On Minimum Sets of 1-Factors Covering a Complete Multipartite Graph

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Received April 20, 2006; Revised January 21, 2008

Published online 15 April 2008 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20303

Abstract: We determine necessary and sufficient conditions for a complete multipartite graph to admit a set of 1-factors whose union is the whole graph and, when these conditions are satisfied, we determine the minimum size of such a set. © 2008 Wiley Periodicals, Inc. J Graph Theory 58: 239–250, 2008

Keywords: *1–factor; 1–factor cover; excessive factorization; excessive index; complete multipartite graph*

1. INTRODUCTION

All graphs considered will be finite, simple and undirected, unless stated otherwise. We denote by $V(G)$ and $E(G)$, respectively, the vertex and edge set of a graph G.

The *order* of G is $|V(G)|$. The *maximum degree* of G will be denoted by $\Delta(G)$ and its *chromatic index* by $\chi'(G)$.

Let G be a graph of even order. A *1-factor* (or *perfect matching*) of G is a 1-regular spanning subgraph, that is, a set of exactly $\frac{|V(G)|}{2}$ independent edges.[1] G is *1-extendable* if every edge of G belongs to at least one 1-factor of G. A *1-factor cover* of G is a set $\mathcal{F}$ of 1-factors of G such that $\cup_{F\in\mathcal{F}} = E(G)$. Notice that G admits a 1-factor cover if and only if it is 1-extendable. If G is 1-extendable, a 1-factor cover of minimum cardinality will be called an *excessive factorization*.

Thus, a *1-factorization* of G is a 1-factor cover $\mathcal{F}$ with the property that all the 1-factors in $\mathcal{F}$ are pairwise disjoint. Any 1-factorization is an excessive factorization, but the converse is obviously not true. For example, the Petersen graph has no 1-factorization, but has an excessive factorization consisting of five 1-factors (see [2]). The graphs which admit an excessive factorization are precisely those that have a 1-factor cover, that is, those that are 1-extendable.

Let G be a 1-extendable graph. The *excessive index* of G, denoted $\chi'_e(G)$, is the size of an excessive factorization of G. We define $\chi'_e(G) = \infty$ if G is not 1-extendable.

Bonisoli [1] and Wallis [6] considered 1-factor covers of the complete graph K_{2n} which do not contain a 1-factorization of K_{2n}.

Bonisoli and Cariolaro [2] introduced the concept of *excessive factorization*, defined the parameter $\chi'_e(G)$, and studied excessive factorizations of regular graphs. They posed a number of open problems and conjectures. A first question is, of course, to determine $\chi'_e(G)$ for any graph G. It is observed in [2] that this problem is NP-hard since, if G is regular and has even order, then $\chi'_e(G) = \Delta(G)$ if and only if G is 1-factorizable, and to determine whether a regular graph G is 1-factorizable is NP-complete. Therefore, we can expect to be able to determine $\chi'_e(G)$ only for some specific classes of graphs.

In this article, we consider the class of *complete multipartite* graphs. Hoffman and Rodger [3] determined the chromatic index of all complete multipartite graphs. Here, we shall determine the excessive index $\chi'_e(G)$ of any complete multipartite graph G.

We will often use, without further reference, the following fact, proved by de Werra [7] and, independently, by McDiarmid [4]. If a multigraph G has a k-edge coloring, that is, if $k \geq \chi'(G)$, then it also has an *equalized* k-edge coloring, namely a k-edge coloring such that each color class has size either $\lfloor\frac{|E(G)|}{k}\rfloor$ or $\lceil\frac{|E(G)|}{k}\rceil$.

We shall also need the concept of *excessive coloring*. An excessive coloring of a graph G is an assignment of (possibly more than one) colors to each of the edges of G such that the edges on which a given color appears are independent (i.e., they form a matching). Thus an excessive coloring can be simply specified

[1] To be precise, a 1-factor F of G is a 1-regular spanning subgraph of G and a perfect matching is the edge set of F, but the two terms are often used interchangeably in the literature.

as a collection of matchings of G whose union is $E(G)$. It is normally interesting to consider this concept when additional restrictions are imposed on the matchings which form the color classes. For example, when each color class is a 1-factor, the corresponding excessive coloring is equivalent to a 1-factor cover.

