PERTURBATION RESULTS RELATED TO PALINDROMIC EIGENVALUE PROBLEMS

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Abstract

We investigate the perturbation of the palindromic eigenvalue problem for the matrix quadratic $P(\lambda) = \lambda^2 A_1^\star + \lambda A_0 + A_1$ with $A_0, A_1 \in \mathcal{C}^{n \times n}$ and $A_0^\star = A_0$ (where $\star = T$ or H). The perturbation of eigenvalues in the context of general matrix polynomials, palindromic pencils, (semi-Schur) anti-triangular canonical forms and differentiation is discussed.

1. Introduction

Consider the matrix quadratic

$$P(\lambda) \equiv \lambda^2 A_1^\star + \lambda A_0 + A_1,$$

where $A_0, A_1 \in \mathcal{C}^{n \times n}$ with $A_0^\star = A_0$ ($\star = T$ or H), and the corresponding *palindromic* quadratic eigenvalue problem

$$P(\lambda)x = 0, \quad x \neq 0. \tag{1.1}$$

In this paper we consider only *regular* matrix polynomials $P(\lambda)$, where "regular" is understood to mean $\det P(\lambda) \not\equiv 0$.

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From the transpose or Hermitian of (1.1), a palindromic eigenvalue problem is seen to possess a spectrum $\sigma(P)$ that contains both λ and its "reciprocal" $1/\lambda^\star$ (with 0 and ∞ considered to be reciprocal to each other). Under favourable conditions, the eigenvalue problem of the original matrix polynomial $P(\lambda)$ has a palindromic linearization of the form $\lambda Z \pm Z^\star$ [6, 10, 16].

We can transform $\lambda Z - Z^\star$ to the form $\nu(-Z) + (-Z)^\star$, with $\nu = -\lambda$. Similarly, $\lambda^2 A_1^\star + \lambda A_0 + A_1$ and $\nu^2 A_1^\star - \nu A_0 + A_1$ define equivalent palindromic eigenvalue problems. As a consequence, and for simplicity of presentation, we shall concentrate on the palindromic eigenvalue problem. Anti-palindromic or odd/even eigenvalue problems [16–18] can be treated similarly.

A solid foundation for the solution of palindromic eigenvalue problems has been laid by Hilliges, D.S. Mackey, N. Mackey, Mehl and Mehrmann (see [13, 16, 17]). An alternative approach to tackling the problem which involves structure-preserving doubling algorithms can be found in [6]. Recently, quadratic eigenvalue problems [21] have attracted much interest. An important example of palindromic eigenvalue problems can be found in the vibration analysis of fast trains; see [13] or [14] for a general introduction and [12] for details. For the general perturbation of eigenvalues for polynomial eigenvalue problems, see [5, 20]; consult also [8, 15] for some related results.

This paper is organized as follows. In Sections 2–5, the perturbation of eigenvalues in terms of general matrix polynomials, palindromic linearizations, the anti-triangular canonical form [16–18] and the semi-Schur anti-triangular canonical form is investigated by means of the Bauer–Fike technique for perturbations of arbitrary size [5]. The derivatives of eigenvalues and eigenvectors of $P(\lambda)$ are considered in Section 6. We summarize our conclusions in Section 7. The results in Section 3–5 are applicable to the more general palindromic pencils $Z - \lambda Z^\star$, which may not be linearizations of any matrix polynomials.

Note that Sun and Stewart's implicit function approach [19] has been applied to palindromic linearizations, general matrix quadratics and palindromic eigenvalue problems in [7], to obtain perturbation results for (simple) eigenvalues and the corresponding deflating subspaces.

It is important to distinguish between the different perturbation techniques. Note that the Bauer–Fike technique allows perturbations of arbitrary size, and the clustering of eigenvalues (and hence the corresponding deflating subspaces) may vary greatly. Consequently, it is meaningless to talk about perturbation of eigenvectors or deflating subspaces in the Bauer–Fike-type perturbation theorems of Sections 2–5. Perturbation results obtained via differentiation or implicit function approaches are valid only for asymptotically small perturbations, but results for deflating subspaces are available. Note that only simple eigenvalues (or the sums and averages of multiple eigenvalues) are differentiable, so generalized derivatives (or subgradients) have to be utilized in general [4]. We remark that the naive differentiation technique employed in Section 6 assumes differentiability, which can only be proved rigorously by using tools such as the implicit function theorem.

Next, we give a few words of warning. Comparison of perturbation results is a risky art. Typically, error bounds and condition numbers are simplified upper bounds of more complicated quantities, and a better (worse) upper bound does not always imply a smaller (bigger) error. Furthermore, optimization of such upper bounds, though often possible, is seldom attempted because of cost or inconvenience, making comparisons of perturbation results even more perilous. Therefore, we do not claim to have found the "best" perturbation results, if such exist at all. We shall quite often interpret perturbation results qualitatively, rather than apply them quantitatively, and we will indicate when things may go wrong or pitfalls to avoid. Nevertheless, our perturbation results, in addition to those in [2, 7, 15], are among the very few that are currently available for palindromic eigenvalue problems, and should be of use in related investigations. Lastly, conditions qualifying when perturbations are large or (asymptotically) small can be written down but are complicated and rarely checked. Again, such perturbation results may have to be used qualitatively rather than quantitatively.

