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Equivalence of permutation polytopes corresponding to strictly supermodular functions

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Abstract

This paper demonstrates a strong equivalence of all permutation polytopes corresponding to strictly supermodular functions.

Keywords: Permutation polytopes; Strictly supermodular functions

1. Introduction

Throughout the paper, let p be a positive integer. A *p-set function* is a real-valued function λ on the subsets of $\{1, \dots, p\}$ with $\lambda(\emptyset) = 0$. A p-set function is called *supermodular* if for every $I, J \subseteq \{1, \dots, p\}$,

$$\lambda(I \cup J) + \lambda(I \cap J) \geq \lambda(I) + \lambda(J); \tag{1.1}$$

λ is called *strictly supermodular* if strict inequality holds in (1.1) whenever the sets I and J are not linearly ordered by set inclusion, that is, $I \nsubseteq J$ and $J \nsubseteq I$. Also, λ is called *(strictly) submodular* if $-\lambda$ is (strictly) supermodular.

Supermodular/submodular functions have extensive applications in optimization and polyhedral combinatorics. Some of the developments have occurred, in parallel, within the studies of specific contexts (at times, with incomplete cross-referencing). One early reference is Choquet [3] who introduced submodular functions when studying Newtonian capacities of subsets in R^3. Edmonds [4], motivated by his work on matroids and their rank functions, began the systematic study of submodularity while Shapley [13] established (independently) important results through his investigation of cores of p-person games. Lovasz [8] provided an extensive survey which lists many applications; in fact, he argued that the role of submodular functions in combinatorial optimization parallels and enforces the role of convex functions in continuous optimization. Additional developments and applications can be found in [1,2,5–7,9,10,12] and the references therein. Some of these applications augment (1.1) with additional requirements, like monotonicity that is used in the definition of *rank functions*.

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Supermodularity gives rise to polyhedra with nice properties. More specifically, for each permutation σ of $\{1,\ldots,p\}$ and $t \in \{1,\ldots,p\}$, let $I_\sigma(t)$ be the integers preceding t according to σ. Given a p-set function λ, each permutation σ defines a vector $\lambda_\sigma \in \mathbb{R}^p$ with $(\lambda_\sigma)_k = \lambda[\{k\} \cup I_\sigma(k)] - \lambda[I_\sigma(k)]$. The *permutation polytope* corresponding to λ, denoted H^λ, is the convex hull of the vectors λ_σ with σ ranging over all permutations of $\{1,\ldots,p\}$. Also, we consider the solution set of the linear system

$$\sum_{j\in I} x_j \geq \lambda(I) \quad \text{for each nonempty subset } I \text{ of } \{1,\ldots,p\}, \quad \text{and} \tag{1.2}$$

$$\sum_{j=1}^{p} x_j = \lambda(\{1,\ldots,p\}), \tag{1.3}$$

which we denote C^λ; in game theoretic terminology this polytope is called the *core* of the game corresponding to λ, see [13]. Shapley [13] proved that when λ is supermodular, the above two polytopes coincide and that the λ_σ's are the vertices of $C^\lambda = H^\lambda$; he also characterized nonempty faces of these polytope in terms of tightness of constraints that appear in (1.2) (see Theorem 3.1). Further algebraic and geometric properties of polytopes corresponding to supermodular functions have been studied extensively (see the aforementioned references).

Example 1. Let $n, n_1, \ldots, n_p$ be positive integers and let $\theta_1, \ldots, \theta_n$ be real numbers such that $\sum_{j=1}^{p} n_j = n$ and $\theta_1 \leq \ldots \leq \theta_n$. For each subset I of $\{1,\ldots,p\}$, let $n_I \equiv \sum_{j\in I} n_j$ and $\lambda(I) \equiv \sum_{i=1}^{n_I} \theta_i = \min\{\sum_{i\in J} \theta_i : J \subseteq \{1,\ldots,p\} \text{and} |J| = n_I\}$. Now, consider subsets I and J of $\{1,\ldots,p\}$. As $n_{I\cup J} = n_I + n_{J\setminus I}$, $n_J = n_{J\setminus I} + n_{I\cap J}$ and $n_I \geq n_{I\cap J}$, we have that

$$\lambda(I\cup J) - \lambda(I) = \sum_{i=n_I+1}^{n_I+n_{J\setminus I}} \theta_i \geq \sum_{i=n_{I\cap J}+1}^{n_{I\cap J}+n_{J\setminus I}} \theta_i = \lambda(J) - \lambda(I\cap J), \tag{1.4}$$

assuring that the p-set function λ is supermodular; further, if the θ_i's are distinct and I and J are not comparable by set inclusion, the above inequalities are strict, assuring that λ is strictly supermodular.

