Productive public expenditures, expectation formations and nonlinear dynamics

Hung-Ju Chen [a,*], Ming-Chia Li [b,1]

[a] *Department of Economics, National Taiwan University, 21 Hsu-Chow Road, Taipei 100, Taiwan*
[b] *Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan*

Abstract

In this paper, we investigate the dynamic properties of an overlapping generations' model with capital accumulation and publicly funded inventions under three different expectations: perfect foresight, myopic expectations and adaptive expectations. We show that considering productive public expenditures in the model will increase the dimension of the dynamical system. To study the dynamic behavior of a high-dimensional dynamical system, we focus on the case when the elasticity of publicly funded invention to output is small and approximate the system by using a one-dimensional dynamical system. This approximation method provides an efficient way to rigorously prove the existence of chaos in high-dimensional dynamical systems. We show that when agents are perfectly foresighted, there exists a unique, nontrivial steady state which is a global attractor. Cycles or even chaos may occur under myopic and adaptive expectations when the inter-temporal elasticity of substitution of consumption is large enough. Furthermore, we find that the impact of fiscal policy is sensible to the expectation formation.
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1. Introduction

There have been growing interests in studying how expectation formations affect the economic behavior in inter-temporal macroeconomic models. Traditional models usually assume

* Corresponding author. Tel.: +866 2 23519641x457; fax: +866 2 23511826.
 E-mail addresses: hjc@ntu.edu.tw (H.-J. Chen), mcli@math.nctu.edu.tw (M.-C. Li).
[1] Tel.: +866 3 5712121x56463; fax: +866 3 5131223.

that agents are perfectly foresighted. However, this assumption does not seem very reasonable in reality. For example, agents would construct their expectations of the future based on current or past information, or simply nothing. We say that agents are myopic if they expect the future situation to be the same as the current situation. Another example is that agents are able to learn from the past experiences and there is a learning process for them when developing their expectations. This is referred as adaptive expectations.

It is illustrated by several studies that complex dynamics or chaos can easily arise in overlapping generations (OLG) models with different expectation formations. The occurrence of multiple equilibria in an OLG model with capital accumulation and perfect foresight has been shown by Galor and Ryder (1989). Benhabib and Day (1982) study an OLG model with myopic expectations and credit expansion. The dynamic transitions in an OLG model under perfect foresight and myopic expectations are compared by Michel and de la Croix (2000). They find that only when the myopic dynamics is monotonic that a study of the myopic dynamics can characterize the perfect foresight dynamics. If the myopic dynamics is not monotonic, cyclical oscillations and chaotic dynamics will appear under myopic expectations. In this situation, economic transitions under myopic and perfect foresight are very different.

The concept of adaptive expectations have been widely used in macroeconomic models.[2] Among these studies, Grandmont (1985) adopts an OLG model without capital accumulation, but with elastic labor supply, to analyze dynamic properties under adaptive expectations. He shows that chaotic motion can arise when the curvature of the utility function is large enough to cause strong income effects. Longo and Valori (2001) develop an OLG model with different expectations to study local properties around steady states and derive stability conditions in each case. Global properties of dynamics in a standard OLG model with different expectations are studied by Chen et al. (in press). It is shown that the dynamics is simple under perfect foresight. However, dynamics will become complicated under myopic and adaptive expectations if the inter-temporal elasticity of substitution of consumption is large enough.

Conventional studies of macroeconomics usually assume that government expenditures are non-productive. The idea of productive public expenditures is introduced into an endogenous-growth model by Barro (1990). The consideration of productive public expenditures will change the returns to scale in a production function of reproducible factors. Following this idea, Antinolfi et al. (2001) develop an OLG model with productive public expenditures and study how the dynamic behavior of the economy is affected by the degree of returns to scale. Their analysis shows that constant returns to reproducible factors is a bifurcation point. However, in order to simplify the analysis, a log-linear utility function is assumed in their study. Unfortunately, this assumption will exclude the role of expectations in determining the dynamic behavior of the economy since the saving function is independent of the expected real interest rate under a log-linear utility function.

In this paper, we construct an OLG model with capital accumulation and publicly funded inventions to study the dynamic properties and the possibility of chaotic motion under perfect foresight, myopic expectations and adaptive expectations. Unlike Antinolfi et al. (2001) who adopt a log-linear utility function, we use a CES utility function in order to study the role of expectation formations in determining the dynamic behavior of the economy. We assume that households need to pay the income tax to the government and the tax revenue is used to support inventive activities (such as constructions, infrastructures and R&D) which are beneficial to the

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[2] See Grandmont (1985, 1998), Chiarella (1988), Hommes (1994, 2002), Brock and Hommes (1997), Longo and Valori (2001) and Chiarella and He (2003).

private production. We show that the inclusion of productive public expenditures will increase the dimension of the dynamical system by one and this will in turn increase the difficulties of the analysis. The dynamic transition of the economy can be represented by two-dimensional dynamical systems (a second-order difference equation) under perfect foresight and myopic expectations. Under adaptive expectations, the dynamical system representing the dynamic behavior of the economy becomes three dimensional (a three-dimensional dynamical system).

Our study indicates that economic dynamics is qualitatively different under different expectation formations of the real interest rate. We find that when agents are perfectly foresighted, there exists a unique, nontrivial steady state which is a global attractor. Dynamics can be rather complex under the other two types of expectation formations within a reasonable range of the tax rate. Furthermore, we show that the inter-temporal elasticity of substitution is an important determinant to the complexity of economic dynamics under myopic and adaptive expectations. A sufficiently large inter-temporal elasticity of substitution will induce cycles or even chaotic motion. Under adaptive expectations, the complexity of dynamics also depends on the learning process of expectation formations. Cycles or chaotic motion will emerge as agents rely more heavily on the current information when forming their expectations. That is, the chaotic dynamics under adaptive expectations is inherited from the chaotic dynamics under myopic expectations. We also show that under myopic and adaptive expectations, the tax rate will affect the dynamic behavior and a sufficiently high tax rate can prevent the economy from chaotic dynamics.

This paper also contributes to the literature on complex dynamics by providing a technique to study the dynamic behavior in a high-dimensional dynamical system in the methodological point of view.[3] Even for a one-dimensional dynamical system, the study of global properties of complex dynamic is not easy. Hence, instead of providing rigorous proofs, most studies of complex dynamics focus only on numerical experiments. The difficulties of examination and tractability of the dynamic behavior under high-dimensional dynamical systems confine researchers to limit their analysis in one-dimensional dynamical systems and there are very few papers studying economic models generating complex dynamics under high-dimensional dynamical systems.

Recently, with more advanced mathematical methods and computer techniques developed, some economists start to explore the possibility of complicated dynamics in two-dimensional dynamical systems by showing either numerical simulation results or solid proofs. When analyzing the dynamic behavior of a one-dimensional dynamical system in economics, the most widely used type of chaos is Li–Yorke chaos due to its easiness of verification. However, the Li–Yorke theorem works only for one-dimensional dynamical systems and we need to use a different type of chaos to study the dynamic behavior for high-dimensional dynamical systems.

