, b_8)\} = \{(1, 7, 3, 6, 8, 2, 4, 5), (1, 3, 8, 4, 7, 5, 2, 6)\}$ or $\{(1, 7, 2, 8, 4, 6, 3, 5), (1, 3, 8, 5, 7, 4, 2, 6)\}$ for $\overline{G}_1$ or $\overline{G}_2$, respectively. Now, let $V(K_4) = \{9, 10, 11, 12\}$. Then the 4-cycle system of $K_4 \vee G'$ is $\{(9, a_1, a_2, a_3), (10, a_3, a_4, a_5), (9, a_5, a_6, a_7), (10, a_7, a_8, a_1), (9, a_2, 10, a_4), (9, a_6, 10, a_8), (11, b_1, b_2, b_3), (12, b_3, b_4, b_5), (11, b_5, b_6, b_7), (12, b_7, b_8, b_1), (11, b_2, 12, b_4), (11, b_6, 12, b_8)\}$.

Finally, we consider the case that G has two components of order 6. Let the two components be H_1 and H_2. If H_1 has x_1 vertices of degree 1 and $x_1 \equiv 2 \pmod 4$ and H_2 has x_2 vertices of degree 1 and $x_2 \equiv 2 \pmod 4$, then the 4-cycle system of $\overline{H}_1 \vee \overline{H}_2$ can be obtained by combining 4-cycle systems of $\overline{H}_1$, $\overline{H}_2$, and $K_{6,6}$, respectively. Therefore, it is left to consider the case when $x_i \equiv 0 \pmod 4$. Clearly, if H_1 (or H_2) has four vertices of degree 1, then we can join these four vertices by a 4-cycle. That is, it suffices to consider the case when H_1 and H_2 are cubic graphs of order 6. This implies that each of $\overline{H}_1$ and $\overline{H}_2$ is either a 6-cycle or a vertex-disjoint union of 3-cycles. Thus, the proof will be concluded by verifying $C_4|C_6 \vee 2C_3$, $C_4|C_6 \vee C_6$, and $C_4|2C_3 \vee 2C_3$.

Starting with $C_6 \vee 2C_3$, we let $V(C_6 \cup 2C_3) = \{a_i, b_i | i \in \mathbb{Z}_6\}$. Then the 4-cycle system of $C_6 \vee 2C_3$ can be obtained as follows: $\{(a_i, b_i, a_{i+3}, b_{i+1}), (a_i, a_{i+1}, b_i, b_{i+2}) | i \in \mathbb{Z}_6\}$. (Note that the 6-cycle is $(a_0, a_1, a_2, a_3, a_4, a_5)$ and $2C_3 = (b_0, b_2, b_4) \cup (b_1, b_3, b_5)$.)

Next, let $V(C_6 \cup C_6) = \{a_i, b_i | i \in \mathbb{Z}_6\}$. The 4-cycle system of $C_6 \vee C_6$ is obtained as follows: $\{(b_0, a_0, a_1, a_2), (b_2, a_2, a_3, a_4), (b_4, a_4, a_5, a_0), (a_1, b_0, b_1, b_2), (a_3, b_2, b_3, b_4), (a_5, b_4, b_5, b_0), (b_0, a_3, b_1, a_4), (b_2, a_0, b_1, a_5), (b_4, a_1, b_1, a_2), (b_3, a_0, b_5, a_1), (b_3, a_2, b_5, a_3), (b_3, a_4, b_5, a_5)\}$.

Finally, let $V(2C_3 \cup 2C_3) = \{a_i, b_i, c_i, d_i | i \in \mathbb{Z}_3\}$. Then the 4-cycle system of $2C_3 \vee 2C_3$ is the following collection of 4-cycles: $\{(a_1, a_2, b_2, b_1), (a_1, a_3, d_3, d_1), (a_1, b_2, c_1, b_3), (a_1, d_2, c_1, d_3), (c_1, c_2, d_2, d_1), (c_1, c_3, b_3, b_1), (a_2, a_3, b_2, b_3), (a_2, d_3, c_3, b_1), (b_3, a_3, b_1, c_2), (b_2, c_2, d_1, c_3), (d_2, d_3, c_2, c_3), (d_1, a_2, d_2, a_3)\}$. □

Moving on to the general setting, we begin with the disconnected case.