2. 1-EXTENDABLE COMPLETE MULTIPARTITE GRAPHS

We adopt the notation $G = K(n_1, n_2, \ldots, n_r)$ to designate a complete multipartite graph with partite sets of size $n_1, n_2, \ldots, n_r$, where $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_r$. We also let $V_1, V_2, \ldots, V_r$ denote the r *partite sets* of G. By definition, for each i, V_i is an independent set of n_i vertices of G which are joined to every vertex in $G - V_i$.

Trivially, the complete bipartite graph $K(m, n)$ is 1-extendable if and only if $m = n$, in which case it actually has a 1-factorization. Therefore, $\chi'_e(K(m, n)) = n$ if $n = m$ and ∞ otherwise. From now on we make the convention that all complete multipartite graphs considered have r partite sets, where $r \geq 3$. The following lemma is probably well known, but we give a full proof for the sake of completeness.

Lemma 1. *The graph $G = K(n_1, n_2, \ldots, n_r)$ has a 1-factor if and only if*

1. $\sum_{i=1}^{r} n_i$ *is even;*
2. $n_1 \leq \sum_{i=2}^{r} n_i$.

Proof. The first of the above conditions is clearly necessary in order for the graph G to have a 1-factor, as G must have even order. To see the necessity of the second, it suffices to see that any 1-factor of G must match the vertices of V_1 (the first partite set) to the vertices of the complement (since the vertices in V_1 are mutually nonadjacent). Hence, the two conditions are necessary. To see the sufficiency, assume both conditions hold. We prove the existence of a 1-factor by induction on k, where $2k = \sum_{i=2}^{r} n_i - n_1$.

If $k = 0$, then we have $n_1 = \sum_{i=2}^{r} n_i$ and a 1-factor of G is easily obtained by matching the vertices of V_1 to the vertices of $V_2 \cup V_3 \cup \cdots \cup V_r$. Assume now that the theorem holds for any G with $\sum_{i=2}^{r} n_i - n_1 < 2k$ and consider the case of a G with $\sum_{i=2}^{r} n_i - n_1 = 2k$. Let $x \in V_r$ and $y \in V_{r-1}$ and consider the edge $e = xy$. We prove that there is a 1-factor of G containing this edge. This is equivalent to proving that the graph $G - x - y$ has a 1-factor. But it is easily seen that the graph $G - x - y$ is complete multipartite with partite sets $V'_1, V'_2, \ldots, V'_r$, where $|V'_i| = n_i$ for all $i \leq r - 2$ and $|V'_{r-1}| = n_{r-1} - 1$ and $|V'_r| = n_r - 1$. Moreover, $G - x - y$ satisfies the inductive hypothesis, since $|V'_2| + |V'_3| + \cdots + |V'_{r-1}| + |V'_r| - |V'_1| = 2k - 2$. Thus, $G - x - y$ has a 1-factor and hence G has the desired 1-factor. ■

Using Lemma 1, it is easy to determine which complete multipartite graphs admit an excessive factorization, as given by the following theorem.

Theorem 1. *The graph $K(n_1, n_2, \ldots, n_r)$ is* 1*-extendable if and only if*

1. $\sum_{i=1}^{r} n_i$ *is even;*
2. $n_1 < \sum_{i=2}^{r} n_i$.

Proof. Let $G = K(n_1, n_2, \ldots, n_r)$ and suppose G is 1-extendable. The first condition follows immediately from the fact that G has a 1-factor. Let e be an edge joining the second and third partite sets. By the fact that G is 1-extendable, there exists a 1-factor F containing e. Clearly, this 1-factor must match the vertices of the first partite set onto the vertices of the complement, but the two vertices which are the endpoint of e are F-saturated. Hence, the condition (2) above (which is clearly equivalent to $n_1 \leq \sum_{i=2}^{r} n_i - 2$, given the parity of G) must hold.

Conversely, suppose G satisfies both conditions above. We prove that G is 1-extendable. Let $e \in E(G)$. We prove the existence of a 1-factor F containing e. Equivalently, we prove that the graph $G - x - y$ has a 1-factor, where $xy = e$. Without loss of generality, we can assume $x \in V_i$, $y \in V_j$, and $i < j$. But then $G - x - y \cong G_1 = K(n_1, n_2, \ldots n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_r)$ and G_1 is easily seen to satisfy the hypotheses of Lemma 1. Hence, G_1 has a 1-factor and G has the desired 1-factor. ■

3. SOME LEMMAS

Theorem 1 gives necessary and sufficient conditions for a complete multipartite graph G to admit an excessive factorization, that is, to satisfy $\chi'_e(G) < \infty$. We are now left with the task of determining precisely $\chi'_e(G)$ for all such graphs.