To prove the existence of chaotic dynamics in high-dimensional systems is a nontrivial task. Medio and Negroni (1996) provide interesting simulation results for the occurrence of chaos in two-dimensional OLG models with various settings. A computer-assisted proof generating homoclinic bifurcation associated with the stable and unstable manifolds of an autarkic steady state in de Vilder (1996) shows that a two-dimensional OLG model with elastic labor supply and Leontief technology can exhibit chaotic dynamics. By locating a so-called Smale's horseshoe in the phase space, a geometric method is used by Feichtinger et al. (1997) to show that a two-dimensional "addiction" model can generate erratic and

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[3] Here we refer a difference equation with the order greater than or equal to 2 as a high-dimensional dynamical system.

unpredictable consumption patterns of the addicted persons. Chen and Li (2008) also use the Smale horseshoe method to show that a two-dimensional cash-in-advance model with persistent habits of both cash-goods and credit-goods consumptions contains chaotic structure. Yokoo (2000) offers a singular perturbation method to prove that a Diamond type two-dimensional OLG model with government debt can exhibit complicated dynamics assured by the presence of a transverse homoclinic point associated to a golden rule steady state. For the existence of the transverse homoclinic point in a two-dimensional system, one needs to first find a transverse homoclinic point for its reduced singular (i.e., one-dimensional) system and then slightly perturb it into the corresponding nonsingular two-dimensional system without destroying the transverse homoclinic point. Onozaki et al. (2003) also use the singular perturbation method to show that a nonlinear cobweb model with adaptive production adjustment will exhibit chaotic dynamics if the singular one-dimensional mapping has negative Schwarzian derivatives and has a homoclinic point. Furthermore, both Yokoo (2000) and Onozaki et al. (2003) also prove the existence of a strange attractor (i.e., observable chaos) due to homoclinic bifurcation.

We choose entropic chaos for our analysis of a high-dimensional dynamical system in this paper. Our goal is to study the stability property of the equilibria if the dynamics is simple and to provide sufficient conditions (with rigorous proofs) for the occurrence of entropic chaos if the dynamics is complex. To achieve this goal, we first reduce the dimension of the dynamical system to one by considering the case when public expenditures are non-productive. We then extend our analysis to the case when the elasticity of publicly funded invention to output is small and apply the perturbation method recently developed by Li and Malkin (2006) and Juang et al. (2005) to approximate the high-dimensional dynamical system by perturbing the one-dimensional dynamical system and verify the existence of entropic chaos (positive topological entropy) under certain conditions. That is, we propose an efficient method to rigorously prove the exhibition of chaos for high-dimensional systems. For the generalized version of a high-dimensional OLG model with productive public expenditures, we only need to examine the occurrence of Li–Yorke ordering (Eq. (19)) or Mitra's ordering (Eq. (20)) for a reduced one-dimensional system, then apply a multi-dimensional perturbation result of Juang et al. (2005) and Li and Malkin (2006) to verify the occurrence of chaotic dynamics in high-dimensional systems under adaptive or myopic expectations. Our methodology can be easily applied to other economic and social scientific models which are high dimensional; moreover, it is free from the constraint of invertibility of the system, which sometimes becomes an obstruction for researchers in proving chaos for high-dimensional dynamical systems.

The remainder of the paper is organized as follows. The next section develops an OLG model with capital accumulation and publicly funded inventions. In Section 3, we consider an economy under perfect foresight and study its dynamic property. In Section 4, we modify the model by assuming that agents have either adaptive expectations or myopic expectations, and show that cycles and chaotic dynamics can appear under certain conditions. The final section concludes our study. Proofs of Proposition 1, Lemma 1, and Theorem 1 are given in the Appendices.

2. The model

We consider an infinite-horizon, discrete time overlapping generations' (OLG) model. Agents live for two periods, each period covering approximately thirty years, corresponding to young agents and old agents. There is no population growth and we normalize the population size to 1. Agents have identical preference and only care about their consumptions in each period. For agents born in period t, we use c_{1t} and c_{2t+1} to denote their consumptions in the first period and

in the second period of life, respectively. We consider a CES utility function:

$$U(c_{1t}, c_{2t+1}) = \begin{cases} \dfrac{c_{1t}^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} + \beta \dfrac{c_{2t+1}^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}, & \text{if } \sigma \neq 1, \\ \log c_{1t} + \beta \log c_{2t+1}, & \text{if } \sigma = 1, \end{cases} \tag{1}$$

where $\sigma > 0$ is the inter-temporal elasticity of substitution of consumption and $\beta \in (0, 1)$ is the discount factor. In this paper, we assume that $\sigma \neq 1$ so that the expectation of the real interest rate can affect the saving decisions of agents.

Each agent is endowed with one unit of time in each period. Young agents use all the time for work to earn the real wage rates (w_t) for consumptions and savings (s_t). When agents become old, they spend all the time for leisure and consume their savings from the previous period. Households need to pay the income tax to the government. We use τ to represent the constant tax rate. The budget constraints for young and old agents are:

$$c_{1t} + s_t \leq (1 - \tau)w_t, \tag{2}$$

$$c_{2t+1} \leq R_{t+1}^e s_t, \tag{3}$$

where R_{t+1}^e represents the expected real interest rate.

The young generation will maximize Eq. (1) subject to Eqs. (2) and (3), together with:

$$c_{1t} \geq 0 \quad \text{and} \quad c_{2t+1} \geq 0.$$

The optimal saving decision given by the young agents is:

$$s_t(w_t, R_{t+1}^e) = \frac{(1 - \tau)w_t}{1 + \beta^{-\sigma}(R_{t+1}^e)^{1-\sigma}}. \tag{4}$$

Eq. (4) shows that when considering a CES utility function with $\sigma \neq 1$, the expectation of the real interest rate will affect the savings of young people. Note that in the study of Antinolfi et al. (2001), they adopt a log-linear preference. Although doing so can simplify the analysis, it ignores the impact of the expected real interest rate on savings because with a log-linear preference, saving function is independent of the expected real interest rate. In the following analysis, we will show that the expected real interest rate plays an important role in determining the dynamic behavior of the economy.

We assume that technology progress is possible. However, unlike studies of R&D models which assume that technological progress occurs due to the monopoly power of firms, we assume that technological inventions are supported by the government. Following Barro (1990), we assume that government runs a balanced budget. In each period, all the tax revenue ($T_t = \tau w_t$) is used to provide public services to improve the technology of private production.[4] These public services include any institution and infrastructure which are beneficial to the productivity of output. The level of technology can be transferred from the old generation to the young generation.[5] However, this transfer may not be complete and we use $\rho \in [0, 1]$ to denote

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[4] This is referred to as productive public expenditures by Barro (1990) and as publicly funded invention by Antinolfi et al. (2001).

[5] See Shell (1966) and Jovanovic and Nyarko (1995). Similar idea can be found in Romer (1990) who argues that there should be an accumulation of "knowledge" for production by stating that "an engineer working today is more productive because he or she can take advantage of all additional knowledge accumulated as design problems which were solved during the last 100 years".

the depreciation rate of knowledge of technology during the transferring process. Using A_t to represent the publicly funded invention, the law of motion of technology is:

$$A_{t+1} = T_t + (1 - \rho)A_t. \tag{5}$$

The production function is:

$$Y_t = g(A_t)f(K_t, L_t),$$

where K_t is the capital and L_t is the labor used for production in period t. The function $g(A_t)$ represents the effects of publicly funded invention (A_t) on the production of output. Following Antinolfi et al. (2001), we assume that the production function is constant returns to scale in K_t and L_t and is homogeneous of degree λ in A_t and K_t.

To be more precise, we assume that $f(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$ exhibits a Cobb–Douglas function with $\alpha \in (0, 1)$ representing the capital share to output. We also assume that $g(A_t) = A_t^{\lambda-\alpha}$. Hence, we consider the following specific production function:

$$Y_t = A_t^{\lambda-\alpha} K_t^\alpha L_t^{1-\alpha}.$$

To reflect the situation that public services are not harmful for production, we assume that $\lambda \geq \alpha$. Note that the difference between λ and α is the elasticity of publicly funded invention to output and represents how efficiently the tax revenue is used by the government. The larger the ($\lambda - \alpha$) is, the more efficiently the tax revenue is used. In an extreme case where $\lambda = \alpha$, the publicly funded invention is not beneficial to the private production and the tax revenue is wasted.