Proposition 2. *Suppose that G is an admissible, odd Δ_3-graph that is disconnected. If $n - 4 \in \mathscr{S}$ then $G \in \mathscr{G}$.*

Proof. If one component of G is a copy of K_4 defined on the vertex set A, then since $n - 4 \in \mathscr{S}$ there exists a 4-cycle system $(V(G)\backslash A, C_1)$ of $K_{n-4} - E(G_1)$, where $G_1 = G[V(G)\backslash A]$. Then $(V(G), C_1 \cup C(V(G)\backslash A, A))$ is a 4-cycle system of $K_n - E(G)$.

Otherwise, each vertex in v is incident with an edge that occurs in no 3-cycle. Since we can assume that G has at most two vertices of degree 1, such an edge exists in each component that joins two vertices of degree 3; let $\{a_1, a_2\}$ and $\{a_3, a_4\}$ be two such edges that occur in different components of G. For $1 \leqslant i \leqslant 4$, let $b_{i,1}$ and $b_{i,2}$ be the other two neighbors of a_i. Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_{i,j} \mid 1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 2\}$.

Since $n-4 \in \mathscr{S}$, by Lemma 1 there exists a 4-cycle system $(V(G)\backslash A, C_1)$ of $K_{n-4} - E(G)$, where G_1 is formed from $G - A$ by adding the edges $\{b_{i,j}, b_{i+2,j}\}$ for $1 \leqslant i, j \leqslant 2$. Since $n-4 \in \mathscr{S}$, there exists a 4-cycle system $(V(G)\backslash A, C_1)$ of $K_{n-4} - E(G)$. Let $C_2 = C_1 \cup \{(a_1, b_{3,1}, b_{1,1}, a_3), (a_1, b_{4,1}, b_{2,1}, a_4), (a_2, b_{3,2}, b_{1,2}, a_4), (a_2, b_{4,2}, b_{2,2}, a_3), (b_{1,1}, a_2, b_{3,1}, a_4), (b_{1,2}, a_2, b_{4,1}, a_3), (b_{2,1}, a_1, b_{4,2}, a_3), (b_{2,2}, a_1, b_{3,2}, a_4)\} \cup C(A, V(G)\backslash(A \cup B))$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$. □

Now we provide the final piece of the puzzle.

Proposition 3. *Let $n \geqslant 14$. Suppose that G is a connected, admissible, odd Δ_3-graph on n vertices. If $n - 4 \in \mathscr{S}$ then $G \in \mathscr{G}$.*

Proof. We consider several cases in turn. Recall that we are assuming that G contains at most two vertices of degree 1.

Case 1: Suppose G contains no 3-cycle and no 4-cycle.

Let a be a vertex of degree 3 with neighbors b_1, b_2, and b_3 that also have degree 3. For $1 \leqslant i \leqslant 3$ let $c_{i,j}$ for $1 \leqslant j \leqslant 2$ be the neighbors of b_i. Being in Case 1, all 10 vertices defined so far must be distinct, and the only possible edges joining those vertices in G must be of the form $\{c_{i,j}, c_{k,l}\}$ where $i \neq k$. Let V_1 be the set of these 10 vertices. Form a graph G_1 from $G[V(G)\backslash\{a, b_1, b_2, b_3\}]$ by adding the edges $\{c_{i,1}, c_{i,2}\}$ for $1 \leqslant i \leqslant 3$. Then by Lemma 1, G_1 is an admissible odd Δ_3-graph, on $n-4$ vertices, so since $n-4 \in \mathscr{S}$ we know $G_1 \in \mathscr{G}$. Let $(V(G) - \{a, b_1, b_2, b_3\}, C_1)$ be a 4-cycle system of $K_{n-4} - E(G_1)$. Define $C_2 = C_1 \cup \{(a, c_{1,1}, b_3, c_{2,1}), (a, c_{1,2}, b_2, c_{3,1}), (a, c_{2,2}, b_1, c_{3,2}), (b_1, b_2, c_{3,2}, c_{3,1}), (b_2, b_3, c_{1,2}, c_{1,1}), (b_1, b_3, c_{2,2}, c_{2,1})\} \cup C(\{a, b_1, b_2, b_3\}, V(G)\backslash V_1)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Case 2: Suppose G contains a 3-cycle but no 4-cycle.