For a partition $\pi = (\pi_1, \ldots, \pi_p)$ of $\{1,\ldots,n\}$ satisfying $|\pi_j| = n_j$ for $j = 1,\ldots,p$, let $\theta_\pi = (\sum_{i\in\pi_1} \theta_i, \ldots, \sum_{i\in\pi_p} \theta_i)$. The convex hull of the θ_π's is referred to as a *(single-shape) partition polytope*. This polytope was shown to coincide with $H^\lambda = C^\lambda$ in [6]. In the special case where $n = p$ (or equivalently, $n_j = 1$ for $j = 1,\ldots,p$), the partition polytope is the convex hull of the coordinate-permutations of $\theta = (\theta_1, \ldots, \theta_p)$ and is referred to as the *generalized permutahedron* corresponding to θ. The *standard permutahedron* is obtained when $\theta = (1,\ldots,p)$, in particular, it is a permutation polytope corresponding to a strictly supermodular set function (see [11,1] and [14, pp. 17–18 and 23]). □

The purpose of the current paper is to demonstrate that from an algebraic and combinatorial perspective all permutation polytopes corresponding to strictly supermodular functions coincide. Specifically, it is shown that all such polytopes are *combinatorially equivalent* (that is, their face-lattices are isomorphic) and they are *algebraically equivalent* (which assures that the sets of tangential hulls of their faces coincide); see Section 2 for formal definitions. One implication of these results is a simple characterization of the vertices and of the facets of these polytopes, characterizations that have been observed by [13,2,4]. The combinatorial equivalence of the face-lattice of permutation polytopes corresponding to restricted classes of strictly supermodular functions has appeared in the literature; for example, [9, Theorem 4.2] establishes this (combinatorial) equivalence for a family of polytopes that arise in some scheduling problems, [1, Section 3] proves the equivalence for permutahedra corresponding to vectors $\theta = (\theta_1, \ldots, \theta_p)$ with distinct θ_i's (see Example 1), and [6, Corollary 2] generalized the equivalence to partition polytopes where the corresponding θ_i's are distinct (again, see Example 1).

Sections 2 and 3 record, respectively, preliminaries about polytopes and about supermodular set functions. Section 4 explores partition polytopes corresponding to strictly supermodular set functions.

2. Preliminaries on polytopes

Given a set $C \subseteq \mathbb{R}^p$, a *linear combination* of points in C is a point x in $\mathbb{R}^p$ of the form $\sum_{i=1}^{k} \alpha_i x^i$ where k is a nonnegative integer, the x^i's are in C and the α_i's are in $\mathbb{R}$. If, in addition, (1) $\sum_{i=1}^{k} \alpha_i = 0$, (2) $\sum_{i=1}^{k} \alpha_i = 1$ and each $\alpha_i \geq 0$, or (3) each $\alpha_i \geq 0$, we call x a (1) *tangential*, (2) *convex* or (3) *conic combination*, respectively (where the empty sum (when $k = 0$) is regarded as the zero vector). The *tangential, convex* and *conic* hulls of C, denoted tng C, conv C and cone C are, respectively, the sets of all corresponding combinations of points of C. In particular, $\text{tng}\,\emptyset = \text{cone}\,\emptyset = \{0\}$ and $\text{conv}\,\emptyset = \emptyset$. The set is *convex* if conv $C = C$, and a *cone* if cone $C = C$. The *dimension* of a convex set $C \subseteq \mathbb{R}^p$ is the number of linearly independent vectors in its tangential hull. A *polytope* in $\mathbb{R}^p$ is the convex hull of finitely many points in $\mathbb{R}^p$. The Main Theorem for Polytopes (e.g., [14, Theorem 1.1, p. 29]) asserts that a subset of $\mathbb{R}^p$ is a polytope if and only if it is bounded and is the solution set of a system of linear inequalities.