2. Bauer–Fike theorem for general matrix polynomials

The unstructured perturbation result for general matrix polynomials, presented in Theorem 2.1 below, may not be directly applicable or satisfactory for palindromic eigenvalue problems. However, it serves as a reference for the structured perturbation results that we consider in later sections. Also, palindromic eigenvalue problems are sometimes perturbed in an unstructured manner; one example is when the QZ algorithm [11] is applied to an associated palindromic linearization $Z - \lambda Z^\star$. The associated perturbation problem has to be treated as an unstructured one, using the theorem below.

We now state, without proof, [5, Theorem 4.2] on the perturbation of eigenvalues of a general matrix polynomial.

THEOREM 2.1. *Consider a regular matrix polynomial*

$$L(\alpha, \beta) \equiv \sum_{j=0}^{l} B_j \alpha^j \beta^{l-j}$$

and its perturbation

$$\widetilde{L}(\alpha, \beta) \equiv \sum_{j=0}^{l} \widetilde{B}_j \alpha^j \beta^{l-j}, \quad \widetilde{B}_j \equiv B_j + \delta B_j \quad (j = 0, \ldots, l).$$

Let (X, T, Z) be a resolvent triple for L (see [5, 10]) which is constructed using some finite and infinite Jordan pairs, J_F and J_∞. For $(\alpha_i, \beta_i) \in \sigma(L)$ and $(\alpha, \beta) \in \sigma(\widetilde{L})$ with the scaling $|\alpha_i|^2 + |\beta_i|^2 = 1 = |\alpha|^2 + |\beta|^2$, the spectral variation of $\widetilde{L}$ from L is defined as

$$s_L(\widetilde{L}) \equiv \max_{(\alpha, \beta)} \{s_{(\alpha, \beta)}\}, \quad s_{(\alpha, \beta)} \equiv \min_i \{|\alpha \beta_i - \beta \alpha_i|\}.$$

Let p be the maximum dimension of the Jordan blocks in J_F or J_∞.

Then, for $\|\cdot\| = \|\cdot\|_\tau$ (where $\tau = 1, 2$ or ∞),

$$s_{(\alpha,\beta)} \leq \max\left\{\theta_1, \theta_1^{1/p}\right\}, \quad \theta_1 \equiv p\mathcal{F}\kappa\Delta, \tag{2.1}$$

where $\mathcal{F} \equiv c_1\sqrt{(l+1)/2}$ with $c_1 = 1$ (for $\tau = 1, 2$) or $c_1 = \sqrt{l}$ (for $\tau = \infty$) and

$$\kappa \equiv \|X\| \cdot \|Z\|, \quad \Delta \equiv \|[\delta B_0, \dots, \delta B_l]\|.$$

Also,

$$s_L(\widetilde{L}) \leq \max\left\{\theta_1, \theta_1^{1/p}\right\}. \tag{2.2}$$

COMMENTS 2.2.

(a) Note that, in the above theorem, we use the representation (α, β) for $\lambda = \alpha/\beta$.

(b) In the palindromic case we have $l = 2$, $B_0 = A_1^\star = B_2^\star$ and $B_1 = A_0 = A_0^\star$. Ultimately, the perturbation of the palindromic eigenvalues is controlled by θ_1 in (2.2), which is in turn dominated by the error term involving $\|[\delta A_0, \delta A_1]\|$. The condition of the eigenvalues will be poor when κ is large or when deflating subspaces for different eigenvalues are getting "close" to each other, making the resolution of the spectrum more and more difficult. Note also that the perturbation in δA_0 may be nonsymmetric, pushing a pair of reciprocal palindromic eigenvalues to ones that are not reciprocal (or approximately reciprocal when δA_0 is small). For a symmetric δA_0, we only have to consider the perturbation of half of the eigenvalues, owing to the palindromic structure.

(c) Based on Theorem 2.1, we can consider the perturbation of a cluster of eigenvalues; for details, see [5, Section 5.2]. A cluster, to be defined later in (2.3), can be one simple eigenvalue, a group of multiple eigenvalues or a group of neighbouring eigenvalues. For $(\alpha, \beta) \notin \sigma(L)$, assume the following decomposition of the resolvent:

$$L(\alpha, \beta)^{-1} = X_1 T_1(\alpha, \beta)^{-1} Z_1 + X_2 T_2(\alpha, \beta)^{-1} Z_2,$$

where $([X_1, X_2], T_1 \oplus T_2, [Z_1, Z_2])$ is a resolvent triple [10] appropriately partitioned into two parts. The eigenvalues in T_1 form a *cluster* when

$$\left\|X_1 T_1(\alpha, \beta)^{-1} Z_1\right\| \gg \left\|X_2 T_2(\alpha, \beta)^{-1} Z_2\right\| \leq \epsilon \left\|X_1 T_1(\alpha, \beta)^{-1} Z_1\right\| \tag{2.3}$$

for some small constant ϵ. Consequently,

$$\left\|L(\alpha, \beta)^{-1}\right\| \leq (1 + \epsilon)\left\|X_1 T_1(\alpha, \beta)^{-1} Z_1\right\|.$$