Let $\theta = (a_1, a_2, a_3)$ be a 3-cycle in G chosen so that the neighbor of a_3 outside θ has degree 3; let this neighbor of a_3 be b. Note that $G[\{a_1, a_2, a_3, b\}]$ has exactly four edges since G has no 4-cycles. Naming further vertices, let $c_1, c_2, c_{3,1}$, and $c_{3,2}$ be neighbors of a_1, a_2, b, and b, respectively. Being in Case 1, all eight vertices defined so far are distinct (for example, $c_1 \neq c_2$; for otherwise (a_1, a_3, a_2, c_1) is a 4-cycle in G). Let V_1 be the set of these eight vertices.

Suppose that c_1 is not adjacent to one of $c_{3,1}$ and $c_{3,2}$, say $c_{3,1}$, and that c_2 is not adjacent to the other, $c_{3,2}$. Then let G_1 be formed from $G - \{a_1, a_2, a_3, b\}$ by adding the edges $\{c_1, c_{3,1}\}$ and $\{c_2, c_{3,2}\}$. By Lemma 1, G_1 is an admissible odd Δ_3-graph, so since $n-4 \in \mathscr{S}$, there exists a 4-cycle system $(V(G)\backslash\{a_1, a_2, a_3, b\}, C_1)$ of $K_{n-4} - E(G)$. Define $C_2 = C_1 \cup \{(a_1, b, c_1, c_{3,1}), (a_2, b, c_2, c_{3,2}), (a_1, c_2, a_3, c_{3,2}), (a_2, c_1, a_3, c_{3,1})\} \cup C(\{a_1, a_2, a_3, b\}, V(G)\backslash V_1)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Next notice that since G contains no 4-cycles, it follows that neither c_1 nor c_2 can be joined to both of $c_{3,1}$ and $c_{3,2}$, nor can they have a common neighbor. Therefore, it now suffices to finish this case by considering the possibility that $\{\{c_1, c_{3,1}\}, \{c_2, c_{3,1}\}\} \subseteq E(G)$ and $\{\{c_1, c_{3,2}\}, \{c_2, c_{3,2}\}\} \cap E(G) = \emptyset$. Then $P = (a_3, b, c_{3,1}, c_2)$ is an induced path in G that satisfies condition (1c) in Lemma 2. So $G \in \mathscr{G}$.

Case 3: Suppose G contains a 4-cycle.

Let (a_1, a_2, a_3, a_4) be a 4-cycle in G. We consider several subcases in turn. Let $A = \{a_1, a_2, a_3, a_4\}$.

(1) Suppose $\{\{a_1, a_3\}, \{a_2, a_4\}\} \subseteq E(G)$. Then $G[A]$ is a component in G, contradicting that G is connected.

(2) Suppose $\{a_1, a_3\} \in E(G)$ and $\{a_2, a_4\} \notin E(G)$. Let b_2 and b_4 be the third neighbors of a_2 and a_4, respectively.

If $b_2 \neq b_4$ and $\{b_2, b_4\} \notin E(G)$ then let G_1 be formed from $G - A$ by adding the edge $\{b_2, b_4\}$. Since $n-4 \in \mathscr{S}$, let $(V(G)\backslash A, C_1)$ be a 4-cycle system of $K_{n-4} - E(G_1)$, and let $C_2 = C_1 \cup \{(a_1, b_2, a_3, b_4), (a_2, a_4, b_2, b_4)\} \cup C(V(G)\backslash(A \cup \{b_2, b_4\}), A)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