Given a polytope P in $\mathbb{R}^p$, we say that a linear inequality $\sum_{j=1}^{p} c_j x_j \leq \gamma$ is *valid* for P if $P \subseteq \{x \in \mathbb{R}^p : \sum_{j=1}^{p} c_j x_j \leq \gamma\}$. A *face* of P is any set of the form $F = P \cap \{x \in \mathbb{R}^p : \sum_{j=1}^{p} d_j x_j = \delta\}$ where $\sum_{j=1}^{p} d_j x_j \leq \delta$ is a valid inequality for P. A face F of P is *proper* if $\emptyset \neq F \neq P$. Faces of dimension 0, 1 and $(\dim P) - 1$ are called *vertices, edges* and *facets*, respectively. For convenience, we refer to a *vertex* not only as a face of dimension zero, but also as the single element that such a face contains. A number of standard results about faces of polytopes are next recorded (see [14, Propositions 2.2 and 2.3, pp. 52–53, and Theorem 2.7 and following discussion, pp. 57–58]).

Proposition 2.1. *Let P be a polytope in $\mathbb{R}^p$. Then:*

(a) *P is the convex hull of its vertices,*
(b) *intersections of faces of P are faces of P,*
(c) *each face of P is the intersection of facets of P,*
(d) *a face F' of P is strictly included in a face F of P if and only if $F' \subseteq F$ and $\dim F' < \dim F$, and*
(e) *if P is a polytope with representation*

$$\sum_{j=1}^{n} B_{kj} x_j \leq b_k \quad \text{for all } k \in \beta, \tag{2.1}$$

where β is a finite index set, then each facet F of P has a representation of the form $F = \{x \in P : \sum_{j=1}^{n} C_{rj} x_j = c_r\}$ for some $r \in \beta$.

With set inclusion as the partial order, the set of faces of a polytope P is known to be a lattice, and we refer to this lattice as the *face-lattice of P*. Two polytopes are *combinatorially-equivalent* if there is a one-to-one dimension-preserving isomorphism of the face-lattice of one onto the face-lattice of the other, where by isomorphism we mean an inclusion-preserving map.

Let P be a polytope in $\mathbb{R}^p$. For each nonempty face F of P, the *normal cone* of F, denoted N_F, is defined by

$$N_F \equiv \{c \in \mathbb{R}^p : F = \arg\max_{x \in P} c^T x\},$$

where $\arg\max_{x \in P} c^T x$ refers to the set of maximizers of the function on P that maps $x \in P$ to $c^T x$ (this definition differs from [14, p. 193], where N_F is defined by $\{c \in \mathbb{R}^p : F \subseteq \arg\max_{x \in P} c^T x\}$). The orthogonal complement $C^\perp$ of a set $C \subseteq \mathbb{R}^p$ is the set $\{x \in \mathbb{R}^p : x^{\mathrm{T}} y = 0 \text{ for each } y \in C\}$. Results about normal cones are next recorded (e.g., [6, Proposition 2, p. 338]).

Proposition 2.2. *Let P be a polytope in $\mathbb{R}^p$.*

(a) *For a nonempty face F of P: N_F is a nonempty cone in $\mathbb{R}^p$, $\text{tng}\, N_F = (\text{tng}\, F)^\perp$ and $\dim F = p - \dim(\text{tng}\, F)^\perp$.*
(b) *For nonempty faces F and G of P: $F \subseteq G$ if and only if $cl\, N_F \supseteq cl\, N_G$.*
(c) *The map $F \to N_F$ is one-to-one and $\{N_F : F$ is a face of $P\}$ partition $\mathbb{R}^p$.*

The *normal fan* of a polytope $P \subseteq \mathbb{R}^d$ is defined by $N(P) \equiv \{N_F : F$ is a nonempty face of $P\}$. Two polytopes are *normally equivalent* if their normal fans coincide. The following result states a known relationship between Normal and combinatorial equivalence (e.g., [6, Proposition 3, p. 338]).

Proposition 2.3. *Normal equivalence of polytopes implies combinatorial equivalence.*

It can be shown that two normally equivalent polytopes have representations as the solution sets of systems of linear inequalities, say $Ax \leq b$ and $A'x \leq b'$, with $A = A'$ and identical parametrization of corresponding faces through tightening inequalities determined by subsets of the set of rows of $A = A'$. Thus, normal equivalence assures related algebraic representation beyond common combinatorial structure.

3. Preliminaries on supermodularity

In this section, we record standard results about supermodular functions used for our development; see [5], where they are given in a more general context. We include some elementary proofs for the sake of completeness.

Henceforth, let λ be a p-set function. We recall the definition of the polytope C^λ defined by (1.2) and (1.3). For each $I \subseteq \{1, \ldots, p\}$, let F_I be the subset of C^λ obtained by tightening the inequality corresponding to I in (1.2), that is,

$$F_I \equiv \left\{ x \in C^\lambda : \sum_{j \in I} x_j = \lambda(I) \right\}; \tag{3.1}$$

of course, each such set is a face of C^λ.