We start by giving some lower bounds. One obvious lower bound is the maximum degree of G, because every edge incident with a vertex of maximum degree must belong to a distinct 1-factor in an excessive factorization. Thus, $\chi'_e(G) \geq \Delta(G)$ holds, not only for complete multipartite graphs, but for all graphs G (in fact an easy argument along the same line also shows that, for all graphs G, $\chi'_e(G) \geq \chi'(G)$, but we shall not need this stronger inequality here).

The next lower bound is less trivial. Let $G = K(n_1, n_2, \ldots, n_r)$. Let V_i be the ith partite set of G. Let E_i be defined as

$$E_i = E(G - V_i).$$

Define

$$\sigma_i(G) = \left\lceil \frac{2|E_i|}{|V(G)| - 2|V_i|} \right\rceil. \tag{1}$$

Since the vertices of V_i are independent in G, any 1-factor F of G must contain exactly n_i edges joining V_i to $G - V_i$. Thus, in particular, the 1-factor F contains exactly $\frac{|V(G)|-2|V_i|}{2}$ edges from E_i. It follows from (1) that any 1-factor cover must contain at least $\sigma_i(G)$ 1-factors. This proves that

$$\chi'_e(G) \geq \max_{1\leq i\leq r} \sigma_i(G).$$

As shown next, the quantity $\max_{1\leq i\leq r} \sigma_i(G)$ is particularly simple to evaluate, since it is always equal to $\sigma_1(G)$.

Proposition 1. *Let $G = K(n_1, n_2, \ldots, n_r)$. Then*

$$\sigma_1(G) = \max_{1\leq i\leq r} \sigma_i(G),$$

where the parameters $\sigma_i(G)$ are defined in (1).

Proof. We start by observing that

$$\sigma_k(G) = \frac{2\sum_{1\leq i<j\leq r; i,j\neq k} n_i n_j}{\sum_{i=1}^r n_i - 2n_k}.$$

Therefore, Proposition 1 follows from the truth of the following inequality involving positive integers $x_1, x_2, \ldots, x_r$, where $x_1 = \max_{1\leq i\leq r} x_i$:

$$\frac{\sum_{2\leq i<j\leq r} x_i x_j}{\sum_{i=1}^r x_i - 2x_1} \geq \frac{\sum_{1\leq i<j\leq r; i,j\neq k} x_i x_j}{\sum_{i=1}^r x_i - 2x_k}. \tag{2}$$

By the arbitrariety of the x_is ($i > 1$), we can assume $k = 2$. Further, if $x_1 = x_2$ there is clearly nothing to prove because the two sides of (2) are in this case identical. Thus, we may assume that $x_1 > x_2$. We have to prove that

$$\left(\sum_{2\leq i<j\leq r} x_i x_j\right)\left(\sum_{i=1}^r x_i - 2x_2\right) \geq \left(\sum_{1\leq i<j\leq r; i,j\neq 2} x_i x_j\right)\left(\sum_{i=1}^r x_i - 2x_1\right).$$

Let $\xi = x_3 + x_4 + \cdots + x_r$ and let $\eta = \sum_{3\leq i<j\leq r} x_i x_j$. Using these notations, we can rewrite the above inequality as

$$(x_2\xi + \eta)(x_1 - x_2 + \xi) \geq (x_1\xi + \eta)(x_2 - x_1 + \xi).$$

Multiplying out, simplifying and rearranging the terms, we obtain

$$(x_1 - x_2)\xi^2 - (x_1^2 - x_2^2)\xi - 2\eta(x_1 - x_2) \leq 0.$$

Dividing by $x_1 - x_2$, which is positive by assumption, we obtain

$$\xi^2 - (x_1 + x_2)\xi - 2\eta \leq 0.$$

But we have

$$\xi^2 = (x_3 + x_4 + \cdots + x_r)^2 = x_3^2 + x_4^2 + \cdots + x_r^2 + 2 \cdot \sum_{3\leq i<j\leq r} x_i x_j = x_3^2 + x_4^2 + \cdots + x_r^2 + 2\eta.$$