The factor prices are:

$$w_t = (1 - \alpha)A_t^{\lambda-\alpha}k_t^\alpha, \tag{6}$$

$$R_t = \alpha A_t^{\lambda-\alpha}k_t^{\alpha-1}, \tag{7}$$

where R_t represents the real interest rate and k_t is the capital per worker.

The capital market clearing condition implies that $k_{t+1} = s_t(w_t, R_{t+1}^e)$. Thus, the dynamic behavior of the economy is characterized by the laws of motion of the level of technology and capital per worker:

$$A_{t+1} = \tau w_t + (1 - \rho)A_t, \tag{8}$$

$$k_{t+1} = \frac{(1 - \tau)w_t}{[1 + \beta^{-\sigma}(R_{t+1}^e)^{1-\sigma}]}. \tag{9}$$

Eq. (9) indicates that the dynamic transition of the economy depends on the expectation of the real interest rate. Most studies of inter-temporal optimization assume that households are perfectly foresighted. However, in reality, households would be myopic or have a learning process for the expectation formations. Hence, in the following sections, we will study the dynamic behavior under three different formations of the expected real interest rate.

As we will see later, the dimension of the dynamical system composed by Eqs. (8) and (9) is higher than 1. An analysis of a high-dimensional dynamical system is much more complicated than the analysis of a one-dimensional dynamical system.[6] The methodology of our analysis is

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[6] The difficulties for the analysis of a high-dimensional dynamical system have also been demonstrated by Antinolfi et al. (2001). Hence, in order to give a formal bifurcation analysis, several assumptions like log-linear preferences and a complete depreciation of the knowledge about technology ($\rho = 1$) are made to simplify the model.

that we first consider a special case of $\lambda = \alpha$. Under this situation, the law of motion of capital per worker (Eq. (9)) is independent of the technological progress. Hence, the dynamic transition of the economy is governed by the difference equation of capital per worker and we can reduce the order of the dynamical system. We then extend our analysis to the case where λ is close to α. In the whole paper, we focus our analysis on the case where λ is close to α and study the dynamic behavior under different expectations.

3. Perfect foresight

In this section, we study the economy where agents are perfectly foresighted. That is,

$$R^e_{t+1} = R_{t+1}.$$

Using Eqs. (6) and (7) to substitute factor prices in Eqs. (8) and (9), the dynamical system representing the economic transition is:

$$A_{t+1} = \tau(1-\alpha)A_t^{\lambda-\alpha}k_t^\alpha + (1-\rho)A_t, \tag{10}$$

$$k_{t+1} = s_t(w_t, R_{t+1}) = \frac{(1-\tau)(1-\alpha)A_t^{\lambda-\alpha}k_t^\alpha}{1 + \beta^{-\sigma}(\alpha A_{t+1}^{\lambda-\alpha}k_{t+1}^{\alpha-1})^{1-\sigma}}. \tag{11}$$

Eqs. (10) and (11) indicate that the economic transition is composed by a two-dimensional dynamical system.

In order to study the global dynamics of the two-dimensional system, we start our analysis for the case when public expenditures are non-productive (that is, $\lambda = \alpha$). Under this circumstance, Eq. (11) is independent of A_t and the order of the dynamical system is reduced to 1. The dynamic transition of the economy is governed by the difference equation of order one in k_t and k_{t+1}:

$$k_{t+1}[1 + \beta^{-\sigma}(\alpha k_{t+1}^{\alpha-1})^{1-\sigma}] = (1-\tau)(1-\alpha)k_t^\alpha. \tag{12}$$

Fig. 1(i) displays the law of motion of capital per worker when $\sigma = 10$. It indicates that dynamics is simple and the nontrivial steady state is stable. Fig. 1(ii) shows that similar to the case of $\sigma = 10$, dynamics of the system $k_t \longmapsto k_{t+1}$ in Eq. (12) with different values of σ ($\sigma = 0.3$ and 3) is also simple and the nontrivial steady state is stable in each case.

We then extend our study to the case when λ is near α and the dynamical system is two dimensional. The following proposition illustrates that there exists a unique, nontrivial steady state which is a global attractor under perfect foresight when λ is near α.

Proposition 1. *If λ is near α, then the economy under perfect foresight, i.e., the dynamics $(A_t, k_t) \longmapsto (A_{t+1}, k_{t+1})$ in Eqs. (10) and (11), has a unique nonzero steady state which is globally attracting.*

Proof. See Appendix A. □

4. Adaptive and myopic expectations

Under adaptive expectation, there is a learning process for households to construct their expectations. To be more specific, we consider a first-order autoregressive adaptive expectation of the real interest rate. That is, R^e_{t+1} is constructed based on current and past information with decreasing weights. With $\mu \in (0, 1]$, R^e_{t+1} can be written as
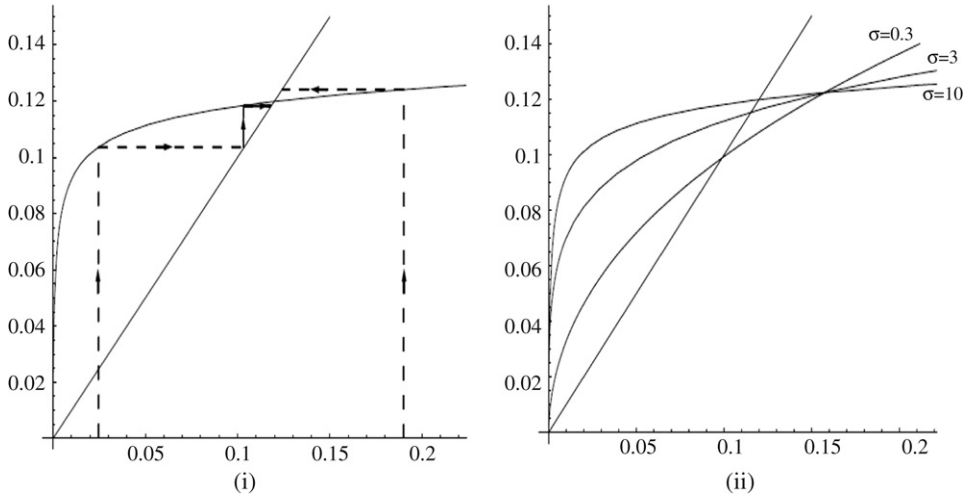
Fig. 1. The graphs of the diagonal line and the system $k_t \longmapsto k_{t+1}$ in Eq. (12) with $\alpha = 1/3$, $\beta = 0.99^{30}$, $\lambda = \alpha$, and $\tau = 0.2$, and (i) $\sigma = 10$, (ii) $\sigma = 0.3, 3, 10$.

$$R^e_{t+1} = \sum_{i=0}^{\infty} \mu(1-\mu)^i R_{t-i} = (1-\mu)R^e_t + \mu R_t. \tag{13}$$

Eq. (13) shows that R^e_{t+1} can be represented by R^e_t and R_t with weights $(1-\mu)$ and μ, respectively. An increase in μ implies that agents depend more heavily on the current information (R_t) when forming their expectations. Note that when $\mu = 1$, $R^e_{t+1} = R_t$ and agents are myopic since they completely rely on current information to form their expectations. Substituting Eq. (13) into Eq. (9), the law of motion of capital per worker can be rewritten as:

$$(1-\tau)(1-\alpha)A^{\lambda-\alpha}_{t+1}k^{\alpha}_{t+1} = k_{t+2}\left\{1 + \beta^{-\sigma}\left[\mu\alpha A^{\lambda-\alpha}_{t+1}k^{\alpha-1}_{t+1} + (1-\mu)\beta^{\frac{\sigma}{1-\sigma}}\right.\right.$$

$$\left.\left. \times \left(\frac{(1-\tau)(1-\alpha)A^{\lambda-\alpha}_t k^{\alpha}_t}{k_{t+1}} - 1\right)^{\frac{1}{1-\sigma}}\right]^{1-\sigma}\right\}. \tag{14}$$

Therefore, dynamic transition of the economy can be expressed by Eq. (10) and (14).