A (possibly empty) sequence $I_1, I_2, \ldots, I_k$ of subsets of $\{1, \ldots, p\}$ is called a *chain* if $\emptyset \subset I_1 \subset I_2 \subset \cdots \subset I_k \subset \{1, \ldots, p\}$, in which case we refer to k as the *length* of the chain. Such a chain is usually augmented with $I_0 \equiv \emptyset$ and $I_{k+1} \equiv \{1, \ldots, p\}$. We say that chain $I'_1, I'_2, \ldots, I'_{k'}$ is a *subchain* of $I_1, I_2, \ldots, I_k$ and that $I_1, I_2, \ldots, I_k$ is a *superchain* of $I'_1, I'_2, \ldots, I'_{k'}$, if $\{I'_1, I'_2, \ldots, I'_{k'}\} \subseteq \{I_1, I_2, \ldots, I_k\}$; we say that $I'_1, I'_2, \ldots, I'_{k'}$ is a *proper subchain* of $I_1, I_2, \ldots, I_k$ and that $I_1, I_2, \ldots, I_k$ is a *proper superchain* of $I'_1, I'_2, \ldots, I'_{k'}$ when the above inclusion is strict. The maximal length of a chain is $p-1$ and every chain has a superchain of length $p-1$. We say that a chain $I_1, I_2, \ldots, I_k$ is a *representing chain* of a subset F of $\mathbb{R}^p$, if $F = \cap_{t=1}^k F_{I_t}$. In this case, F is a face of H^λ (as an intersection of F_I's).

The following theorem rearranges results in [13, Theorems 2, 3 and 5] (see Section 1 for the definition of H^λ).

Theorem 3.1. *Suppose that λ is supermodular. Then $H^\lambda = C^\lambda$ and the nonempty faces of this polytope are precisely the sets represented by chains.*

A triplet (I, K, J) of subset of $\{1, \ldots, p\}$ is called λ-*flat* if $I \subset K \subset J$ and

$$\lambda(I) + \lambda(J) = \lambda(K) + \lambda(L) \quad \text{where } L = I \cup (J \setminus K). \tag{3.2}$$

We observe that when λ is supermodular, strict supermodularity means that there exist no λ-flat triplets. The next lemma provides a necessary condition and a sufficient condition for flatness; the results and their proofs appear in [7, Lemma 3.1].

Lemma 3.2. *Suppose that λ is supermodular and I and J are subsets of $\{1, \ldots, p\}$.*

(a) *If $(I \cap J, I, I \cup J)$ is λ-flat, then $F_{I \cap J} \cap F_{I \cup J} \subseteq F_I \cap F_J$.*
(b) *If $F_I \cap F_J \neq \emptyset$, then $F_{I \cap J} \cap F_{I \cup J} = F_I \cap F_J$ and either I and J are linearly ordered by set inclusion or $(I \cap J, I, I \cup J)$ is λ-flat.*

The next two lemmas provide important tools for understanding the chain-representation of nonempty faces of permutation polytopes corresponding to supermodular set functions. The first lemma and its proof appear in [7, Lemma 3.3].

Lemma 3.3. *Suppose that λ is supermodular. Let $I_1, \ldots, I_k$ be a chain and let $I'_1, \ldots, I'_{k'}$ be nonempty proper subsets of $\{1, \ldots, p\}$ with $F' = (\cap_{t=1}^k F_{I_t}) \cap (\cap_{t=1}^{k'} F_{I'_t}) \neq \emptyset$. Then there exists a superchain of $I_1, \ldots, I_k$ which is a representing chain of F'.*

A variant of the next lemma appears in [7, Lemma 4.4] with a stronger assumption about λ and a stronger condition in (c).

Lemma 3.4. *Suppose that λ is supermodular on subsets of $\{1, \ldots, p\}$, $I_1, \ldots, I_k$ is a chain and $s \in \{1, \ldots, k\}$. Then the following are equivalent:*

(a) $\cap_{t=1}^k F_{I_t} = \cap_{t=1, t\neq s}^k F_{I_t}$.
(b) $F_{I_{s-1}} \cap F_{I_{s+1}} \subseteq F_{I_s}$, *and*
(c) (I_{s-1}, I_s, I_{s+1}) *is λ-flat.*

Proof. (b) ⇒ (a): This implication is trite.