Thus, the problem is reduced to showing that

$$x_3^2 + x_4^2 + \cdots + x_r^2 \leq (x_1 + x_2)(x_3 + x_4 + \cdots + x_r).$$

Using the assumption that $x_1 = \max_{1 \leq i \leq r} x_i$ and the fact that

$$x_3^2 + x_4^2 + \cdots + x_r^2 \leq x_1(x_3 + x_4 + \cdots + x_r) < (x_1 + x_2)(x_3 + x_4 + \cdots + x_r),$$

we conclude the proof. ■

Thus, we have two nontrivial lower bounds on $\chi'_e(G)$, one is $\Delta(G)$ and the other is $\sigma_1(G)$. Neither of the two is necessarily worse or better than the other. For example, if $G = K(5, 4, 3)$ then $\sigma_1(G) = 12 > \Delta(G) = 9$, but if $G = K(4, 4, 4, 2)$ then $\sigma_1(G) = 11 < \Delta(G) = 12$.

Let

$$\tau(G) = \max\{\sigma_1(G), \Delta(G)\}.$$

By what we have just proved, we have

Lemma 2. *Let G be a 1-extendable complete r-partite graph. Then*

$$\chi'_e(G) \geq \tau(G).$$

In the next section, we shall prove that the above inequality is indeed an equality by proving the following theorem.

Theorem 2. *Let G be a 1-extendable complete r-partite graph. Then*

$$\chi'_e(G) = \tau(G) = \max\{\sigma_1(G), \Delta(G)\}.$$

We shall use the following lemma, which can be seen as an extension of Hall's Theorem. It generalizes a theorem of Bondy ([5, Theorem 13.3, p.109]). *Inter alia*, it completely solves the problem of characterizing the excessive colorings of $G - V_1$ which extend to 1-factor covers of G, for any complete multipartite graph G.

Lemma 3. *Let $\mathcal{C}$ be a set (of colors) and let s, t be positive integers, with $s \leq t \leq |\mathcal{C}|$. Let $T = \{y_1, y_2, \ldots, y_t\}$ be a set of cardinality t and let $S = \{x_1, x_2, \ldots, x_s\}$ be a set (disjoint from T) of cardinality s. For each $y \in T$, let $L(y) \subset \mathcal{C}$ be a set of colors. Let, for each $\alpha \in \mathcal{C}$, $T_\alpha = \{y \in T; | \alpha \in L(y)\}$. Consider the complete bipartite graph X with bipartition (T, S). There exists an excessive coloring ψ of X with color set $\mathcal{C}$ such that, for each $\alpha \in \mathcal{C}$, the color class corresponding to α is a perfect matching from S to T_α if and only if the following conditions are satisfied:*

1. *every $\alpha \in \mathcal{C}$ is contained in precisely s sets of the family $\{L(y) \mid y \in T\}$;*
2. *$|L(y)| \geq s$ (for all $y \in T$).*

Proof. Assume that there exists an excessive coloring as in the statement of the lemma. Then clearly condition (1) is satisfied since the existence of a perfect matching from S to T_α implies $|T_\alpha| = s$. Condition (2) follows from the fact that every $y \in T$ is incident with s edges in the graph X, and each of these edges must be assigned at least one distinct color by ψ, this color being in $L(y)$. Thus, the two conditions above are necessary for the existence of ψ. We now show that they are sufficient.

Consider the bipartite graph B_1 with bipartition $(T, \mathcal{C})$, where there is an edge between y and α if and only if $\alpha \in L(y)$. By assumption, $\deg_{B_1}(\alpha) = s$ for all $\alpha \in \mathcal{C}$ and $\deg_{B_1}(y) \geq s$ for all $y \in T$.

Let B_2 be a spanning subgraph of B_1 such that

$$\deg_{B_2}(y) = s \text{ for all } y \in T.$$

Since B_2 is bipartite and $\Delta(B_2) = s$, by Kőnig's Theorem B_2 has an s-edge coloring π with colors $\{1, 2, \ldots, s\}$. We now define an edge coloring θ of X as follows: if y is joined in B_2 to color α by an edge colored j, we color the edge yx_j of X by color α. It is easy to see that the coloring θ is well defined, and that it is in fact a proper edge coloring of X. To obtain the required excessive coloring of X, it is now sufficient, for each color α, to extend arbitrarily the color class C_α corresponding to α to a perfect matching from S to T_α. ■

Instead of proving Theorem 2 directly, in the next section we shall prove the following theorem, whose equivalence with Theorem 2 will be established below.