From Eq. (10), we can derive that:

$$k_t = \left[\frac{A_{t+1} - (1-\rho)A_t}{\tau(1-\alpha)A^{\lambda-\alpha}_t}\right]^{\frac{1}{\alpha}}. \tag{15}$$

Substituting Eq. (15) for terms t, $t+1$, and $t+2$ into Eq. (14), we can obtain the difference equation of order three in A_t, A_{t+1}, A_{t+2}, and A_{t+3} as follows:

$$\frac{(1-\tau)}{\tau}\left[A_{t+2} - (1-\rho)A_{t+1}\right]$$

$$
= \left(\frac{A_{t+3} - (1-\rho)A_{t+2}}{\tau(1-\alpha)A_{t+2}^{\lambda-\alpha}} \right)^{\frac{1}{\alpha}} \left\{ 1 + \beta^{-\sigma} \left[\mu\alpha \left(\frac{1}{\tau(1-\alpha)} \right)^{\frac{\alpha-1}{\alpha}} \right. \right.
$$

$$
\times \left[A_{t+2} - (1-\rho)A_{t+1} \right]^{\frac{\alpha-1}{\alpha}} A_{t+1}^{\frac{\lambda-\alpha}{\alpha}} + (1-\mu)\beta^{\frac{\sigma}{1-\sigma}}
$$

$$
\times \left. \left. \left(\frac{(1-\tau)}{\tau} \left[A_{t+1} - (1-\rho)A_t \right]^{\frac{\alpha-1}{\alpha}} \left[\tau(1-\alpha)A_t^{\lambda-\alpha} \right]^{\frac{1}{\alpha}} \right)^{\frac{1}{1-\sigma}} \right]^{1-\sigma} \right\} . \tag{16}
$$

Hence, the economic dynamics is represented by a three-dimensional dynamical system.

To examine the possibility of entropic chaos (see Definition 1) in the three-dimensional system, we first check Mitra's ordering (Eq. (20)) for a reduced one-dimensional system which approximates the three-dimensional system, then apply a multi-dimensional perturbation result of Juang et al. (2005) and Li and Malkin (2006) to obtain the existence of entropic chaos for the three-dimensional system. To reduce the three-dimensional system represented by Eq. (16) to a one-dimensional system, we consider the case when $\lambda = \alpha$, $\rho = 1$ and $\mu = 1$. Then Eq. (16) can be written as the following equation:

$$
A_{t+3} = f_\sigma(A_{t+2}), \tag{17}
$$

where $f_\sigma : [0, \infty) \to [0, \infty)$ is a one-parameter family of functions defined by

$$
f_\sigma(x) = \frac{Gx^\alpha}{\left[1 + \beta^{-1} \left(\frac{x}{H} \right)^{\frac{(1-\alpha)(\sigma-1)}{\alpha}} \right]^\alpha}, \quad \text{where} \begin{cases} G = \tau^{1-\alpha}(1-\alpha)(1-\tau)^\alpha, \\ H = \tau(1-\alpha)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}, \end{cases} \tag{18}
$$

with parameter σ.

When complex dynamics appears, even the analysis of a one-dimensional dynamical system is not easy. Because of the convenience of verification, the most popular type of chaotic motions studied in one-dimensional dynamical systems is the Li–Yorke chaos.[7] The Li–Yorke Theorem (Li and Yorke (1975), Theorem 1) says that for any continuous interval map h, if there is a point p such that

$$
h^3(p) < p < h(p) < h^2(p), \tag{19}
$$

then h exhibits Li–Yorke chaos, where h^s denote the composition of s copies of h. The Proposition 2.3 of Mitra (2001) demonstrates that for any unimodal map h, if there exist a modal point x_c and a fixed point $\bar{x}$ satisfying

$$
h^2(x_c) < x_c \quad \text{and} \quad h^3(x_c) < \bar{x}, \tag{20}
$$

then h^2 exhibits Li–Yorke chaos.

For the reduced one-dimensional system f_σ in Eq. (18), Fig. 2(i) indicates that the inequalities of Eq. (20) are satisfied for f_σ with $\sigma = 10$, so it exhibits Li–Yorke chaos. Fig. 2(ii) shows that

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[7] Definition of *Li–Yorke chaos:* A map $h : I \to I$ is said to exhibit *Li–Yorke chaos* on an interval I if the following conditions are satisfied: (A) h has periodic points of all periods; here by a *periodic point p of period t*, we mean that $h^t(p) = p$ and $h^i(p) \neq p$ for $0 < i < t$; (B) there exists an uncountable set $S \subset I$ such that (B1) if $x, y \in S$ with $x \neq y$ then $\lim_{t\to\infty} \sup |h^t(x) - h^t(y)| > 0$ and $\lim_{t\to\infty} \inf |h^t(x) - h^t(y)| = 0$; (B2) if $x \in S$ and $y \in I$ is periodic then $\lim_{t\to\infty} \sup |h^t(x) - h^t(y)| > 0$.
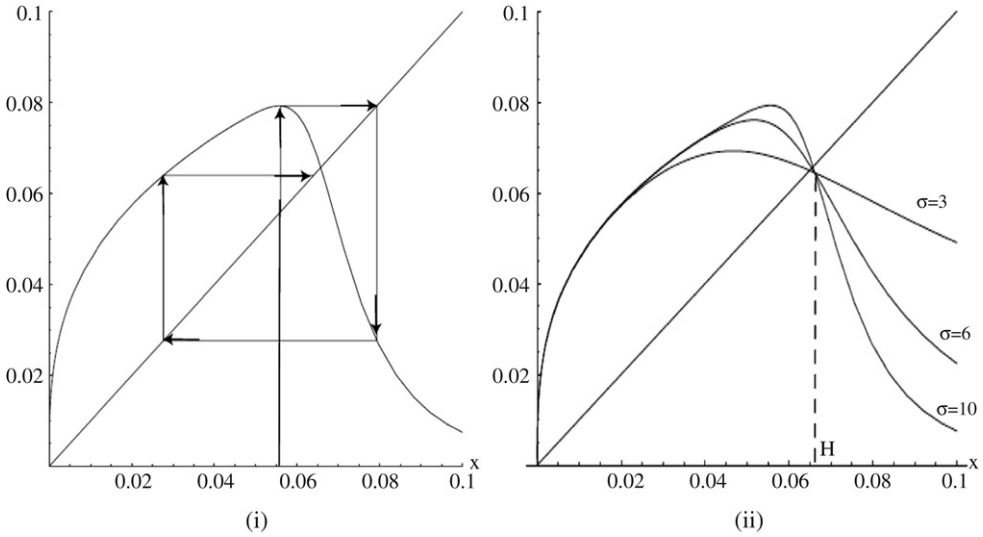
Fig. 2. (i) The graphs of $y = f_{10}(x)$ and $y = x$, and the first three iterates of an initial point under f_{10}; (ii) the graphs of $f_\sigma(x)$ with $\sigma = 3, 6, 10$ and $\alpha = 1/3$, $\beta = 0.99^{30}$, and $\tau = 0.2$.

for different values of σ, the function f_σ attains the same value at a particular point which is $H = \tau(1 - \alpha)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}$. That is, for any $\sigma_1, \sigma_2 > 0$, we have that $f_{\sigma_1}(H) = f_{\sigma_2}(H)$. This fact will play a role in the proof of the following lemma, which gives a sufficient condition of the occurrence of chaotic dynamics for the second iterate of f_σ.