(c) ⇒ (b): Suppose that (I_{s-1}, I_s, I_{s+1}) is λ-flat. Let $J_s \equiv I_{s-1} \cup (I_{s+1} \setminus I_s)$. As $I_s \cap J_s = I_{s-1}$ and $I_s \cup J_s = I_{s+1}$, the λ-flatness of (I_{s-1}, I_s, I_{s+1}) and part (a) of Lemma 3.2 imply that $F_{I_{s-1}} \cap F_{I_{s+1}} \subseteq F_{I_s} \cap F_{J_s} \subseteq F_{I_s}$.

(a) ⇒ (c): Suppose that (a) holds. Again, let $J_s \equiv I_{s-1} \cup (I_{s+1} \setminus I_s)$. As $I_1, \ldots, I_{s-1}, J_s, I_{s+1}, \ldots, I_k$ is a chain, $F \equiv (\cap_{t\neq 1, t\neq s}^k F_{I_t}) \cap F_{J_s}$ is a nonempty face (Theorem 3.1) which is contained in the face $F' \equiv \cap_{t=1,t\neq s}^k F_{I_t} = \cap_{t=1}^k F_{I_t}$. It follows that $\emptyset \neq F \subseteq F_{I_s} \cap F_{J_s}$. As I_s and J_s are not linearly ordered by set inclusion, $I_s \cap J_s = I_{s-1}$ and $I_s \cup J_s = I_{s+1}$, part (b) of Lemma 3.2 implies that (I_{s-1}, I_s, I_{s+1}) is λ-flat. □

4. Permutation polytopes corresponding to strictly supermodular functions

Lemma 4.1. *Suppose that λ is strictly supermodular on subsets of* $\{1, \ldots, p\}$. *Then:*

(a) *if* $I_1, \ldots, I_k$ *are distinct subsets of* $\{1, \ldots, p\}$ *with* $\cap_{t=1}^k F_{I_t} \neq \emptyset$, *then* $I_1, \ldots, I_k$ *are linearly ordered under set inclusion,*

(b) *if* $I_1, \ldots, I_k$ *are distinct subsets of* $\{1, \ldots, p\}$ *with* $\cap_{t=1}^k F_{I_t} \neq \emptyset$, *then for every* $s \in \{1, \ldots, k\}$, $\cap_{t=1,t\neq s}^k F_{I_t} \neq \cap_{t=1}^k F_{I_t}$, *and*

(c) *if* $I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}$ *are subsets of* $\{1, \ldots, p\}$ *with* $\cap_{t=1}^k F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} \neq \emptyset$, *then* $k = k'$ *and* $\{I_1, \ldots, I_k\} = \{I'_1, \ldots, I'_{k'}\}$.

Proof. (a) Let $I_1, \ldots, I_k$ be distinct subsets of $\{1, \ldots, p\}$ with $F \equiv \cap_{t=1}^k F_{I_t} \neq \emptyset$. We will show that all pairs of distinct sets in the list are linearly ordered by set inclusion. Indeed, let r and s be two distinct elements in $\{1, \ldots, k\}$. Then $F_{I_r} \cap F_{I_s} \supseteq F \neq \emptyset$. As λ is strictly supermodular, there exist no λ-flat triplets, and part (b) of Lemma 3.2 implies that I_r and I_s are linearly ordered by set inclusion.

(b) The strict supermodularity of λ assures that there are no λ-flat triplets. Consequently, the equivalence (a) ⇔ (c) in Lemma 3.4 implies (b).

(c) Suppose that $I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}$ are subsets of $\{1, \ldots, p\}$ with $F \equiv \cap_{t=1}^k F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} \neq \emptyset$. By part (a), the sequences $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are each linearly ordered by set inclusion. Hence, by possibly permuting the sets in each group we may assume that $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are chains. Consider an enumeration of the distinct sets in $\{I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}\}$, say $I''_1, \ldots, I''_{k''}$. Then $\cap_{t=1}^{k''} F_{I''_t} = F \neq \emptyset$ and, again by part (a), after possible permutation we may assume that $I''_1, \ldots, I''_{k''}$ is a chain. As $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are subchains of $I''_1, \ldots, I''_{k''}$ with $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} = F \neq \emptyset$, part (b) implies that $\{I''_1, \ldots, I''_{k''}\} = \{I_1, \ldots, I_k\} = \{I'_1, \ldots, I'_{k'}\}$. □

The next theorem and its corollary establish a one-to-one correspondence of chains and nonempty faces of H^λ where λ is any given strictly supermodular set function.