If $b_2 \neq b_4$ and $\{b_2, b_4\} \in E(G)$, then let c_2 and c_4, be the third neighbors of b_2 and b_4, respectively. If $c_2 = c_4$ then let d be the third neighbor of c_2, and let e_1 and e_2 be neighbors of d; apply Lemma 2 with $P = (e_1, d, c_2, b_2)$ using condition (1b) or (1c) depending on whether $\{e_1, e_2\} \in E(G)$ or not. If one of b_2 and b_4 has degree 1, then let the other, b_x (so $x \in 2, 4$) be adjacent to c which is adjacent to d_1 and d_2; apply Lemma 2 to $P = (d_1, c, b_x, a_x)$ using condition (1c) or (1a) depending on whether or not $\{d_1, d_2\} \in E(G)$. So now suppose that $c_2 \neq c_4$. If c_2 or c_4 has degree 1, then proceed as above. If either c_2 or c_4, say c_x (so $x \in \{2, 4\}$) has a neighbor $d \notin \{c_2, c_4\}$ that is not also adjacent to the other c vertex, then apply Lemma 2 with $P = (d, c_x, b_x, a_x)$ using condition (1a). Otherwise c_2 and c_4 have a common neighbor d that has a third neighbor e that has two further neighbors f_1 and f_2 that are different from all the vertices defined so far (since $n \geqslant 14$ and G is connected); apply Lemma 2 to $P = (f_1, e, d, c_2)$ using condition (1a–c), depending on whether $\{c_2, c_4\} \in E(G)$ and/or $\{f_1, f_2\} \in E(G)$.

If $b_2 = b_4$ then let c be the third neighbor of b_2. If c has degree 1 in G then we have just defined a component in G, contradicting the assumption that G is connected. If c has degree 3, then let the neighbors of c be d_1 and d_2. Then $P = (a_2, b_2, c, d_1)$ satisfies condition (1a) or (1c) of Lemma 2, depending on whether d_1 and d_2 are adjacent or not. So again the result follows.

(3) We can now assume that A induces a 4-cycle in G. Let B be the set of vertices in $V(G)\backslash A$ that are adjacent to vertices in A. We consider the values of $|B|$ in turn. For $1 \leqslant i \leqslant 4$ let b_i be adjacent to a_i; so $B = \{b_i | 1 \leqslant i \leqslant 4\}$ where $b_1, \ldots, b_4$ need not all be distinct.

(3i) Suppose that $|B| = 4$. First suppose that the complement of $G[B]$ contains a 1-factor. Without loss of generality we may assume that either both $\{b_1, b_3\}$ and $\{b_2, b_4\}$ are not edges in G, or both $\{b_1, b_4\}$ and $\{b_2, b_3\}$ are not edges in G.

If both the edges $\{b_1, b_3\}$ and $\{b_2, b_4\}$ do not occur in G then form G_1 by adding them to $G[V(G)\backslash A]$. Since $n-4 \in \mathscr{S}$, let $(V(G)\backslash A, C_1)$ be a 4-cycle system of $K_{n-4} - E(G_1)$. Let $C_2 = C_1 \cup \{(a_1, a_3, b_1, b_3), (a_2, a_4, b_2, b_4), (a_1, b_2, a_3, b_4), (a_2, b_1, a_4, b_3)\} \cup C(V(G)\backslash(A \cup B), A)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

If both the edges $\{b_1, b_4\}$ and $\{b_2, b_3\}$ do not occur in G then form G_1 by adding them to $G[V(G)\backslash A]$. Since $n-4 \in \mathscr{S}$, there exists a 4-cycle system $(V(G)\backslash A, C_1)$ of $K_{n-4} - E(G_1)$. Let $\{z_1, z_2\} \subset V(G)\backslash(A \cup B)$. Let $C_2 = C_1 \cup \{(a_1, b_4, b_1, a_3), (a_2, b_3, b_2, a_4), (b_2, a_3, b_4, z_1), (b_4, a_2, b_1, z_2), (b_1, a_4, b_3, z_1), (b_3, a_1, b_2, z_2)\} \cup C(V(G)\backslash(A \cup B), A)$, where z_1 and z_2 are any two vertices in $V(G)\backslash(A \cup B)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

If the complement of $G[B]$ contains no 1-factor then since $G[B]$ has maximum degree at most 2 it must contain a 3-cycle, say (b_2, b_3, b_4). Since G is connected, let c_1 and c_2 be the two other neighbors of b_1. Then $P = (c_1, b_1, a_1, a_2)$ satisfies condition (1c) or (1a) of Lemma 2, depending on whether c_1 and c_2 are adjacent or not. So again the result follows.

(3ii) Suppose that $|B| = 3$. We can assume that either $b_3 = b_4$ or that $b_2 = b_4$.