Lemma 1. *Let f_σ be as in Eq.* (18). *If $1 - \frac{\alpha(1+\beta)}{1-\alpha} < \tau < 1 - \frac{\alpha\beta}{1-\alpha}$, then for all large σ, the map f_σ^2 exhibits Li–Yorke chaos.*

Proof. See Appendix B. $\square$

We have shown that chaotic dynamics in the sense of Li and Yorke can occur when σ is large enough in the special case which is one dimensional. We then turn to examine the possibility of chaotic motion in a more general case represented by Eq. (16) which is three dimensional. Since the Li–Yorke theorem works only for one-dimensional dynamical systems, we study the possibility of entropic chaos for the high-dimensional case represented by Eq. (16). Note that in one-dimensional dynamical systems, the existence of Li–Yorke chaos also implies the presence of entropic chaos. We consider the Li–Yorke chaos instead of entropic chaos in the special case because of its easiness of verification. One may also use a criterion for entropic chaos in the special case but doing so will complicate the analysis.

The definitions of topological entropy and entropic chaos are given as follows; refer to Robinson (1999):

Definition 1. Let $g : X \to X$ be a continuous map on a space X with metric d. For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $S \subset X$ is called an (n, ϵ)-separated set for g if for every pair of points $x, y \in S$ with $x \neq y$, there exists an integer k with $0 \leq k < n$ such that $d(g^k(x), g^k(y)) > \epsilon$. The topological entropy of g is defined to be

$$h_{\text{top}}(g|X)$$

$$= \lim_{\epsilon \to 0, \epsilon > 0} \limsup_{n \to \infty} \frac{\log(\max\{\#(S) : S \subset X \text{is an } (n, \epsilon) - \text{separated set for } g\})}{n},$$

where $\#(S)$ is the cardinality of elements of S.

We say that g has entropic chaos on X if $h_{\text{top}}(g|X) > 0$.

Topological entropy has played a fundamental role in the theory of chaos. Its concept is a mathematical formulation of exponential divergence of nearby initial conditions. It describes the total exponential complexity of the orbit structure with a single number in a rough but expressive way. The topological entropy is positive for chaotic systems and is zero for non-chaotic systems. For any continuous interval map h, one has the relation between Li–Yorke chaos and entropic chaos, due to Misiurewicz (1979), that h must exhibit entropic chaos if h^t exhibits Li–Yorke chaos for some t. Therefore, Lemma 1 implies that the reduced one-dimensional system (18) exhibits entropic chaos.

We then approximate the three-dimensional dynamical system by using the one-dimensional dynamical system. In the following theorem, we give a sufficient condition for the existence of entropic chaos under adaptive expectations for λ near α, ρ near 1, and μ close to 1.

Theorem 1. *Let* $1 - \frac{\alpha(1+\beta)}{1-\alpha} < \tau < 1 - \frac{\alpha\beta}{1-\alpha}$. *If* σ *is sufficiently large, then for all* λ *near* α, ρ *near 1 and* μ *near 1, the economy under adaptive expectations, i.e., Eqs.* (10) *and* (14)*, exhibits entropic chaos.*

Proof. See Appendix C.   □

Theorem 1 gives a sufficient condition of the presence of chaotic motion. It shows that within a reasonable range of the tax rate, entropic chaos will occur if the inter-temporal elasticity of substitution is large enough or if the weight of the current information is large enough in the expectation formation.[8] Fig. 3 presents the bifurcation diagrams of A_t and k_t with varying σ when $\mu = 0.95$, $\lambda - \alpha = 0.05$ and $\rho = 0.95$. From Fig. 3, we can see that an increase of σ will induce deterministic cycles and the system will get into the region of chaotic motion when σ is large enough.

Fig. 4 shows the bifurcation diagrams of A_t and k_t with varying value of μ given the same parameter values of μ, $\lambda - \alpha$ and ρ as those in Fig. 3. It displays that if μ is small, there is a unique positive steady state. As μ increases, agents will put a higher weight on the current real interest rate when forming their expectations and cycles and complex dynamics start to emerge with an increase in μ. A sufficiently large value of μ will cause chaos.

When $\mu = 1$, agents are myopic and the dynamical system becomes two dimensional. In fact, the requirement of a sufficiently large value of μ implies that the chaotic motion of the dynamical system under adaptive expectations is inherited from the chaotic behavior of the dynamical system under myopic expectations. On the other hand, an increase in the weight of previous expectations $(1 - \mu)$ in the expectation formation can help reduce the complexity of the dynamic motion from chaos to cyclical movements, or even to simple dynamics.[9] When

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[8] For example, when $\alpha = \frac{1}{3}$ and $\beta = 0.99^{30}$, the range of τ for the occurrence of chaos in Theorem 1 is between 13.01% and 63.01% if $(\lambda - \alpha)$, $(1 - \rho)$ and $(1 - \mu)$ are sufficiently small.

[9] Notice that this result in an OLG model differs from the findings in the cobweb models where there is only simple dynamics under myopic expectations while chaotic dynamics may appear under adaptive expectations (see Chiarella (1988) and Hommes (1994)).
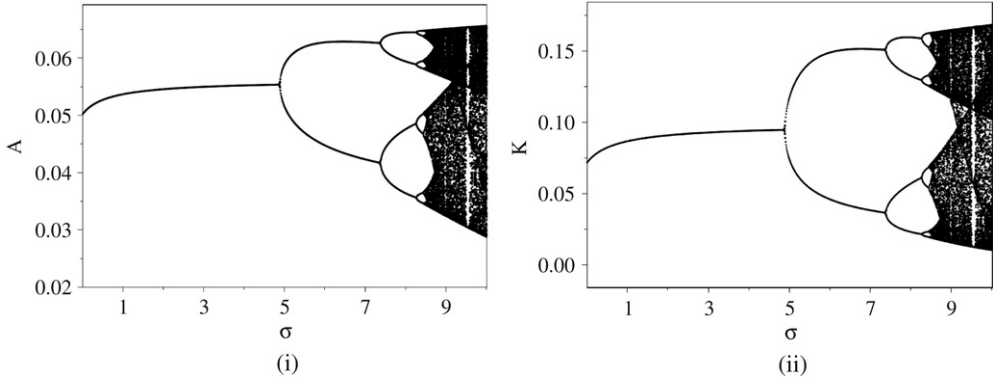
Fig. 3. The bifurcation diagram on varying σ with $\alpha = 1/3, \beta = 0.99^{30}, \lambda = \alpha + 0.05, \rho = 0.95, \tau = 0.2$, and $\mu = 0.95$ for (i) A_t and (ii) K_t.
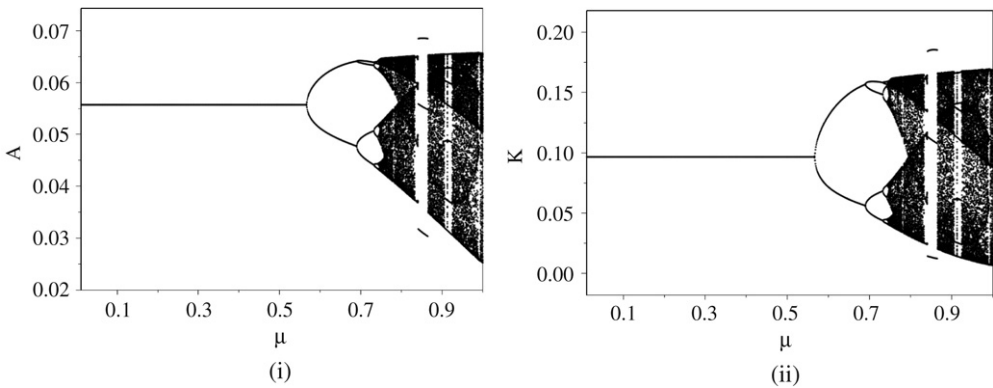

Fig. 4. The bifurcation diagram on varying μ with $\alpha = 1/3, \beta = 0.99^{30}, \lambda = \alpha + 0.05, \rho = 0.95, \tau = 0.2$, and $\sigma = 10$ for (i) A_t and (ii) K_t.

the system becomes myopic dynamics, an immediate consequence of Theorem 1 is given in the following corollary:

Corollary 1. *Let* $1 - \frac{\alpha(1+\beta)}{1-\alpha} < \tau < 1 - \frac{\alpha\beta}{1-\alpha}$. *If* σ *is sufficiently large, then for all* λ *near* α *and* ρ *near 1, the economy under myopic expectations exhibits entropic chaos.*

Corollary 1 provides a sufficient condition for the presence of ergodic chaos under myopic expectations and it demonstrates the important role of the inter-temporal elasticity of substitution in determining the existence of complex dynamics.