Arguments and techniques similar to those used in proving Theorem 2.1 can then be applied to $L(\alpha, \beta) + \delta L(\alpha, \beta)$, so that

$$(1 + \epsilon)\left\|X_1 T_1(\alpha, \beta)^{-1} Z_1\right\|\|\delta L(\alpha, \beta)\| \geq \left\|L(\alpha, \beta)^{-1}\delta L(\alpha, \beta)\right\| \geq 1$$

and

$$(1 + \epsilon)\kappa_1 \|\delta L(\alpha, \beta)\| \geq \left\| T_1(\alpha, \beta)^{-1} \right\|^{-1}, \quad \kappa_1 \equiv \|X_1\| \|Z_1\|.$$

Replacing $\|T_1(\alpha, \beta)^{-1}\|$ by an upper bound (as in Appendix A where T_1 is in Jordan or Kronecker form) yields results similar to those in Theorem 2.1, but for the cluster in T_1 rather than the whole spectrum $\sigma(L)$. Here p will be the size of the largest Jordan block associated with the cluster in T_1. Ignoring higher-order terms in ϵ, the perturbation results will then involve κ_1 instead of κ. The price to pay for the sharper result is the restriction that the perturbation δL has to be small (in the sense of (2.3)), whereas in Theorem 2.1 it can be arbitrary.

(d) In (c) above, when T_1 contains a simple eigenvalue, κ_1 will be the product of the norms of the corresponding left- and right-eigenvectors. Similarly, for a group of multiple eigenvalues, the corresponding condition number will be the product of the norms of the corresponding left- and right-eigenvectors (or deflating subspaces). Analogous condition numbers can be obtained for clusters of eigenvalues.

(e) Obviously, for large perturbations with $\theta_1 > 1$, we have $\max\{\theta_1, \theta_1^{1/p}\} = \theta_1$. On the other hand, when $\theta_1 < 1$, which is usually the case in (c) above, the maximum occurs at $\theta_1^{1/p}$. Furthermore, when the perturbation is asymptotically small, p in (2.1) equals the size of the Jordan block associated with (α_k, β_k), where the minimum in $s_{(\alpha, \beta)} \equiv \min_i \{|\alpha\beta_i - \beta\alpha_i|\}$ occurs at $i = k$. (In fact, a perturbation can be considered "small" when this correct pairing occurs; see the proof in Theorem 3.1.) Notice that the pth root is a common feature in perturbation results for multiple eigenvalues.

(d) A feature of the Bauer–Fike-type perturbation result is that one starts with a perturbed eigenvalue (α, β) whose spectral variation from a nearby unperturbed eigenvalue (α_i, β_i) is bounded. As the size of the perturbation is unrestricted, there may well be unperturbed eigenvalues that are not paired up with any perturbed eigenvalues.

3. Bauer–Fike theorem for palindromic pencils

For the pencil $\lambda Z - Z^\star$, we can work from the Kronecker canonical form

$$Q_1^\star \left(\lambda Z - Z^\star \right) Q_2 = \lambda \Lambda_+ - \Lambda_-, \quad Q_1 = [P_1, P_2], \quad Q_2 = [P_2, P_1],$$

where

$$\Lambda_+ = \begin{bmatrix} \Lambda & \\ & I \end{bmatrix}, \quad \Lambda_- = \begin{bmatrix} I & \\ & \Lambda \end{bmatrix}, \quad \Lambda = \text{diag}\{J_1, \ldots, J_N\},$$

with J_i being the Jordan block for λ_i on or inside the unit circle.

We have the following Bauer–Fike perturbation result.

THEOREM 3.1. *Consider the palindromic pencil $L \equiv \beta Z - \alpha Z^\star$ with the above Kronecker canonical form. Let $\tilde{Z} = Z + \delta Z$, $\tilde{L} \equiv \beta \tilde{Z} - \alpha \tilde{Z}^\star$, $(\alpha_i, \beta_i) \in \sigma(L)$ and*

$(\alpha, \beta) \in \sigma(\widetilde{L})$, *with the scaling* $|\alpha_i|^2 + |\beta_i|^2 = 1 = |\alpha|^2 + |\beta|^2$. *The spectral variation of* $\widetilde{L}$ *from* L *is defined as*

$$s_L(\widetilde{L}) \equiv \max_{(\alpha, \beta)}\{s_{(\alpha, \beta)}\}, \quad s_{(\alpha, \beta)} \equiv \min_i\{|\alpha\beta_i - \beta\alpha_i|\}.$$

Then, for any Hölder norm $\|\cdot\|$,

$$s_{(\alpha, \beta)} \leq \max\left\{\theta_2, \theta_2^{1/p}\right\}, \quad \theta_2 \equiv c_2\kappa_2 \|\delta Z\|,$$

where κ_2 *is defined as in* (3.2), p *is the size of the largest Jordan block in* Λ *and* $c_2 = 2\sqrt{2(|\alpha|^2 + |\beta|^2)} = 2\sqrt{2}$.