Theorem 4.2. *Suppose that λ is strictly supermodular on the subsets of* $\{1, \ldots, p\}$. *Then for every nonempty face* F *of* H^λ *there is a unique collection* $\{I_1, \ldots, I_k\}$ *of distinct subsets of* $\{1, \ldots, p\}$ *with* $F = \cap_{t=1}^k F_{I_t}$; *further, in such a representation the subsets* $I_1, \ldots, I_k$ *are linearly ordered under set inclusion, that is, with possible relabeling* $I_1, \ldots, I_k$ *is a chain.*

Proof. Nonempty faces of H^λ have representations as intersections $\cap_{t=1}^k F_{I_t}$ (parts (c) and (e) of Proposition 2.1), and by part (a) of Lemma 4.1, the sets $I_1, \ldots, I_k$ in such a representation are linearly ordered by set inclusion. The uniqueness of these representations follows from part (c) of Lemma 4.1. □

Corollary 4.3. *Suppose that λ is strictly supermodular on subsets of* $\{1, \ldots, p\}$. *Then chains are in one-to-one correspondence with the nonempty faces of* H^λ, *with chain* $I_1, \ldots, I_k$ *mapped into the face* $\cap_{t=1}^k F_{I_t}$.

Proof. Theorem 3.1 asserts that for a chain $I_1, \ldots, I_k$, $\cap_{t=1}^k F_{I_t}$ is a nonempty face of $C^\lambda = H^\lambda$ and each nonempty face has such a representation. The fact that the correspondence of chains onto faces is one-to-one follows from Theorem 4.2. □

For a subset I of $\{1, \ldots, p\}$ we let e^I be the characteristic vector of I, that is, e^I is the vector in $\mathbb{R}^p$ with $(e^I)_j = 1$ if $j \in I$ and $(e^I)_j = 0$ if $j \in \{1, \ldots, p\} \setminus I$. Also, for a chain, $I_1, \ldots, I_k$, let

$$L(I_1, \ldots, I_k) = \left\{ z \in \mathbb{R}^p : \left(e^{I_t}\right)^{\mathrm{T}} z = 0 \text{ for } t = 1, \ldots, k+1 \right\} \tag{4.1}$$

and

$$N(I_1, \ldots, I_k) \equiv \left\{ \sum_{t=1}^{k+1} \beta_t e^{I_t} : \beta_t < 0 \text{ for } t = 1, \ldots, k \text{ and } \beta_{k+1} \in \mathbb{R} \text{ (unrestricted)} \right\}. \tag{4.2}$$

Theorem 4.4. *Suppose that λ is strictly supermodular on the subsets of $\{1, \ldots, p\}$, and let F be a face of H^λ with representing chain $I_1, \ldots, I_k$. Then:*

(a) $\dim F = p - 1 - k$,
(b) $\operatorname{tng} F = L(I_1, \ldots, I_k)$, *and*
(c) $N_F = N(I_1, \ldots, I_k)$.

Proof. As $I_1, \ldots, I_k$ is a chain, the vectors $e^{I_1}, \ldots, e^{I_{k+1}}$ are obviously linearly independent. Now, the set $L(I_1, \ldots, I_k)$ is the orthogonal complement of the subspace spanned by the vectors $e^{I_1}, \ldots, e^{I_{k+1}}$; as these vectors are linearly independent, it follows that $\dim L(I_1, \ldots, I_k) = p - k - 1$ and $[L(I_1, \ldots, I_k)]^\perp$ is the linear span of $e^{I_1}, \ldots, e^{I_{k+1}}$.

For $t = 1, \ldots, k+1$ and $x \in F_{I_t}$, $(e^{I_t})x = \lambda(I_t)$. As $\operatorname{tng} F = \{\alpha(x - y) : x, y \in F \text{ and } \alpha \in \mathbb{R}\}$, for each $z \in \operatorname{tng} F = \operatorname{tng} [\cap_{t=1}^k F_{I_t}]$ and $t = 1, \ldots, k+1$, $\left(e^{I_t}\right)^T z = 0$. Thus, $\operatorname{tng} F \subseteq \cap_{t=1}^{k+1} \{z \in \mathbb{R}^p : (e^{I_t})^{\mathrm{T}} z = 0\} = L\{I_1, \ldots, I_k\}$. Thus, $\dim F = \dim(\operatorname{tng} F) \le \dim L(I_1, \ldots, I_k) = p - k - 1$.