Theorem 1 also implies that the tax rate can affect the complexity of the dynamic behavior. To examine the effects of the tax rate on dynamic properties of the economy, the bifurcation diagrams of k_t with varying τ under adaptive expectations and myopic expectations are respectively given in Fig. 5(i) and (ii). Fig. 5(i) shows that under adaptive expectations, chaotic motion and cycles appear when τ is (roughly) between 10% and 35%. On the other hand, Fig. 5(ii) shows that chaotic motion or cycles will occur if τ is (roughly) less than 33%. These two figures indicate that the occurrence of chaotic motion is sensible to the expectation formation (or the value of μ). Comparing Fig. 5(i) with Fig. 5(ii), we find that under myopic
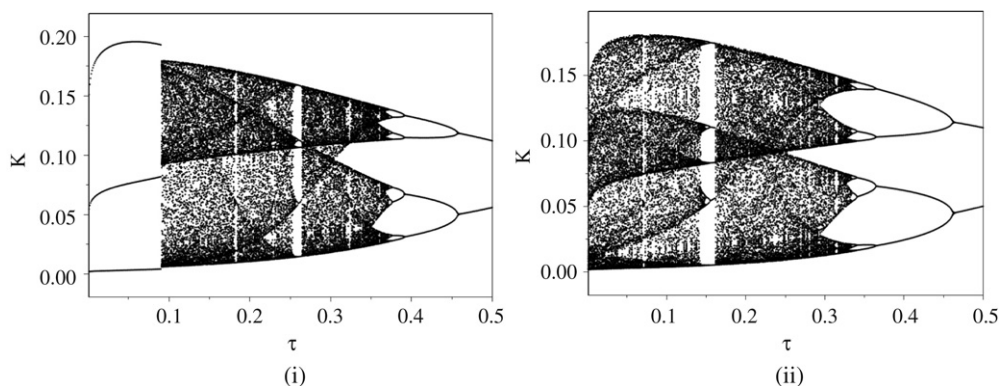
Fig. 5. The bifurcation diagram for k_t on varying τ with $\alpha = 1/3$, $\beta = 0.99^{30}$, $\lambda = \alpha + 0.05$, $\rho = 0.95$, and $\sigma = 10$ for (i) adaptive expectations with $\mu = 0.95$, and (ii) myopic expectations.

expectations, government can avoid chaotic dynamics only by setting the tax rate to a sufficiently high level. However, under adaptive expectations, government can prevent the economy from chaotic dynamics either by increasing the tax rate to a sufficiently high level or by decreasing it to a sufficiently low level. This implies that the effects of fiscal policies could be quite different under myopic and adaptive expectations.

5. Conclusion

In this paper, we study the dynamic behavior in an OLG model with capital accumulation and publicly funded inventions under three different types of expectations: perfect foresight, myopic expectations and adaptive expectations. Since including the productive public expenditures into the model will increase the dimension of the dynamical system, we propose a new approach to study the dynamic properties of high-dimensional dynamical systems generated by economic models. We first reduce the dimension of the dynamical system by assuming that public expenditures are not productive ($\lambda = \alpha$). Then we expand our analysis to the case where λ is close to α and approximate this high-dimensional dynamical system ($\lambda > \alpha$) by using the low-dimensional dynamical system ($\lambda = \alpha$). In the literature of economic dynamics, there are very few papers studying the global property of a high-dimensional dynamical system due to its difficulties of examination and tractability. With this new technique, economists can explore many other interesting issues without restricting their models to one-dimensional dynamical systems.

Our results indicate that qualitative properties of the dynamic behavior are very different under different expectations. We show that under perfect foresight, the nonlinear dynamics will contain a unique, nontrivial steady state which is a global attractor. The dynamics may become rather complex under myopic and adaptive expectations. In both cases, the complexity of the dynamics is affected by the tax rate. Under myopic expectations, cycles will emerge as the inter-temporal elasticity of substitution increases and chaotic dynamics can occur when the inter-temporal elasticity of substitution is large enough. Under adaptive expectations, both the inter-temporal elasticity of substitution and the weight of past information when forming expectations are important to determine the complexity of economic dynamics. Cycles and chaotic motion will emerge as the inter-temporal elasticity of substitution increases or as households depend more on the current information when constructing their expectations. Hence,

our work emphasizes the importance of the expectation formation when constructing inter-temporal macroeconomic models.

Acknowledgements

Appendix A

Proof of Proposition 1. First, we consider the case $\rho \neq 1$. We obtain that (A_t, k_t) is a function of (A_{t+1}, k_{t+1}) in an explicit form as follows. Multiplying Eq. (10) by $(1 - \tau)$ gives us:

$$(1 - \tau)(1 - \alpha)A_t^{\lambda-\alpha}k_t^{\alpha} = \frac{(1 - \tau)A_{t+1} - (1 - \tau)(1 - \rho)A_t}{\tau}. \tag{21}$$

By plugging Eq. (21) into Eq. (11), we get:

$$A_t = \frac{1}{(1 - \tau)(1 - \rho)}[(1 - \tau)A_{t+1} - \tau k_{t+1} - \tau k_{t+1}\beta^{-\sigma}(\alpha A_{t+1}^{\lambda-\alpha}k_{t+1}^{\alpha-1})^{1-\sigma}]. \tag{22}$$

Plugging Eq. (22) back into Eq. (10) gives us:

$$k_t = \left\{ \frac{(1 - \tau)^{\lambda-\alpha-1}(1 - \rho)^{\lambda-\alpha}[k_{t+1} + k_{t+1}\beta^{-\sigma}(\alpha A_{t+1}^{\lambda-\alpha}k_{t+1}^{\alpha-1})^{1-\sigma}]}{(1 - \alpha)[(1 - \tau)A_{t+1} - \tau k_{t+1} - \tau k_{t+1}\beta^{-\sigma}(\alpha A_{t+1}^{\lambda-\alpha}k_{t+1}^{\alpha-1})^{1-\sigma}]^{\lambda-\alpha}} \right\}^{\frac{1}{\alpha}}. \tag{23}$$

Thus the backward dynamics is governed by Eqs. (22) and (23).