Theorem 3. *Let $G = K(n_1, n_2, \ldots, n_r)$ be a 1-extendable complete r-partite graph and let $H = K(n_2, n_3, \ldots, n_r)$. Then there exists an excessive coloring ϕ of H with exactly $\tau(G)$ colors such that each color class misses exactly n_1 vertices of H and each vertex of H misses at least n_1 colors.*

Using Lemma 3, we now prove the equivalence between Theorems 2 and 3.

Lemma 4. *Theorem 2 holds for G if and only if Theorem 3 does.*

Proof. Assume Theorem 2 is true for the graph $G = K(n_1, n_2, \ldots, n_r)$. Let ψ be an excessive factorization of G. Then ψ, when viewed as an excessive coloring, consists of $\tau(G)$ color classes. Let $H = G - V_1$, where V_1 is the largest partite set of G. Then $H \cong K(n_2, n_3, \ldots, n_r)$. Clearly, the restriction ϕ of ψ to $E(H)$ makes Theorem 3 true for the graph G. Hence, if Theorem 2 holds for G then Theorem 3 does. For the converse, let G and H be as above and assume Theorem 3 holds for G. By Theorem 3, there exists an excessive coloring ϕ of H with $\tau(G)$ color classes such that each color class is a matching missing exactly n_1 vertices of H and each vertex of H misses at least n_1 colors. We now extend ϕ to an excessive coloring of G as follows. Let $\mathcal{C}$ be the color set of ϕ. For each vertex $v \in V(H)$, let $L(v)$ be the set of colors missing at v, that is, the set of colors which do not appear on any of the edges incident with v. By the conditions satisfied by ϕ, $|L(v)| \geq n_1$ for all $v \in V(H)$ and every color $\alpha \in \mathcal{C}$ appears on exactly n_1 of the sets $L(v)$, $v \in V(H)$. Therefore, the conditions of Lemma 3 are satisfied by the color set $\mathcal{C}$, the family of sets $\mathcal{L} = \{L(v) \mid v \in V(H)\}$ and the set $S = V_1$. By Lemma 3, there exists an excessive coloring ψ of the complete bipartite graph U with bipartition $(V(H), V_1)$

such that, for every color α, the color class corresponding to α is a perfect matching from S to T_α, where T_α is the set of vertices in $V(H)$ which are missing (with respect to ϕ) color α. Therefore, it is easily seen that the map ρ defined by

$$\rho(e) = \begin{cases} \phi(e) & \text{if } e \in E(H), \text{ and} \\ \psi(e) & \text{if } e \in E(U). \end{cases}$$

is an excessive coloring of G which uses $\tau(G)$ colors and such that every color class is a 1-factor of G. Therefore, ρ is a 1-factor cover of G consisting of $\tau(G)$ 1-factors, and, by Lemma 2, this number is necessarily the minimum, which proves that ρ is an excessive factorization of G. Thus, Theorem 2 holds for G. This concludes the proof of Lemma 4. ■

4. THE MAIN RESULT

We now prove Theorem 2 for all those complete multipartite graphs for which $\sigma_1(G) > \Delta(G)$ by proving the following.

Theorem 4. *Let G be a 1-extendable complete multipartite graph such that $\sigma_1(G) > \Delta(G)$. Then $\chi'_e(G) = \sigma_1(G)$.*

Proof. Let $G = K(n_1, n_2, \ldots, n_r)$ and let $H = G - V_1 \cong K(n_2, n_3, \ldots, n_r)$, where V_1 is the largest partite set of G, and assume that G is 1-extendable and $\sigma_1(G) > \Delta(G)$. By Lemma 4, it will suffice to show that Theorem 3 holds for G. Thus, we need to find an excessive coloring ϕ of H with exactly $\sigma_1(G)$ color classes, each of which misses exactly n_1 vertices of H and with respect to which each vertex of H misses at least n_1 colors. Let $m = \frac{|V(H)|-n_1}{2}$. Notice that $\sigma_1(G) = \lceil \frac{|E(H)|}{m} \rceil$. Let $m_1 = |E(H)| - (\sigma_1(G) - 1)m$. Notice that $0 < m_1 \leq m$.

We prove that there exists an edge-coloring of H with $\sigma_1(G) - 1$ color classes of size m and 1 color class of size m_1.