Also,

$$s_L(\widetilde{L}) \leq \max\left\{\theta_2, \theta_2^{1/p}\right\}.$$

PROOF. Applying the techniques in [5], we consider the singular matrix

$$Q_1^\star\Big[\beta(Z + \delta Z) - \alpha(Z + \delta Z)^\star\Big]Q_2$$
$$= (\beta\Lambda_+ - \alpha\Lambda_-)\Big[I + (\beta\Lambda_+ - \alpha\Lambda_-)^{-1}Q_1^\star(\beta\delta Z - \alpha\delta Z^\star)Q_2\Big],$$

which implies that

$$\kappa_2\left\|(\beta\Lambda_+ - \alpha\Lambda_-)^{-1}\right\|\left\|\beta\delta Z - \alpha\delta Z^\star\right\| \geq 1 \tag{3.1}$$

where

$$\kappa_2 \equiv \|Q_1\|\,\|Q_2\|. \tag{3.2}$$

From Appendix A, we have the upper bound

$$\left\|(\beta\Lambda_+ - \alpha\Lambda_-)^{-1}\right\| \leq c_0\max\left\{|z_i|^{-1}, |z_i|^{-p_i}\right\},$$

where $z_i \equiv \alpha\beta_i - \beta\alpha_i$, p_i is the size of the Jordan block associated with (α_i, β_i) and $c_0 \leq 2$. Substituting this bound into (3.1), we obtain

$$\kappa_2 c_2\|\delta Z\| \max\left\{|z_i|^{-1}, |z_i|^{-p_i}\right\} \geq 1 \quad \Rightarrow \quad \min\left\{|z_i|, |z_i|^{p_i}\right\} \leq \kappa_2 c_2\|\delta Z\|.$$

The conclusions of the theorem then follow. □

COMMENTS 3.2.

(a) Note that for the 2-norm or the (Frobenius) F-norm,

$$\theta_2 \leq \sqrt{2(|\alpha|^2 + |\beta|^2)}\,p\|\delta Z\|.$$

With the scaling $|\alpha|^2 + |\beta|^2 = 1 = |\alpha_i|^2 + |\beta_i|^2$, $s_{(\alpha, \beta)}$ becomes the chordal metric [5, 11] and $\theta_2 \leq \sqrt{2}\,p\,\|\delta Z\|$. Although the F-norm is not a Hölder norm, the corresponding results can be deduced from the 2-norm results.

(b) Comments similar to those labelled (c)–(f) after Theorem 2.1 apply for Theorem 3.1. The proof of the results for clusters of eigenvalues, using the partitioning of an appropriate resolvent, is similar and will not be repeated here.

(c) Details associated with palindromic "linearizations" can be found in [16]. Obviously, results in this section are applicable to general palindromic pencils which may not be linearizations of matrix polynomials.

4. Bauer–Fike theorem for anti-triangular form

From [18], we have the following anti-triangular canonical form for $\star = T$.

THEOREM 4.1. *Let* $Z - \lambda Z^\star$ *be a regular* $n \times n$ *palindromic pencil. There exists a unitary* $U \in \mathcal{C}^{n \times n}$ *such that* $U^\star Z U = (m_{ij})$ *with* $m_{ij} = 0$ *for* $i + j \leq n + 1$ *(that is,* $U^\star Z U$ *is anti-triangular, with zero elements in the upper left corner).*

Note that the result for $\star = H$ can easily be obtained by extending the proofs of [18, Lemma 2.2 and Theorem 2.3]. Note also that we are only interested in the case where $Z - \lambda Z^\star$ is regular, which does not hold in [18].

The eigenvalues of the palindromic pencil $Z - \lambda Z^\star$ are

$$\frac{m_{1,n}}{m_{n,1}^\star}, \frac{m_{2,n-1}}{m_{n-1,2}^\star}, \ldots, \frac{m_{i,n-i+1}}{m_{n-i+1,i}^\star}, \ldots, \frac{m_{n-i+1,i}}{m_{i,n-i+1}^\star}, \ldots, \frac{m_{n-1,2}}{m_{2,n-1}^\star}, \frac{m_{n,1}}{m_{1,n}^\star}.$$

Note that n will be even when considering a linearization of a palindromic quadratic pencil [16], but the results in this section hold for any n.