In order to study the dynamic behavior for λ close to α, we shall first consider the boundary case when $\lambda = \alpha$, then apply the structural stability theorem to prove that simple dynamics occurs for λ near α. Assuming that $\lambda = \alpha$, the backward dynamics becomes:

$$A_t = \frac{1}{(1 - \tau)(1 - \rho)}[(1 - \tau)A_{t+1} - \tau k_{t+1} - \tau k_{t+1}\beta^{-\sigma}(\alpha k_{t+1}^{\alpha-1})^{1-\sigma}], \tag{24}$$

$$k_t = \left\{ \frac{1}{(1 - \tau)(1 - \alpha)}k_{t+1}\left[1 + \beta^{-1}\left(\frac{k_{t+1}^{1-\alpha}}{\alpha\beta}\right)^{\sigma-1}\right] \right\}^{\frac{1}{\alpha}}. \tag{25}$$

From Eq. (25), one can easily check that the backward subsystem $k_{t+1} \longmapsto k_t$ has a globally repelling fixed point, say $\bar{k}$. Putting $\bar{k}$ into Eq. (24), we have the globally repelling fixed point, denoted by $\bar{A}$, for the subsystem $A_{t+1} \mapsto A_t$. Thus $(\bar{A}, \bar{k})$ is the unique nonzero fixed point for Eqs. (10) and (11), and it is globally attracting for the forward dynamics. By using the structurally stable theorem (Robinson, 1999), one has the same dynamic phenomena for the systems with λ near α.

For the case when $\rho = 1$, we also use the structural stability theorem while the systems are parametrized by λ around the center α. Let $\lambda = \alpha$. Then Eq. (11) becomes Eq. (25) again and

the backward subsystem $k_{t+1} \mapsto k_t$ has a globally repelling fixed point $\bar{k}$. Putting $\bar{k}$ into Eq. (10) with $\rho = 1$, we have the globally repelling fixed point $\bar{A}$ for the subsystem $A_{t+1} \mapsto A_t$. Therefore, for all λ near α, the forward dynamics has a unique nonzero fixed point which is globally attracting. $\square$

Appendix B

Proof of Lemma 1. It is sufficient to show that the inequalities in Eq. (20) are satisfied for f_σ; indeed, Mitra's result implies that f_σ^2 exhibits Li–Yorke chaos.

First, we observe some basic properties of f_σ. Note that f_σ is a family of continuous functions with parameter $\sigma > 0$, and both G and H are independent of σ (G and H are given in Eq. (18)). It is easy to see that $f_\sigma(H) = \frac{GH^\alpha}{[1+\beta^{-1}]^\alpha}$ is constant for all σ; see Fig. 2(ii). If $x < H$ fixed, then $f_\sigma(x)$ is strictly increasing and approaches Gx^α provided that σ increases and goes to ∞, and if $x > H$ fixed, then $f_\sigma(x)$ is strictly decreasing and tends to 0 provided that σ increases and goes to ∞.

First, we prove the uniqueness of the critical point for f_σ. From elementary calculation, we have that:

$$f_\sigma'(x) = \frac{G\left\{\alpha\beta - [\sigma(1-\alpha)-1]\left(\frac{x}{H}\right)^{\frac{(1-\alpha)(\sigma-1)}{\alpha}}\right\}}{\beta x^{1-\alpha}\left[1+\beta^{-1}\left(\frac{x}{H}\right)^{\frac{(1-\alpha)(\sigma-1)}{\alpha}}\right]^{1+\alpha}}.$$

Let $\sigma > \frac{1}{1-\alpha}$. Then f_σ has a unique critical point at:

$$x_c = H\left[\frac{\alpha\beta}{\sigma(1-\alpha)-1}\right]^{\frac{\alpha}{(1-\alpha)(\sigma-1)}}$$

and f_σ is strictly increasing on $(0, x_c]$ and is strictly decreasing on $[x_c, \infty)$.

Second, we prove that f_σ has a unique nonzero steady state. Let $g(x) = f_\sigma(x)/x$. Since

$$g'(x) = -\frac{f_\sigma(x)}{x^2}\left\{\frac{1-\alpha\beta+[\sigma(1-\alpha)-1]\left(\frac{x}{H}\right)^{\frac{(1-\alpha)(\sigma-1)}{\alpha}}}{1+\beta^{-1}\left(\frac{x}{H}\right)^{\frac{(1-\alpha)(\sigma-1)}{\alpha}}}\right\} < 0,$$

for all $x > 0$, g is strictly decreasing; hence g is a one-to-one function. Because $\lim_{x\to 0} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$, as $x \in (0, \infty)$ increases the value of $g(x)$ strictly decreases. By the intermediate value theorem, there exists a unique point, namely $\bar{x}$, such that $g(\bar{x}) = 1$. Thus, $\bar{x}$ is the unique nonzero steady state for f_σ; moreover, $f_\sigma(x) > x$ for $0 < x < \bar{x}$, and $f_\sigma(x) < x$ for $x > \bar{x}$. Note that there is no explicit expression of $\bar{x}$ although it is the solution of the equation $f_\sigma(x) = x$.

Third, we decide the ordering of the critical point x_c and the steady state $\bar{x}$. By L'Hospital's rule, we have:

$$\lim_{\sigma\to\infty}(f_\sigma(x_c) - x_c)$$

$$= \lim_{\sigma\to\infty}\left\{\frac{GH^\alpha\left[\frac{\alpha\beta}{\sigma(1-\alpha)-1}\right]^{\frac{\alpha}{(1-\alpha)(\sigma-1)}}}{\left[1+\beta^{-1}\left(\frac{\alpha\beta}{\sigma(1-\alpha)-1}\right)\right]^\alpha} - H\left[\frac{\alpha\beta}{\sigma(1-\alpha)-1}\right]^{\frac{\alpha}{(1-\alpha)(\sigma-1)}}\right\}$$

$$= GH^\alpha - H$$

$$= \tau(1-\alpha)(\alpha\beta)^{\frac{\alpha^2}{(1-\alpha)}}[(1-\tau)^\alpha(1-\alpha)^\alpha - (\alpha\beta)^\alpha].$$

Since $\tau < 1 - \frac{\alpha\beta}{1-\alpha}$, we get that if σ is large then $f_\sigma(x_c) > x_c$. Hence, combining this with the conclusion about $\bar{x}$ in the previous paragraph, we have $0 < x_c < \bar{x}$. Moreover, since $\lim_{x \to 0^+} f_\sigma(x) = 0$ and f_σ is strictly increasing on $(0, x_c]$, there is a unique point x_p in $(0, x_c)$ which is a preimage $\bar{x}$ of under f_σ, i.e., $f_\sigma(x_p) = \bar{x}$.

We further decide the ordering of the critical point x_c and the steady state H. Since $\tau > 1 - \frac{\alpha(1+\beta)}{1-\alpha}$, we have that for any $\sigma > 0$:

$$f_\sigma(H) - H = \frac{GH^\alpha}{(1+\beta^{-1})^\alpha} - H$$

$$= \tau(1-\alpha)(\alpha\beta)^{\frac{\alpha^2}{(1-\alpha)}}[(1-\tau)^\alpha(1-\alpha)^\alpha - (\alpha\beta+\alpha)^\alpha] < 0.$$

Hence, $\bar{x} < H$.

It is not difficult to see that $\lim_{\sigma \to \infty}(f_\sigma(x_c) - H) = \lim_{\sigma \to \infty}(f_\sigma(x_c) - x_c)$. Thus for $\tau < 1 - \frac{\alpha\beta}{1-\alpha}$, we have that if σ is large then $H < f_\sigma(x_c)$. So far, we have shown that if $\sigma > 0$ is large, then $0 < x_p < x_c < \bar{x} < H < f_\sigma(x_c)$.

Let $\varepsilon > 0$ be so small that $\max\{\varepsilon, G\varepsilon^\alpha\} < \min\{\frac{H}{2}, x_c\}$. By the L'Hospital's rule, the critical point x_c of f_σ is strictly increasing and tends to H as σ increases and goes to ∞. Together with the facts above, we can take $\sigma > 0$ so large that $\frac{H}{2} < x_c < H$, $H < f_\sigma(x_c)$, and $f_\sigma^2(x_c) < \varepsilon$. Then $\varepsilon < x_p$; otherwise, $\bar{x} = f_\sigma(x_p) \leq f_\sigma(\varepsilon) < G\varepsilon^\alpha < \frac{H}{2} < x_c$; a contradiction. By strict monotonicity of f_σ on $(0, x_c]$, we get that $f_\sigma^3(x_c) = f_\sigma(f_\sigma^2(x_c)) < f_\sigma(\varepsilon) < f_\sigma(x_p)$.