By assumption G has a 1-factor, and hence H has a matching of size m. Let M_1 be a matching in H of size m_1. Consider the graph $H - M_1$. We have

$$\chi'(H - M_1) \leq \chi'(H) \leq \chi'(G) = \Delta(G) \leq \sigma_1(G) - 1.$$

Hence, there exists a $(\sigma_1(G) - 1)$-edge coloring of $H - M_1$. Notice that $|E(H - M_1)| = (\sigma_1(G) - 1)m$. But then there exists an *equalized* $(\sigma_1(G) - 1)$-edge coloring of $H - M_1$, so that each color class has size exactly m.

Putting back the matching M_1 as an additional color class, we have the required edge coloring of H. But now, in order to obtain an excessive coloring of H as in Theorem 3, we just need to extend the color class M_1 to an arbitrary color class (matching) of size m of H. The excessive coloring ϕ thus defined is such that any vertex v of H of degree $\deg_H(v)$ "sees" at most $\deg_H(v) + 1$ colors, because the only possible edges with multiple colors are those in M_1. Thus, at any vertex of H there are at most

$$\sigma_1(G) - (\deg_H(v) + 1) \geq \sigma_1(G) - \Delta(H) - 1 \geq \Delta(G) - \Delta(H) = n_1$$

colors missing. Notice that each color class of ϕ is a matching of size m and hence (by the definition of m) misses exactly n_1 vertices of H. Therefore, ϕ verifies Theorem 3 for G, and hence (by Lemma 4) it verifies Theorem 2 for G. This terminates the proof. ■

To terminate the proof of Theorem 2, we need to settle the case $\sigma_1(G) \leq \Delta(G)$. The following notation will be helpful in the sequel. If Z is a graph, $v \in V(Z)$ and k is an integer, the *k-deficiency* of v in Z is the quantity

$$k\_\mathrm{def}(v) = k - \deg_Z(v)$$

and the *k-deficiency* of Z is defined as

$$k\_\mathrm{def}(Z) = \sum_{v \in V(Z)} k\_\mathrm{def}(v) = \sum_{v \in V(Z)} (k - \deg_Z(v)).$$

Notice that, if $k = \Delta(G)$, then the k-deficiency of Z is usually called *deficiency* of Z and denoted by $\mathrm{def}(Z)$.

Proposition 2. *Let $G = K(n_1, n_2, \ldots, n_r)$ and let $H = G - V_1 \cong K(n_2, n_3, \ldots, n_r)$. Then the following two conditions are equivalent:*

1. $\sigma_1(G) \leq \Delta(G)$*;*
2. $\Delta(G)\_\mathrm{def}(H) \geq n_1 \Delta(G)$.

Proof. Condition 1 is equivalent to

$$\frac{2|E(H)|}{|V(H)| - n_1} \leq \Delta(G),$$

that is,

$$2|E(H)| \leq (|V(H)| - n_1)\Delta(G),$$

that is,

$$\sum_{v \in V(H)} \deg_H(v) \leq (|V(H)| - n_1)\Delta(G),$$

that is,

$$\sum_{v \in V(H)} (\deg_H(v) - \Delta(G)) \leq -n_1 \Delta(G).$$

Changing sign, we have

$$\Delta(G)\_\mathrm{def}(H) \geq n_1 \Delta(G)$$

which is condition 2. ■

We will need the following lemma.

Lemma 5. *Let $G = K(n_1, n_2, \ldots, n_r)$ be a 1-extendable complete multipartite graph such that $\sigma_1(G) \leq \Delta(G)$ and let $H = G - V_1 \cong K(n_2, n_3, \ldots, n_r)$. Assume* *that there exists a multigraph H^* containing H as a spanning subgraph such that H^* is obtained by replicating some of the existing edges of H but without adding edges between nonadjacent vertices of H. Suppose furthermore that $\Delta(H^*) = \Delta(H)$ and $\Delta(G)\_\mathrm{def}(H^*) \leq n_1\Delta(G)$. Then $\chi'_e(G) = \Delta(G)$.*