Let N be the strict lower right triangular part of $U^\star Z U$. Reorganize the anti-triangular form in Theorem 4.1 into upper triangular form

$$\mathcal{P}_n U^\star (Z - \lambda Z^\star) U = (D_1 + N_1) - \lambda (D_2 + N_2) \tag{4.1}$$

with the order-reversing permutation matrix $\mathcal{P}_n = [e_n, e_{n-1}, \ldots, e_1]$; here

$$D_2 = \text{diag}\{m_{1,n}, m_{2,n-1}, \ldots, m_{n-1,2}, m_{n,1}\},$$
$$D_1 = \mathcal{P}_n D_2 \mathcal{P}_n = \text{diag}\{m_{n,1}, m_{n-1,2}, \ldots, m_{2,n-1}, m_{1,n}\},$$

where $N_1 = \mathcal{P}_n N$ and $N_2 = \mathcal{P}_n N^\star$ are strictly upper triangular.

Using the Schur-like form in (4.1), we can prove the following perturbation result for a palindromic pencil.

THEOREM 4.2. *Consider the palindromic pencil* $L \equiv \beta Z - \alpha Z^\star$, *with* N *being the strict lower right triangular part of its anti-triangular canonical form. Let* $\tilde{Z} = Z + \delta Z$, $\tilde{L} \equiv \beta \tilde{Z} - \alpha \tilde{Z}^\star$, $(\alpha_i, \beta_i) \in \sigma(L)$ *and* $(\alpha, \beta) \in \sigma(\tilde{L})$. *Assume the scaling* $|\alpha|^2 + |\beta|^2 = 1 = |\alpha_i|^2 + |\beta_i|^2$. *Then, for any Hölder norm* $\| \cdot \|$,

$$s_{(\alpha,\beta)} \leq \sqrt{2} \, c_0 \, \max\left\{\theta_3, \, c_3 \theta_3^{1/p}\right\}, \quad \theta_3 \equiv \|\delta Z\|$$

for some $p \leq n$ *and* $c_0 \equiv \min\{2, \, p\}$, $c_3 \equiv \sqrt{2} \, \|N\|^{1-1/p}$.
   Also,

$$s_L(\tilde{L}) \leq \sqrt{2} c_0 \max\left\{\theta_3, \, c_3 \theta_3^{1/p}\right\}.$$

PROOF. Consider the singular matrix

$$
\beta(Z+\delta Z) - \alpha(Z+\delta Z)^\star
$$
$$
= (U^\star)^{-1}\mathcal{P}_n\Big[\beta(D_1+N_1+\mathcal{P}_nU^\star\delta ZU) - \alpha(D_2+N_2+\mathcal{P}_nU^\star\delta Z^\star U)\Big]U^H
$$
$$
= (U^\star)^{-1}\mathcal{P}_n[\beta(D_1+N_1) - \alpha(D_2+N_2)]
$$
$$
\times \Big\{I + [\beta(D_1+N_1) - \alpha(D_2+N_2)]^{-1}\mathcal{P}_nU^\star(\beta\delta Z - \alpha\delta Z^\star)U\Big\}U^H.
$$

It is easy to see that

$$
\left\|[\beta(D_1+N_1) - \alpha(D_2+N_2)]^{-1}\right\|\left\|\beta\delta Z - \alpha\delta Z^\star\right\|
$$
$$
\geq \left\|[\beta(D_1+N_1) - \alpha(D_2+N_2)]^{-1}\mathcal{P}_nU^\star(\beta\delta Z - \alpha\delta Z^\star)U\right\| \geq 1. \quad (4.2)
$$

Note that $[\beta(D_1+N_1) - \alpha(D_2+N_2)]$ is assumed to be nonsingular, otherwise the results in the theorem become trivial.

With $z \equiv \min_{(\alpha_i,\beta_i)} \|\beta D_1 - \alpha D_2\| = \min_{(\alpha_i,\beta_i)} |\beta\alpha_i - \alpha\beta_i|$, $\tilde{D} \equiv \beta D_1 - \alpha D_2$ and $\tilde{N} \equiv \beta N_1 - \alpha N_2$, we have

$$
M \equiv [\beta(D_1+N_1) - \alpha(D_2+N_2)]^{-1}
$$
$$
= \Big[I - (\beta D_1 - \alpha D_2)^{-1}(\beta N_1 - \alpha N_2)\Big]^{-1}(\beta D_1 - \alpha D_2)^{-1} = \Big(I - \tilde{D}^{-1}\tilde{N}\Big)^{-1}\tilde{D}^{-1}.
$$

As $\tilde{D}^{-1}\tilde{N}$ is nilpotent, there exists some $p \leq n$ such that

$$
\Big(I - \tilde{D}^{-1}\tilde{N}\Big)^{-1} = I + \tilde{D}^{-1}\tilde{N} + \cdots + \Big(\tilde{D}^{-1}\tilde{N}\Big)^{p-1}
$$

and we obtain

$$
\|M\| \leq \left\|\tilde{N}\right\|^{-1}\eta^{-1} \equiv z^{-1}\Big(1 + \left\|\tilde{N}\right\|z^{-1} + \cdots + \left\|\tilde{N}\right\|^{p-1}z^{-p+1}\Big). \quad (4.3)
$$