Finally, we summarize the ordering of all the relevant values as follows:

$$0 < x_c < \bar{x} < f_\sigma(x_c), \quad 0 < f_\sigma^2(x_c) < \varepsilon < x_c \quad \text{and} \quad f_\sigma^3(x_c) < f_\sigma(\varepsilon) < f_\sigma(x_p) = \bar{x}.$$

Therefore, the inequalities in (20) with the modal point x_c are satisfied; hence f_σ^2 exhibits Li–Yorke chaos. $\square$

Appendix C

The following theorem is a simple version of Theorem 3 of Juang et al. (2005). Also refer to Theorem 3.3 of Li and Malkin (2006).

Theorem 2. *Let S be a subset of $\mathbb{R}^n$ with $n \geq 1$ and $\gamma_0 \in S$. Consider a difference equation of order m in the form*

$$\Phi_\gamma(A_t, \ldots, A_{t+m}) = 0, \quad t \geq 0, \tag{26}$$

where $\gamma \in S$ is a parameter and the real-valued function Φ_γ is defined on a cube $[B, C]^{m+1} \subset \mathbb{R}^{m+1}$ with constants $B < C$. Assume that (i) *Φ_γ is C^1 on $[B, C]^m$ for each $\gamma \in S$;* (ii) *the function $\gamma \mapsto \Phi_\gamma$ is continuous on S; and* (iii) *for $i = 1, \ldots, m+1$, the function $\gamma \mapsto \partial_i \Phi_\gamma$ is continuous on S, where $\partial_i \Phi_\gamma$ are the partial derivatives of Φ_γ with respect to the ith variable. Suppose that for $\gamma = \gamma_0$, the difference equation (26) is reduced to a difference equation of order one in the form $A_{t+m} - \varphi(A_{t+m-1}) = 0, t \geq 0$, where $\varphi : [B, C] \to \mathbb{R}$ is a C^2 function with*

positive topological entropy.[10] *Let Y_γ be the set of solutions for Eq.* (26), *i.e. the set of sequences $\underline{A} = (A_0, A_1, A_2, \ldots)$ such that for any $t \geq 0$,*

1. $A_t \in [B, C]$; *and*
2. $m + 1$ *consecutive components* $A_t, \ldots, A_{t+m}$ *of $\underline{A}$ satisfy Eq.* (26).

Let θ be the shift map on Y_γ, i.e., $\theta(\underline{A}) = \underline{A}'$, where $A'_t = A_{t+1}$ for all $t \geq 0$.

Then there exists a neighborhood U of γ_0 in S such that for any $\gamma \in U$, there is a closed θ-invariant subset Γ_γ of Y_γ in the product topology such that $h_{\text{top}}(\theta|\Gamma_\gamma) > 0$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We will apply Theorem 2 together with Lemma 1 to prove the desired result. Fix α, β and τ such that $1 - \frac{\alpha(1+\beta)}{1-\alpha} < \tau < 1 - \frac{\alpha\beta}{1-\alpha}$. Denote $\gamma_0 = (\alpha, 1, 1)$ and let $T = \{\gamma = (\lambda, \rho, \mu) : \lambda \geq \alpha, 0 \leq \rho \leq 1, 0 < \mu \leq 1\}$. Then T is a subset of $\mathbb{R}^3$ and $\gamma_0 \in T$. Let C be a positive number which is much larger than GH^α, where G and H are given in Eq. (18). Derived from Eq. (16) with $\sigma \geq 1$, for γ in a neighborhood of γ_0 in T, we define $\Phi_\gamma : [0, C]^4 \to \mathbb{R}$ by

$$\Phi_\gamma(A_t, A_{t+1}, A_{t+2}, A_{t+3})$$

$$= A_{t+3} - (1 - \rho)A_{t+2} - \frac{\tau(1-\alpha)A_{t+2}^{\lambda-\alpha}\left\{\frac{(1-\tau)}{\tau}[A_{t+2} - (1-\rho)A_{t+1}]\right\}^\alpha}{\{1 + \beta^{-\sigma}[P + Q]^{1-\sigma}\}^\alpha}, \quad (27)$$

where

$$P = \mu\alpha\left(\frac{1}{\tau(1-\alpha)}\right)^{\frac{\alpha-1}{\alpha}}[A_{t+2} - (1-\rho)A_{t+1}]^{\frac{\alpha-1}{\alpha}}A_{t+1}^{\frac{\lambda-\alpha}{\alpha}},$$

$$Q = (1-\mu)\beta^{\frac{\sigma}{1-\sigma}}\left(\frac{(1-\tau)}{\tau}[A_{t+1} - (1-\rho)A_t]^{\frac{\alpha-1}{\alpha}}[\tau(1-\alpha)A_t^{\lambda-\alpha}]^{\frac{1}{\alpha}}\right)^{\frac{1}{1-\sigma}}.$$

Thus if $\gamma = \gamma_0$ then $\Phi_\gamma(A_t, A_{t+1}, A_{t+2}, A_{t+3}) = A_{t+3} - f_\sigma(A_{t+2})$, where f_σ is given in Lemma 1. In the proof of Lemma 1, we have shown that $\lim_{\sigma \to \infty}(f_\sigma(x_c)) = GH^\alpha$, the global maximum of f_σ, where x_c is the critical point of f_σ. Thus for C larger than GH^α, if γ is in a neighborhood S of γ_0 in T, then $\Phi_\gamma(A_t, A_{t+1}, A_{t+2}, A_{t+3})$ is well defined for all $t \geq 0$. For $\gamma \in S$, let Y_γ be the set of solutions of the difference equation:

$$\Phi_\gamma(A_t, A_{t+1}, A_{t+2}, A_{t+3}) = 0, \quad t \geq 0. \quad (28)$$

Then Y_γ is the set of equilibrium paths for Eq. (28). Let θ be the shift map on Y_γ, then the dynamics of θ is equivalent to the one of $A_t \mapsto A_{t+1}$ for Eq. (28) by the conjugacy $\{A_t\}_{t\geq 0} \mapsto A_0$.

It is clear that Φ_γ is C^1 on $[0, C]^4$ for each $\gamma \in S$ and that the function $\gamma \mapsto \Phi_\gamma$ is continuous on S and so are $\gamma \mapsto \partial_i \Phi_\gamma$ for $i = 1, 2, 3, 4$. Now taking $\gamma \to \gamma_0$, the definition of Φ_γ implies that Eq. (28) is reduced to a difference equation of order one in which the limit function is $A_{t+3} - f_\sigma(A_{t+2}) = 0$. In Lemma 1, we have shown that if σ is large, then f_σ^2 exhibits Li–Yorke chaos. Due to Misiurewicz (1979), we get that the topological entropy $h_{\text{top}}(f_\sigma) > 0$.

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[10] In Juang et al. (2005) and Li and Malkin (2006), $[B, C]$ is replaced by $[B, C] \setminus V$ for some open set $V \subset [B, C]$ so that therein the function φ is allowed to be not differentiable or even not continuous at some interior points of V. Here we consider the simple case when $V = \emptyset$.

By Theorem 2, we get that for all $\gamma = (\lambda, \rho, \mu)$ near $\gamma_0 = (\alpha, 1, 1)$ enough, there is a closed θ-invariant subset Γ_γ of Y_γ in the product topology, such that $h_{\text{top}}(\theta|\Gamma_\gamma) > 0$. Therefore, the dynamics of the economy system has entropic chaos.   □

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