Next, the chain $I_1, \ldots, I_k$ has a superchain of length $p - 1$ (just like any other chain). Thus, there exist subsets $J_{k+1}, \ldots, J_{p-1}$ of $\{1, \ldots, p\}$ such that $I_1, \ldots, I_k, J_{k+1}, \ldots, J_{p-1}$ are distinct and linearly ordered by set inclusion. For $s = k, k+1, \ldots, p-1$, let $F_s = F \cap (\cap_{t=k+1}^s F_{J_t})$. We then have that

$$F = F_k \supset F_{k+1} \supset \cdots \supset F_{p-1};$$

the weak inclusions are trite and the strict inclusions follow from the unique representation of faces by chains established in Corollary 4.3. It follows that

$$\dim F = \dim F_k > \dim F_{k+1} > \cdots > \dim F_{p-1}$$

(part (d) of Proposition 2.1), implying that $\dim F \ge p - 1 - k$. As we have seen that $\dim F \le p - 1 - k$, it follows that $\dim(\operatorname{tng} F) = \dim F = p - k - 1 = \dim L(I_1, \ldots, I_k)$, proving (a). Further, as $\operatorname{tng} F$ and $L(I_1, \ldots, I_k)$ are linear subspaces of $\mathbb{R}^p$ that are linearly ordered by set inclusion ($\operatorname{tng} F \subseteq L(I_1, \ldots, I_k)$) and have the same dimension, we conclude that they are equal. So, (b) has been verified.

Finally, to prove (c), let c be a vector in $N(I_1, \ldots, I_k)$ and we will show that $c \in N_F$. As $c \in N(I_1, \ldots, I_k)$, c has a representation $\sum_{t=1}^{k+1} \beta_t e^{I_t}$ with $\beta_1, \ldots, \beta_k$ negative and β_{k+1} unrestricted. For $x \in H^\lambda$, $c^T x = \sum_{t=1}^{k+1} \beta_t (e^{I_t})^{\mathrm{T}} x = \sum_{t=1}^{k+1} \beta_t (\sum_{j \in I_t} x_j) \le \sum_{t=1}^{k+1} \beta_t \lambda(I_t)$ and equality holds if and only if $\sum_{j \in I_t} x_j = \lambda(I_t)$ for $t = 1, \ldots, k$; so $\arg\max_{x \in H^\lambda} c^T x = \cap_{t=1}^k F_{I_t} = F$, that is, $c \in N_F$. To see the reverse inclusion, let $c \in N_F$ and we will show that $c \in N(I_1, \ldots, I_k)$. As N_F is a cone, $N_F \subseteq \operatorname{tng} N_F$ (for $u \in N_F$, $u = 2u - u \in \operatorname{tng} N_F$). It follows from Proposition 2.2 (part (e)), the established part (b) of the current theorem, and the first paragraph of the current proof that $N_F \subseteq \operatorname{tng} N_F = (\operatorname{tng} F)^\perp = [L(I_1, \ldots, I_k)]^\perp = \{\sum_{t=1}^{k+1} \beta_t e^{I_t} : \beta_t \in \mathbb{R} \text{ for } t = 1, \ldots, k+1\}$. So, c has a representation $c = \sum_{t=1}^{k+1} \beta_t e^{I_t}$ and it remains to show that $\beta_1, \ldots, \beta_k$ are negative. Fix $s \in \{1, \ldots, k\}$ and let $F^s \equiv (\cap_{t=1, t \ne s}^k F_{I_t})$. The unique chain-representation of faces assures that $F = \cap_{t=1}^k F_{I_t} \subset F^s$, so there is a vector x^s in $F^s \setminus F$. As $c \in N_F$, $F = \arg\max_{x \in H^\lambda} c^T x$; further, for all $x \in F = \cap_{t=1}^k F_{I_t}$, $c^T x = \sum_{t=1}^{k+1} \beta_t (e^{I_t})^T x = \sum_{t=1}^{k+1} \beta_t (\sum_{j \in I_t} x_j) = \sum_{t=1}^{k+1} \beta_t \lambda(I_t)$. As $x^s \notin F$ and $x^s \in F^s = \cap_{t=1, t \ne s}^k F_{I_t}$, we have $\sum_{t=1}^{k+1} \beta_t \lambda(I_t) > c^{\mathrm{T}} x^s = \sum_{t=1, t \ne s}^{k+1} \beta_t (\sum_{j \in I_t} x_j^s) + \beta_s (\sum_{j \in I_t} x_j^s) = \sum_{t=1, t \ne s}^{k+1} \beta_t \lambda(I_t) + \beta_s (\sum_{j \in I_t} x_j^s)$, and therefore $\beta_s \lambda(I_s) > \beta_s (\sum_{j \in I_t} x_j^s)$. As $\sum_{j \in I_t} x_j^s \ge \lambda(I_s)$ it follows that $\beta_s < 0$. $\square$