With $x \equiv \|\tilde{N}\|^{-1}z$, we have the polynomial $P(x) \equiv x^p - \eta(1 + x + \cdots + x^{p-1}) = 0$, as in Appendix A or [5]. The only positive root x of P satisfies

$$
x \equiv \left\|\tilde{N}\right\|^{-1}z \leq c_0 \max\Big\{\eta,\ \eta^{1/p}\Big\}.
$$

As (4.2) and (4.3) imply

$$
\sqrt{2}\,\|\delta Z\| \geq \|\beta\delta Z - \alpha\delta Z^\star\| \geq \|M\|^{-1} \geq \left\|\tilde{N}\right\|\eta,
$$

this and the upper bound in (4.3) then lead to

$$
z \leq c_0\left\|\tilde{N}\right\|\max\Big\{\eta,\ \eta^{1/p}\Big\} \leq \sqrt{2}\,c_0 \max\Big\{\|\delta Z\|,\ \left\|\tilde{N}\right\|^{1-1/p}\|\delta Z\|^{1/p}\Big\}.
$$

As $\|\tilde{N}\| \leq \sqrt{2}\,\|N\|$, the results in the theorem then follow. $\qquad\square$

With the chosen scaling $|\alpha|^2 + |\beta|^2 = 1 = |\alpha_i|^2 + |\beta_i|^2$, $s_{(\alpha,\beta)}$ equals the chordal metric. Note also that p is the integer for which $\tilde{N}^k \neq 0$ for all $0 \leq k < p$ and $\tilde{N}^p = 0$.

5. Bauer–Fike theorem for semi-Schur anti-triangular form

A refinement of Theorem 4.2, with a smaller value for p, can be proved. We first refine the decomposition in Theorem 4.1.

THEOREM 5.1. *Let $Z - \lambda Z^\star$ be a regular palindromic pencil. There exist nonsingular $U, V \in \mathcal{C}^{n \times n}$ such that*

$$V^\star Z U = \text{anti-diag}\{M_1, \ldots, M_r\}, \quad M_j = D_j + N_j, \tag{5.1}$$

where M_j is anti-triangular with anti-diagonal elements in $D_j = \text{anti-diag}\{\lambda_j, \ldots, \lambda_j\}$.

PROOF. The proof is similar to the standard transformation of a Schur decomposition to the corresponding Jordan canonical form. It suffices to show that it is possible to transform the anti-triangular form $(\tilde{U}^\star Z \tilde{U}, \tilde{U}^\star Z^\star \tilde{U})$ in Theorem 4.1 to anti-block-diagonal form, so that

$$\begin{bmatrix} I & 0 \\ P^\star & I \end{bmatrix} \tilde{U}^\star Z \tilde{U} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ P^\star & I \end{bmatrix} \begin{bmatrix} 0 & T_1 \\ T_2 & T_{12} \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} & T_1 \\ T_2 & \end{bmatrix},$$

$$\begin{bmatrix} I & 0 \\ P^\star & I \end{bmatrix} \tilde{U}^\star Z^\star \tilde{U} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ P^\star & I \end{bmatrix} \begin{bmatrix} 0 & T_2^\star \\ T_1^\star & T_{12}^\star \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} & T_2^\star \\ T_1^\star & \end{bmatrix}$$

when T_1 and T_2 have nonintersecting spectra. Multiplying out the above equations produces

$$\phi(P, Q) \equiv (T_2 Q + P^\star T_1, \ T_1^\star Q + P^\star T_2^\star) = -(T_{12}, \ T_{12}^\star), \tag{5.2}$$

which is uniquely solvable [3]. □

Similar to Theorem 5.1, but with p being bounded by the maximum size of M_j, we now have the following refined version of Theorem 4.2.

THEOREM 5.2. *Consider the palindromic pencil $L \equiv \beta Z - \alpha Z^\star$ with its semi-Schur anti-triangular canonical form (5.1). Let $\tilde{Z} = Z + \delta Z$, $\tilde{L} \equiv \beta \tilde{Z} - \alpha \tilde{Z}^\star$, $(\alpha_i, \beta_i) \in \sigma(L)$ and $(\alpha, \beta) \in \sigma(\tilde{L})$. Assume the scaling $|\alpha|^2 + |\beta|^2 = 1 = |\alpha_i|^2 + |\beta_i|^2$. Then, for any Hölder norm $\|\cdot\|$,*

$$s_{(\alpha,\beta)} \leq \sqrt{2}\, c_0 \ \max\left\{\theta_4, \ c_4 \theta_4^{1/p}\right\}, \quad \theta_4 \equiv \kappa_4 \|\delta Z\|$$

with $\kappa_4 \equiv \|U\| \|V\|$, $c_0 \equiv \min\{2, \ p\}$ and $c_4 \equiv \sqrt{2} \ \max_j \|N_j\|^{1-1/p}$.