Proof. Let H, H^* be as in the statement of Lemma 5. The condition

$$\Delta(G)\_\mathrm{def}(H^*) \leq n_1\Delta(G)$$

is equivalent (arguing as in Proposition 2) to

$$|E(H^*)| \geq \frac{1}{2}(|V(H)| - n_1)\Delta(G). \tag{3}$$

By possibly removing some of the edges from H^*, we can assume that the sign of equality holds in (3) and hence that

$$|E(H^*)| = \frac{1}{2}(|V(H)| - n_1)\Delta(G), \tag{4}$$

which is equivalent to

$$\Delta(G)\_\mathrm{def}(H^*) = n_1\Delta(G). \tag{5}$$

Let $v \in V(H) = V(H^*)$. By assumption, the edges incident with v which are in H^* but not in H are at most

$$\Delta(H) - \deg_H(v) \leq \Delta(H) - (|V(H)| - n_1) = n_1 - n_r \leq n_1 - 1,$$

so that (denoting by $\mu(H^*)$ the maximum multiplicity of the edges of H^*) we have $\mu(H^*) \leq n_1$, since H is a simple graph. But then, by Vizing's Theorem, we have

$$\chi'(H^*) \leq \Delta(H^*) + \mu(H^*) \leq \Delta(H) + n_1 = \Delta(G).$$

Thus, H^* is $\Delta(G)$-edge colorable. But then, in particular, H^* has an equalized $\Delta(G)$-edge coloring, which we denote by φ. It follows by (4) that each color class of φ contains exactly $\frac{1}{2}(|V(H)| - n_1)$ edges. Since $\Delta(H^*) = \Delta(H) = \Delta(G) - n_1$, it follows that every vertex of H^* misses at least n_1 of the colors given by φ.

Let ψ be the excessive coloring of H obtained by assigning to the edge $xy \in E(H)$ all the colors assigned by φ to the edges $xy \in E(H^*)$. Then clearly ψ is an excessive coloring of H using $\Delta(G)$ colors, such that each color class contains exactly $\frac{1}{2}(|V(H)| - n_1)$ edges and each vertex misses at least n_1 colors. Thus, ψ satisfies Theorem 3 and hence, by Lemma 4, Theorem 2 holds for G, as we wanted. ■

We are ready to prove the following theorem, which, together with Theorem 4, proves Theorem 2.

Theorem 5. *Let G be a 1-extendable complete multipartite graph such that $\sigma_1(G) \leq \Delta(G)$. Then $\chi'_e(G) = \Delta(G)$.*

Proof. By Lemma 5, it suffices to prove the existence of a multigraph H^* as specified in the statement of Lemma 5.

Let H^* be a maximal multigraph which is a spanning supergraph of H obtained by replicating existing edges of H without introducing edges between nonadjacent vertices of H and such that $\Delta(H^*) = \Delta(H)$. We prove that H^* satisfies the conditions of Lemma 5.

Claim 1. *There can be at most one partite set V_i of H^* containing vertices of degree less than $\Delta(H)$.*

This is obvious since otherwise we could add to H^* an edge by replicating an existent edge xy of H^* without violating the constraints on the maximum degree but contradicting the maximality of H^*.

Conclusion. If all the vertices in $V(H^*)$ have degree $\Delta(H)$ there is clearly nothing to prove, since then

$$\Delta(G)\_\mathrm{def}(H^*) = n_1|V(H)| \leq n_1\Delta(G)$$

and all the conditions of Lemma 5 are satisfied.

Thus, we can assume (by Claim 1) that there is exactly one partite set V_i of H^* containing vertices of degree less than $\Delta(H)$.

But then

$$\begin{aligned}\Delta(H)\_\mathrm{def}(H^*) &= \sum_{v\in V_i}(\Delta(H) - \deg_{H^*}(v)) \\ &\leq \sum_{v\in V_i}(\Delta(H) - \deg_H(v)) = n_i(n_i - n_r).\end{aligned}$$

Hence

$$\Delta(G)\_\mathrm{def}(H^*) \leq n_i(n_i - n_r) + n_1(|V(G)| - n_1). \tag{6}$$

Using the fact that $n_1 \geq n_i \geq n_r$, it is easily seen that

$$n_i(n_i - n_r) \leq n_1(n_1 - n_r).$$

Hence, using (6), we see that

$$\Delta(G)\_\mathrm{def}(H^*) \leq n_1(n_1 - n_r) + n_1(|V(G)| - n_1) = n_1(|V(G)| - n_r) = n_1\Delta(G).$$

Therefore, H^* satisfies all the conditions of Lemma 5 and hence Theorem 5 is proved. ■

Proof of Theorem 2. This follows immediately from Theorems 4 and 5.

ACKNOWLEDGMENT

The authors are grateful to the anonymous referees for their comments on the first version of this article.

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