Table 1
The λ_σ's for Example 2

Determining Permutation σ	λ_σ
({1}, {2}, {3})	$(1, 3, 6)^{\mathrm{T}}$
({1}, {3}, {2})	$(1, 4, 5)^{\mathrm{T}}$
({2}, {1}, {3})	$(2, 2, 6)^{\mathrm{T}}$
({2}, {3}, {1})	$(2, 2, 6)^{\mathrm{T}}$
({3}, {1}, {2})	$(2, 4, 4)^{\mathrm{T}}$
({3}, {2}, {1})	$(2, 4, 4)^{\mathrm{T}}$

Theorem 4.4 implies that the vertices of a partition polytope corresponding to a strictly supermodular set function are represented by the $p!$ chains of length p (corresponding to the permutations of $\{1, \ldots, p\}$) and the facets are represented by chains of length 1 (corresponding to the $2^p - 2$ nontrivial subsets of $\{1, \ldots, p\}$; see [13,2].

Recall the definition of permutahedra in Example 1.

Corollary 4.5. *All permutation polytopes corresponding to strictly supermodular functions are both normally and combinatorially equivalent; in particular, each is both normally and combinatorially equivalent to the standard permutahedron.*

Proof. Let λ be strictly supermodular on the subsets of $\{1, \ldots, p\}$ and let F be a face of H^λ. Recall the notation $N(P)$ for the normal fan of a polytope P. By Corollary 4.3 and part (c) of Theorem 4.4, F has a unique representing chain, say $I_1, \ldots, I_k$, and $N(I_1, \ldots, I_k) = N_F$. We conclude that $N(H^\lambda) \subseteq N \equiv \{N(I_1, \ldots, I_k) : I_1, \ldots, I_k$ is a chain on $\{1, \ldots, p\}\}$. To see that this inclusion holds as equality observe that if $I_1, \ldots, I_k$ is a chain on $\{1, \ldots, p\}$, the above arguments show that $N(I_1, \ldots, I_k) = N_F$ for $F = \cap_{t=1}^{k} F_{I_t}$. Thus, with λ strictly supermodular, $N(H^\lambda) = N$ is independent of λ, that is, all such permutation polytopes H^λ are normally equivalent. As normal equivalence of polytopes implies combinatorial equivalence (Proposition 2.3), we also conclude that all of these polytopes are combinatorially equivalent. The last conclusion of the corollary now follows from the fact that the standard permutahedron is a permutation polytope corresponding to a strictly supermodular function (see Example 1). □

In view of Corollary 4.5, one may conjecture that generalized permutahedra (see Example 1) generate all permutation polytopes, at least in the sense that each permutation polytope is combinatorially equivalent to some generalized permutahedron. But, the next example demonstrates that this is not the case.

Example 2. Suppose that $p = 3$ and λ is defined on subsets of $\{1, 2, 3\}$ with $\lambda(\{1\}) = 1$, $\lambda(\{2\}) = 2$, $\lambda(\{3\}) = 4$, $\lambda(\{1, 2\}) = 4$, $\lambda(\{1, 3\}) = 6$, $\lambda(\{2, 3\}) = 8$, $\lambda(\{1, 2, 3\}) = 10$. It is easy to verify that λ is supermodular. It is noted that λ is not strictly supermodular as $\lambda(\{1, 3\}) + \lambda(\{2, 3\}) = 14 = \lambda(\{3\}) + \lambda(\{1, 2, 3\})$.

Table 1 lists the λ_σ's with σ ranging over the six permutations over $\{1, 2, 3\}$; it shows that there are precisely four distinct vectors. As neither is in the convex hull of the other three, we have from part (a) of Proposition 2.2 that these are the vertices of H^λ. Thus, H^λ has exactly four distinct vertices.

A generalized permutahedron in $\mathbb{R}^3$ is determined by 3 parameters, say η_1, η_2 and η_3; the generalized permutahedron is then the convex hull of the six coordinate-permutations of the vector (η_1, η_2, η_3). It is easy to verify that each coordinate-permutation of the vector (η_1, η_2, η_3) is a vertex and the number of such vectors is six when the η_j's are distinct, three if two η_j's coincide and are different from the third, and one if all η_j's coincide. As H^λ has four vertices, in either case the generalized permutahedron is not combinatorially equivalent to H^λ, implying that it is neither normally equivalent to H^λ. □

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