For sufficiently small perturbations, p is the size of the Schur block M_j associated with (α_i, β_i). In general we have $p = p^*$, the maximum size of the Schur blocks M_j. Also,

$$s_L(\widetilde{L}) \leq \sqrt{2}\, c_0 \, \max\left\{\theta_4, \, c_4 \theta_4^{1/p^*}\right\}.$$

PROOF. The proof is exactly the same as that for Theorem 4.2, except that the $\|M\|$ in (4.3) now equals the maximum of the norms of its diagonal blocks. The same argument can be followed in a similar fashion as in the proof of Theorem 4.2, using the diagonal block at which the maximum occurs.

When the perturbation is small enough, this maximum (nearly infinite) occurs at the same block associated with (α_k, β_k) at which $\min_i\{|\alpha\beta_i - \beta\alpha_i|\}$ occurs. This gives a sharper perturbation result, with a smaller p which is just the size of the diagonal block associated with (α_k, β_k). □

Comments similar to those labelled (c)–(f) after Theorem 2.1 apply for the above theorem. Like Theorem 3.1, when we consider sufficiently small perturbations there will be a one-to-one correspondence between the original and the perturbed eigenvalues. The above perturbation bounds can be proved for a particular eigenvalue (α_i, β_i) with the condition number κ_4 replaced by $\|U_j\| \|V_j\|$. In addition, instead of considering one particular eigenvalue, we can consider a group of neighbouring eigenvalues together. This will increase p or the size of the corresponding semi-Schur block M_j, but will improve the condition of the linear operator ϕ in (5.2) as well as κ_4.

6. Perturbation by differentiation

The results in this section are quoted from [7].

Without establishing differentiability or the existence of asymptotic expansions (which can be achieved by using the implicit function approach), perturbation results can be obtained via simple differentiation. See [1] for more details of this approach.

For some fixed $z \neq 0$, consider the palindromic eigenvalue problem

$$P(\lambda, \rho)x(\rho) = 0, \quad P(\lambda, \rho) \equiv \lambda(\rho)^2 A_1^\star(\rho) + \lambda(\rho)A_0(\rho) + A_1(\rho)$$

with the scaling $z^\star x(\rho) - 1 = 0$, where ρ is the perturbation parameter, $A_0(0) = A_0$ and $A_1(0) = A_1$. We shall use notation $(\cdot)_\rho$ and $(\cdot)_\lambda$ to denote the corresponding partial derivatives. For a simple eigenvalue λ, differentiation produces, at $\rho = 0$,

$$\lambda_\rho = -\frac{y^\star P_\rho x}{y^\star P_\lambda x} = -\frac{y^\star P_\rho x}{y^\star (2\lambda A_1^\star + A_0)x} \tag{6.1}$$

and

$$Px_\rho = -(\lambda_\rho P_\lambda + P_\rho)x, \quad z^\star x_\rho = 0.$$

Upon choosing $z = y(0)$ (the left-eigenvector corresponding to $\lambda(0)$) we obtain, at $\rho = 0$,

$$x_\rho = -P^\dagger(\lambda_\rho P_\lambda + P_\rho)x,$$

where $P^\dagger$ denotes the Penrose generalized inverse [11] of P.

The usual conclusions can be drawn: the right-eigenvector x will be rotated through a big angle, even for a small perturbation, when $\|P^\dagger\|$ is large, that is, when the separation between λ and the other eigenvalues is fine. This happens, of course, when the assumption of simplicity for the eigenvalue is close to collapsing.

Note that for palindromic eigenvalue problems with $\star = T$, the eigenvalues $\lambda = \pm 1$ may be multiple and nondifferentiable; thus a more sophisticated approach, like the one in [4], is required.

For perturbation results obtained through the application of Sun and Stewart's approach [9, 19] in terms of the implicit function theorem, see [7]. Asymptotic perturbation series for the eigenvalues and the deflating subspaces have been derived.

7. Conclusions

Bauer–Fike-type perturbation results for general matrix polynomials, palindromic linearizations and (semi-Schur) anti-triangular canonical forms have been discussed for perturbations of arbitrary size. These perturbation results complement the ones for asymptotic perturbations given in [7]. Consistent results for simple eigenvalues and their corresponding eigenvectors were obtained using simple differentiation. These results indicate, not surprisingly, that the perturbations of an eigenvalue λ and its corresponding deflating subspace $\mathcal{S}_\lambda$ are proportional, respectively, to the size of the perturbation and the reciprocal of the gap between $\mathcal{S}_\lambda$ and other deflating subspaces. Condition numbers are typically proportional to the products of the norms of the left- and right-eigenvectors or deflating subspaces.

Appendix A. Bounding $\|(\beta\Lambda_+ - \alpha\Lambda_-)^{-1}\|$

For any Hölder norm, given

$$\Lambda_+ = \begin{bmatrix} \Lambda & \\ & I \end{bmatrix}, \quad \Lambda_- = \begin{bmatrix} I & \\ & \Lambda \end{bmatrix}, \quad \Lambda = \mathrm{diag}\,\{J_1, \ldots, J_N\}$$

with the eigenvalues of J_i all on or inside the unit circle, we have

$$\|(\beta\Lambda_+ - \alpha\Lambda_-)^{-1}\| = \max_i \|M_i\|,$$

where $M_i \equiv (\alpha I - \beta J_i)^{-1} \in \mathcal{R}^{p_i \times p_i}$.