First suppose that $b_3 = b_4$. If neither $\{b_1, b_3\}$ nor $\{b_2, b_3\}$ are edges in G then let G_1 be formed by adding them to $G[V(G)\backslash A]$. Since $n-4 \in \mathscr{S}$ there exists a 4-cycle system $(V(G)\backslash A, C_1)$ of $K_{n-4} - E(G_1)$. Let $C_2 = C_1 \cup \{(a_1, a_3, b_2, b_3), (a_2, a_4, b_1, b_3), (b_1, a_2, z, a_3), (b_2, a_1, z, a_4)\} \cup C(V(G)\backslash(A \cup B \cup \{z\}), A)$, where z is any vertex in $V(G)\backslash(A \cup B)$. Then $(V(G), C_2)$ is a 4-cycle system of $K_n - E(G)$.

Otherwise, if $b_3 = b_4$ then it now follows that one of $\{b_1, b_3\}$ or $\{b_2, b_3\}$ must be an edge in G (we can assume it is $\{b_2, b_3\}$ by symmetry). Either b_1 has degree 3 or it has degree 1 in G. If b_1 has degree 3, then let its two other neighbors be c_1 and c_2 (possibly $c_2 = b_2$). If $d_G(b_1) = 3$ and $c_2 \neq b_2$ then $P = (c_1, b_1, a_1, a_4)$ satisfies condition (1a) or (1c) of Lemma 2 depending on whether c_1 is adjacent to c_2 or not. If $d_G(b_1) = 3$ and $c_2 = b_2$ then since G is connected, c_1 must have two neighbors d_1 and d_2; then $P = (d_1, c_1, b_1, a_1)$ satisfies condition (1c) or (1a) of Lemma 2 depending on whether d_1 is adjacent to d_2 or not. If b_1 has degree 1, then let c be the remaining neighbor of b_2, and let d_1 and d_2 be the other two neighbors of c (all vertices defined in this situation are necessarily distinct because for each vertex all three incident edges have been defined). Then $P = (d_1, c, b_2, a_2)$ satisfies condition (1c) or (1a) of Lemma 2 depending on whether d_1 is adjacent to d_2 or not, so the result follows.

Next suppose that $b_2 = b_4$. If either $\{b_1, b_2\}$ or $\{b_2, b_3\}$ is in G (by symmetry we assume $\{b_2, b_3\} \in E(G)$) then (a_2, a_3, a_4, b_2) is a 4-cycle in G where $N(a_2, a_3, a_4, b_2)$ contains only two further vertices, namely a_1 and b_3; so this case is handled when we consider $|B| = 2$. Therefore, we can assume that neither $\{b_1, b_2\}$ nor $\{b_2, b_3\}$ is an edge in G. If $d_G(b_1) = 1 = d_G(b_3)$ then let c be the remaining neighbor of b_2, and let d_1 and d_2 be the other neighbors of c. Then $P = (d_1, c, b_2, a_4)$ satisfies condition (1c) or (1a) of Lemma 2 depending on whether d_1 and d_2 are adjacent or not. So suppose $d_G(b_1) = 3$. Let c_1 and c_2 be neighbors of b_1, and let c_3 be the other neighbor of b_2, where possibly $c_2 = c_3$. Then $P = (c_1, b_1, a_1, a_2)$ satisfies condition (1c) or (1a) of Lemma 2, depending on whether c_1 and c_2 are adjacent or not.

(3iii) Suppose that $|B| = 2$. Then we have three cases to consider.

Suppose that $\{a_1, b_1\}$, $\{a_2, b_1\}$, $\{a_3, b_3\}$, and $\{a_4, b_3\}$ are edges in G. Since G is connected, $\{b_1, b_3\} \notin E(G)$, so let the remaining neighbors of b_1 and b_3 be c_1 and c_3, respectively. If $c_1 = c_3$ then $P = (a_1, b_1, c_1, d)$ is a path satisfying condition (1c) of Lemma 2, where d is the third neighbor of c_1. If $c_1 \neq c_3$ then $P = (a_1, b_1, c_1, d_1)$ satisfies condition (1b) or (1c) of Lemma 2, where $d_1 \notin \{b_1, c_3\}$ is a neighbor of c_1.