Thus it is sufficient to consider the bound for $\|M_i\|$. Let $z_i \equiv \alpha\beta_i - \beta\alpha_i$, with α_i and β_i being diagonal elements of I and J_i, respectively. (Here, $\alpha_i = 1$ and $|\beta_i| \le 1$; the case where $M_i \equiv (\alpha J_i - \beta I)^{-1}$ can be treated similarly, and the symmetric notations used here can be adapted easily.)

We have the Toeplitz matrix

$$
M_i \equiv
\begin{bmatrix}
z_i & -\beta & & & \\
 & z_i & -\beta & & 0 \\
 & & \ddots & \ddots & \\
 0 & & & z_i & -\beta \\
 & & & & z
\end{bmatrix}^{-1}
$$

$$
=
\begin{bmatrix}
z_i^{-1} & \beta z_i^{-2} & \beta^2 z_i^{-3} & \cdots & \beta^{p_i-1} z_i^{-p_i} \\
 & z_i^{-1} & \beta z_i^{-2} & \cdots & \beta^{p_i-2} z_i^{-p_i+1} \\
 & & z_i^{-1} & \ddots & \vdots \\
 0 & & & \ddots & \beta z_i^{-2} \\
 & & & & z_i^{-1}
\end{bmatrix}.
$$

We then have

$$
\|M_i\| \le c_1 \cdot \max\left\{ |z_i|^{-1}, \, |z_i|^{-p_i} \right\}.
$$

Finally, p_i and $|z_i|$ can be replaced by $p \equiv \max p_i$ and $z \equiv \min_i |\alpha\beta_i - \beta\alpha_i|$, respectively, with $c_1 = p$.

Alternatively, potentially sharper bounds can be obtained, with c_1 replaced by c_0. If $M_i^{-1} = z_i[I_{q_i} - z_i^{-1}N^{(i)}]$ for a nilpotent $N^{(i)}$ such that $[N^{(i)}]^{q_i} = 0$, then

$$
M_i = z_i^{-1}\sum_{j=0}^{q_i-1} z_i^{-1}\left[N^{(i)}\right]^j, \quad \|M_i\| \le \eta^{-1} \equiv |z_i|^{-1}\sum_{j=0}^{q_i-1} |z_i|^{-j}.
$$

For simplicity, let $x = |z_i|$ and $m = q_i$. The above definition of η leads to the polynomial

$$
P_m(x) \equiv x^m - \eta(1 + x + \cdots + x^{m-1}).
$$

Descartes' sign rule (*La Géométrie* 1637) then implies that $P_m(x)$ has at most one positive real root. As $P_m(0) = -\eta < 0$ and $P_m(x) > 0$ as $x \to \infty$, any positive number x^* for which $P_m(x^*) > 0$ is an upper bound of the unique real positive root of $P_m(x)$. Simple inspection leads to the upper bounds $x^* = c_0\eta$ when $\eta > 1$, and $x^* = c_0\eta^{1/m}$ when $\eta \le 1$, with $c_0 = \min\{2, m\}$. Consequently, $c_0 = 1$ when $m = 1$ (with $x = \eta$), and $c_0 = 2$ when $m > 1$.

The details are as follows. When $c_0\eta > \eta \ge 1$ and $m > 1$,

$$
\begin{aligned}
P_m(c_0\eta) &= (c_0\eta)^m - \eta\frac{(c_0\eta)^m - 1}{c_0\eta - 1} \\
&= \frac{c_0^{m+1}\eta^{m+1} - c_0^m\eta^m - c_0^m\eta^{m+1} + \eta}{c_0\eta - 1} \\
&= \frac{(c_0^m\eta^{m+1} - c_0^m\eta^m) + (c_0^{m+1}\eta^{m+1} - c_0^m\eta^{m+1} - c_0^m\eta^{m+1}) + \eta}{c_0\eta - 1} \ge 0,
\end{aligned}
$$

because $c_0^{m+1}\eta^{m+1} - c_0^m\eta^{m+1} - c_0^m\eta^{m+1} = (c_0 - 2)c_0^m\eta^{m+1} = 0$. Thus $c_0\eta$ is an upper bound of the root x of P_m when $\eta \geq 1$.

When $\eta < 1$ and $m > 1$,

$$P_m(c_0\eta^{1/m}) \geq c_0^m\eta - \eta(1 + c_0 + \cdots + c_0^m) = c_0^m\eta - \eta\frac{c_0^m - 1}{c_0 - 1} = \eta > 0.$$

Thus $c_0\eta^{1/m}$ is an upper bound of the root x of P_m when $\eta < 1$.

Finally,

$$|z_i| \leq c_0 \max\{\eta, \eta^{-q_i}\} \quad \Rightarrow \quad \|M_i\| \leq \eta^{-1} \leq c_0 \max\left\{|z_i|^{-1}, |z_i|^{-q_i}\right\},$$

and the result follows, with c_1 replaced by the sharper c_0.

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