Next, suppose that $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_1\}$, and $\{a_4, b_2\}$ are edges in G. Since G is connected, $\{b_1, b_2\} \notin E(G)$. Let the remaining neighbors of b_1 and b_2 and c_1 and c_2, respectively. Then the rest of this case follows the previous case, using $P = (a_1, b_1, c_1, d_1)$ and Lemma 2 to obtain the result.

Finally, suppose that b_1 is adjacent to a_1, a_2, and a_3, and $\{a_4, b_4\} \in E(G)$. Then $P = (c_4, b_4, a_4, a_3)$ satisfies condition (1c) or (1a) of Lemma 2, where $c_4 \neq a_4$ is a neighbor of b_4. □

We can now collect all the pieces.

Theorem 1. *Let G be a graph on n vertices, where n is even and $\Delta(G) \leqslant 3$. Then there exists a 4-cycle system of $K_n - E(G)$ if and only if*

1. *all vertices in G have odd degree,*
2. *4 divides $n(n-1)/2 - |E(G)|$, and*
3. *G is not one of the two graphs of order 8 described in Proposition 1.*

Proof. Since the necessity is clear, we prove the sufficiency. So let G satisfy conditions (1–3) of this theorem. If $n \leqslant 12$ then the result follows from Proposition 1. So suppose that $n \geqslant 14$. Using induction, suppose that $m \in \mathscr{S}$ for all m satisfying $10 \leqslant m \leqslant n-2$. If G is disconnected then the result follows from Proposition 2. If G is connected then the result follows from Proposition 3. □

Acknowledgment

The first two authors would like to express their thanks to the Discrete and Statistical Sciences Department at Auburn University for their support of the visit during which this research was completed. The third author wishes to thank the Department of Mathematics at Tamkang University for their kind support for a visit during which this research was begun.

References

[1] B. Alspach, H. Gavlas, Cycle decompositions of K_n and K_{n-I}, J. Combin. Theory B 81 (2001) 77–99.
[2] D.J. Ashe, H.L. Fu, C.A. Rodger, A solution to the forest leave problem for partial 6-cycle systems, Discrete Math. 281 (2004) 27–41.

[3] D.J. Ashe, H.L. Fu, C.A. Rodger, All 2-regular leaves of partial 6-cycle systems, Ars Combin. 76 (2005) 129–150.
[4] D. Bryant, Hamilton cycle rich two-factorizations of complete graphs, J.Combin. Design 12 (2) (2004) 147–155.
[5] D. Bryant, Y. Chang, C.A. Rodger, R. Wei, Two-dimensional balanced sampling plans excluding contiguous units, Comm. Statist. Theory Methods 31 (2002) 1441–1455.
[6] H. Buchanan II, Graph factors and hamiltonian decompositions, Ph.D. Dissertation, West Virginia University, 1997.
[7] C.J. Colbourn, C.C. Lindner, C.A. Rodger, Neighbor designs and m-wheel systems, J. Statist. Plann. Inference 27 (1991) 335–340.
[8] C.J. Colbourn, A. Rosa, Quadratic leaves of maximal partial triple systems, Graphs Combin. 2 (1986) 317–337.
[9] H.L. Fu, C.A. Rodger, Forest leaves and four-cycles, J. Graph Theory 33 (2000) 161–166.
[10] H.L. Fu, C.A. Rodger, Four-cycle systems with two-regular leaves, Graphs Combin. 17 (2001) 457–461.
[11] A.S. Hedayat, C.R. Rao, J. Stufken, Sampling plans excluding contiguous units, J. Statist. Plann. Inference 19 (1988) 159–170.
[12] C.C. Lindner, C.A. Rodger, Decomposition into Cycles II: Cycle Systems, Contemporary Design Theory, Wiley-Interscience Series Discrete Mathematics Optimization, Wiley, New York, 1992 pp. 325–369.
[13] D.H. Rees, Some designs of use in serology, Biometrics 23 (1967) 779–791.
[14] C.A. Rodger, in: C.J. Colbourn, J.H. Dinitz (Eds.), Cycle Systems, The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996, pp. 266–269.
[15] C.A. Rodger, Hamilton decomposable graphs with specified leaves, Graphs Combin. 20 (2004) 541–543.
[16] M. Sajna, Cycle decompositions. III. Complete graphs and fixed length cycles, J. Combin. Design 10 (2002) 27–78.
[17] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory B 30 (1981) 75–81.